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À Claire et à mes parents.

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Comme on lit dans le ciel les étoiles lointaines
Que serais-je sans toi que ce balbutiement
Que serais-je sans toi qu'un cœur au bois dormant...

Author's bibliography

The present manuscript is devoted to subelliptic PDEs. To keep the presentation as unified as possible, I decided not to present the works written during my PhD which are not directly related to this subject.

Articles and preprints presented in the manuscript

- [Let20b], *Subelliptic wave equations are never observable*, pending revisions for publication in *Analysis and Partial Differential Equations* (Hal link).
- [LS20], *Observability of Baouendi-Grushin-type equations through resolvent estimates*, with Chenmin Sun, accepted in *Journal de l'Institut Mathématique de Jussieu* (Hal link).
- [FL21], *Observability and controllability for the Schrödinger equation on quotients of groups of Heisenberg type*, with Clotilde Fermanian Kammerer, published in *Journal de l'École Polytechnique - Mathématiques* (Tome 8, 2021) (Hal link).
- [Let20a], *Quantum limits of sub-Laplacians via joint spectral calculus*, (Hal link).
- [Let21b], *Propagation of singularities for subelliptic wave equations*, (Hal link).
- [CL21], *Propagation of well-prepared states along Martinet singular geodesics*, with Yves Colin de Verdière, accepted in *Journal of Spectral Theory* (Hal link).

The paper [Let21a], to appear in *Actes du séminaire de théorie spectrale et géométrie* (Hal link) surveys the first three above articles.

Articles and preprints not presented

- [Let19], *From internal to pointwise control for the 1D heat equation and minimal control time*, published in *Systems and Control Letters* (2019) (Hal link).
- [Let20c], *Catching all geodesics of a manifold with moving balls and application to controllability of the wave equation*, (Hal link).
- [LL], *Uniform controllability of waves from thin domains*, with Matthieu Léautaud (work in progress).

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Chapter 1

Introduction

"A thing of beauty is a joy for ever."
John Keats, *Endymion.*

In this first chapter, we present the main results of this manuscript, just focusing on their motivations and statements, and not on their proofs. Along the way, we review basic facts concerning subelliptic PDEs and sub-Riemannian geometry.

The original results of this thesis are presented in boxes, in order to distinguish them from previously known results.

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1.1 Generalities

1.1.1 Motivations: Linear PDEs and sub-Riemannian geometry

A Partial Differential Equation (PDE) is an equation which relates the partial derivatives of a function of several variables. As for ordinary differential equations, their unknowns are functions. Partial Differential Equations are ubiquitous in science: they are one of the mathematical grounds on which the modern description of many physical phenomena relies. For instance, sound, diffusion, heat, waves, elasticity, electrostatics or else electrodynamics can be described with PDEs. Well-known PDEs include Einstein equations in general relativity, the Euler and Navier-Stokes equations in fluid mechanics, the Boltzmann equation in kinetic theory, the Yang-Mills equations in particle physics...

One of the beauties of PDEs is that they interact since their birth in the 18^{th} century with many other branches of mathematics: with differential geometry for example through minimal surfaces and the Atiyah-Singer theorem, with algebraic topology through the pioneering work of Leray on sheaves, with numerical calculus through numerical schemes, with probabilities through the Feynman-Kacs formula or stochastic PDEs, ...

In the second half of the 20^{th} century, among other developments, a certain class of PDEs has been intensively studied: the so-called "subelliptic PDEs". As most PDEs, their roots are to be found in physics: the Russian mathematician Andrei Kolmogorov, in his study of the motion of colliding particles [Kol34], was probably the first to notice that the equation he wrote down was indeed hypoelliptic[1]. Then, Lars Hörmander, followed by dozens of other mathematicians, undertook their systematic study; let us only mention the works [Hor67] and [RS76] as illustrations.

In the years 1980-1990, while the field of subelliptic PDEs was progressively becoming less active, another more geometric branch of mathematics has undergone an important development: sub-Riemannian geometry. Starting from the study of the Heisenberg group, it focused on the geometry of balls, shortest paths and isoperimetric sets in these particular geometries where not all directions play the same role. Its interest relied also on its links with control theory, a very active field of "applied" mathematics which serves, among other, to park cars or to design motions of rockets.

In the same way as Riemannian geometry is the natural geometric framework for elliptic PDEs, sub-Riemannian geometry became the natural geometric framework for subelliptic PDEs. But sub-Riemannian geometers focused their attention mostly on the heat equation, leaving aside other natural linear PDEs such as subelliptic wave equations and Schrödinger equations. Indeed, these two linear equations give rise to particularly strong subelliptic effects whose analysis required new approaches.

The present thesis aims at partly bridging this gap: it uses the new tools brought by sub-Riemannian geometry to shed a different light on linear[2] subelliptic wave and Schrödinger equations. Our initial target, finally partially achieved, was to understand the exact role played by "abnormal minimizers", discovered by Montgomery in 1991 (see [Mon94]), in their propagation. But along the way, since this initial question was difficult, we were led to other problems, in the control theory of subelliptic linear PDEs and concerning eigenfunctions of sub-Laplacians, whose solutions involved tools coming not only from sub-Riemannian geometry, but also from non-commutative harmonic analysis and semiclassical analysis.

[1]We will come back later to the relation between hypoellipticity and subellipticity.

[2]Although people working in the field of PDEs are particularly interested in non-linear effects, we focus in this manuscript on linear PDEs since in the subelliptic world, even linear PDEs are still imperfectly understood.

1.1.2 Organization of the manuscript

The present manuscript is organized as follows.

Chapter 1 aims at presenting the main results of this manuscript, just focusing on their motivations and statements, and not on their proofs. It is partly inspired by the survey [Let21a].

Then, the first part of our manuscript, composed of Chapters 2, 3 and 4, addresses problems in control/observability of subelliptic PDEs. They illustrate the slowdown of energy propagation of solutions of subelliptic PDEs in directions needing brackets to be generated.

- In Chapter 2, we prove that subelliptic wave equations are never observable. The proof uses tools coming from sub-Riemannian geometry, namely the privileged coordinates and the nilpotentization procedure of [RS76]. This Chapter essentially follows the article [Let20b].
- In Chapter 3, we establish a resolvent estimate with the tools of semiclassical analysis. This resolvent estimate implies controllability results for subelliptic Schrödinger, heat and damped wave equations. This Chapter essentially follows the article [LS20].
- Chapter 4 lies at the intersection of two usually distinct fields: semiclassical analysis and non-commutative harmonic analysis. Using ideas steming from [FF21], we construct semiclassical measures adapted to the non-commutative framework provided by quotients of Heisenberg-type groups, and we use them to prove a controllability results. This Chapter essentially follows the preprint [FL21].

In the second part, we focus on the propagation of singularities in subelliptic wave equations.

- In Chapter 5, we revisit the paper [Mel86] by R. Melrose, providing a full proof of the main theorem on propagation of singularities for subelliptic wave equations, and linking this result with sub-Riemannian geometry. This theorem asserts that singularities of subelliptic wave equations only propagate along null-bicharacteristics and abnormal extremal lifts of singular curve. We also derive new consequences of Melrose's result. This Chapter essentially follows the preprint [Let21b].
- In Chapter 6, in the context of the subelliptic wave equation with Martinet sub-Laplacian, we construct explicit examples of solutions whose singularities propagate along abnormal extremal lifts of singular curves. We find that singularities can propagate at any speed between 0 and 1, which is in strong contrast with the usual propagation of singularities at speed 1 for wave equations with elliptic Laplacian. This Chapter essentially follows the preprint [CL21].

A third part is devoted to the study of high-frequency eigenfunctions of some sub-Laplacians.

- In Chapter 7, we describe the behaviour of high-frequency eigenfunctions of some sub-Laplacians, using the joint spectral theory of various operators. In some particular cases, we are able to describe all "Quantum Limits" of the sub-Laplacians. This Chapter essentially follows the preprint [Let20a].

Finally, we gathered in Chapter 8 open questions which seem of particular interest.

The rest of the introduction is organized as follows: in Section 1.2, we introduce sub-Laplacians and define the notions of hypoellipticity and subellipticity; in Section 1.3 we present our main results related to control/observability of subelliptic PDEs; in Section 1.4 we explain our results about propagation of singularities of subelliptic wave equations; and finally in Section 1.5 we describe our results concerning eigenfunctions of sub-Laplacians.

1.2 Subelliptic Partial Differential Equations

1.2.1 Sub-Laplacians and sub-Riemannian geometry

Sub-Laplacians are a natural generalization of the usual Laplacian in the Euclidean space, and of the Laplace-Beltrami operator in Riemannian manifolds. They are also called "Hörmander sums of squares" since they were studied a lot by Hörmander (see [Hor67], [Hor07c, Chapter XXVII]) and they take the simple form of a sum of squares of vector fields.

Let $n \in \mathbb{N}^*$ and let M be a smooth connected compact manifold of dimension n, with or without boundary. Let μ be a smooth volume on M. We consider $m \geqslant 1$ smooth vector fields $X_1, \ldots, X_m$ on M which are not necessarily independent, and we assume that the following Hörmander condition holds (see [Hor67]):

$$\begin{gathered}\text{The vector fields } X_1, \ldots, X_m \text{ and their iterated brackets } [X_i, X_j], [X_i, [X_j, X_k]], \text{ etc.} \\ \text{span the tangent space } T_xM \text{ at every point } x \in M.\end{gathered} \tag{1.1}$$

We consider the sub-Laplacian Δ defined by

$$\Delta = -\sum_{i=1}^{m} X_i^* X_i = \sum_{i=1}^{m} X_i^2 + \operatorname{div}_\mu(X_i) X_i \tag{1.2}$$

where the star designates the transpose in $L^2(M, \mu)$ and the divergence with respect to μ is defined by $L_X \mu = (\operatorname{div}_\mu X)\mu$, where L_X stands for the Lie derivative. Up to a lower order term, a sub-Laplacian is thus a "sum of squares". The domain $D(\Delta)$ is the completion in $L^2(M, \mu)$ of the set of all $u \in C_c^\infty(M)$ for the norm $\|(\mathrm{Id} - \Delta)u\|_{L^2}$.

There is a natural geometry associated to such operators, called sub-Riemannian geometry, which is an extension of the usual Riemannian geometry. We shall describe its foundations, and refer the reader to the books [Mon02] and [ABB19] for comprehensive treatments of sub-Riemannian geometry.

We set

$$\mathcal{D} = \operatorname{Span}(X_1, \ldots, X_m) \subset TM$$

which is called the *distribution* associated to the vector fields $X_1, \ldots, X_m$. For $x \in M$, we denote by $\mathcal{D}_x$ the distribution $\mathcal{D}$ taken at point x. Note that $\mathcal{D}$ does not necessarily have constant rank. When $\mathcal{D} = TM$, the operator Δ is elliptic.

We also introduce the *metric* g on $\mathcal{D}$ defined at any $x \in M$ by

$$g_x(v, v) = \inf \left\{ \sum_{i=1}^{m} u_i^2 \mid v = \sum_{i=1}^{m} u_i X_i(x) \right\}.$$

This is a Riemannian metric on $\mathcal{D}$. We call $(M, \mathcal{D}, g)$ a *sub-Riemannian structure.*

In the general case where $\mathcal{D} \subsetneq TM$, the set $TM \setminus \mathcal{D}$ can be understood as the directions where the metric g takes the value $+\infty$. A well-known theorem, due to Chow and Rashevskii, asserts that any two points can be joined by a path, i.e., a continuous function $\gamma : [0,1] \to M$ with derivative $\dot\gamma(t)$ contained in $\mathcal{D}_{\gamma(t)}$ for almost any $t \in [0,1]$. In other words, the sub-Riemannian distance

$$d_g(x_0, x_1) = \inf \left\{ \int_0^1 \sqrt{g_{\gamma(t)}(\dot\gamma(t), \dot\gamma(t))}dt \mid \gamma(0) = x_0, \gamma(1) = x_1,\ \dot\gamma(t) \in \mathcal{D}_{\gamma(t)} \text{ a.s. for } t \in [0,1] \right\}$$

is finite for any $x_0, x_1 \in M$.

When moving in a sub-Riemannian structure, $\mathcal{D}$ should be understood as the "set of allowed directions for the motion", and, although it is not possible to move directly in directions of $TM \setminus \mathcal{D}$, Chow-Rashevskii's theorem asserts that any two points can be joined by a path. This is due to "indirect motions", that is, paths which describe spirals turning around a fixed forbidden direction of $TM \setminus \mathcal{D}$ and thus advancing in this direction (although indirectly).

Definition 1.1. *The step k of a sub-Riemannian structure $(M, \mathcal{D}, g)$ is the least integer $k \in \mathbb{N}$ such that $\mathcal{D}^k = TM$, where, for $j \in \mathbb{N}^*$, $\mathcal{D}^j$ is defined through the recursive relation $\mathcal{D}^j = \mathcal{D}^{j-1} + [\mathcal{D}, \mathcal{D}^{j-1}]$ and $\mathcal{D}^1 = \mathcal{D}$.*

Remark 1.2. More generally, the step k_x can be defined at any point $x \in M$, just by considering $\mathcal{D}^j_x$ instead of $\mathcal{D}^j$ in the above definition.

Examples

We now give a few examples of sub-Laplacians which we shall study in the sequel.

Example 1.3. On $M = \mathbb{R}_x \times \mathbb{R}_y$, we set $\Delta_G = \partial_x^2 + x^2\partial_y^2$. This sub-Laplacian is the so-called Baouendi-Grushin operator, sometimes unproperly called simply Grushin operator (see [Gar17, Section 11]). In this case, $\mathcal{D} = \mathrm{Span}(\partial_x, x\partial_y)$ and $\mathcal{D}^2 = \mathrm{Span}(\partial_x, \partial_y) = TM$. In particular, $\mathcal{D} = TM$ outside the line $\{x = 0\}$. Also, μ is the Lebesgue measure. The structure $(M, \mathcal{D}, g)$ has step 2 on the line $\{x = 0\}$ and step 1 outside this line. Since this sub-Riemannian structure is "Riemannian" outside this line, the Baouendi-Grushin operator is sometimes called "almost-Riemannian".

Example 1.4. Given $d \in \mathbb{N}^*$, one can also define a sub-Laplacian arising from the Heisenberg group $\mathbf{H}_d$ of dimension $2d+1$. Recall that the Heisenberg group $\mathbf{H}_d$ is $\mathbb{R}^{2d+1}$ endowed with the group law $(x, y, z)\cdot(x', y', z') := (x+x', y+y', z+z'+\frac{1}{2}\sum_{j=1}^d (x_j y_j' - x_j' y_j))$, where $x, y, x', y' \in \mathbb{R}^d$ and $z, z' \in \mathbb{R}$. Let

$$X_j = \partial_{x_j} - \frac{y_j}{2}\partial_z, \quad Y_j = \partial_{y_j} + \frac{x_j}{2}\partial_z, \qquad \text{for } j = 1, \dots, d.$$

We define the sub-Lapacian

$$\Delta_{\mathbf{H}_d} = \sum_{j=1}^d X_j^2 + Y_j^2.$$

Since $[X_j, Y_j] = \partial_z$ for any j, this sub-Laplacian is naturally associated with a step 2 sub-Riemannian structure.

Example 1.5. Contact sub-Laplacians arise from a particular type of sub-Riemannian structures of step 2. We assume that the vector fields $X_1, \dots, X_m$ span a distribution $\mathcal{D}$ which is a contact distribution over M, i.e., M has odd dimension $n = 2m+1$ and there exists a 1-form α on M with $\mathcal{D} = \mathrm{Ker}(\alpha)$ and $\alpha \wedge (d\alpha)^m \neq 0$ at any point of M. Then, for any smooth volume μ, the sub-Laplacian Δ is called a contact sub-Laplacian. A typical example is given by the Heisenberg sub-Laplacian $\Delta_{\mathbf{H}^d}$ defined above.

Example 1.6. Magnetic Laplacians are also sub-Laplacians. An example of magnetic Laplacian is the following: in $\mathbb{R}^3$ with coordinates x, y, z, we consider the two vector fields $X_1 = \partial_x - A_x(x, y)\partial_z$ and $X_2 = \partial_y - A_y(x, y)\partial_z$ where A_x, A_y are functions which do not depend on z. The magnetic Laplacian is then $\Delta = X_1^2 + X_2^2$. The 1-form $A = A_x dx + A_y dy$ is called the connection form, and the 2-form $B = dA$ is called the magnetic field. The modulus $|b|$ of the function b defined by the relation $B = b\, dx \wedge dy$ is called the intensity of the magnetic field.

Example 1.7. On $M = \mathbb{R}^3$, we set $\Delta = \partial_x^2 + (\partial_y + x^2\partial_z)^2$. This sub-Laplacian is called the "Martinet sub-Laplacian". The associated distribution $\mathcal{D} = \mathrm{Span}(\partial_x, \partial_y + x^2\partial_z)$ is of step 3 since $\mathcal{D}^2 = \mathrm{Span}(\partial_x, \partial_y, x\partial_z)$ and $\mathcal{D}^3 = TM$. The Martinet sub-Laplacian is a particular case of magnetic Laplacian with magnetic field $B = -2xdx \wedge dy$. The line $\{x = 0\}$ where B vanishes is an example of "singular geodesic" (also called "abnormal minimizer", see Definition 1.24), and the Martinet distribution is one the simplest to exhibit such geodesics.

1.2.2 Hypoellipticity and subellipticity

Two notions are often used to qualify the smoothing properties of sub-Laplacians: hypoellipticity and subellipticity. Here, we briefly recall their definitions and explain why they are not exactly equivalent.

Definition 1.8. *A (pseudo-)differential operator A with C^∞ coefficients in M is* hypoelliptic *near $x \in M$ if for all $u \in \mathcal{D}'(M)$, if $Au \in C^\infty$ near x, then $u \in C^\infty$ near x.*

Hypoellipticity appeared naturally in the work of Kolmogorov [Kol34] on the motion of colliding particles when he wrote down the equation

$$\partial_t u - \mathcal{L}u = f \qquad \text{where} \qquad \mathcal{L} = x\partial_y + \partial_x^2.$$

Indeed, the operator $\mathcal{L}$ is hypoelliptic.

Definition 1.9. *A formally selfadjoint (pseudo-)differential operator $A : C^\infty(M) \to C^\infty(M)$ of order 2 is said to be* subelliptic *if there exist $s, C > 0$ such that*

$$\|u\|_{H^s(M)}^2 \leqslant C((Au, u)_{L^2(M)} + \|u\|_{L^2(M)}^2) \tag{1.3}$$

for any $u \in C^\infty(M)$.

Since M is compact, the subellipticity of the selfadjoint operator A implies that its resolvent is compact, and as a consequence, its spectrum is discrete.

Using (1.1), Hörmander was able to prove that any sub-Laplacian Δ is hypoelliptic (see [Hor67] and [HN05, Chapter 2]). His proof relies on the fact that Δ is subelliptic; indeed, the optimal s in (1.3) is $1/k$, where k is the step of the associated sub-Riemannian structure, as proved by Rotschild and Stein [RS76, Theorem 17 and estimate (17.20)].

Conversely, note that an hypoelliptic "sum of squares" (i.e., an operator of the form (1.2) which is hypoelliptic) does not necessarily satisfy the Lie bracket assumption (1.1): given a smooth function $a : \mathbb{R} \to \mathbb{R}$ vanishing at infinite order at 0 but with $a(s) > 0$ for $s \neq 0$, the sub-Laplacian $\Delta = \partial_{x_1}^2 + a(x_1)^2\partial_{x_2}^2$ on $\mathbb{R}^2_{x_1x_2}$ is hypoelliptic although (1.1) fails (see [Fed71] and [Mor78]).

Let us finally mention that some operators A satisfy the property that if Au is real-analytic, then u is real-analytic: they are called *analytic hypoelliptic*. The so-called Trèves conjecture describes a possible link between analytic hypoellipticity of an operator and the absence of singular geodesics (see [Tre99] for the conjecture and [ABM18] for more recent results).

1.2.3 The characteristic cone Σ

In all our results, a central role is played by the characteristic cone Σ, which we now define. Let us consider a general sub-Laplacian Δ given by (1.2). We set

$$g^* = \sigma_P(-\Delta) \in C^\infty(T^*M) \tag{1.4}$$

where σ_P denotes the principal symbol of a pseudodifferential operator (see Appendix A). This is the Hamiltonian naturally associated to the non-holonomic system defined by $X_1, \ldots, X_m$. We denote by

$$\Sigma = (g^*)^{-1}(0) = \mathcal{D}^\perp \subset T^*M \tag{1.5}$$

the characteristic cone (where $\perp$ is in the sense of duality).

The cotangent bundle T^*M is then composed of two regions:

- $T^*M \setminus \Sigma$ is the "elliptic part", where $g^* \neq 0$. In some sense, the sub-Laplacian acts as an elliptic operator in this region of the phase-space;
- Σ, the characteristic cone, is the place where "truly subelliptic" phenomena show up.

It is a tautology to say that the existence of the characteristic cone Σ is responsible for all properties which differ between elliptic operators and "truly subelliptic" operators (for which $\Sigma \neq \{0\}$).

1.3 Main results on control of subelliptic PDEs

The first series of result we shall present in this manuscript concerns the control of subelliptic PDEs.

The problem of (exact) controllability of PDEs, which has been intensively studied in the past decades, is the following: given a manifold M, a subset $\omega \subset M$, a time $T > 0$ and an operator A acting on functions on M, the study of exact controllability consists in determining whether, for any initial state u_0 and any final state u_1, there exists f such that the solution of

$$\partial_t u = Au + \mathbf{1}_\omega f, \qquad u_{|t=0} = u_0 \tag{1.6}$$

in M is equal to u_1 at time T. Here, $\mathbf{1}_\omega$ is the characteristic function of ω. In other words, exact controllability holds if it is possible, starting from any initial state, to reach any final state just acting on ω during a time T. The general answer depends on the time T, the control set ω, the operator A, and the functional spaces in which u_0, u_1 and f live. This problem is relevant in many physical situations: typical examples are the control of the temperature of a room by a heater, or the acoustic insulation of a room just by acting on a small part of it.

By duality (Hilbert Uniqueness Method, see [Lio88]), the exact controllability property is equivalent to some inequality of the form

$$\exists C_{T,\omega} > 0, \ \forall u_0, \qquad \|u_0\|^2 \leqslant C_{T,\omega} \int_0^T \|\mathbf{1}_\omega u(t)\|^2 dt, \tag{1.7}$$

where u is the solution of the adjoint equation $(\partial_t + A^*)u = 0$ with initial datum u_0 (here again, one should specify functional spaces). This is called an observability inequality. In other words, controllability holds if and only if any solution of $(\partial_t + A^*)u = 0$ can be detected from ω, in a "quantitative way" which is measured by the constant $C_{T,\omega}$. In the sequel, we focus our attention on equations of wave-type, Schrödinger-type or heat-type:

$$(\partial_{tt}^2 - \mathcal{L})u = 0 \qquad \text{(Wave-type)}, \tag{1.8}$$

$$(i\partial_t - \mathcal{L})u = 0 \qquad \text{(Schrödinger-type)}, \tag{1.9}$$

$$(\partial_t - \mathcal{L})u = 0 \qquad \text{(Heat-type)} \tag{1.10}$$

for various time-independent operators $\mathcal{L}$ on M.[3] By duality, all the observability results presented here imply exact controllability results as explained above, but we won't state them for the sake of simplicity.

1.3.1 Observability of classical PDEs: known results

Let us present a first series of results, dating back to the 1990's, which concern the observability problem in case M is a compact Riemannian manifold with a metric g and with boundary $\partial M \neq \emptyset$, $\mathcal{L} = \Delta_g$ is the Laplace-Beltrami operator on (M, g) and the equation is one of the three equations (1.8), (1.9) or (1.10), with Dirichlet boundary conditions $u|_{\partial M} = 0$. We deal with these three problems in this order, following the chronology of the results.

Throughout this section, (M, g) is a fixed manifold with boundary $\partial M \neq \emptyset$ and $\mathcal{L} = \Delta_g$. In this section, the notation dx stands for the associated Riemannian volume $dx = d\mathrm{vol}_g(x)$.

Remark 1.10. Because of the physical nature of the problems studied in control/observability theory, most equations are set in compact manifolds, and this introduction is no exception to the rule. Together with the hypoellipticity, the compactness of the underlying manifold implies that all sub-Laplacians have a compact resolvent, and thus a discrete spectrum.

Observability of the Riemannian wave equation

Let us start with the wave equation (1.8) with initial data $(u_{t=0}, \partial_t u_{|t=0}) = (u_0, u_1) \in H^1(M) \times L^2(M)$ and Dirichlet boundary conditions. The energy of a solution, which is conserved along the flow, is

$$E(u(t)) = \int_M (|\nabla_g u(t,x)|^2 + |\partial_t u(t,x)|^2) dx$$

which is in particular equal to the initial energy $\|\nabla u_0\|^2_{L^2(M)} + \|u_1\|^2_{L^2(M)}$. Let $T > 0$ and ω be a measurable subset. The observability inequality reads as follows:

$$E(u(0)) \leqslant C \int_0^T \int_\omega |\partial_t u(t,x)|^2 dx dt. \tag{1.11}$$

Note that the left hand-side of (1.11) is the initial energy, and not the final energy[4]

We set $P = \partial_{tt}^2 - \Delta_g$ (which is a second-order pseudo-differential operator), whose principal symbol is

$$p_2(t, \tau, x, \xi) = -\tau^2 + g^*(x, \xi)$$

with τ the dual variable of t and g^* the principal symbol of $-\Delta_g$. In $T^*(\mathbb{R} \times M)$, the Hamiltonian vector field H_{p_2} associated with p_2 is given by $H_{p_2} f = \{p_2, f\}$ (see Appendix A.1). Since $H_{p_2} p_2 = 0$, we get that p_2 is constant along the integral curves of H_{p_2}. Thus, the characteristic set $\mathcal{C}(p_2) = \{p_2 = 0\}$ is preserved under the flow of H_{p_2}. Null-bicharacteristics are then defined as the maximal integral curves of H_{p_2} which live in $\mathcal{C}(p_2)$. In other words, the null-bicharacteristics

[3] The wave equation involves a ∂_{tt}^2 term, and thus does not enter, strictly speaking, the framework given by equation (1.6). However, it is possible to give a common framework for all three equations, at the cost of being a bit more abstract. See [Cor07, Section 2.3] for a general introduction.

[4] The HUM method tells us that the observability inequality is (1.7), which makes appear the final energy since it concerns the adjoint problem, but we can then use the conservation of energy to obtain that the observability inequality is equivalent to (1.11).

are the maximal solutions of

$$\begin{cases} \dot{t}(s) = -2\tau(s)\,, \\ \dot{x}(s) = \nabla_\xi g^*(x(s), \xi(s))\,, \\ \dot{\tau}(s) = 0\,, \\ \dot{\xi}(s) = -\nabla_x g^*(x(s), \xi(s))\,, \\ \tau^2(0) = g^*(x(0), \xi(0)). \end{cases} \tag{1.12}$$

It is well-known that the projection $x(s)$ of a bicharacteristic ray $(x(s), \xi(s))$ traveled at speed 1 is a geodesic in M, i.e., a curve which realizes the minimal distance between any two of its points which are close enough.

Let us also mention the fact that at the boundary of M, the above definition of null-bicharacteristics has to be completed (yielding the so-called Melrose-Sjöstrand flow): due to trajectories which "graze" along the boundary, one cannot always define the null-bicharacteristics which touch the boundary by reflexion, and we refer the reader to [MS78b] and [LLTT17] for more on this subject. In these papers, a notion of "generalized bicharacteristics" is defined, which explains how to define bicharacteristics at the boundary. For us, this will only be useful to give a precise statement for Theorem 1.12.

Definition 1.11. *Let $T > 0$ and $\omega \subset M$ be a measurable subset. We say that the Geometric Control Condition holds in time T in ω, and we write $(GCC)_{\omega,T}$, if for any projection γ of a bicharacteristic ray traveled at speed* 1*, there exists $t \in (0, T)$ such that $\gamma(t) \in \omega$.*

The following result states that the observability of (1.8) is (more or less) equivalent to the geometric condition $(GCC)_{\omega,T}$. It illustrates the finite speed of propagation for waves.

Theorem 1.12 ([BLR92], [BG97], [HPT19])**.** *Assume that $\omega \neq \emptyset$ is open and that $(GCC)_{\omega,T}$ holds. Assume also that no generalized bicharacteristic has a contact of infinite order with $(0, T) \times \partial M$. Then* (1.11) *holds, i.e., the wave equation* (1.8) *is observable in time T on ω. Conversely, if the wave equation* (1.8) *is observable in time T, then $(GCC)_{T,\overline{\omega}}$ holds, where $\overline{\omega}$ denotes the closure of ω.*

Note that the second statement in the last theorem is not the exact converse of the first one, since it involves the closure $\overline{\omega}$ and not simply ω. This is due to the phenomenon of grazing rays: if there exists a ray γ which does not enter ω but which touches the boundary $\partial\omega$, so that the geometric control condition is not satisfied, it can however happen (notably if the flow is "stable" close from the ray) that observability holds, see [Leb92a, Section VI.B] for an example.

Considering solutions of (1.8) of the form $e^{it\sqrt{\lambda}}\varphi$ where φ is an eigenfunction of $-\Delta_g$ corresponding to the eigenvalue λ, the following result follows from Theorem 1.12:

Corollary 1.13. *Assume that $\omega \neq \emptyset$ is open and that there exists $T > 0$ such that $(GCC)_{\omega,T}$ holds. Then, for any eigenfunction φ of $-\Delta_g$, there holds*

$$\int_\omega |\varphi(x)|^2 dx \geqslant C \int_M |\varphi(x)|^2 dx.$$

In particular, supp$(\varphi) = M$.

All the observability inequalities stated in this introduction yield similar lower bounds, but we will not state them thereafter.

Remark 1.14 (Gaussian beams)**.** The fact that $(GCC)_{\overline{\omega},T}$ is a necessary condition for observability can be understood as follows. If $(GCC)_{\overline{\omega},T}$ does not hold, then let $\gamma : [0,T] \to M$ be a geodesic which does not enter $\overline{\omega}$. By compactness, there exists $\varepsilon > 0$ such that $\gamma_{|[0,T]}$ does not meet an ε-neighborhood of $\overline{\omega}$. Then, one can construct a sequence of solutions $(u_n)_{n\in\mathbb{N}}$ of the wave equation whose initial energy $E(u_n(0))$ is normalized to 1, and with energy $E(u(t))$ localized around $\gamma(t)$ at any time $t \in [0,T]$: quantitatively, the energy of u_n outside a tubular neighborhood of γ of size ε tends to 0 as $n \to +\infty$. This disproves the observability inequality (1.11). The sequence $(u_n)_{n\in\mathbb{N}}$, if taken as a Gaussian profile centered at a point describing γ, is called a Gaussian beam.

Observability of the Riemannian Schrödinger equation

For the Schrödinger equation (1.9), the observability inequality reads as follows:

$$\|u_0\|^2_{L^2(M)} \leqslant C \int_0^T \int_\omega |u(t,x)|^2 dx dt. \tag{1.13}$$

As for the wave equation (1.8), the L^2-norm of the solution is preserved along the flow, so that $\|u(T)\|_{L^2} = \|u_0\|_{L^2}$. A sufficient condition for observability is the following:

Theorem 1.15 ([Leb92b] and Appendix of [DGL06])**.** *Assume that $\omega \neq \emptyset$ is open and that $(GCC)_{\omega,T'}$ holds for some $T' > 0$. Then* (1.13) *holds, i.e., the Schrödinger equation* (1.9) *is observable in any time $T > 0$ on ω.*

The interplay between T' and T in the above result is due to the fact that the Schrödinger equation "propagates at infinite speed" so that no matter how large T' is, observability holds in any time $T > 0$ if $(GCC)_{\omega,T'}$ holds. This contrasts with the finite speed of propagation of the wave equation.

The converse of the above theorem, namely to find necessary conditions on (ω, T) for (1.13) to hold, is notoriously a difficult problem. The main results in this direction are for the torus (see [Jaf90], [BZ12], [AM14]), and in Riemannian manifolds with negative curvature (see [DJN19]), where (1.13) holds for any non-empty open subset ω and any time $T > 0$. Indeed, it is expected that if the geodesic flow of the background geometry is unstable, solutions of (1.9) are more "delocalized" than those of (1.8) for example. See also the case of the disk [ALM16].

Observability of the Riemannian heat equation

Let us end with the heat equation. The observability inequality reads as follows:

$$\|u(T)\|^2_{L^2(M)} \leqslant C \int_0^T \int_\omega |u(t,x)|^2 dx dt. \tag{1.14}$$

Theorem 1.16 ([LR95])**.** *Let $\omega \neq \emptyset$ be open and $T > 0$. Then* (1.14) *holds, i.e., the heat equation* (1.10) *is observable in time T on ω.*

Note that no geometric condition on ω is required in this case. This result illustrates the infinite speed of propagation of the heat equation.

The works presented hereafter address that same problem of observability of linear PDEs, but with focus on subelliptic PDEs, meaning that the Laplace-Beltrami operator is replaced in these PDEs by a sub-Laplacian.

1.3.2 Observability of subelliptic PDEs: known results

This section is devoted to stating results which were previously known in the literature about controllability/observability of subelliptic PDEs. All PDEs we consider are well-posed in natural energy spaces which we do not systematically recall.

Subelliptic heat equations

Let us start with the result proved in [BCG14], which concerns the heat equation (1.10) where $\mathcal{L} = \Delta_\gamma$ is the following generalized Baouendi-Grushin operator:

Example 1.17. For $\gamma \geqslant 0$ (not necessarily an integer), we consider $\Delta_\gamma = \partial_x^2 + |x|^{2\gamma}\partial_y^2$ on the manifold $M = (-1,1)_x \times \mathbb{T}_y$. When $\gamma \in \mathbb{N}$, the associated sub-Riemannian structure has step $k = \gamma + 1$.

The two main differences with Example 1.3 are the following: there is an additional degree of freedom $\gamma \in \mathbb{R}^+$, and Example 1.17 is posed on a compact manifold, which is natural in observability problems as already mentioned in Remark 1.10,

The open subset of observation $\omega \subset (-1,1) \times \mathbb{T}$ considered in [BCG14] is a vertical strip of the form $(a,b) \times \mathbb{T}$ where $0 < a < b < 1$. The observability inequality is (1.14), with the modification that u runs over the set of solutions of (1.10) with $\mathcal{L} = \Delta_\gamma$. The authors prove the following result, to be compared with Theorem 1.16:

Theorem 1.18 ([BCG14])**.** *Let $\gamma > 0$ and ω be as above. Then*

- *If $\gamma \in (0,1)$, then for any $T > 0$,* (1.14) *holds;*
- *If $\gamma = 1$, i.e., $\Delta_\gamma = \Delta_G$, then there exists $T_0 > 0$ such that* (1.14) *holds if $T > T_0$ and does not hold if $T < T_0$;*
- *If $\gamma > 1$, then, for any $T > 0$,* (1.14) *fails.*

Koenig studied the observability of (1.10) with $\mathcal{L} = \Delta_G$, but for another geometry of the observation set ω: this time, it is a horizontal band of the form $(-1,1) \times I$ where I is a proper open subset of $\mathbb{T}$.

Theorem 1.19 ([Koe17])**.** *Let $\omega = (-1,1) \times I$ where I is a proper open subset of $\mathbb{T}$. Then* (1.14) *fails for any $T > 0$.*

Although the observability properties of the heat equation driven by general hypoelliptic operators are still mysterious, we list here a few works addressing this question. The recent works [Lis20], [BDE20] and [DK20] continue and generalize the analysis of [BCG14] and [Koe17] on the control of the Baouendi-Grushin heat equation. Besides, [BC17] establishes the existence of a minimal time of observability, as in the second point of Theorem 1.18, for the heat equation driven by the Heisenberg sub-Laplacian of Example 1.4 (with $d = 1$). Let us finally mention the papers [DR20] and [BP18] which also deal with controllability issues for hypoelliptic parabolic equations.

The above theorems show that some subelliptic heat equations driven by simple sub-Laplacians require a larger time to be observable than the usual Riemannian heat equation, and observability may even fail in any time $T > 0$. As we will see, this is a very general phenomenon for subelliptic evolution PDEs, at least for subelliptic wave equations and (some) Schrödinger-type equations. Our results, however, do not shed any new light on subelliptic heat equations, which remain mysterious due to the lack of "general arguments" which would not rely on geometric and analytic features specific to very particular sub-Laplacians.

Approximate observability of subelliptic PDEs

Recently, Laurent and Léautaud have studied the observability of subelliptic PDEs but with focus on a different notion of observability, called approximate observability. The next paragraph is devoted to a brief description of their results (see [LL20]).

Let us consider a sub-Laplacian Δ as in (1.2), with associated sub-Riemannian structure $(M, \mathcal{D}, g)$. We assume that the manifold M (assumed to have no boundary, $\partial M = \emptyset$), the smooth volume μ and the vector fields X_i are all *real-analytic*. For $s \in \mathbb{R}$, the operator $(1-\Delta)^{\frac{\ell}{2}}$ is defined thanks to functional calculus, and we consider the (adapted) Sobolev spaces

$$\mathcal{H}^\ell(M) = \{u \in \mathcal{D}'(M),\ (1-\Delta)^{\frac{\ell}{2}} u \in L^2(M)\}$$

with the associated norm $\|u\|_{\mathcal{H}^\ell(M)} = \|(1-\Delta)^\ell u\|_{L^2(M)}$.

Theorem 1.20 ([LL20])**.** *Let ω be a non-empty open subset of M and let $T > \sup_{x\in M} d_g(x,\omega)$. We denote by k the step of the sub-Riemannian structure $(M, \mathcal{D}, g)$. Then there exist $c, C > 0$ such that*

$$\|(u_0,u_1)\|_{\mathcal{H}^1\times L^2} \leqslant C e^{c\Lambda^k} \|u\|_{L^2((-T,T)\times\omega)}, \qquad \text{with } \Lambda = \frac{\|(u_0,u_1)\|_{\mathcal{H}^1\times L^2}}{\|(u_0,u_1)\|_{L^2\times\mathcal{H}^{-1}}} \tag{1.15}$$

for any solution u of (1.8) *on $(-T,T)$ such that $(u, \partial_t u)_{|t=0} = (u_0,u_1) \in \mathcal{H}^1(M) \times L^2(M)$.*

The above result in particular implies unique continuation (and quantifies it): if $u = 0$ in $(-T,T)\times\omega$, then $u \equiv 0$. However, the exact observability inequality which we shall study (see (1.11)) is a stronger requirement than (1.15), in particular because of the presence of the "typical frequency of the datum" Λ in the right-hand side of (1.15). The techniques used for proving Theorem 1.20 are totally different from those we present in the sequel.

Observability of Baouendi-Grushin Schrödinger equation

The recent work [BS19] is the first one dealing with exact observability of a subelliptic Schrödinger equation, namely in the context of Example 1.3 (on $(-1,1)\times\mathbb{T}$ instead of $\mathbb{R}\times\mathbb{R}$) with observation set given by a horizontal band as in Theorem 1.19. The observability inequality is given by (1.13), except that u runs over the solutions of the Schrödinger equation driven by the sub-Laplacian Δ_G.

Theorem 1.21 ([BS19])**.** *Let $M = (-1,1)\times\mathbb{T}$ and $\Delta_G = \partial_x^2 + x^2\partial_y^2$. Let $\omega = (-1,1)\times I$ where $I \subsetneq \mathbb{T}$ is open. Let $T_0 = \mathscr{L}(\omega)$ be the length of the maximal sub-interval contained in $\mathbb{T}\setminus I$. Then, the observability property* (1.13) *holds if and only if $T > T_0$.*

Again, this result shows the existence of a minimal time of control which contrasts with the "infinite speed of propagation" illustrated by Theorem 1.15. Its proof relies on fine semi-classical analysis.

Non-linear subelliptic PDEs

Although this thesis is devoted only to *linear* subelliptic PDEs, let us say a word about *non-linear* subelliptic PDEs. To study the cubic Grushin-Schrödinger equation $i\partial_t u - (\partial_x^2 + x^2\partial_y^2)u = |u|^2 u$, Patrick Gérard and Sandrine Grellier introduced a toy model, the cubic Szegö equation, which models the interactions between the nonlinearity and the lack of dispersivity of the linear equation (already visible in the above Theorem 1.21). In [GG10], they put this equation into a Hamiltonian framework and classify the traveling waves for this equation.

1.3.3 Main results

Let us now present the main results contained in the papers [Let20b], [LS20] and [FL21]. All of them illustrate the slowdown of propagation of evolution PDEs in directions transverse to the distribution: in a nutshell, observability will require a much longer time to hold for subelliptic PDEs than for elliptic ones, and this time will be even larger when the step k of the underlying sub-Riemannian structure is larger. All our results are summarized in Figure 1.1 at the end of this section.

First main result

We start with a general result on subelliptic wave equations. Let $\Delta = -\sum_{i=1}^m X_i^* X_i$ be a sub-Laplacian, where the adjoint denoted by star is taken with respect to a volume μ on M, which is assumed to have a boundary $\partial M \neq \emptyset$.[5] The sub-Riemannian gradient is defined by the formula

$$\nabla^{\mathrm{sR}}\phi = \sum_{i=1}^m (X_i\phi)X_i.$$

Consider the wave equation

$$\begin{cases} \partial_{tt}^2 u - \Delta u = 0 \quad \text{in } (0,T)\times M \\ u = 0 \quad \text{on } (0,T)\times \partial M, \\ (u_{|t=0}, \partial_t u_{|t=0}) = (u_0, u_1) \end{cases} \tag{1.16}$$

where $T > 0$, and the initial data (u_0, u_1) are in an appropriate energy space. The natural energy of a solution u of the sub-Riemannian wave equation (1.16) is

$$E(u(t,\cdot)) = \frac{1}{2}\int_M \left(|\partial_t u(t,x)|^2 + |\nabla^{\mathrm{sR}} u(t,x)|^2\right) d\mu(x).$$

Observability holds in time T_0 on ω if there exists $C > 0$ such that for any solution u of (1.16),

$$E(u(0)) \leqslant C \int_0^{T_0}\int_\omega |\partial_t u(t,x)|^2 d\mu(x)dt. \tag{1.17}$$

Theorem 1: [Let20b]

Let $T_0 > 0$ and let $\omega \subset M$ be a measurable subset. We assume that there exist $1 \leqslant i, j \leqslant m$ and x in the interior of $M\backslash\omega$ such that $[X_i, X_j](x) \notin \mathcal{D}_x$. Then the subelliptic wave equation (1.16) is not exactly observable on ω in time T_0.

Theorem 1 can be reformulated as follows: subelliptic wave equations are never observable. The condition that there exists x (in the interior of $M \setminus \omega$) such that $[X_i, X_j](x) \notin \mathcal{D}_x$ means that Δ is not elliptic at x; this assumption is absolutely necessary since otherwise, in $M \setminus \omega$, (1.16) would be a wave equation with elliptic Laplacian, and its observability properties would depend on the GCC, as stated in Theorem 1.12.

The key ingredient in the proof of Theorem 1 is that the GCC fails for any time $T_0 > 0$: in other words, there exist geodesics which spend a time greater than T_0 outside ω. Then, the Gaussian beam construction described in Remark 1.14 allows to contradict the observability inequality (1.17).

[5] This assumption is not necessary, since Theorem 1 also works for manifolds without boundary, but this would require to introduce a slightly different notion of observability.

Second main result

Our second main result, obtained in collaboration with Chenmin Sun, sheds a different light on Theorem 1. For this second statement, we consider the generalized Baouendi-Grushin operator of Example 1.17, i.e., $\Delta_\gamma = \partial_x^2 + |x|^{2\gamma}\partial_y^2$ on $(-1,1)_x \times \mathbb{T}_y$.

We assume that $\gamma \geqslant 1$ (not necessarily an integer). We consider the Schrödinger-type equation with Dirichlet boundary conditions

$$\begin{cases} i\partial_t u - (-\Delta_\gamma)^s u = 0 \\ u_{|t=0} = u_0 \in L^2(M) \\ u_{|x=\pm 1} = 0 \end{cases} \tag{1.18}$$

where $s \in \mathbb{N}$ is a fixed integer. Given an open subset $\omega \subset M$, we say that (3.2) is observable in time $T_0 > 0$ in ω if

$$\exists C > 0,\ \forall u_0 \in L^2(M), \qquad \|u_0\|^2_{L^2(M)} \leqslant C \int_0^{T_0} \|e^{-it(-\Delta_\gamma)^s} u_0\|^2_{L^2(\omega)} dt. \tag{1.19}$$

We define

$$T_{\mathrm{ob}} = \inf\{T_0 > 0,\ (1.19) \text{ holds}\},$$

with the convention that $T_{\mathrm{ob}} = +\infty$ if there does not exist $T_0 > 0$ such that (1.19) holds. Note that T_{ob} depends on s, γ and ω.

Our result roughly says that observability holds if and only if the subellipticity (measured by the step $\gamma + 1$ in case $\gamma \in \mathbb{N}$), is not too strong compared to the strength of propagation s:

Theorem 2: [LS20]

Assume that $\gamma \geqslant 1$ and $s \in \mathbb{N}$. Let $I \subsetneq \mathbb{T}_y$ be a strict open subset, and $\omega = (-1,1)_x \times I$.

1. If $\frac{1}{2}(\gamma + 1) < s$, then $T_{\mathrm{ob}} = 0$;
2. If $\frac{1}{2}(\gamma + 1) = s$, then $0 < T_{\mathrm{ob}} < +\infty$;
3. If $\frac{1}{2}(\gamma + 1) > s$, then $T_{\mathrm{ob}} = +\infty$.

The case $s = 1/2$ corresponds to wave equations. Strictly speaking, it is not covered by Theorem 2 since s is assumed to belong to $\mathbb{N}$ in this theorem, but we see that for any positive γ, this case is roughly related to Point (3), and we thus recover the intuition given by Theorem 1 that subelliptic wave equations should not be observable. The case $\gamma = s = 1$ allows to recover Theorem 1.21, except that we do not find with our method the critical time $\mathscr{L}(\omega)$. Let us also notice that if $\gamma \in \mathbb{N}$, since $\gamma + 1$ is the step of the sub-Laplacian Δ_γ, the number $\frac{1}{2}(\gamma + 1)$ appearing in Theorem 2 coincides with the exponent known as the gain of Sobolev derivatives in subelliptic estimates (see Section 1.2.2).

Third main result

Finally, our third main result, obtained in collaboration with Clotilde Fermanian Kammerer, illustrates how tools coming from noncommutative harmonic analysis can be used to analyze sub-Laplacians and the associated evolution equations. Our main message is that *a pseudodifferential calculus "adapted to the sub-Laplacian" can be used to prove controllability and observability results for subelliptic PDEs* (instead of the usual pseudodifferential calculus used for example to prove Theorem 2). As we will see, in the present context, once defined this natural pseudodifferential calculus and the associated semi-classical measures (which relies essentially on functional analysis arguments), observability results follow quite directly.

To relate this last result to the previous ones, let us say that it is roughly linked to the critical case $s = \gamma = 1$ of Point 2 of Theorem 2, i.e., to the case where subelliptic effects are exactly balanced by the strength of propagation of the equation. Indeed, we consider the usual Schrödinger equation ($s = 1$) in some particular non-commutative Lie groups, called H-type groups, which have step 2 (corresponding to $\gamma = 1$ for Baouendi-Grushin operators). As in Point 2 of Theorem 2, we establish that under some geometric conditions on the set of observation ω, observability holds if and only if time is sufficiently large. The main difference with Theorem 2 relies in the tools used for the proof, which could lead to different generalizations. For example, the tools employed in this section allow to handle the case with analytic potential, see (1.20) below. Also, with these tools, we could imagine to prove observability results for higher-step nilpotent Lie groups, but it requires to know explicit formulas for their representations, since they determine the propagation properties of the semi-classical measures we construct.

To keep the presentation as simple as possible, we will present our last result only for the Heisenberg groups $\mathbf{H}_d$ of Example 1.4, and not for general H-type groups (which are handled in [FL21]). By doing so, we avoid defining general H-type groups for the moment, while keeping the main message of this work, namely the use of noncommutative harmonic analysis for proving observability inequalities.

Let us explain how to get compact quotients of the Heisenberg group (as required in Remark 1.10). Using the notations of Example 1.4, we consider the left-quotient of $\mathbf{H}_d$ by the discrete subgroup $\widetilde{\Gamma} = (\sqrt{2\pi}\mathbb{Z})^{2d} \times \pi\mathbb{Z}$, which yields a compact manifold $M = \widetilde{\Gamma}\backslash\mathbf{H}_d$. The vector fields X_j, Y_j are left-invariant and can be thus considered as vector fields on the quotient manifold M. This allows to consider the sub-Laplacian Δ_M as acting on functions on M.

We consider the equation

$$i\partial_t u + \frac{1}{2}\Delta_M u + \mathbb{V}u = 0 \tag{1.20}$$

on M, where $\mathbb{V}$ is an analytic function defined on M. The factor $\frac{1}{2}$ in front of Δ_M plays no role, we put it here just to keep the same conventions as in [FL21].

The Schrödinger equation (1.20) is observable in time T on the measurable set U if there exists a constant $C_{T,U} > 0$ such that

$$\forall u_0 \in L^2(M), \quad \|u_0\|^2_{L^2(M)} \leqslant C_{T,U} \int_0^T \left\| e^{it(\frac{1}{2}\Delta_M + \mathbb{V})} u_0 \right\|^2_{L^2(U)} dt. \tag{1.21}$$

Recall that Theorem 1.15 asserts that, in the Riemannian setting and without potential, the observability of the Schrödinger equation is implied by the Geometric Control Condition (GCC), which says that any trajectory of the geodesic flow enters U within time T. Here, one can also define a sub-Riemannian geodesic flow (see (1.12)) but in some directions of the phase space, called degenerate directions in the sequel, it vanishes due to the fact that $\Sigma \neq \{0\}$ (see Section 1.2.3). In these degenerate directions, we thus need to replace GCC by another condition. In the case of the Heisenberg group $\mathbf{H}_d$, there is only one such direction, thought of as "vertical" since it is related to the ∂_z vector field.

The Heisenberg group $\mathbf{H}_d$ comes with a Lie algebra $\mathfrak{g}$. Via the exponential map

$$\mathrm{Exp} : \mathfrak{g} \to \mathbf{H}_d$$

which is a diffeomorphism from $\mathfrak{g}$ to $\mathbf{H}_d$, one identifies $\mathbf{H}_d$ and $\mathfrak{g}$ as a set and a manifold. Moreover, $\mathfrak{g}$ is equipped with a vector space decomposition

$$\mathfrak{g} = \mathfrak{v} \oplus \mathfrak{z}\,,$$

such that $[\mathfrak{v},\mathfrak{v}] = \mathfrak{z} \neq \{0\}$ and $\mathfrak{z}$ (of dimension 1) is the center of $\mathfrak{g}$. We define a scalar product on $\mathfrak{z}$ by saying that ∂_z has norm 1, which allows to identify $\mathfrak{z}$ to its dual $\mathfrak{z}^*$. We define the scalar product on $\mathfrak{v}$ by saying that the $2d$ vector fields

$$X_j = \partial_{x_j} - \frac{y_j}{2}\partial_s, \quad Y_j = \partial_{y_j} + \frac{x_j}{2}\partial_s, \qquad j = 1, \dots, d \tag{1.22}$$

form an orthonormal basis, denoted by V.

We consider the "vertical" flow map (also called "Reeb", in honor of Georges Reeb) on $M \times \mathfrak{z}^*$:

$$\Phi_0^s : (x, \lambda) \mapsto (\mathrm{Exp}(sd\mathcal{Z}^{(\lambda)}/2)x, \lambda), \qquad s \in \mathbb{R}$$

where, for $\lambda \in \mathfrak{z}$, $\mathcal{Z}^{(\lambda)}$ is the element of $\mathfrak{z}$ defined by $\lambda(\mathcal{Z}^{(\lambda)}) = |\lambda|$ (or equivalently, $\mathcal{Z}^{(\lambda)} = \lambda/|\lambda|$ after identification of $\mathfrak{z}$ and $\mathfrak{z}^*$). We introduce the following H-type geometric control condition.

(H-GCC) The measurable set U satisfies **H-type GCC** in time T if

$$\forall (x, \lambda) \in M \times (\mathfrak{z}^* \setminus \{0\}), \;\; \exists s \in (0, T), \;\; \Phi_0^s((x, \lambda)) \in U \times \mathfrak{z}^*.$$

The flow Φ_0^s thus replaces the geodesic flow in the degenerate direction.

Definition 1.22. *We denote by $T_{\mathrm{GCC}}(U)$ the infimum of all $T > 0$ such that H-type GCC holds in time T (and we set $T_{\mathrm{GCC}}(U) = +\infty$ if H-type GCC does not hold in any time).*

We also consider the additional assumption:

(A) For any $(x, \omega) \in M \times \mathfrak{v}^*$ such that $|\omega| = 1$, there exists $s \in \mathbb{R}$ such that $\mathrm{Exp}(s\omega \cdot V)x \in U$. Here, $\omega \cdot V = \sum_{j=1}^{2d} \omega_j V_j$ where ω_j denote the coordinates of ω in the dual basis of V and it is assumed that $\sum_{j=1}^{2d} \omega_j^2 = 1$.

More explicitly, denoting by (x, y, t) the elements of $\mathbf{H}_d$, we have

$$\Phi_0^s(x, y, t, \lambda) = \left(x, y, t + s\frac{d}{2}\,\mathrm{sgn}(\lambda), \lambda\right), \quad s \in \mathbb{R}$$

and

$$\mathrm{Exp}\big(s\sum_{j=1}^{d}(a_jX_j + b_jY_j)\big)(x, y, t) = \Big(x + sa, y + sb, t + \frac{s}{2}(x \cdot b - y \cdot a)\Big), \quad s \in \mathbb{R}.$$

These trajectories are the lifts in $\mathbf{H}_d$ of the geodesics of $\mathbf{T}^{2d}$.

Theorem 3: [FL21]

Let $U \subset M$ be open and denote by $\overline{U}$ its closure.

1. Assume that U satisfies **(A)** and that $T > T_{\mathrm{GCC}}(U)$, then the observability inequality (1.21) holds.

2. Assume $T \leqslant T_{\mathrm{GCC}}(\overline{U})$, then the observability inequality (1.21) fails.

This statement looks like Theorem 1.12 which holds for elliptic waves. In some sense, "the Schrödinger equation in Heisenberg groups looks like an elliptic wave equation", a phenomenon which was already pointed out by authors studying Strichartz estimates, see [BGX00] and [BFG16] for example.

Let us also say that, as already mentioned, Theorem 3 holds more generally in quotients of Heisenberg-type groups.

Summary

To conclude, let us draw a table summing up most of the results presented in this introduction:

	Elliptic	Step 2	Step $2s$	Step $> 2s$
Waves and half-waves ($s = 1/2$)	T_{inf} (under GCC)	∞	∞	∞
Schrödinger ($s = 1$)	0 (under GCC)	T_{inf}	∞	∞
Generalized Schrödinger ($s > 1$)	0 (under GCC)	0	T_{inf}	∞
Heat	0	T_{inf} or ∞	?	?

Figure 1.1: Observability of subelliptic PDEs depending on the step.
If the results are established only in particular cases, they are in blue. The first line is covered by Theorems 1.12 and 1, the second line by Theorems 1.15, 2 and 3, the third line by Theorem 2 and the fourth line by Theorems 1.16, 1.18 and 1.19.

1.4 Main results on propagation of singularities

Our second series of results concerns propagation of singularities for subelliptic wave equations. Thus, instead of addressing *propagation of energy* issues as in the observability inequalities of Section 1.3.3, we focus here on *propagation of regularity/singularity*. It turns out that this requires a deeper understanding of the characteristic cone Σ introduced in Section 1.2.3.

Recall that the celebrated propagation of singularities theorem describes the wave-front set $WF(u)$ of a distributional solution u to a partial differential equation $Pu = f$ in terms of the principal symbol p of P: it says that, if p is real, then $WF(u) \setminus WF(f) \subset p^{-1}(0)$, and that, if additionally the characteristics are simple ($p = 0 \Rightarrow dp \neq 0$ outside the null section), then $WF(u) \setminus WF(f)$ is invariant under the bicharacteristic flow induced by the Hamiltonian vector field H_p of p.

This result was first proved in [DH72, Theorem 6.1.1] and [Hor71a, Proposition 3.5.1]. However, it leaves open the case where the characteristics of P are not simple. In this case, the difficulty is that at doubly characteristic points, H_p vanishes, and thus the above theorem is void. The results presented in this section seek to bridge this gap.

1.4.1 A general result

In a short and impressive paper [Mel86], Melrose sketched the proof of an analogous propagation of singularities result for the wave operator $P = D_t^2 - A$ when A is a self-adjoint non-negative real second-order differential operator which is only subelliptic (for example, A can be a sub-Laplacian, of the form (1.2)). Such operators P are typical examples for which there exist double characteristic points, namely the points for which $\tau = \sigma_P(A) = 0$. When A is a sub-Laplacian, $\{\sigma_P(A) = 0\}$ is exactly the characteristic cone Σ introduced in (1.5).

Restated in the language of sub-Riemannian geometry, Melrose's result [Mel86, Theorem 1.8] asserts that singularities of subelliptic wave equations propagate only along null-bicharacteristics (solutions of (1.12), with g^* replaced by $\sigma_P(A)$) and along *abnormal extremal lifts of singular curves* (see Definition 1.24). The propagation along null-bicharacteristics corresponds to singularities outside Σ, and it is indeed implied by the classical "elliptic" theorem recalled above. But the true novelty of [Mel86] is the characterization of the propagation inside Σ.

Despite the potential scope of this result, we did not find in the literature any other paper quoting it. The proof provided in [Mel86] is indeed very sketchy, and it took us months to understand the statement and to reconstruct the full proof. But this paper contains truly new ideas, if it is read with the magnifying glass of sub-Riemannian geometry (which was only at a very early stage of development at the time of publication of [Mel86], in 1986).

The fact that singularities inside Σ propagate along abnormal extremal lifts of singular curves must be explained. Singular curves are something specific to sub-Riemannian geometry: non-trivial examples of singular curves do not exist in Riemannian geometry, and they even exist only in quite specific sub-Riemannian structures (e.g., in the Martinet structure of Example 1.7, but not in Heisenberg groups). They are central objects in control theory and played a key role in the discovery of so-called abnormal minimizers in sub-Riemannian geometry (see [Mon94], [Mon02]). Many open problems in sub-Riemannian geometry revolve around singular curves, see [Agr14].

In this introductory section, we state Melrose's theorem in the case where A is a sub-Laplacian, and we postpone the general statement of Melrose for subelliptic self-adjoint non-negative real second-order differential operators to Chapter 5.

We consider $A = -\Delta$ the opposite of a subelliptic sub-Laplacian on a manifold X. Take care that in this section as well as in Chapter 5, the manifold is X and not M (as in the previous sections), since M denotes indeed something different, see (1.24). These choices are made to be coherent with the original notations of [Mel86].

The wave equation under study is

$$\begin{aligned} (D_t^2 - A)u = Pu = 0 \qquad &\text{in } \mathbb{R} \times X, \\ u = u_0, \quad \partial_t u = u_1 \quad &\text{at} \quad t = 0 \end{aligned} \tag{1.23}$$

where $D_t = \frac{1}{i}\partial_t$. We denote by $a = \sigma_P(A)$ the principal symbol of A (which is indeed equal to g^*, introduced in (1.4)), and by $p = \sigma_P(P) = \tau^2 - a$ that of P. We also consider

$$M = T^*(\mathbb{R} \times X) \setminus 0 \tag{1.24}$$

and we denote by ω the canonical symplectic form on M (see Appendix A.1 for the sign conventions concerning the Hamiltonian and symplectic formalism). In the next paragraphs, we define some cones Γ_m which generalize the bicharacteristic directions at points where the Hamiltonian vector field H_p defined by $\omega(H_p, \cdot) = -dp(\cdot)$ vanishes.

We set

$$M_+ = \{m \in M,\ p(m) \geqslant 0, \tau \geqslant 0\}, \qquad M_- = \{m \in M,\ p(m) \geqslant 0, \tau \leqslant 0\};$$

in particular, $M_+ \cup M_- = \{p \geqslant 0\}$. Let

$$\Sigma_{(2)} = M_+ \cap M_- = \{m \in M, \tau = a = 0\}.$$

(the last equality follows from the fact that $a \geqslant 0$).

For $m \in M_+ \setminus \Sigma_{(2)}$, we set

$$\Gamma_m = \mathbb{R}^+ \cdot H_p(m) \subset T_m M,$$

where H_p is the Hamiltonian vector field of p verifying $\omega(H_p, Z) = -dp(Z)$ for any smooth vector field Z. To extend this definition to $M_- \setminus \Sigma_{(2)}$, for $(t, \tau, \alpha) \in M_- \setminus \Sigma_{(2)}$, we set

$$\Gamma_m = \Gamma_{m'} \qquad \text{where} \qquad m' = (t, -\tau, \alpha) \in M_+.$$

At $m \in \Sigma_{(2)}$, the Hamiltonian vector field $H_p(m)$ vanishes, but the Hessian a_m of a is well-defined: it is a quadratic form on T_mM. For $m \in \Sigma_{(2)}$, we set

$$\begin{aligned} \Gamma_m &= \mathbb{R}^+(\partial_t + B), \\ B &= \{b \in \ker(a_m)^{\perp_{\omega_0}},\ g(d\pi(b)) \leqslant 1\}. \end{aligned} \tag{1.25}$$

Here, $\perp_{\omega_0}$ designates the symplectic orthogonal with respect to the canonical symplectic form ω_0 on T^*X.

Definition 1.23. *A null-ray for p is a Lipschitz curve*

$$\gamma : I \to \{m \in M;\ p(m) = 0\}$$

defined on some interval $I \subset \mathbb{R}$ with (set-valued) derivative $\gamma'(s) \subset \Gamma_{\gamma(s)}$ for all $s \in I$. In particular, null-rays live in $\{p = 0\}$.

Theorem 4: [Mel86], [Let21b]

Let $t \mapsto u(t)$ be a solution of (1.23). For any $t > 0$, if $(x, \xi) \in WF(u(0))$ then there exists $(y, \eta) \in WF(u(-t)) \cup WF(\partial_t u(-t))$ such that (y, η) and (x, ξ) can be joined by a null-ray of length t.

It follows from the definition of the cones Γ_m that there are two types of null-rays:

- Those contained in $M \backslash \Sigma_{(2)}$: they are the usual null-bicharacteristics, for which $\tau^2 = a \neq 0$.
- Those contained in $\Sigma_{(2)}$, for which $\tau^2 = a = 0$.

A null-ray of the second type is tangent to the cones Γ_m defined by (1.25). The vector b in (1.25) belongs to both $\ker(a_m)^{\perp_{\omega_0}}$ (by definition) and $\ker(a_m)$ (since the null-ray is contained in $\Sigma_{(2)}$). Therefore it is a characteristic curve, in the sense of the following definition. In this definition, we take the notation $\mathcal{D}^\perp$, equivalent to Σ (see (1.5)), to insist on the fact that the characteristic cone depends only on the distribution, and not on the metric. Also, $\overline{\omega_0}$ denotes the restriction to $\mathcal{D}^\perp$ of the canonical symplectic form ω_0 on T^*X.

Definition 1.24. *A characteristic curve for $\mathcal{D}$ is an absolutely continuous curve $t \mapsto \lambda(t) \in \mathcal{D}^\perp$ that never intersects the zero section of $\mathcal{D}^\perp$ and that satisfies*

$$\dot{\lambda}(t) \in \ker(\overline{\omega_0}(\lambda(t)))$$

for almost every t. The projection of $\lambda(t)$ onto X, which is an horizontal curve[6] *for $\mathcal{D}$, is called a singular curve, and the corresponding characteristic an abnormal extremal lift of that curve.*

With this definition, Theorem 4 can be reformulated as follows:

Singularities of subelliptic wave equations propagate only along usual null-bicharacteristics, and also along abnormal extremal lifts of singular curves at speeds between 0 and 1.

Theorem 4 has consequences on the singularities of the wave kernel, which we now explain. By the spectral theorem, for any $t \in \mathbb{R}$, the self-adjoint operator

$$G(t) = A^{-1/2} \sin(tA^{1/2})$$

[6] i.e., $d\pi(\dot{\lambda}(t)) \in \mathcal{D}_{\lambda(t)}$ for almost every t, where $\pi : T^*X \to X$ denotes the canonical projection.

is a well-defined operator bounded on $L^2(X)$, in fact it maps $L^2(X)$ into the domain $\mathscr{D}(A^{1/2})$. Together with the self-adjoint operator $G'(t) = \cos(tA^{1/2})$, this allows to solve the Cauchy problem (1.23) by

$$u(t,x) = G'(t)u_0 + G(t)u_1.$$

For $(u_0, u_1) \in \mathscr{D}(A^{1/2}) \times L^2(X)$, we have $u \in C^0(\mathbb{R}; \mathscr{D}(A^{1/2})) \cap C^1(\mathbb{R}; L^2(X))$. Then, the Schwartz kernel $K_G \in \mathcal{D}'(\mathbb{R} \times X \times X)$ of G is defined by

$$\forall v \in C^\infty(X), \qquad K_G(t)v(x) = \int_X K_G(t,x,y)v(y)dy.$$

Theorem 4 implies the following inclusion.

Theorem 5: [Mel86], [Let21b]

We have

$$\begin{aligned} WF(K_G) \subset \{&(t,x,y,\tau,\xi,-\eta) \in T^*(\mathbb{R} \times X \times X) \setminus 0; \\ &\text{there is a null ray from } (0,\tau,y,\eta) \text{ to } (t,\tau,x,\xi)\}. \end{aligned} \tag{1.26}$$

In turn, we can deduce from Theorem 5 the following corollary in the spirit of the trace formula of Duistermaat and Guillemin [DG75].

Corollary 1: [Let21b]

Fix $x, y \in X$ with $x \neq y$. We denote by $\mathscr{L}$ the set of lengths of normal geodesics from x to y and by T_s the minimal length of a singular curve joining x to y. Then $\mathscr{G} : t \mapsto K_G(t,x,y)$ is well-defined as a distribution on $(-T_s, T_s)$, and

$$\text{sing supp}(\mathscr{G}) \subset 0 \cup \mathscr{L} \cup -\mathscr{L}.$$

Theorem 4 raises the following problem:

Is it really possible that singularities propagate along singular curves, and if yes, at which speed?

Our next results answer this question.

1.4.2 Propagation along singular curves in the Martinet case

Let us consider the *Martinet sub-Laplacian* (see Example 1.7)

$$\Delta = X_1^2 + X_2^2$$

on $\mathbb{R}^3$, where

$$X_1 = \partial_x, \qquad X_2 = \partial_y + x^2\partial_z.$$

The Martinet half-wave equation is

$$i\partial_t u - \sqrt{-\Delta}u = 0 \tag{1.27}$$

on $\mathbb{R}_t \times \mathbb{R}^3$, with initial datum $u(t=0) = u_0$.

We denote by $\lambda_1(\mu)$ the lowest eigenvalue of

$$-d_x^2 + (\mu + x^2)^2 \tag{1.28}$$

over $\mathbb{R}$, and we set

$$F(\mu) = \sqrt{\lambda_1(\mu)}.$$

In the next result, the curve $\gamma : t \mapsto (0,t,0)$ plays a particular role. Indeed, it is one of the simplest examples of singular curves[7].

Theorem 6: [CL21]

For any bounded union of intervals I, there exists $U(t)u_0$ solution of (1.27) such that for any $t \in \mathbb{R}$, we have

$$WF(U(t)u_0) = \{(0,y,0;\ 0,0,\lambda) \in T^*\mathbb{R}^3,\ \lambda > 0,\ y \in tF'(I)\}\ . \tag{1.29}$$

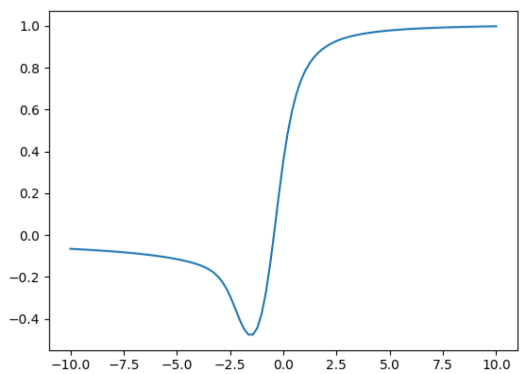
In particular,

$$\text{Sing Supp}(U(t)u_0) = \{(0,y,0) \in \mathbb{R}^3,\quad y \in tF'(I)\}. \tag{1.30}$$

Theorem 7 means that

$$\begin{gathered}\textit{singularities propagate along the singular curve } \gamma\\ \textit{at speeds given by } F'(I).\end{gathered} \tag{1.31}$$

But the graph of F', restricted to the interval $\mu \in (-10, 10)$, is the following



and we have in particular:

Proposition 1.25. *There holds $F'(\mathbb{R}) = [a, 1)$ for some $-1 < a < 0$.*

Together with (1.31), and choosing I adequately, this implies the following informal statement.

Theorem 7: [CL21]

Any value between 0 and 1 can be realized as a speed of propagation of singularities along the singular curve γ.

This result is in strong contrast with the propagation of singularities along geodesics *at speed* 1 for wave equations with elliptic Laplacian.

[7]One can check that an abnormal extremal lift is given by $\xi \equiv \eta \equiv 0$ and $\zeta \neq 0$, where ξ, η, ζ are the dual variables of x, y, z.

The proof of Theorem 7 can be summarized in a few lines. First, recalling that

$$-\Delta = -\partial_x^2 - (\partial_y + x^2\partial_z)^2,$$

we take the Fourier transform in the variables y, z (with dual variables η, ζ): we get the operator

$$H_{\eta,\zeta} = -d_x^2 + (\eta + x^2\zeta)^2.$$

Its lowest eigenvalue is denoted by α_1, and the corresponding eigenfunction by $\psi_{\eta,\zeta}(x)$. For $f(x, y, z) = \psi_{\eta,\zeta}(x)e^{iy\eta+iz\zeta}$, we have $-\Delta f = \alpha_1 f$.

We make the following changes of variables: $\mu = \frac{\eta}{\zeta^{1/3}}$, $\psi_{\eta,\zeta}(x) = \psi_\mu(\zeta^{1/3}x)$ and $\alpha_1 = \zeta^{2/3}\lambda_1(\mu)$. We get

$$(-d_x^2 + (\mu + x^2)^2)\psi_\mu = \lambda_1(\mu)\psi_\mu,$$

so that $\lambda_1(\mu)$ and ψ_μ are indeed the lowest eigenvalue and the corresponding eigenfunction of (1.28). All in all, we deduce for any $\mu, \zeta \in \mathbb{R}$ a solution of (1.27):

$$v_{\mu,\zeta}(t, x, y, z) = e^{-it\zeta^{1/3}F(\mu)}e^{iy\zeta^{1/3}\mu+iz\zeta}\psi_\mu(\zeta^{1/3}x).$$

Making linear combinations of the previous solutions for different $\mu, \zeta \in \mathbb{R}$, we obtain a new solution

$$(U(t)u_0)(x, y, z) = \iint_{\mathbb{R}^2} Y(\zeta)\phi(\mu)\underbrace{e^{-it\zeta^{1/3}F(\mu)}e^{iy\zeta^{1/3}\mu+iz\zeta}\psi_\mu(\zeta^{1/3}x)}_{\text{solution for fixed } \mu,\zeta}\, d\mu d\zeta.$$

We choose $Y \in C^\infty(\mathbb{R}, [0, 1])$ to be a truncation function: $Y(\zeta) = 0$ for $\zeta \leqslant 1$ and $Y(\zeta) = 1$ for $\zeta \geqslant 2$. The function ϕ will be specified a bit later.

We finally answer the question: *How does this solution propagate ?* Since the speed of propagation is given by the group velocity, we differentiate the phase $-\zeta^{1/3}(tF(\mu) - y\mu) + z\zeta$ with respect to ζ and μ.

- Differentiating with respect to ζ, we find that the critical points satisfy the relation $z = -\frac{1}{3}\zeta^{-2/3}(tF(\mu) - y\mu)$. Since singularities are created only in the regime where $\zeta\gamma 1$, the speed in z is 0 (of course, these arguments are very rough and have to be justified).
- Differentiating now the phase with respect to μ, we find that the critical points satisfy the relation $tF'(\mu) = y$. Therefore, the speed in y is $F'(\mu)$, for μ belonging to Support(ϕ). Hence, we choose ϕ so that $I =$ Support(ϕ).

This gives the intuition for Theorem 7. Mathematically, the above reasoning is justified by the stationary phase method, see Chapter 6.

To conclude this section, let us mention that it might be possible to extend Theorem 7 to more general geometries, starting with non-flat Martinet metrics and non-flat quasi-contact metrics, which are other examples where singular curves show up.

1.5 Main results on eigenfunctions of sub-Laplacians

1.5.1 Generalities about Quantum Limits

Our last series of results concerns eigenfunctions of sub-Laplacians in the high-frequency limit. A typical problem is the description of their Quantum Limits (QL), i.e., the measures which are weak limits of a subsequence of squares of eigenfunctions. In the sequel, we denote by $\Delta_{g,\mu}$ a general sub-Laplacian of the form (1.2), since the notation Δ will denote a particular sub-Laplacian (see (1.40)).

Under the assumption (1.1), $\Delta_{g,\mu}$ is hypoelliptic (see [Hor67]), and if moreover M is compact[8], then $\Delta_{g,\mu}$ has a compact resolvent and there exists a sequence of (real-valued) eigenfunctions of $-\Delta_{g,\mu}$ associated to the eigenvalues in increasing order $0 = \lambda_1 < \lambda_2 \leqslant \ldots$ (with $\lambda_j \to +\infty$ as $j \to +\infty$) which is orthonormal for the $L^2(M,\mu)$ scalar product. Our main purpose here is to describe the possible behaviours of the sequence of probability measures $|\varphi_k|^2 d\mu$ when $(\varphi_k)_{k\in\mathbb{N}^*}$ is a sequence of normalized eigenfunctions of $-\Delta_{g,\mu}$ with associated eigenvalue tending to $+\infty$, for particular sub-Laplacians $\Delta_{g,\mu}$, typically by describing its weak limits (in the sense of duality with continuous functions).

There is a phase-space extension of these weak limits whose behaviour is also of interest. Let us recall the following definition (see [Ger91a]):

Definition 1.26. *Let $(u_k)_{k\in\mathbb{N}^*}$ be a bounded sequence in $L^2(M)$ and weakly converging to 0. We call microlocal defect measure of $(u_k)_{k\in\mathbb{N}^*}$ any Radon measure ν on S^*M such that for any $a \in \mathscr{S}^0(M)$, there holds*

$$(Op(a)u_{\sigma(k)}, u_{\sigma(k)}) \underset{k\to+\infty}{\longrightarrow} \int_{S^*M} a\, d\nu$$

for some extraction σ. Here, $(\cdot,\cdot)$ denotes the $L^2(M,\mu)$ scalar product, $\mathscr{S}^0(M)$ is the space of classical symbols of order 0, and $Op(a)$ is the Weyl quantization of a (see Appendix A).

Microlocal defect measures are useful tools for studying the (asymptotic) concentration and oscillation properties of sequences, and they are necessarily non-negative.

Definition 1.27. *Given a sequence $(\varphi_k)_{k\in\mathbb{N}^*}$ of eigenfunctions of $-\Delta_{g,\mu}$ with $\|\varphi_k\|_{L^2(M,\mu)} = 1$, we call Quantum Limit (QL) any microlocal defect measure of $(\varphi_k)_{k\in\mathbb{N}^*}$.*

Remark 1.28. Since for any $k \in \mathbb{N}^*$, φ_k is normalized, any QL is a probability measure on S^*M.

Quantum Limits of Riemannian Laplacians.

Let us first state known properties of Quantum Limits in the case where $\Delta_{g,\mu} = \Delta_g$ is the Laplace-Beltrami operator of a Riemannian manifold (M,g). It is known that any Quantum Limit ν of Δ_g is then invariant under the geodesic flow $\exp(t\vec{H})$, where $\vec{H}$ denotes temporarily the Hamiltonian vector field associated to $(g^*)^{1/2}$: there holds $\exp(t\vec{H})\nu = 0$ for any $t \in \mathbb{R}$. To see it, we note that for any sequence $(\varphi_k)_{k\in\mathbb{N}^*}$ consisting of normalized eigenfunctions of $-\Delta_g$, there holds

$$(\exp(-it\sqrt{-\Delta_g})\mathrm{Op}(a)\exp(it\sqrt{-\Delta_g})\varphi_k, \varphi_k)_{L^2} = (\mathrm{Op}(a)\varphi_k, \varphi_k)_{L^2} \tag{1.32}$$

for any $t \in \mathbb{R}$, any $k \in \mathbb{N}^*$ and any classical symbol $a \in \mathscr{S}^0(M)$. It follows from Egorov's theorem that $\exp(-it\sqrt{-\Delta_g})\mathrm{Op}(a)\exp(it\sqrt{-\Delta_g})$ is a pseudodifferential operator of order 0 with principal symbol $a \circ \exp(t\vec{H})$, which in turn implies $\exp(t\vec{H})\nu = 0$.

Conversely, not any probability measure which is invariant under the geodesic flow is a Quantum Limit. The description of all Quantum Limits of a given Riemannian manifold is indeed a long-standing question. Over the years, a particular attention has been drawn towards Riemannian manifolds whose geodesic flow is ergodic since in this case, up to extraction of a

[8]Note that we left the notation M of Section 1.4 and came back to the notation M of Section 1.2.

density-one subsequence, the set of Quantum Limits is reduced to the Liouville measure, a phenomenon which is called Quantum Ergodicity (see for example [Shn74], [Col85], [Zel87]). More recently, the results [Ana08] and [DJ18] gave more precise results in the negative curvature case, using in the first case the notion of metric entropy and in the second one the fractal uncertainty principle. For compact arithmetic surfaces, a detailed study of invariant measures lead to the resolution of the Quantum Unique Ergodicity conjecture for these manifolds, meaning that the extraction of a density-one subsequence in the Quantum Ergodicity result is even not necessary for these particular manifolds ([Lin06]). In manifolds which have a degenerate spectrum, the set of Quantum Limits is generally richer: see for example [Jak97] for the description of Quantum Limits on flat tori or [ALM16] for the case of the disk. Also, the Quantum Limits of the sphere $\mathbb{S}^d$ equipped with its canonical metric have been fully characterized in [JZ96]. And Quantum Ergodicity results have been also established on other mathematical objects such as quantum graphs, see [Col15].

Quantum Limits of sub-Laplacians

The structure and the invariance properties of the Quantum Limits of sub-Laplacians is more complicated than that of Riemannian Laplacians. In the sequel, we make the identification

$$S^*M = U^*M \cup S\Sigma \tag{1.33}$$

where S^*M is the cosphere bundle (i.e., the sphere bundle of T^*M), $U^*M = \{g^* = 1\}$ is a cylinder bundle and $S\Sigma$, the sphere bundle of Σ, consists of the points at infinity of the compactification of U^*M. We also denote by $\exp(t\vec{g^*})$ the sub-Riemannian geodesic flow, which is the Hamiltonian flow of g^*. Note that while working on U^*M, it is equivalent to consider $\exp(t\vec{g^*})$ or $\exp(t\vec{s})$ where $s = \sqrt{g^*}$, since both flows coincide on U^*M. Indeed, $\exp(t\vec{s})$ is the flow which shows up from the application of Egorov's theorem after the computation (1.32).

In comparison with the Riemannian case, the invariance of Quantum Limits of $\Delta_{g,\mu}$ under the sub-Riemannian geodesic flow $\exp(t\vec{g^*})$ is still true, but it does not say anything about the part of the QL lying in Σ since the geodesic flow is stationary at such points. Indeed, we note that the above computation (1.32) does not work anymore for general sub-Laplacians since $\sqrt{-\Delta_{g,\mu}}$ is not a pseudodifferential operator near its characteristic manifold Σ, and hence Egorov's theorem does not apply. Therefore, it is interesting to determine other invariance properties for this part of the QL.

In [CHT18, Theorem B], it was proved that for any sub-Laplacian $\Delta_{g,\mu}$, any of its Quantum Limit ν can be decomposed as a sum $\nu = \beta\nu_0 + (1-\beta)\nu_\infty$ of mutually singular probability measures, where ν_0 is supported in the "elliptic part" U^*M and is invariant under the sub-Riemannian geodesic flow $\exp(t\vec{g^*})$, and ν_∞ is supported in $S\Sigma$ (and its invariance properties are far more difficult to establish, as will be seen below). It was also proved that for "most" QLs, $\nu_0 = 0$, and therefore ν_∞ is the "main part" of the QL. The precise statement is the following.

Proposition 1.29. *[CHT18, Theorem B] Let $(\varphi_k)_{k\in\mathbb{N}^*}$ be an orthonormal Hilbert basis of $L^2(M,\mu)$ consisting of eigenfunctions of $-\Delta_{g,\mu}$ associated with the eigenvalues $(\lambda_k)_{k\in\mathbb{N}^*}$ labeled in increasing order. Let ν be a QL associated with $(\varphi_k)_{k\in\mathbb{N}^*}$. Using the identification $S^*M = U^*M \cup S\Sigma$ (see (1.33)), the probability measure ν can be written as the sum $\nu = \beta\nu_0 + (1-\beta)\nu_\infty$ of two mutually singular measures with $\nu_0, \nu_\infty \in \mathscr{P}(S^*M)$, $\beta \in [0,1]$ and*

(1) $\nu_0(S\Sigma) = 0$ and ν_0 is invariant under the sub-Riemannian geodesic flow $\exp(t\vec{g^*})$;

(2) ν_∞ is supported on $S\Sigma$. Moreover, ***in the 3D contact case****, ν_∞ is invariant under the lift to $S\Sigma$ of the Reeb flow.*[9]

[9] See [CHT18] for a definition of the Reeb flow.

Moreover, there exists a density-one sequence $(k_\ell)_{\ell\in\mathbb{N}}$ of positive integers such that, if ν is a QL associated with a subsequence of $(k_\ell)_{\ell\in\mathbb{N}}$, then the support of ν is contained in $S\Sigma$, i.e., $\beta = 0$ in the previous decomposition.

In [CHT18], the authors also proved Weyl laws (i.e., results "in average" on eigenfunctions), and Quantum Ergodicity properties (i.e., equidistribution of Quantum Limits under an ergodicity assumption) for 3D contact sub-Laplacians.

The Quantum Limits of H-type (or Heisenberg-type) sub-Laplacians were also implicitly studied in [FF21] thanks to a detailed study of the Schrödinger flow. There are some relations between the results of [FF21], which work for H-type sub-Laplacians, and our results, although the precise statements are not the same and the techniques are totally different: semiclassical analysis and non-commutative harmonic analysis for [FF21] (as in the third result of Section 1.3.3) versus joint spectral calculus for our work. We will come back to it in Chapter 7.

We end this section with the definition of joint microlocal defect measures, which is needed to state our results in the next section.

Definition 1.30. *Let $(u_k)_{k\in\mathbb{N}^*}, (v_k)_{k\in\mathbb{N}^*}$ be bounded sequences in $L^2(M)$ such that u_k and v_k weakly converge to 0 as $k \to +\infty$. We call joint microlocal defect measure of $(u_k)_{k\in\mathbb{N}^*}$ and $(v_k)_{k\in\mathbb{N}^*}$ any Radon measure ν_{joint} on S^*M such that for any $a \in \mathscr{S}^0(M)$, there holds*

$$(Op(a)u_{\sigma(k)}, v_{\sigma(k)}) \underset{k\to+\infty}{\longrightarrow} \int_{S^*M} a d\nu_{joint}$$

for some extraction σ.

In case $u_k = v_k$ for any $k \in \mathbb{N}^*$, we recover the microlocal defect measures of Definition 1.26. Note that joint microlocal defect measures are not necessarily non-negative, and that joint Quantum Limits (defined as joint microlocal defect measures of two sequences of eigenfunctions) are not necessarily invariant under the geodesic flow, even in the Riemannian case.

1.5.2 Main results

Our results seek to describe the measure ν_∞ (see Proposition 1.29 above) in other cases than the 3D contact case handled in [CHT18]. We restrict our attention to particular sub-Laplacians, for which, despite their lack of ellipticity, techniques of joint (elliptic) spectral calculus apply thanks to additional commutativity assumptions.

Let us fix sub-Laplacian $\Delta_{g,\mu}$ given by (1.2) and denote by $\mathcal{D}$ the associated distribution. We make the following assumption:

Assumption (A). There exist $Z_1, \ldots, Z_m$ smooth global vector fields on M such that:

(i) At any point $x \in M$ where $\mathcal{D}_x \neq T_xM$, the vector fields $Z_1(x), \ldots, Z_m(x)$ complete $\mathcal{D}_x$ into a basis of T_xM (in particular, they are independent);

(ii) For any $1 \leqslant i, j \leqslant m$, there holds $[\Delta_{g,\mu}, Z_i] = [Z_i, Z_j] = 0$.

Assumption (A) is satisfied for example in the following cases:

- For Baouendi-Grushin-type sub-Laplacians, see Examples 1.3 and 1.17. In this case, $m = 1$ and $Z_1 = \partial_y$.
- For H-type sub-Laplacians, in particular for sub-Laplacians defined on the $(2d + 1)$-dimensional Heisenberg group (see Example 1.4). In this case, the vector fields Z_j form a basis of the center of the associated Lie algebra.

- For the Martinet sub-Laplacian, see Example 1.7. In this case, $m = 1$ and $Z_1 = \partial_z$.
- For manifolds obtained as products of the previous examples (and associated sub-Laplacians obtained by sum), since Assumption (A) is stable by product. For example, for the quasi-contact sub-Laplacian $\partial_x^2 + (\partial_y - x\partial_z)^2 + \partial_w^2$.

Of course, in the above examples, M is not compact, but it is possible to take adequate quotients and get sub-Laplacians defined on the quotients and still satisfying Assumption (A).

Let us introduce some notations of symplectic geometry. We denote by ω the canonical symplectic form on the cotangent bundle T^*M of M. In local coordinates (q,p) of T^*M, we have $\omega = dp \wedge dq$. Given a smooth Hamiltonian function $h : T^*M \to \mathbb{R}$, we denote by $\vec{h}$ the corresponding Hamiltonian vector field on T^*M, defined by $\omega(\vec{h}, \cdot) = -dh(\cdot)$ (in other parts of the manuscript, this Hamiltonian vector field is denoted by H_h, see Appendix A.1). Given any smooth vector field V on M, we denote by h_V the Hamiltonian function (momentum map) on T^*M associated with V, defined in local coordinates by $h_V(q,p) = p(V(q))$. The Hamiltonian flow $\exp(t\vec{h}_V)$ of h_V projects onto the integral curves of V.

Quantum Limits under Assumption (A)

In all the sequel, we consider a sub-Laplacian $\Delta_{g,\mu}$ satisfying Assumption (A). Let

$$\mathcal{P} = \mathcal{P}(\{1, \dots, m\})$$

be the set of all subsets of $\{1, \dots, m\}$. We write Σ as a disjoint union

$$\Sigma = \bigcup_{\mathcal{J} \in \mathcal{P}} \Sigma_{\mathcal{J}} \tag{1.34}$$

where, for $\mathcal{J} \in \mathcal{P}$, $\Sigma_{\mathcal{J}}$ is defined as the set of points $(q,p) \in \Sigma$ with

$$\left(h_{Z_j}(q,p) \neq 0\right) \Leftrightarrow (j \in \mathcal{J}).$$

Note that (1.34) is a disjoint union and that the $\Sigma_{\mathcal{J}}$ are non-empty, thanks to point (i) in Assumption (A) together with the fact that $\pi(\Sigma) = \{x \in M,\ \mathcal{D}_x \neq T_xM\}$ where $\pi : T^*M \to M$ is the canonical projection.

Our first main result on eigenfunctions states that it is possible to split any QL into several pieces which can be studied separately, and which come from well-characterized parts of the associated sequence of eigenfunctions.

The operator

$$E = -\Delta_{g,\mu} + \sum_{j=1}^{m} Z_j^* Z_j$$

is elliptic. According to Assumption (A), the following decomposition holds:

$$L^2(M) = \bigoplus \mathcal{H}_{\alpha,\beta_1,\dots,\beta_m} \tag{1.35}$$

where, for any $f \in \mathcal{H}_{\alpha,\beta_1,\dots,\beta_m}$, we have

$$-\Delta_{g,\mu} f = \alpha^2 f, \qquad Z_j^* Z_j f = \beta_j^2 f, \qquad Ef = \left(\alpha^2 + \sum_{j=1}^{m} \beta_j^2\right) f.$$

The main idea is that there is a correspondance between the decomposition (1.34) of Σ and the decomposition (1.35) of $L^2(M)$ in the limit where $\alpha^2+\sum_{j=1}^m \beta_j^2 \to +\infty$. Let us explain this point. For $n\in\mathbb{N}^*$, let $\chi_n\in C_c^\infty(\mathbb{R},[0,1])$ such that $\chi_n(x)=1$ for $|x|\leqslant \frac{1}{2n}$ and $\chi_n(x)=0$ for $|x|\geqslant \frac{1}{n}$. Then, for any $\mathcal{J}\in\mathcal{P}\setminus\emptyset$ and any $n\in\mathbb{N}^*$, we consider

$$P_n^{\mathcal{J}}=\chi_n\left(\frac{-\Delta_{g,\mu}}{E}\right)\prod_{i\notin\mathcal{J}}\chi_n\left(\frac{Z_i^*Z_i}{E}\right)\prod_{j\in\mathcal{J}}(1-\chi_n)\left(\frac{Z_j^*Z_j}{E}\right). \tag{1.36}$$

Similarly, we consider

$$P_n^{\emptyset}=(1-\chi_n)\left(\frac{-\Delta_{g,\mu}}{E}\right). \tag{1.37}$$

These operators form a "microlocal partition of unity", i.e., $\sum_{\mathcal{J}\in\mathcal{P}} P_n^{\mathcal{J}}=1$. Moreover, we have the following properties:

- If $\mathcal{J}\neq\emptyset$, then $\sigma_P(P_n^{\mathcal{J}})\to \mathbf{1}_{\Sigma_{\mathcal{J}}}$ pointwise as $n\to+\infty$.
- If $\mathcal{J}=\emptyset$, then $\sigma_P(P_n^{\mathcal{J}})\to \mathbf{1}_{U^*M}$ pointwise as $n\to+\infty$.

Therefore, when applied to an (eigen)function, and as $n\to+\infty$, the operators $P_n^{\mathcal{J}}$ allow to cut this function into small pieces whose "microlocalization" is known. These ideas are close to those of [Col79, Theorem 0.6], which deals with the joint spectrum of commuting pseudodifferential operators whose sum of squares is elliptic.

Therefore, the following result is not a surprise.

Theorem 8: [Let20a]

Let $\Delta_{g,\mu}$ satisfy Assumption (A). We assume that $(\varphi_k)_{k\in\mathbb{N}^*}$ is a normalized sequence of eigenfunctions of $-\Delta_{g,\mu}$ with associated eigenvalues $\lambda_k\to+\infty$. Then, up to extraction of a subsequence, one can decompose

$$\varphi_k=\varphi_k^{\emptyset}+\sum_{\mathcal{J}\in\mathcal{P}\setminus\{\emptyset\}}\varphi_k^{\mathcal{J}}, \tag{1.38}$$

with the following properties:

- The sequence $(\varphi_k)_{k\in\mathbb{N}^*}$ has a unique Quantum Limit ν;
- For any $\mathcal{J}\in\mathcal{P}$ and any $k\in\mathbb{N}^*$, $\varphi_k^{\mathcal{J}}$ is an eigenfunction of $-\Delta_{g,\mu}$ with eigenvalue λ_k;
- Using the identification $S^*M=U^*M\cup S\Sigma$ (see (1.33)), the sequence $(\varphi_k^{\emptyset})_{k\in\mathbb{N}^*}$ admits a unique microlocal defect measure $\beta\nu^{\emptyset}$, where $\beta\in[0,1]$, $\nu^{\emptyset}\in\mathscr{P}(S^*M)$ and $\nu^{\emptyset}(S\Sigma)=0$, and, for any $\mathcal{J}\in\mathcal{P}\setminus\{\emptyset\}$, the sequence $(\varphi_k^{\mathcal{J}})_{k\in\mathbb{N}^*}$ also admits a unique microlocal defect measure $\nu^{\mathcal{J}}$, having all its mass contained in $S\Sigma_{\mathcal{J}}$;
- For any $\mathcal{J}\neq\mathcal{J}'\in\mathcal{P}$, the joint microlocal defect measure of the sequences $(\varphi_k^{\mathcal{J}})_{k\in\mathbb{N}^*}$ and $(\varphi_k^{\mathcal{J}'})_{k\in\mathbb{N}^*}$ vanishes. As a consequence,

$$\nu=\beta\nu^{\emptyset}+\sum_{\mathcal{J}\in\mathcal{P}\setminus\{\emptyset\}}\nu^{\mathcal{J}} \tag{1.39}$$

and the sum in (1.39) is supported in $S\Sigma$.

In this statement, we separated the emptyset from the other subsets $\mathcal{J} \in \mathcal{P}\backslash\{\emptyset\}$ to emphasize on the concentration of $\beta\nu^{\emptyset}$ on U^*M, while the rest of the measure ν in (1.39) is supported in $S\Sigma$. This is purely artificial, since one could have included $\beta\nu^{\emptyset}$ into the sum over $\mathcal{J}$. Besides, the notation $\nu^{\emptyset}$ used above corresponds to the notation ν_0 in [CHT18] (see Proposition 1.29 above): we changed it to get a unified notation for the different parts of the QL, namely $\nu^{\emptyset}$ and $\nu^{\mathcal{J}}$.

Products of flat contact sub-Laplacians

Our second main result gives much more information on Quantum Limits, but it works only for a very specific family of sub-Laplacians, which in particular satisfy Assumption (A). In order to define these sub-Laplacians, let us first recall the definition of the 3D Heisenberg group. This definition is not exactly the same as in Example 1.4 (with $d = 1$), but these two definitions yield isomorphic groups. Endow $\mathbb{R}^3$ with the product law

$$(x, y, z) \star (x', y', z') = (x + x', y + y', z + z' - xy').$$

With this law, $\widetilde{\mathbf{H}} = (\mathbb{R}^3, \star)$ is a Lie group, which is isomorphic to the group of matrices

$$\left\{ \begin{pmatrix} 1 & x & -z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix}, \ x, y, z \in \mathbb{R} \right\}$$

endowed with the standard product law on matrices.

We consider the left quotient $\mathbf{H} = \Gamma\backslash\widetilde{\mathbf{H}}$ where $\Gamma = (\sqrt{2\pi}\mathbb{Z})^2 \times 2\pi\mathbb{Z}$ is a cocompact subgroup of $\widetilde{\mathbf{H}}$ (meaning that $\mathbf{H}$ is compact). Note that $\mathbf{H}$ is not homeomorphic to $\mathbb{T}^2 \times \mathbb{S}^1$ since its fundamental group is Γ. The vector fields on $\mathbf{H}$

$$X = \partial_x \quad \text{and} \quad Y = \partial_y - x\partial_z$$

are left invariant, and we consider $\Delta_{\mathbf{H}} = X^2 + Y^2$ the associated sub-Laplacian (here μ is the Lebesgue measure $\mu = dxdydz$ and (X, Y) is orthonormal for g).

Then, we consider the product manifold $\mathbf{H}^m$ and the associated sub-Laplacian Δ for some integer $m \geqslant 2$, that is

$$\Delta = \Delta_{\mathbf{H}} \otimes (\mathrm{Id})^{\otimes m-1} + \mathrm{Id} \otimes \Delta_{\mathbf{H}} \otimes (\mathrm{Id})^{m-2} + \ldots + (\mathrm{Id})^{\otimes m-1} \otimes \Delta_{\mathbf{H}}, \qquad (1.40)$$

which is a second-order pseudodifferential operator. Below, we give an expression (1.41) for Δ which is more tractable.

Note that these sub-Laplacians are not contact sub-Laplacians (in the sense of Example 1.5): they are products of 3D Heisenberg sub-Laplacians.

In the sequel, we fix once for all an integer $m \geqslant 2$.

Remark 1.31. If $(\varphi_k)_{k\in\mathbb{N}^*}$ denotes an orthonormal Hilbert basis of $L^2(\mathbf{H})$ consisting of eigenfunctions of $-\Delta_{\mathbf{H}}$, then

$$\{\varphi_{k_1} \otimes \ldots \otimes \varphi_{k_m} \mid k_1, \ldots, k_m \in \mathbb{N}^*\}$$

is an orthonormal Hilbert basis of $L^2(\mathbf{H}^m)$ consisting of eigenfunctions of $-\Delta$. However, there exist orthonormal Hilbert bases of $L^2(\mathbf{H}^m)$ which cannot be put in this tensorized form.

In order to give a precise statement of our second main result on eigenfunctions, it is necessary to introduce a decomposition of the sub-Laplacian Δ defined by (1.40). Taking coordinates (x_j, y_j, z_j) on the j-th copy of $\mathbf{H}$, we can write

$$\Delta = \sum_{j=1}^{m} (X_j^2 + Y_j^2) \tag{1.41}$$

with $X_j = \partial_{x_j}$ and $Y_j = \partial_{y_j} - x_j\partial_{z_j}$. We note that Δ satisfies Assumption (A) (for $Z_j = \partial_{z_j}$ for $j = 1, \dots, m$). In other words, the operator

$$E = -\Delta + \sum_{j=1}^{m} Z_j^* Z_j$$

is elliptic.

Let us briefly describe Σ (defined by (1.5)) for the sub-Laplacian Δ. Denoting by (q, p) the canonical coordinates in $T^*\mathbf{H}^m$, i.e., $q = (x_1, y_1, z_1, \dots, x_m, y_m, z_m)$ and $p = (p_{x_1}, p_{y_1}, p_{z_1}, \dots, p_{x_m}, p_{y_m}, p_{z_m})$, we obtain that

$$\Sigma = \left\{(q,p) \in T^*\mathbf{H}^m \;\mid\; p_{x_j} = p_{y_j} - x_j p_{z_j} = 0 \text{ for any } 1 \leqslant j \leqslant m\right\},$$

which is isomorphic to $\mathbf{H}^m \times \mathbb{R}^m$. Above any point $q \in \mathbf{H}^m$, the fiber of Σ is of dimension m, and therefore, above any point $q \in \mathbf{H}^m$, $S\Sigma$ consists of an $(m-1)$-dimensional sphere.

For $1 \leqslant j \leqslant m$, we consider the operator $R_j = \sqrt{\partial_{z_j}^* \partial_{z_j}}$ and we make a Fourier expansion with respect to the z_j-variable in the j-th copy of $\mathbf{H}$. On the eigenspaces corresponding to non-zero modes of this Fourier decomposition, we define the operator $\Omega_j = -R_j^{-1}\Delta_j = -\Delta_j R_j^{-1}$ where $\Delta_j = X_j^2 + Y_j^2$. For example, $-\Delta$ acts as

$$-\Delta = \sum_{j=1}^{m} R_j \Omega_j \tag{1.42}$$

on any eigenspace of $-\Delta$ on which $R_j \neq 0$ for any $1 \leqslant j \leqslant m$.

The operator Ω_j, seen as an operator on the j-th copy of $\mathbf{H}$, is an harmonic oscillator, having in particular eigenvalues $2n+1$, $n \in \mathbb{N}$ (see [CHT18, Section 3.1]). Moreover, the operators Ω_i (considered this time as operators on $\mathbf{H}^m$) commute with each other and with the operators R_j.

Instead of the simple decomposition (1.35) with respect to the Z_j (or, here, the R_j), our second result requires a decomposition of $L^2(\mathbf{H}^m)$ with respect to the operators R_j and Ω_j. This decomposition is not easy to write down, and we postpone the full details to Chapter 7, but we explain here the guiding lines.

Thanks to Theorem 8, it is possible to fix $\mathcal{J} \in \mathcal{P}$ and to focus only on $\varphi_k^{\mathcal{J}}$ (with the notations of this theorem). We decompose the action of $-\Delta$ on functions microlocalized in this part of the phase space as a sum of the operators

$$\sum_{j \in \mathcal{J}} \omega_j R_j$$

where ω_j accounts for the eigenvalue of Ω_j. Any of these first-order operators, is proportional to an operator of the form

$$R_s^{\mathcal{J}} = \sum_{j \in \mathcal{J}} s_j R_j$$

where $s=(s_j)_{j\in\mathcal{J}}$ is in the simplex

$$\mathbf{S}_{\mathcal{J}}=\left\{s=(s_j)\in\mathbb{R}_+^{\mathcal{J}},\ \sum_{j\in\mathcal{J}}s_j=1\right\}.$$

This encourages us to introduce, for $s\in\mathbf{S}_{\mathcal{J}}$,

$$\rho_s^{\mathcal{J}}(q,p)=(\sigma_P(R_s))_{|\Sigma_{\mathcal{J}}} \tag{1.43}$$

where σ_P denotes the principal symbol (see Appendix A). These Hamiltonians are homogeneous of degree 1, and they replace $\sqrt{g^*}$ (see around (1.32)) in the invariance properties of the QLs.

Indeed, we will prove that ν_∞, introduced in Proposition 1.29, belongs to the set (see the footnote[10] for the notations)

$$\begin{aligned}\mathscr{P}_{S\Sigma}=\Bigg\{\nu_\infty=\sum_{\mathcal{J}\in\mathcal{P}\setminus\{\emptyset\}}\nu^{\mathcal{J}}\in\mathscr{P}(S^*\mathbf{H}^m),\quad \nu^{\mathcal{J}}=\int_{\mathbf{S}_{\mathcal{J}}}\nu_s^{\mathcal{J}}\,dQ^{\mathcal{J}}(s),\\ \text{where } Q^{\mathcal{J}}\in\mathscr{M}_+(\mathbf{S}_{\mathcal{J}}),\ \nu_s^{\mathcal{J}}\in\mathscr{P}(S^*\mathbf{H}^m),\\ \nu_s^{\mathcal{J}}(S^*\mathbf{H}^m\setminus S\Sigma_{\mathcal{J}})=0 \text{ and, for } Q^{\mathcal{J}}\text{-almost any } s\in\mathbf{S}_{\mathcal{J}}\ ,\ \vec{\rho}_s^{\,\mathcal{J}}\nu_s^{\mathcal{J}}=0\Bigg\}\end{aligned} \tag{1.44}$$

In a few words, (1.44) means that any measure $\nu_\infty\in\mathscr{P}_{S\Sigma}$ is supported in $S\Sigma$, and that its invariance properties are given separately on each set $S\Sigma_{\mathcal{J}}$ (for $\mathcal{J}\in\mathcal{P}\setminus\{\emptyset\}$). Its restriction to any of these sets, denoted by $\nu^{\mathcal{J}}$, can be disintegrated with respect to $\mathbf{S}_{\mathcal{J}}$, and for any $s\in\mathbf{S}_{\mathcal{J}}$, there is a corresponding measure $\nu_s^{\mathcal{J}}$ which is invariant under the flow $e^{t\vec{\rho}_s^{\,\mathcal{J}}}$.

Our second main result on eigenfunctions is the following:

Theorem 9: [Let20a]

Let $(\varphi_k)_{k\in\mathbb{N}^*}$ be an orthonormal Hilbert basis of $L^2(\mathbf{H}^m)$ consisting of eigenfunctions of $-\Delta$ associated with the eigenvalues $(\lambda_k)_{k\in\mathbb{N}^*}$ labeled in increasing order. Let ν be a Quantum Limit associated to the sequence $(\varphi_k)_{k\in\mathbb{N}^*}$. Then, using the identification (1.33), we can write ν as the sum of two mutually singular measures $\nu=\beta\nu^\emptyset+(1-\beta)\nu_\infty$, with $\nu^\emptyset,\nu_\infty\in\mathscr{P}(S^*\mathbf{H}^m)$, $\beta\in[0,1]$ and

(1) $\nu^\emptyset(S\Sigma)=0$ and $\nu^\emptyset$ is invariant under the sub-Riemannian geodesic flow $e^{t\vec{g}^*}$;

(2) $\nu_\infty\in\mathscr{P}_{S\Sigma}$.

Moreover, there exists a density-one sequence $(k_\ell)_{\ell\in\mathbb{N}}$ of positive integers such that, if ν is a QL associated with a subsequence of $(k_\ell)_{\ell\in\mathbb{N}}$, then the support of ν is contained in $S\Sigma$, i.e., $\beta=0$ in the previous decomposition.

The reason why we consider here only orthonormal bases is to give a sense to the density-one subsequence of the last part of the statement. However, the first part of the statement is true for any sequence of normalized eigenfunctions of $-\Delta$ with eigenvalues tending to $+\infty$.

[10] The notation $\mathscr{M}_+(E)$ (respectively $\mathscr{P}(E)$) denotes the set of non-negative Radon measures (respectively Radon probability measures) on a given separated space E. The notation $S\Sigma_{\mathcal{J}}$ designates the set of points (q,p) of $S\Sigma$ which have null (homogeneous) coordinate p_{z_i} for any $i\notin\mathcal{J}$ and non-null p_{z_j} for $j\in\mathcal{J}$. Note that this set is, in general, neither open nor closed.

Note that Theorem 9 holds for *any* orthonormal Hilbert basis of $L^2(\mathbf{H}^m)$ consisting of eigenfunctions of $-\Delta$, and not only for the bases described in Remark 1.31.

The converse of Theorem 9 holds too, in the following sense:

Theorem 10: [Let20a]

Let $\nu_\infty \in \mathscr{P}_{S\Sigma}$. Then ν_∞ is a Quantum Limit associated to some sequence of normalized eigenfunctions of $-\Delta$ with eigenvalues tending to $+\infty$.

Theorem 10 and Point (2) of Theorem 9 serve as substitutes to Point (2) of Proposition 1.29 for the sub-Laplacians Δ on $\mathbf{H}^m$. Together, Theorem 9 and Theorem 10 yield a classification of (nearly) all Quantum Limits of Δ.

The particular geometry of the QLs of Δ. As already recalled, the QLs of Riemannian Laplacians are invariant under the geodesic flow: in some sense, this means that for any $(x, \xi) \in T^*M$, the QL near (x, ξ) "is invariant in the direction given by ξ". The above Proposition 1.29 for 3D contact sub-Laplacians, and the result of [FF21, Theorem 2.10(ii)(2)] for H-type groups extend this intuition to these sub-Laplacians. But Theorems 9 and 10 show that such a property is not true for any sub-Laplacian: there exist QLs of Δ and points $(x, \xi) \in \mathbf{H}^m$ such that the QL near (x, ξ) is not invariant in the direction ξ, but in some other direction of the cotangent bundle (parametrized by $s \in \mathbf{S}$). This fact will be highlighted again along the proof of Theorem 10.

The next chapters present our works [Let20b, LS20, FL21, Let21b, CL21, Let20a]. The presentation may very occasionally differ from the versions published (or submitted for publication). To keep each chapter as self-contained as possible, we kept within these chapters nearly all reminders and comments, even when they were slightly redundant with the information already provided in the introduction. In each chapter, the concluding section provides "supplementary material", which corresponds to appendices in the original preprints.

Chapter 2

Subelliptic wave equations are never observable

"Un bon contrôle, c'est la moitié d'un but."
Michel Platini.

This chapter is adapted from [Let20b]. Its main object is the proof of Theorem 1.

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It is well-known that observability (and, by duality, controllability) of the elliptic wave equation, i.e., with a Riemannian Laplacian, in time T_0 is almost equivalent to the Geometric Control Condition (GCC), which stipulates that any geodesic ray meets the control set within time T_0. We show that in the subelliptic setting, GCC is never verified, and that subelliptic wave equations are never observable in finite time. More precisely, given any subelliptic Laplacian $\Delta = -\sum_{i=1}^m X_i^* X_i$ on a manifold M, and any measurable subset $\omega \subset M$ such that $M \backslash \omega$ contains in its interior a point q with $[X_i, X_j](q) \notin \text{Span}(X_1, \ldots, X_m)$ for some $1 \leqslant i, j \leqslant m$, we show that for any $T_0 > 0$, the wave equation with subelliptic Laplacian Δ is not observable on ω in time T_0.

The proof is based on the construction of sequences of solutions of the wave equation concentrating on geodesics (for the associated sub-Riemannian distance) spending a long time in $M \backslash \omega$. As a counterpart, we prove a positive result of observability for the wave equation in the Heisenberg group, where the observation set is a well-chosen part of the phase space.

2.1 Introduction

2.1.1 Setting

Let $n \in \mathbb{N}^*$ and let M be a smooth connected compact manifold of dimension n with a non-empty boundary ∂M. Let μ be a smooth volume on M. We consider $m \geqslant 1$ smooth vector fields $X_1, \ldots, X_m$ on M which are not necessarily independent, and we assume that the following Hörmander condition holds (see [Hor67]):

The vector fields $X_1, \ldots, X_m$ and their iterated brackets $[X_i, X_j], [X_i, [X_j, X_k]]$, etc. span the tangent space T_qM at every point $q \in M$.

We consider the sub-Laplacian Δ defined by

$$\Delta = -\sum_{i=1}^m X_i^* X_i = \sum_{i=1}^m X_i^2 + \text{div}_\mu(X_i) X_i$$

where the star designates the transpose in $L^2(M, \mu)$ and the divergence with respect to μ is defined by $L_X \mu = (\text{div}_\mu X)\mu$, where L_X stands for the Lie derivative. Then Δ is hypoelliptic (see [Hor67, Theorem 1.1]).

We consider Δ with Dirichlet boundary conditions and the domain $D(\Delta)$ which is the completion in $L^2(M, \mu)$ of the set of all $u \in C_c^\infty(M)$ for the norm $\|(\text{Id} - \Delta)u\|_{L^2}$. We also consider the operator $(-\Delta)^{\frac{1}{2}}$ with domain $D((-\Delta)^{\frac{1}{2}})$ which is the completion in $L^2(M, \mu)$ of the set of all $u \in C_c^\infty(M)$ for the norm $\|(\text{Id} - \Delta)^{\frac{1}{2}} u\|_{L^2}$.

Consider the wave equation

$$\begin{cases} \partial_{tt}^2 u - \Delta u = 0 \quad \text{in } (0, T) \times M \\ u = 0 \quad \text{on } (0, T) \times \partial M, \\ (u_{|t=0}, \partial_t u_{|t=0}) = (u_0, u_1) \end{cases} \tag{2.1}$$

where $T > 0$. It is well-known (see for example [GR15, Theorem 2.1], [EN99, Chapter II, Section 6]) that for any $(u_0, u_1) \in D((-\Delta)^{\frac{1}{2}}) \times L^2(M)$, there exists a unique solution

$$u \in C^0(0, T; D((-\Delta)^{\frac{1}{2}})) \cap C^1(0, T; L^2(M)) \tag{2.2}$$

to (2.1) (in a mild sense).

We set

$$\|v\|_{\mathcal{H}} = \left(\int_M |\nabla^{\mathrm{sR}} v(x)|^2 d\mu(x)\right)^{\frac{1}{2}}. \tag{2.3}$$

where, for any $\phi \in C^\infty(M)$,

$$\nabla^{sR}\phi = \sum_{i=1}^{m} (X_i\phi)X_i$$

is the horizontal gradient. Note that ∇^{sR} is the formal adjoint of $(-\mathrm{div}_\mu)$ in $L^2(M,\mu)$, and that $\Delta = \mathrm{div}_\mu \circ \nabla^{\mathrm{sR}}$. Note also that $\|v\|_{\mathcal{H}} = \|(-\Delta)^{\frac{1}{2}} v\|_{L^2(M,\mu)}$.

The natural energy of a solution is

$$E(u(t,\cdot)) = \frac{1}{2}(\|\partial_t u(t,\cdot)\|^2_{L^2(M,\mu)} + \|u(t,\cdot)\|^2_{\mathcal{H}}).$$

If u is a solution of (2.1), then

$$\frac{d}{dt}E(u(t,\cdot)) = 0,$$

and therefore the energy of u at any time is equal to

$$\|(u_0,u_1)\|^2_{\mathcal{H}\times L^2} = \|u_0\|^2_{\mathcal{H}} + \|u_1\|^2_{L^2(M,\mu)}.$$

In this chapter, we investigate exact observability for the wave equation (2.1).

Definition 2.1. *Let $T_0 > 0$ and $\omega \subset M$ be a μ-measurable subset. The subelliptic wave equation* (2.1) *is exactly observable on ω in time T_0 if there exists a constant $C_{T_0}(\omega) > 0$ such that, for any $(u_0,u_1) \in D((-\Delta)^{\frac{1}{2}}) \times L^2(M)$, the solution u of* (2.1) *satisfies*

$$\int_0^{T_0}\int_\omega |\partial_t u(t,x)|^2 d\mu(x)dt \geqslant C_{T_0}(\omega)\|(u_0,u_1)\|^2_{\mathcal{H}\times L^2}. \tag{2.4}$$

2.1.2 Main result

Our main result is the following.

Theorem 2.2. *Let $T_0 > 0$ and let $\omega \subset M$ be a measurable subset. We assume that there exist $1 \leqslant i,j \leqslant m$ and q in the interior of $M \backslash \omega$ such that $[X_i,X_j](q) \notin \mathrm{Span}(X_1(q),\ldots,X_m(q))$. Then the subelliptic wave equation* (2.1) *is not exactly observable on ω in time T_0.*

Consequently, using a duality argument (see Section 2.4.2), we obtain that exact controllability does not hold either in any finite time.

Definition 2.3. *Let $T_0 > 0$ and $\omega \subset M$ be a measurable subset. The subelliptic wave equation* (2.1) *is exactly controllable on ω in time T_0 if for any $(u_0,u_1) \in D((-\Delta)^{\frac{1}{2}}) \times L^2(M)$, there exists $g \in L^2((0,T_0)\times M)$ such that the solution u of*

$$\begin{cases} \partial^2_{tt}u - \Delta u = \mathbf{1}_\omega g \quad \textit{in } (0,T_0)\times M \\ u = 0 \quad \textit{on } (0,T_0)\times \partial M, \\ (u_{|t=0}, \partial_t u_{|t=0}) = (u_0,u_1) \end{cases} \tag{2.5}$$

satisfies $u(T_0,\cdot) = 0$.

Corollary 2.4. *Let $T_0 > 0$ and let $\omega \subset M$ be a measurable subset. We assume that there exist $1 \leqslant i,j \leqslant m$ and q in the interior of $M \backslash \omega$ such that $[X_i,X_j](q) \notin \mathrm{Span}(X_1(q),\ldots,X_m(q))$. Then the subelliptic wave equation* (2.1) *is not exactly controllable on ω in time T_0.*

In what follows, we denote by $\mathcal{D}$ the set of all vector fields that can be decomposed as linear combinations with smooth coefficients of the X_i:

$$\mathcal{D} = \mathrm{Span}(X_1, \dots, X_m) \subset TM.$$

$\mathcal{D}$ is called the *distribution* associated to the vector fields $X_1, \dots, X_m$. For $q \in M$, we denote by $\mathcal{D}_q \subset T_qM$ the distribution $\mathcal{D}$ taken at point q.

The assumptions of Theorem 2.2 are satisfied as soon as the interior U of $M \setminus \omega$ is non-empty and $\mathcal{D}$ has constant rank $< n$ in U. Indeed, under these conditions, we can argue by contradiction: assume that for any $q \in U$ and any $1 \leqslant i, j \leqslant m$, there holds $[X_i, X_j](q) \in \mathrm{Span}(X_1(q), \dots, X_m(q)) = \mathcal{D}_q$. Then we have $[\mathcal{D}, \mathcal{D}] \subset \mathcal{D}$ in U, i.e., $\mathcal{D}$ is involutive. By Frobenius's theorem, $\mathcal{D}$ is then completely integrable, which contradicts Hörmander's condition.

The following examples show that the assumptions of Theorem 2.2 are also satisfied in some non-constant rank cases:

Example 2.5. In the Baouendi-Grushin case, for which $X_1 = \partial_{x_1}$ and $X_2 = x_1\partial_{x_2}$ are vector fields on $(-1,1)_{x_1} \times \mathbb{T}_{x_2}$ where $\mathbb{T} = \mathbb{R}/\mathbb{Z}$, the corresponding sub-Laplacian $\Delta = X_1^2 + X_2^2$ (here, $\mu = dx_1dx_2$ for simplicity) is elliptic outside of the singular submanifold $S = \{x_1 = 0\}$. Therefore, the corresponding subelliptic wave equation is observable on any open subset containing S (with some finite minimal time of observability, see [BLR92]), but according to Theorem 2.2, it is not observable in any finite time on any subset ω such that the interior of $M \setminus \omega$ has a non-empty intersection with S.

Example 2.6. In the Martinet case, the vector fields are $X_1 = \partial_{x_1}$ and $X_2 = \partial_{x_2} + x_1^2\partial_{x_3}$ on $(-1,1)_{x_1} \times \mathbb{T}_{x_2} \times \mathbb{T}_{x_3}$, and the corresponding sub-Laplacian is $\Delta = X_1^2 + X_2^2$ (again, $\mu = dx_1dx_2dx_3$ for simplicity). Then, we have $[X_1, X_2] = 2x_1\partial_{x_3}$. The only points at which this bracket belongs to the distribution $\mathrm{Span}(X_1, X_2)$ are the points for which $x_1 = 0$. Since this set of points has empty interior, the assumptions of Theorem 2.2 are satisfied as soon as $M \setminus \omega$ has non-empty interior.

Remark 2.7. The assumption of compactness on M is not necessary: we may remove it, and just require that the subelliptic wave equation (2.1) in M is well-posed. It is for example the case if M is complete for the sub-Riemannian distance induced by $X_1, \dots, X_m$ since Δ is then essentially self-adjoint ([Str86]).

Remark 2.8. Theorem 2.2 remains true if M has no boundary. In this case, the equation (2.1) is well-posed in a space slightly smaller than (2.2): a condition of null average has to be added since non-zero constant functions on M are solutions of (2.1), see Section 2.1.5. The observability inequality of Theorem 2.2 remains true in this space of solutions: anticipating the proof, we notice that the spiraling normal geodesics of Proposition 2.17 still exist (since their construction is purely local), and we subtract to the initial datum u_0^k of the localized solutions constructed in Proposition 2.16 their spatial average $\int_M u_0^k d\mu$.

Remark 2.9. Thanks to abstract results (see for example [Mil12]), Theorem 2.2 remains true when the subelliptic wave equation (2.1) is replaced by the subelliptic half-wave equation $\partial_t u + i\sqrt{-\Delta}u = 0$ with Dirichlet boudary conditions.

2.1.3 Ideas of the proof

In the sequel, we call "normal geodesic"[1] the projection on M of a bicharacteristic (parametrized by time) for the principal symbol of the wave equation (2.1). We will give a more detailed

[1] This terminology is common in sub-Riemannian geometry, and it is justified by the fact that we can naturally associate to the vector fields $X_1, \dots, X_m$ a metric structure on M for which these projected paths are geodesics, see [Mon02].

definition in Section 2.1.4.

The proof of Theorem 2.2 mainly requires two ingredients:

1. There exist solutions of the free subelliptic wave equation (2.1) whose energy concentrates along any given normal geodesic;

2. There exist normal geodesics which "spiral" around curves transverse to $\mathcal{D}$, and which therefore remain arbitrarily close to their starting point on arbitrarily large time-intervals.

Combining the two above facts, the proof of Theorem 2.2 is straightforward (see Section 2.4.1). Note that the first point follows from the general theory of propagation of complex Lagrangian spaces, while the second point is the main novelty of this work.

Since our construction is purely local (meaning that it does not "feel" the boundary and only relies on the local structure of the vector fields), we can focus on the case where there is a (small) open neighborhood V of the origin O such that $V \subset M \backslash \omega$, and $[X_i, X_j](O) \notin \mathcal{D}_O$ for some $1 \leqslant i, j \leqslant m$. In the sequel, we assume it is the case.

Let us give an example of vector fields where the spiraling normal geodesics used in the proof of Theorem 2.2 are particularly simple. We consider the three-dimensional manifold with boundary $M_1 = (-1,1)_{x_1} \times \mathbb{T}_{x_2} \times \mathbb{T}_{x_3}$, where $\mathbb{T} = \mathbb{R}/\mathbb{Z} \approx (-1,1)$ is the 1D torus. We endow M_1 with the vector fields $X_1 = \partial_{x_1}$ and $X_2 = \partial_{x_2} - x_1\partial_{x_3}$. This is the "Heisenberg manifold with boundary". We endow M_1 with an arbitrary smooth volume μ. The normal geodesics we consider are given by

$$\begin{aligned} x_1(t) &= \varepsilon \sin(t/\varepsilon) \\ x_2(t) &= \varepsilon \cos(t/\varepsilon) - \varepsilon \\ x_3(t) &= \varepsilon(t/2 - \varepsilon \sin(2t/\varepsilon)/4). \end{aligned} \tag{2.6}$$

They spiral around the x_3 axis $x_1 = x_2 = 0$.

Here, one should think of ε as a small parameter. In the sequel, we denote by x_ε the normal geodesic with parameter ε.

Clearly, given any $T_0 > 0$, for ε sufficiently small, we have $x_\varepsilon(t) \in V$ for every $t \in (0, T_0)$. Our objective is to construct solutions u^k of the subelliptic wave equation (2.1) such that $\|(u_0^k, u_1^k)\|_{\mathcal{H}\times L^2} = 1$ and the energy of $u^k(t,\cdot)$ concentrates outside of an open set V_t containing $x_\varepsilon(t)$, i.e.,

$$\int_{M_1\backslash V_t} \left(|\partial_t u^k(t,x)|^2 + |\nabla^{\mathrm{sR}} u^k(t,x)|^2\right) d\mu(x)$$

tends to 0 as $k \to +\infty$ uniformly with respect to $t \in (0, T_0)$. As a consequence, the observability inequality (2.4) fails.

The construction of solutions of the free wave equation whose energy concentrates on geodesics is classical in the elliptic (or Riemannian) case: these are the so-called Gaussian beams, for which a construction can be found for example in [Ral82]. Here, we adapt this construction to our subelliptic (sub-Riemannian) setting, which does not raise any problem since the normal geodesics we consider stay in the elliptic part of the operator Δ. It may also be directly justified with the theory of propagation of complex Lagrangian spaces (see Section 2.2).

In the case of general vector fields $X_1, \dots, X_m$, the existence of spiraling normal geodesics also has to be justified. For that purpose, we first approximate $X_1, \dots, X_m$ by their *nilpotent approximations*, and we then prove that for the latters, such a family of spiraling normal geodesics exists, as in the Heisenberg case.

2.1.4 Normal geodesics

In this section, we explain in more details what *normal geodesics* are. As said before, they are natural extensions of Riemannian geodesics since they are projections of bicharacteristics.

We denote by $S^m_{\text{phg}}(T^*((0,T)\times M))$ the set of polyhomogeneous symbols of order m with compact support and by $\Psi^m_{\text{phg}}((0,T)\times M)$ the set of associated polyhomogeneous pseudodifferential operators of order m whose distribution kernel has compact support in $(0,T)\times M$ (see Appendix A).

We set $P = \partial^2_{tt} - \Delta \in \Psi^2_{\text{phg}}((0,T)\times M)$, whose principal symbol is

$$p_2(t,\tau,x,\xi) = -\tau^2 + g^*(x,\xi)$$

with τ the dual variable of t and g^* the principal symbol of $-\Delta$. For $\xi \in T^*M$, we have (see Appendix A)

$$g^* = \sum_{i=1}^m h^2_{X_i}.$$

Here, given any smooth vector field X on M, we denoted by h_X the Hamiltonian function (momentum map) on T^*M associated with X defined in local (x,ξ)-coordinates by $h_X(x,\xi) = \xi(X(x))$.

In $T^*(\mathbb{R}\times M)$, the Hamiltonian vector field $\vec{H}_{p_2}$ associated with p_2 is given by $\vec{H}_{p_2} f = \{p_2, f\}$ where $\{\cdot,\cdot\}$ denotes the Poisson bracket (see Appendix A). Since $\vec{H}_{p_2} p_2 = 0$, we get that p_2 is constant along the integral curves of $\vec{H}_{p_2}$. Thus, the characteristic set $\mathcal{C}(p_2) = \{p_2 = 0\}$ is preserved by the flow of $\vec{H}_{p_2}$. Null-bicharacteristics are then defined as the maximal integral curves of $\vec{H}_{p_2}$ which live in $\mathcal{C}(p_2)$. In other words, the null-bicharacteristics are the maximal solutions of

$$\begin{cases} \dot{t}(s) = -2\tau(s)\,, \\ \dot{x}(s) = \nabla_\xi g^*(x(s),\xi(s))\,, \\ \dot{\tau}(s) = 0\,, \\ \dot{\xi}(s) = -\nabla_x g^*(x(s),\xi(s))\,, \\ \tau^2(0) = g^*(x(0),\xi(0)). \end{cases} \tag{2.7}$$

This definition needs to be adapted when the null-bicharacteristic meets the boundary ∂M, but in the sequel, we only consider solutions of (2.7) on time intervals where $x(t)$ does not reach ∂M.

In the sequel, we take $\tau = -1/2$, which gives $g^*(x(s),\xi(s)) = 1/4$. This also implies that $t(s) = s + t_0$ and, taking t as a time parameter, we are led to solve

$$\begin{cases} \dot{x}(t) = \nabla_\xi g^*(x(t),\xi(t))\,, \\ \dot{\xi}(t) = -\nabla_x g^*(x(t),\xi(t))\,, \\ g^*(x(0),\xi(0)) = \frac{1}{4}. \end{cases} \tag{2.8}$$

In other words, the t-variable parametrizes null-bicharacteristics in a way that they are traveled at speed 1.

Remark 2.10. In the subelliptic setting, the co-sphere bundle S^*M can be decomposed as $S^*M = U^*M \cup S\Sigma$, where $U^*M = \{g^* = 1/4\}$ is a cylinder bundle, $\Sigma = \{g^* = 0\}$ is the characteristic cone and $S\Sigma$ is the sphere bundle of Σ (see [CHT18, Section 1]).

We denote by $\phi_t : S^*M \to S^*M$ the (normal) geodesic flow defined by $\phi_t(x_0,\xi_0) = (x(t),\xi(t))$, where $(x(t),\xi(t))$ is a solution of the system given by the first two lines of (2.8) and initial conditions (x_0,ξ_0). Note that any point in $S\Sigma$ is a fixed point of ϕ_t, and that the other normal geodesics are traveled at speed 1 since we took $g^* = 1/4$ in U^*M (see Remark 2.10).

The curves $x(t)$ which solve (2.8) are geodesics (i.e. local minimizers) for a sub-Riemannian metric g (see [Mon02, Theorem 1.14]).

2.1.5 Observability in some regions of phase-space

We have explained in Section 2.1.3 that the existence of solutions of the subelliptic wave equation (2.1) concentrated on spiraling normal geodesics is an obstruction to observability in Theorem 2.2. Our goal in this section is to state a result ensuring observability if one "removes" in some sense these normal geodesics.

For this result, we focus on a version of the Heisenberg manifold described in Section 2.1.3 which has *no boundary*. This technical assumption avoids us using boundary microlocal defect measures in the proof, which, in this sub-Riemannian setting, are difficult to handle. As a counterpart, we need to consider solutions of the wave equation with null initial average, in order to get well-posedness.

We consider the Heisenberg group G, that is $\mathbb{R}^3$ with the composition law

$$(x_1, x_2, x_3) \star (x_1', x_2', x_3') = (x_1 + x_1', x_2 + x_2', x_3 + x_3' - x_1 x_2').$$

Then $X_1 = \partial_{x_1}$ and $X_2 = \partial_{x_2} - x_1\partial_{x_3}$ are left invariant vector fields on G. Since $\Gamma = \sqrt{2\pi}\mathbb{Z} \times \sqrt{2\pi}\mathbb{Z} \times 2\pi\mathbb{Z}$ is a co-compact subgroup of G, the left quotient $M_H = \Gamma\backslash G$ is a compact three dimensional manifold and, moreover, X_1 and X_2 are well-defined as vector fields on the quotient. We call M_H endowed with the vector fields X_1 and X_2 the "Heisenberg manifold without boundary". Finally, we define the Heisenberg Laplacian $\Delta_H = X_1^2 + X_2^2$ on M_H. Since $[X_1, X_2] = -\partial_{x_3}$, it is a hypoelliptic operator. We endow M_H with an arbitrary smooth volume μ.

We introduce the space

$$L_0^2 = \left\{ u_0 \in L^2(M_H), \int_{M_H} u_0 \, d\mu = 0 \right\}$$

and we consider the operator Δ_H whose domain $D(\Delta_H)$ is the completion in L_0^2 of the set of all $u \in C_c^\infty(M_H)$ with null-average for the norm $\|(\mathrm{Id} - \Delta_H)u\|_{L^2}$. Then, $-\Delta_H$ is definite positive and we consider $(-\Delta_H)^{\frac{1}{2}}$ with domain $D((-\Delta_H)^{\frac{1}{2}}) = \mathcal{H}_0 := L_0^2 \cap \mathcal{H}(M_H)$. The wave equation

$$\begin{cases} \partial_{tt}^2 u - \Delta_H u = 0 \quad \text{in } \mathbb{R} \times M_H \\ (u_{|t=0}, \partial_t u_{|t=0}) = (u_0, u_1) \in D((-\Delta_H)^{\frac{1}{2}}) \times L_0^2 \end{cases} \tag{2.9}$$

admits a unique solution $u \in C^0(\mathbb{R}; D((-\Delta_H)^{\frac{1}{2}})) \cap C^1(\mathbb{R}; L_0^2)$.

We note that $-\Delta_H$ is invertible in L_0^2. The space $\mathcal{H}_0$ is endowed with the norm $\|u\|_{\mathcal{H}}$ (defined in (2.3) and also equal to $\|(-\Delta_H)^{\frac{1}{2}} u\|_{L^2}$), and its topological dual $\mathcal{H}_0'$ is endowed with the norm $\|u\|_{\mathcal{H}_0'} := \|(-\Delta_H)^{-\frac{1}{2}} u\|_{L^2}$.

We note that $g^*(x, \xi) = \xi_1^2 + (\xi_2 - x_1\xi_3)^2$ and hence the null-bicharacteristics are solutions of

$$\begin{aligned} \dot{x}_1(t) &= 2\xi_1, & \dot{\xi}_1(t) &= 2\xi_3(\xi_2 - x_1\xi_3), \\ \dot{x}_2(t) &= 2(\xi_2 - x_1\xi_3), & \dot{\xi}_2(t) &= 0, \\ \dot{x}_3(t) &= -2x_1(\xi_2 - x_1\xi_3), & \dot{\xi}_3(t) &= 0. \end{aligned} \tag{2.10}$$

The spiraling normal geodesics described in Section 2.1.3 correspond to $\xi_1 = \cos(t/\varepsilon)/2$, $\xi_2 = 0$ and $\xi_3 = 1/(2\varepsilon)$. In particular, the constant ξ_3 is a kind of rounding number reflecting the fact that the normal geodesic spirals at a certain speed around the x_3 axis. Moreover, ξ_3 is preserved under the flow (somehow, the Heisenberg flow is completely integrable), and this property plays a key role in the proof of Theorem 2.11 below and justifies that we state it only for the Heisenberg manifold (without boundary).

As said above, normal geodesics corresponding to a large momentum ξ_3 are precisely the ones used to contradict observability in Theorem 2.2. We expect to be able to establish observability if we consider only solutions of (2.1) whose ξ_3 (in a certain sense) is not too large. This is the purpose of our second main result.

Set

$$V_\varepsilon = \left\{(x,\xi) \in T^*M_H : |\xi_3| > \frac{1}{\varepsilon}(g_x^*(\xi))^{1/2}\right\}$$

Note that since ξ_3 is constant along null-bicharacteristics, V_ε and its complementary V_ε^c are invariant under the bicharacteristic equations (2.10).

In the next statement, we call horizontal strip the periodization under the action of the co-compact subgroup Γ of a set of the form

$$\{(x_1,x_2,x_3)\ :\ (x_1,x_2) \in [0,\sqrt{2\pi})^2,\ x_3 \in I\}$$

where I is a strict open subinterval of $[0,2\pi)$.

Theorem 2.11. *Let $B \subset M_H$ be an open subset and suppose that B is sufficiently small, so that $\omega = M_H \backslash B$ contains a horizontal strip. Let $a \in S^0_{phg}(T^*M_H)$, $a \geqslant 0$, such that, denoting by $j : T^*\omega \to T^*M_H$ the canonical injection,*

$$j(T^*\omega) \cup V_\varepsilon \subset Supp(a) \subset T^*M_H,$$

and in particular a does not depend on time. There exists $\kappa > 0$ such that for any $\varepsilon > 0$ and any $T \geqslant \kappa\varepsilon^{-1}$, there holds

$$C\|(u(0),\partial_t u(0))\|^2_{\mathcal{H}_0\times L^2_0} \leqslant \int_0^T |(Op(a)\partial_t u, \partial_t u)_{L^2}|\,dt \ +\ \|(u(0),\partial_t u(0))\|^2_{L^2_0\times\mathcal{H}_0'} \tag{2.11}$$

for some $C = C(\varepsilon,T) > 0$ and for any solution $u \in C^0(\mathbb{R}; D((-\Delta_H)^{\frac{1}{2}}))\ \cap\ C^1(\mathbb{R}; L^2_0)$ of (2.9).

The term $\|(u_0,u_1)\|^2_{L^2\times\mathcal{H}_0'}$ in the right-hand side of (2.11) cannot be removed, i.e. our statement only consists in a *weak* observability inequality. Indeed, the usual way to remove such terms is to use a unique continuation argument for eigenfunctions φ of Δ, but here it does not work since $\mathrm{Op}(a)\varphi = 0$ does not imply in general that $\varphi \equiv 0$ in the whole manifold, even if the support of a contains $j(T^*\omega)$ for some non-empty open set ω: in some sense, there is no "pseudodifferential unique continuation argument".

2.1.6 Comments on the existing literature

Elliptic and subelliptic waves. The exact controllability/observability of the elliptic wave equation is known to be almost equivalent to the so-called Geometric Control Condition (GCC) (see [BLR92]) that any geodesic enters the control set ω within time T. In some sense, our main result is that GCC is not verified in the subelliptic setting, as soon as $M\backslash\omega$ contains in its interior a point x at which Δ is "truly subelliptic". For the elliptic wave equation, in many geometrical situations, there exists a minimal time $T_0 > 0$ such that observability holds only for $T \geqslant T_0$: when there exists a geodesic $\gamma : (0,T_0) \to M$ traveled at speed 1 which does not meet $\overline{\omega}$, one constructs a sequence of initial data $(u_0^k, u_1^k)_{k\in\mathbb{N}^*}$ of the wave equation whose associated microlocal defect measure is concentrated on $(x_0, \xi_0) \in S^*M$ taken to be the initial conditions for the null-bicharacteristic projecting onto γ. Then, the associated sequence of solutions $(u^k)_{k\in\mathbb{N}^*}$ of the wave equation has an associated microlocal defect measure ν which is invariant under the geodesic flow: $\vec{H}_p\nu = 0$ where $\vec{H}_p$ is the Hamiltonian flow associated to the principal symbol p of the wave operator. In particular, denoting by $\pi : T^*M \to M$ the canonical projection, $\pi_*\nu$ gives no mass to ω since γ is contained in $M \setminus \overline{\omega}$, and this proves that observability cannot hold.

In the subelliptic setting, the invariance property $\vec{H}_p\nu = 0$ does not give any information on ν on the characteristic manifold Σ, since $\vec{H}_p = -2\tau\partial_t + \vec{g}^*$ vanishes on Σ. This is related to the lack of information on propagation of singularities in this characteristic manifold, see the main theorem of [Las82]. If one instead tries to use the propagation of the microlocal defect measure for subelliptic half-wave equations, one is immediately confronted with the fact that $\sqrt{-\Delta}$ is not a pseudodifferential operator near Σ.

This is why, in this chapter, we used only the elliptic part of the symbol g^* (or, equivalently, the strictly hyperbolic part of p_2), where the propagation properties can be established, and then the problem is reduced to proving geometric results on normal geodesics.

Subelliptic Schrödinger equations. The recent article [BS19] deals with the same observability problem, but for subelliptic Schrödinger equations: namely, the authors consider the (Baouendi)-Grushin Schrödinger equation $i\partial_t u - \Delta_G u = 0$, where $u \in L^2((0,T) \times M_G)$, $M_G = (-1,1)_x \times \mathbb{T}_y$ and $\Delta_G = \partial_x^2 + x^2\partial_y^2$ is the Baouendi-Grushin Laplacian. Given a control set of the form $\omega = (-1,1)_x \times \omega_y$, where ω_y is an open subset of $\mathbb{T}$, the authors prove the existence of a minimal time of control $\mathcal{L}(\omega)$ related to the maximal height of a horizontal strip contained in $M_G\backslash\omega$. The intuition is that there are solutions of the Baouendi-Grushin Schrödinger equation which travel along the degenerate line $x = 0$ at a finite speed: in some sense, along this line, the Schrödinger equation behaves like a classical (half)-wave equation. What we want here is to explain in a few words why there is a minimal time of observability for the Schrödinger equation, while the wave equation is never observable in finite time as shown by Theorem 2.2.

The plane $\mathbb{R}^2_{x,y}$ endowed with the vector fields ∂_x and $x\partial_y$ also admits normal geodesics similar to the 1-parameter family q_ε, namely, for $\varepsilon > 0$,

$$\begin{aligned} x(t) &= \varepsilon \sin(t/\varepsilon) \\ y(t) &= \varepsilon(t/2 - \varepsilon \sin(2t/\varepsilon)/4) \end{aligned}$$

These normal geodesics, denoted by γ_ε, also "spiral" around the line $x = 0$ more and more quickly as $\varepsilon \to 0$, and so we might expect to construct solutions of the Baouendi-Grushin Schrödinger equation with energy concentrated along γ_ε, which would contradict observability when $\varepsilon \to 0$ as above for the Heisenberg wave equation.

However, we can convince ourselves that it is not possible to construct such solutions: in some sense, the dispersion phenomena of the Schrödinger equation exactly compensate the lengthening of the normal geodesics γ_ε as $\varepsilon \to 0$ and explain that even these Gaussian beams may be observed in ω from a certain minimal time $\mathcal{L}(\omega) > 0$ which is uniform in ε.

To put this argument into a more formal form, we consider the solutions of the bicharacteristic equations for the Baouendi-Grushin Schrödinger equation $i\partial_t u - \Delta_G u = 0$ given by

$$\begin{aligned}
x(t) &= \varepsilon \sin(\xi_y t) \\
y(t) &= \varepsilon^2 \xi_y \left(\frac{t}{2} - \frac{\sin(2\xi_y t)}{4\xi_y} \right) \\
\xi_x(t) &= \varepsilon \xi_y \cos(\xi_y t) \\
\xi_y(t) &= \xi_y.
\end{aligned}$$

It follows from the hypoellipticity of Δ_G (see [BS19, Section 3] for a proof) that

$$|\xi_y|^{1/2} \lesssim \sqrt{-\Delta_G} = (|\xi_x|^2 + x^2|\xi_y|^2)^{1/2} = \varepsilon|\xi_y|.$$

Therefore $\varepsilon^2|\xi_y| \gtrsim 1$, and hence $|y(t)| \gtrsim t$, independently from ε and ξ_y. This heuristic gives the intuition that a minimal time $\mathcal{L}(\omega)$ is required to detect all solutions of the Baouendi-Grushin Schödinger equation from ω, but that for $T_0 > \mathcal{L}(\omega)$, no solution is localized enough to stay in $M \backslash \omega$ during the time interval $(0, T_0)$. Roughly speaking, the frequencies of order ξ_y travel at speed $\sim \xi_y$, which is typical for a dispersion phenomenon. This picture is very different from the one for the wave equation (which we consider in this chapter) for which no dispersion occurs.

With similar ideas, in [LS20], the interplay between the subellipticity effects measured by the non-holonomic order of the distribution $\mathcal{D}$ (see Section 2.3.1) and the strength of dispersion of Schrödinger-type equations was investigated. More precisely, for $\Delta_\gamma = \partial_x^2 + |x|^{2\gamma}\partial_y^2$ on $M = (-1,1)_x \times \mathbb{T}_y$, and for $s \in \mathbb{N}$, the observability properties of the Schrödinger-type equation $(i\partial_t - (-\Delta_\gamma)^s)u = 0$ were shown to depend on the value $\kappa = 2s/(\gamma+1)$. In particular it is proved that, for $\kappa < 1$, observability fails for any time, which is consistent with the present result, and that for $\kappa = 1$, observability holds only for sufficiently large times, which is consistent with the result of [BS19]. The results of [LS20] are somehow Schrödinger analogues of the results of [BCG14] which deal with a similar problem for the Baouendi-Grushin heat equation.

General bibliographical comments. Control of subelliptic PDEs has attracted much attention in the last decade. Most results in the literature deal with subelliptic parabolic equations, either the Baouendi-Grushin heat equation ([Koe17], [DK20], [BDE20]) or the heat equation in the Heisenberg group ([BC17], see also references therein). The paper [BS19] is the first to deal with a subelliptic Schrödinger equation and the present work is the first to handle exact controllability of subelliptic wave equations.

A slightly different problem is the *approximate* controllability of hypoelliptic PDEs, which has been studied in [LL20] for hypoelliptic wave and heat equations. Approximate controllability is weaker than exact controllability, and it amounts to proving "quantitative" unique continuation results for hypoelliptic operators. For the hypoelliptic wave equation, it is proved in [LL20] that for $T > 2\sup_{x\in M}(\mathrm{dist}(x,\omega))$ (here, dist is the sub-Riemannian distance), the observation of the solution on $(0,T)\times\omega$ determines the initial data, and therefore the whole solution.

2.1.7 Organization of the chapter

In Section 2.2, we construct exact solutions of the subelliptic wave equation (2.1) concentrating on any given normal geodesic. First, in Section 2.2.1, we show that, given any normal geodesic $t \mapsto x(t)$ which does not hit ∂M in the time interval $(0,T)$, it is possible to construct a sequence $(v_k)_{k\in\mathbb{N}}$ of *approximate solutions* of (2.1) whose energy concentrates along $t \mapsto x(t)$ during the time interval $(0,T)$ as $k \to +\infty$. By "approximate", we mean here that $\partial_{tt}^2 v_k - \Delta v_k$ is small, but not necessarily exactly equal to 0. In Section 2.2.1, we provide a first proof for this construction using the classical propagation of complex Lagrangian spaces. An other proof using a Gaussian beam approach is provided in Section 2-A.1. Then, in Section 2.2.2, using this sequence $(v_k)_{k\in\mathbb{N}}$, we explain how to construct a sequence $(u_k)_{k\in\mathbb{N}}$ of *exact* solutions of $(\partial_{tt}^2-\Delta)u=0$ in M with the same concentration property along the normal geodesic $t\mapsto x(t)$.

In Section 2.3, we prove the existence of normal geodesics which spiral in M, spending an arbitrarily large time in $M\backslash\omega$. These normal geodesics generalize the example described in Section 2.1.3 for the Heisenberg manifold with boundary. The proof proceeds in two steps: first, we show that it is sufficient to prove the result in the so-called "nilpotent case" (Section 2.3.2), and then we prove it in the nilpotent case (Section 2.3.3).

In Section 2.4.1, we use the results of Section 2.2 and Section 2.3 to conclude the proof of Theorem 2.2. In Section 2.4.2, we deduce Corollary 2.4 by a duality argument. Finally, in Section 2.4.3, we prove Theorem 2.11.

2.2 Gaussian beams along normal geodesics

2.2.1 Construction of sequences of approximate solutions

We consider a solution $(x(t),\xi(t))_{t\in[0,T]}$ of (2.8) on M. We shall describe the construction of solutions of

$$\partial_{tt}^2 u-\Delta u=0 \tag{2.12}$$

on $[0,T]\times M$ with energy

$$E(u(t,\cdot)):=\frac{1}{2}\int_M \left(|\partial_t u(t,x)|^2+|\nabla^{\mathrm{sR}}u(t,x)|^2\right)d\mu(x)$$

concentrated along $x(t)$ for $t\in[0,T]$. The following proposition, which is inspired by [Ral82] and [MZ02], shows that it is possible, at least for approximate solutions of (2.12).

Proposition 2.12. *Fix $T>0$ and let $(x(t),\xi(t))_{t\in[0,T]}$ be a solution of* (2.8) *(in particular $g^*(x(0),\xi(0))=1/4$) which does not hit the boundary ∂M in the time-interval $(0,T)$. Then there exist $a_0,\psi\in C^2((0,T)\times M)$ such that, setting, for $k\in\mathbb{N}$,*

$$v_k(t,x)=k^{\frac{n}{4}-1}a_0(t,x)e^{ik\psi(t,x)}$$

the following properties hold:

- *v_k is an approximate solution of* (2.12)*, meaning that*

$$\|\partial_{tt}^2 v_k-\Delta v_k\|_{L^1((0,T);L^2(M))}\leqslant Ck^{-\frac{1}{2}}. \tag{2.13}$$

- *The energy of v_k is bounded below with respect to k and $t\in[0,T]$:*

$$\exists A>0,\forall t\in[0,T],\quad \liminf_{k\to+\infty}E(v_k(t,\cdot))\geqslant A. \tag{2.14}$$

- *The energy of v_k is small off $x(t)$: for any $t\in[0,T]$, we fix V_t an open subset of M for the initial topology of M, containing $x(t)$, so that the mapping $t\mapsto V_t$ is continuous (V_t is chosen sufficiently small so that this makes sense in a chart). Then*

$$\sup_{t\in[0,T]}\int_{M\backslash V_t}\left(|\partial_t v_k(t,x)|^2+|\nabla^{sR}v_k(t,x)|^2\right)d\mu(x)\underset{k\to+\infty}{\to}0. \tag{2.15}$$

Remark 2.13. The construction of approximate solutions such as the ones provided by Proposition 2.12 is usually done for strictly hyperbolic operators, that is operators with a principal symbol p_m of order m such that the polynomial $f(s) = p_m(t, q, s, \xi)$ has m distinct real roots when $\xi \neq 0$ (see for example [Ral82]). The operator $\partial_{tt}^2 - \Delta$ is not strictly hyperbolic because g^* is degenerate, but our proof shows that the same construction may be adapted without difficulty to this operator *along normal bicharacteristics*. This is due to the fact that along normal bicharacteristics, $\partial_{tt}^2 - \Delta$ is indeed strictly hyperbolic (or equivalently, Δ is elliptic). It was already noted by [Ral82] that the construction of Gaussian beams could be done for more general operators than strictly hyperbolic ones, and that the differences between the strictly hyperbolic case and more general cases arise while dealing with propagation of singularities. Also, in [Hor07b, Chapter 24.2], it was noticed that "since only microlocal properties of p_2 are important, it is easy to see that hyperbolicity may be replaced by $\nabla_\xi p_2 \neq 0$".

Hereafter we provide two proofs of Proposition 2.12. The first proof is short and is actually quite straightforward for readers acquainted with the theory of propagation of complex Lagrangian spaces, once one has noticed that the solutions of (2.8) which we consider live in the *elliptic part* of the principal symbol of $-\Delta$. For the sake of completeness, and because this also has its own interest, we provide in Section 2-A.1 a second proof, longer but more elementary and accessible without any knowledge of complex Lagrangian spaces; it relies on the construction of Gaussian beams in the subelliptic context. The two proofs follow parallel paths, and indeed, the computations which are only sketched in the first proof are written in full details in the second proof, given in Section 2-A.1.

First proof of Proposition 2.12. The construction of Gaussian beams, or more generally of a WKB approximation, is related to the transport of complex Lagrangian spaces along bicharacteristics, as reported for example in [Hor07b, Chapter 24.2] and [Ivr19, Volume I, Part I, Chapter 1.2]. Our proof follows the lines of [Hor07b, pages 426-428].

A usual way to solve (at least approximately) evolution equations of the form

$$Pu = 0 \tag{2.16}$$

where P is a hyperbolic second order differential operator with real principal symbol and C^∞ coefficients is to search for oscillatory solutions

$$v_k(x) = k^{\frac{n}{4}-1} a_0(x) e^{ik\psi(x)}. \tag{2.17}$$

In this expression as in the rest of the proof, we suppress the time variable t. Thus, we use $x = (x_0, x_1, \ldots, x_n)$ where $x_0 = t$ in the earlier notations, and we set $x' = (x_1, \ldots, x_n)$. Similarly, we take the notation $\xi = (\xi_0, \xi_1, \ldots, \xi_n)$ where $\xi_0 = \tau$ previously, and $\xi' = (\xi_1, \ldots, \xi_n)$. The bicharacteristics are parametrized by s as in (2.7), and without loss of generality, we only consider bicharacteristics with $x(0) = 0$ at $s = 0$, which implies in particular $x_0(s) = s$ because of our choice $\tau^2(s) = g^*(x(s), \xi(s)) = 1/4$.

Taking charts of M, we can assume $M \subset \mathbb{R}^n$. The precise argument for reducing to this case is written at the end of Section 2-A.1. Also, in the sequel, $P = \partial_{tt}^2 - \Delta$.

Plugging the Ansatz (2.17) into (2.16), we get

$$Pv_k = (k^{\frac{n}{4}+1} A_1 + k^{\frac{n}{4}} A_2 + k^{\frac{n}{4}-1} A_3) e^{ik\psi} \tag{2.18}$$

with

$$\begin{aligned}
A_1(x) &= p_2\left(x, \nabla\psi(x)\right) a_0(x) \\
A_2(x) &= L a_0(x) \\
A_3(x) &= \partial_{tt}^2 a_0(x) - \Delta a_0(x).
\end{aligned}$$

and L is a transport operator given by

$$La_0 = \frac{1}{i}\sum_{j=0}^{n} \frac{\partial p_2}{\partial \xi_j}(x, \nabla\psi(x))\frac{\partial a_0}{\partial x_j} + \frac{1}{2i}\left(\sum_{j,k=0}^{n} \frac{\partial^2 p_2}{\partial \xi_j \partial \xi_k}(x, \nabla\psi(x))\frac{\partial^2 \psi}{\partial x_j \partial x_k}\right) a_0. \tag{2.19}$$

In order for v_k to be an approximate solution of P, we are first led to cancel the higher order term in (2.18), i.e.,

$$f(x) := p_2(x, \nabla\psi(x)) = 0 \tag{2.20}$$

which we solve for initial conditions

$$\psi(0, x') = \psi_0(x'), \qquad \nabla\psi_0(0) = \xi'(0) \quad \text{and} \quad \psi_0(0) = 0 \tag{2.21}$$

(i.e., we fix such a ψ_0, and then we solve (2.20) for ψ). Indeed, it will be sufficient for our purpose for (2.20) to be verified at second order along the curve $x(s)$, i.e., $D_x^\alpha f(x(s)) = 0$ for any $|\alpha| \leqslant 2$ and any s. For that, we first notice that the choice $\nabla\psi(x(s)) = \xi(s)$ ensures that (2.20) holds at orders 0 and 1 along the curve $s \mapsto x(s)$ (see Section 2-A.1 for detailed computations). Now, we explain how to choose $D^2\psi(x(s))$ adequately in order for (2.20) to hold at order 2.

We use the decomposition of p_2 into

$$p_2(x_0, x', \xi_0, \xi') = -(\xi_0 - r(x', \xi'))(\xi_0 + r(x', \xi')) + R(x', \xi')$$

where $r = \sqrt{g^*}$ in a conic neighborhood of $(0, \xi(0))$. Note that $\sqrt{g^*}$ is smooth in small conic neighborhoods of $(0, \xi(0))$ since $g^*(0, \xi(0)) = 1/4 \neq 0$. Indeed, g^* is elliptic along the whole bicharacteristic since $g^*(x(t), \xi(t)) = 1/4$ is preserved by the bicharacteristic flow. The rest term $R(x', \xi')$ is smooth and microlocally supported far from the bicharacteristic, i.e., $R(x', \xi') = 0$ for any $(x', \xi') \in T^*M$ in a conic neighborhood of $(x'(s), \xi'(s))$ for $s \in [0, T]$.

We consider the bicharacteristic γ_+ starting at $(0, 0, r(0, \xi'(0)), \xi'(0))$ and the bicharacteristic γ_- starting at $(0, 0, -r(0, \xi'(0)), \xi'(0))$.

We denote by $\Phi^\pm(x_0, y', \eta')$ the solution of the Hamilton equations with Hamiltonian $H_\pm(x_0, x', \xi') = \xi_0 \mp r(x', \xi')$ and initial datum $(x', \xi') = (y', \eta')$ at $x_0 = 0$. In other words, $\Phi^\pm(x_0, y', \eta') = e^{x_0 \vec{H}_\pm}(0, y', \eta')$. Then, for any s, $\Phi(s, \cdot)$ is well-defined and symplectic from a neighborhood of $(0, \xi'(0))$ to a neighborhood of $H_\pm(s, 0, \xi'(0))$.

The solution $\psi(s, \cdot)$ of (2.20) and (2.21) is equal to 0 on $\gamma_\pm$ and $\nabla\psi(s, \cdot)$ is obtained by the transport of the values of $\nabla\psi_0$ by $\Phi^\pm(s, \cdot)$. In other words, to compute $\nabla\psi(s, \cdot)$, one transports the Lagrangian sub-space $\Lambda_0 = \{(x', \nabla\psi_0(x'))\}$ along the Hamiltonian flow $\vec{H}_\pm$ during a time s, which yields $\Lambda_s \subset T^*M$, and then, if possible, one writes Λ_s under the form $\{(x', \nabla_{x'}\psi(s, x'))\}$, which gives $\nabla_{x'}\psi(s, x')$. The trouble is that the solution is only local in time: when $x' \mapsto \pi(\Phi^\pm(s, x', \nabla\psi_0(x')))$ ceases to be a diffeomorphism (conjugate point), where $\pi : T^*M \to M$ is the canonical projection, we see that the process described above does not work (appearance of caustics). In the language of Lagrangian spaces, $\Lambda_0 = \{(x', \nabla\psi_0(x'))\} \subset T^*M$ is a Lagrangian subspace and, since $\Phi^\pm(s, \cdot)$ is a symplectomorphism, $\Lambda_s = \Phi^\pm(s, \Lambda_0)$ is Lagrangian as well. If $\pi_{|\Lambda_s}$ is a local diffeomorphism, one can locally describe Λ_s by $\Lambda_s = \{(x', \nabla_{x'}\psi(s, x'))\} \subset T^*M$ for some function $\psi(s, \cdot)$, but blow-up happens when $\operatorname{rank}(d\pi_{|\Lambda_s}) < n$ (classical conjugate point theory), and such a $\psi(s, \cdot)$ may not exist.

However, if the phase ψ_0 is complex, quadratic, and satisfies the condition $\operatorname{Im}(D^2\psi_0) > 0$, where $D^2\psi_0$ denotes the Hessian, no blow-up happens, and the solution is global in time. Let us explain why. Indeed, $\Lambda_0 = \{(x', \nabla\psi_0(x'))\}$ then lives in the complexification of the tangent space T^*M, which may be thought of as $\mathbb{C}^{2(n+1)}$. We take coordinates (y, η) on $T^*\mathbb{R}^{n+1}$ or $T^*\mathbb{C}^{n+1}$ and we consider the symplectic forms defined by $\sigma = \sum dy_j \wedge d\eta_j$ and $\sigma_{\mathbb{C}} = \sum dy_j \wedge \overline{d\eta_j}$.

Because of the condition $\mathrm{Im}(D^2\psi_0) > 0$, Λ_0 is called a "strictly positive Lagrangian space" (see [Hor07b, Definition 21.5.5]), meaning that $i\sigma_{\mathbb{C}}(v,v) > 0$ for v in the tangent space to Λ_0. For any s, the symplectic forms σ and $\sigma_{\mathbb{C}}$ are preserved by $\Phi(s,\cdot)$, meaning that $\Phi(s,\cdot)_*\sigma = \sigma$ and $\Phi(s,\cdot)_*\sigma_{\mathbb{C}} = \sigma_{\mathbb{C}}$, therefore $\sigma = 0$ on the tangent space to Λ_s, and $i\sigma_{\mathbb{C}}(v,v) > 0$ for v tangent to Λ_s. It precisely means that Λ_s is also a strictly positive Lagrangian space. Then, by [Hor07b, Proposition 21.5.9], we know that there exists $\psi(s,\cdot)$ complex and quadratic with $\mathrm{Im}(D^2\psi(s,\cdot)) > 0$ such that $\Lambda_s = \{(x', \nabla_{x'}\psi(s,x'))\}$ (to apply [Hor07b, Proposition 21.5.9], recall that for $\varphi(x') = \frac{1}{2}(Ax', x')$, there holds $\nabla\varphi(x') = Ax'$). In other words, the key point in using complex phases is that strictly positive Lagrangian spaces are parametrized by complex quadratic phases φ with $\mathrm{Im}(D^2\varphi) > 0$, whereas real Lagrangian spaces were not parametrized by real phases (see explanations above). This parametrization is a diffeomorphism from the Grassmannian of strictly positive Lagrangian spaces to the space of complex quadratic phases with φ with $\mathrm{Im}(D^2\varphi) > 0$. Hence, the phase

$$\psi(s,y') = \nabla_{x'}\psi(x(s)) \cdot (y' - x'(s)) + \frac{1}{2}(y' - x'(s)) \cdot D^2_{x'}\psi(s, x'(s))(y' - x'(s))$$

for $s \in [0,T]$ and $y' \in \mathbb{R}^n$ is smooth and for this choice, (2.20) is satisfied at second order along $s \mapsto x(s)$ (the rest $R(x', \xi')$ plays no role since it vanishes in a neighborhood of $s \mapsto x(s)$).

Then, we note that A_2 vanishes along the bicharacteristic if and only if $La_0(x(s)) = 0$ (see also [Hor07b, Equation (24.2.9)]). According to (2.19), this turns out to be a linear transport equation on $a_0(x(s))$, with leading coefficient $\nabla_\xi p_2(x(s), \xi(s))$ different from 0. Given $a \neq 0$ at $(t = 0, x' = x'(0))$, this transport equation has a solution $a_0(x(s))$ with initial datum a, and, by Cauchy uniqueness, $a_0(x(s)) \neq 0$ for any s. We can choose a_0 in a smooth (and arbitrary) way outside the bicharacteristic. We choose it to vanish outside a small neighborhood of this bicharacteristic, so that no boundary effect happens.

With these choices of ψ and a_0, the bound (2.13) then follows from the following result whose proof is given in [Ral82, Lemma 2.8].

Lemma 2.14. *Let $c(x)$ be a function on $\mathbb{R}^{n+1}$ which vanishes at order $S-1$ on a curve Γ for some $S \geqslant 1$. Suppose that Supp $c \cap \{|x_0| \leqslant T\}$ is compact and that Im $\psi(x) \geqslant ad(x)^2$ on this set for some constant $a > 0$, where $d(x)$ denotes the distance from the point $x \in \mathbb{R}^{d+1}$ to the curve Γ. Then there exists a constant C such that*

$$\int_{|x_0|\leqslant T} \left| c(x)e^{ik\psi(x)} \right|^2 dx \leqslant Ck^{-S-n/2}.$$

Let us now sketch the end of the proof, which is given in Section 2-A.1 in full details. We apply Lemma 2.14 to $S = 3$, $c = A_1$ and to $S = 1$, $c = A_2$, and we get

$$\|\partial^2_{tt} v_k - \Delta v_k\|_{L^1(0,T;L^2(M))} \leqslant C(k^{-\frac{1}{2}} + k^{-\frac{1}{2}} + k^{-1}),$$

which implies (2.13). The bounds (2.14) and (2.15) follow from the facts that $\mathrm{Im}(D^2\psi(s,\cdot)) > 0$ and $v_k(x) = k^{\frac{n}{4}-1} a_0(x) e^{ik\psi(x)}$. □

Remark 2.15. An interesting question would be to understand the delocalization properties of the Gaussian beams constructed along normal geodesics in Proposition 2.12. Compared with the usual Riemannian case done for example in [Ral82], there is a new phenomenon in the sub-Riemannian case since the normal geodesic $x(t)$ (or, more precisely, its lift to S^*M) may approach the characteristic manifold $\Sigma = \{g^* = 0\}$ which is the set of directions in which Δ is not elliptic. In finite time T as in our case, the lift of the normal geodesic remains far from Σ, but it may happen as $T \to +\infty$ that it goes closer and closer to Σ. The question is then to understand the link between the delocalization properties of the Gaussian beams constructed along such a normal geodesic, and notably the interplay between the time T and the semi-classical parameter $1/k$.

2.2.2 Construction of sequences of exact solutions in M

In this section, using the approximate solutions of Proposition 2.2.1, we construct *exact* solutions of (2.12) whose energy concentrates along a given normal geodesic of M which does not meet the boundary ∂M during the time interval $[0,T]$.

Proposition 2.16. *Let $(x(t),\xi(t))_{t\in[0,T]}$ be a solution of* (2.8) *in M (in particular $g^*(x(0),\xi(0)) = 1/4$) which does not meet ∂M. Let $\theta \in C_c^\infty([0,T]\times M)$ with $\theta(t,\cdot)\equiv 1$ in a neighborhood of $x(t)$ and such that the support of $\theta(t,\cdot)$ stays at positive distance of ∂M.*

Suppose $(v_k)_{k\in\mathbb{N}}$ is constructed along $x(t)$ as in Proposition 2.12 and u_k is the solution of the Cauchy problem

$$\begin{cases} (\partial_{tt}^2-\Delta)u_k = 0 \quad \text{in } (0,T)\times M, \\ u_k = 0 \quad \text{in } (0,T)\times\partial M, \\ u_{k|t=0} = (\theta v_k)_{|t=0}, \ \partial_t u_{k|t=0} = [\partial_t(\theta v_k)]_{|t=0}. \end{cases}$$

Then:

- *The energy of u_k is bounded below with respect to k and $t\in[0,T]$:*

$$\exists A>0, \forall t\in[0,T], \quad \liminf_{k\to+\infty} E(u_k(t,\cdot)) \geqslant A. \tag{2.22}$$

- *The energy of u_k is small off $x(t)$: for any $t\in[0,T]$, we fix V_t an open subset of M for the initial topology of M, containing $x(t)$, so that the mapping $t\mapsto V_t$ is continuous (V_t is chosen sufficiently small so that this makes sense in a chart). Then*

$$\sup_{t\in[0,T]} \int_{M\setminus V_t} \left(|\partial_t u_k(t,x)|^2 + |\nabla^{sR} u_k(t,x)|^2\right) d\mu(x) \underset{k\to+\infty}{\to} 0. \tag{2.23}$$

Proof of Proposition 2.16. Set $h_k = (\partial_{tt}^2-\Delta)(\theta v_k)$. We consider w_k the solution of the Cauchy problem

$$\begin{cases} (\partial_{tt}^2-\Delta)w_k = h_k \text{ in } (0,T)\times M, \\ w_k = 0 \ \text{ in } (0,T)\times\partial M, \\ (w_{k|t=0},\partial_t w_{k|t=0}) = (0,0)\,. \end{cases} \tag{2.24}$$

Differentiating $E(w_k(t,\cdot))$ and using Gronwall's lemma, we get the energy inequality

$$\sup_{t\in[0,T]} E(w_k(t,\cdot)) \leqslant C\left(E(w_k(0,\cdot)) + \|h_k\|_{L^1(0,T;L^2(M))}\right).$$

Therefore, using (2.13), we get $\sup_{t\in[0,T]} E(w_k(t,\cdot)) \leqslant Ck^{-1}$. Since $u_k = \theta v_k - w_k$, we obtain that

$$\lim_{k\to+\infty} E(u_k(t,\cdot)) = \lim_{k\to+\infty} E((\theta v_k)(t,\cdot)) = \lim_{k\to+\infty} E(v_k(t,\cdot))$$

for every $t\in[0,T]$ where the last equality comes from the fact that θ and its derivatives are bounded and $\|v_k\|_{L^2} \leqslant Ck^{-1}$ when $k\to+\infty$. Using (2.14), we conclude that (2.22) holds.

To prove (2.23), we observe similarly that

$$\begin{aligned}&\sup_{t\in[0,T]}\int_{M\setminus V_t}\left(|\partial_t u_k(t,x)|^2+|\nabla^{\mathrm{sR}}u_k(t,x)|^2\right)d\mu(x)\\ &\leqslant C\sup_{t\in[0,T]}\left(\int_{M\setminus V_t}\left(|\partial_t v_k(t,x)|^2+|\nabla^{\mathrm{sR}}v_k(t,x)|^2\right)d\mu(x)\right)+Ck^{-\frac{1}{2}}\\ &\to 0\end{aligned}$$

as $k\to+\infty$, according to (2.15). It concludes the proof of Proposition 2.16. □

2.3 Existence of spiraling normal geodesics

The goal of this section is to prove the following proposition, which is the second building block of the proof of Theorem 2.2, after the construction of localized solutions of the subelliptic wave equation (2.1) done in Section 2.2.

We say that $X_1,\ldots,X_m$ satisfies the property $(\mathbf{P})$ at $q\in M$ if the following holds:

$(\mathbf{P})$ *For any open neighborhood V of q, for any $T_0>0$, there exists a non-stationary normal geodesic $t\mapsto x(t)$, traveled at speed 1, such that $x(t)\in V$ for any $t\in[0,T_0]$.*

Proposition 2.17. *At any point $q\in M$ such that there exist $1\leqslant i,j\leqslant m$ with $[X_i,X_j](q)\notin\mathcal{D}_q$, property $(\mathbf{P})$ holds.*

In Section 2.3.1, we define the so-called nilpotent approximations $\widehat{X}_1^q,\ldots,\widehat{X}_m^q$ at a point $q\in M$, which are first-order approximations of $X_1,\ldots,X_m$ at $q\in M$ such that the associated Lie algebra $\mathrm{Lie}(\widehat{X}_1^q,\ldots,\widehat{X}_m^q)$ is nilpotent. Roughly, we have $\widehat{X}_i^q\approx X_i(q)$, but low order terms of $X_i(q)$ are not taken into account for defining $\widehat{X}_i^q$, so that the high order brackets of the $\widehat{X}_i^q$ vanish (which is not generally the case for the X_i). These nilpotent approximations are good local approximations of the vector fields $X_1,\ldots,X_m$, and their study is much simpler.

The proof of Proposition 2.17 splits into two steps: first, we show that it is sufficient to prove the result in the nilpotent case (Section 2.3.2), then we handle this simpler case (Section 2.3.3).

2.3.1 Nilpotent approximation

In this section, we recall the construction of the nilpotent approximations $\widehat{X}_1^q,\ldots,\widehat{X}_m^q$. The definitions we give are classical, and the reader can refer to [ABB19, Chapter 10] and [Jea14, Chapter 2] for more material on this section. This construction is related to the notion of tangent space in the Gromov-Hausdorff sense of a sub-Riemannian structure $(M,\mathcal{D},g)$ at a point $q\in M$; the tangent space is defined intrinsically (meaning that it does not depend on a choice of coordinates or of local frame) as an equivalence class under the action of sub-Riemannian isometries (see [Bel96], [Jea14]).

Sub-Riemannian flag. We define the sub-Riemannian flag as follows: we set $\mathcal{D}^0=\{0\}$, $\mathcal{D}^1=\mathcal{D}$, and, for any $j\geqslant 1$, $\mathcal{D}^{j+1}=\mathcal{D}^j+[\mathcal{D},\mathcal{D}^j]$. For any point $q\in M$, it defines a flag

$$\{0\}=\mathcal{D}_q^0\subset\mathcal{D}_q^1\subset\ldots\subset\mathcal{D}_q^{r-1}\subsetneq\mathcal{D}_q^{r(q)}=T_qM.$$

The integer $r(q)$ is called the non-holonomic order of $\mathcal{D}$ at q, and it is equal to 2 everywhere in the Heisenberg manifold for example. Note that it depends on q, see Example 2.5 in Section 2.1.2 (the Baouendi-Grushin example).

For $0 \leqslant i \leqslant r(q)$, we set $n_i(q) = \dim \mathcal{D}_q^i$, and the sequence $(n_i(q))_{0\leqslant i\leqslant r(q)}$ is called the growth vector at point q. We set $\mathcal{Q}(q) = \sum_{i=1}^{r(q)} i(n_i(q) - n_{i-1}(q))$, which is generically the Hausdorff dimension of the metric space given by the sub-Riemannian distance on M (see [Mit85]). Finally, we define the non-decreasing sequence of weights $w_i(q)$ for $1 \leqslant i \leqslant n$ as follows. Given any $1 \leqslant i \leqslant n$, there exists a unique $1 \leqslant j \leqslant n$ such that $n_{j-1}(q) + 1 \leqslant i \leqslant n_j(q)$. We set $w_i(q) = j$. For example, for any q in the Heisenberg manifold, $w_1(q) = w_2(q) = 1$ and $w_3(q) = 2$: indeed, the coordinates x_1 and x_2 have "weight 1", while the coordinate x_3 has "weight 2" since ∂_{x_3} requires a bracket to be generated.

Regular and singular points. We say that $q \in M$ is regular if the growth vector $(n_i(q'))_{0\leqslant i\leqslant r(q')}$ at q' is constant for q' in a neighborhood of q. Otherwise, q is said to be singular. If any point $q \in M$ is regular, we say that the structure is equiregular. For example, the Heisenberg manifold is equiregular, but not the Baouendi-Grushin example.

Non-holonomic orders. The non-holonomic order of a smooth germ of function is given by the formula

$$\mathrm{ord}_q(f) = \min\{s \in \mathbb{N} : \exists i_1, \ldots, i_s \in \{1, \ldots, m\} \text{ such that } (X_{i_1} \ldots X_{i_s} f)(q) \neq 0\}$$

where we adopt the convention that $\min \emptyset = +\infty$.

The non-holonomic order of a smooth germ of vector field X at q, denoted by $\mathrm{ord}_q(X)$, is the real number defined by

$$\mathrm{ord}_q(X) = \sup\{\sigma \in \mathbb{R} : \mathrm{ord}_q(Xf) \geqslant \sigma + \mathrm{ord}_q(f), \quad \forall f \in C^\infty(q)\}.$$

For example, there holds $\mathrm{ord}_q([X,Y]) \geqslant \mathrm{ord}_q(X) + \mathrm{ord}_q(Y)$ and $\mathrm{ord}_q(fX) \geqslant \mathrm{ord}_q(f) + \mathrm{ord}_q(X)$. As a consequence, every X which has the property that $X(q') \in \mathcal{D}_{q'}^i$ for any q' in a neighborhood of q is of non-holonomic order $\geqslant -i$.

Privileged coordinates. Locally around $q \in M$, it is possible to define a set of so-called "privileged coordinates" of M (see [Bel96]).

A family $(Z_1, \ldots, Z_n)$ of n vector fields is said to be adapted to the sub-Riemannian flag at q if it is a frame of T_qM at q and if $Z_i(q) \in \mathcal{D}_q^{w_i(q)}$ for any $i \in \{1, \ldots, n\}$. In other words, for any $i \in \{1, \ldots, r(q)\}$, the vectors $Z_1, \ldots, Z_{n_i(q)}$ at q span $\mathcal{D}_q^i$.

A system of privileged coordinates at q is a system of local coordinates $(x_1, \ldots, x_n)$ such that

$$\mathrm{ord}_q(x_i) = w_i, \qquad \text{for } 1 \leqslant i \leqslant n. \tag{2.25}$$

In particular, for privileged coordinates, we have $\partial_{x_i} \in \mathcal{D}_q^{w_i(q)} \backslash \mathcal{D}_q^{w_i(q)-1}$ at q, meaning that privileged coordinates are adapted to the flag.

Example: exponential coordinates of the second kind. Choose an adapted frame $(Z_1, \ldots, Z_n)$ at q. It is proved in [Jea14, Appendix B] that the inverse of the local diffeomorphism

$$(x_1, \ldots, x_n) \mapsto \exp(x_1 Z_1) \circ \cdots \circ \exp(x_n Z_n)(q)$$

defines privileged coordinates at q, called exponential coordinates of the second kind.

Dilations. We consider a chart of privileged coordinates at q given by a smooth mapping $\psi_q : U \to \mathbb{R}^n$, where U is a neighborhood of q in M, with $\psi_q(q) = 0$. For every $\varepsilon \in \mathbb{R}\backslash\{0\}$, we consider the dilation $\delta_\varepsilon : \mathbb{R}^n \to \mathbb{R}^n$ defined by

$$\delta_\varepsilon(x) = (\varepsilon^{w_i(q)} x_1, \ldots, \varepsilon^{w_n(q)} x_n)$$

for every $x = (x_1, \ldots, x_n)$. A dilation δ_ε acts also on functions and vector fields on $\mathbb{R}^n$ by pull-back: $\delta_\varepsilon^* f = f \circ \delta_\varepsilon$ and $\delta_\varepsilon^* X$ is the vector field such that $(\delta_\varepsilon^* X)(\delta_\varepsilon^* f) = \delta_\varepsilon^*(Xf)$ for any $f \in C^1(\mathbb{R}^n)$. In particular, for any vector field X of non-holonomic order k, there holds $\delta_\varepsilon^* X = \varepsilon^{-k} X$.

Nilpotent approximation. Fix a system of privileged coordinates $(x_1, \ldots, x_n)$ at q. Given a sequence of integers $\alpha = (\alpha_1, \ldots, \alpha_n)$, we define the weighted degree of $x^\alpha = x_1^{\alpha_1} \ldots x_n^{\alpha_n}$ to be $w(\alpha) = w_1(q)\alpha_1 + \ldots + w_n(q)\alpha_n$. Coming back to the vector fields $X_1, \ldots, X_m$, we can write the Taylor expansion

$$X_i(x) \sim \sum_{\alpha,j} a_{\alpha,j} x^\alpha \partial_{x_j}. \tag{2.26}$$

Since $X_i \in \mathcal{D}$, its non-holonomic order is necessarily -1, hence there holds $w(\alpha) \geqslant w_j(q) - 1$ if $a_{\alpha,j} \neq 0$. Therefore, we may write X_i as a formal series

$$X_i = X_i^{(-1)} + X_i^{(0)} + X_i^{(1)} + \ldots$$

where $X_i^{(s)}$ is a homogeneous vector field of degree s, meaning that

$$\delta_\varepsilon^* (\psi_q)_* X_i^{(s)} = \varepsilon^s (\psi_q)_* X_i^{(s)}.$$

We set $\widehat{X}_i^q = (\psi_q)_* X_i^{(-1)}$ for $1 \leqslant i \leqslant m$. Then $\widehat{X}_i^q$ is homogeneous of degree -1 with respect to dilations, i.e., $\delta_\varepsilon^* \widehat{X}_i^q = \varepsilon^{-1} \widehat{X}_i^q$ for any $\varepsilon \neq 0$. Each $\widehat{X}_i^q$ may be seen as a vector field on $\mathbb{R}^n$ thanks to the coordinates $(x_1, \ldots, x_n)$. Moreover,

$$\widehat{X}_i^q = \lim_{\varepsilon \to 0} \varepsilon \delta_\varepsilon^* (\psi_q)_* X_i$$

in C^∞ topology: all derivatives uniformly converge on compact subsets. For $\varepsilon > 0$ small enough we have

$$X_i^\varepsilon := \varepsilon \delta_\varepsilon^* (\psi_q)_* X_i = \widehat{X}_i^q + \varepsilon R_i^\varepsilon$$

where R_i^ε depends smoothly on ε for the C^∞ topology (see also [ABB19, Lemma 10.58]). An important property is that $(\widehat{X}_1^q, \ldots, \widehat{X}_m^q)$ generates a nilpotent Lie algebra of step $r(q)$ (see [Jea14, Proposition 2.3]).

The nilpotent approximation of $X_1, \ldots, X_m$ at q is then defined as $\widehat{M}^q \simeq \mathbb{R}^n$ endowed with the vector fields $\widehat{X}_1^q, \ldots, \widehat{X}_m^q$. It is important to note that the nilpotent approximation depends on the initial choice of privileged coordinates. For an explicit example of computation of nilpotent approximation, see [Jea14, Example 2.8].

2.3.2 Reduction to the nilpotent case

In this section, we show the following

Lemma 2.18. *Let $X_1, \ldots, X_m$ be smooth vector fields on M satisfying Hörmander's condition, and let $q \in M$. If the property* (**P**) *holds at point $0 \in \mathbb{R}^n$ for the nilpotent approximation $\widehat{X}_1^q, \ldots, \widehat{X}_m^q$, then the property* (**P**) *holds at point q for $X_1, \ldots, X_m$.*

Note that the above lemma is true for any nilpotent approximation $\widehat{X}_1^q, \ldots, \widehat{X}_m^q$ at q, i.e., for any choice of privileged coordinates (see Section 2.3.1).

Proof of Lemma 2.18. We use the notation h_Z for the momentum map associated with the vector field Z (see Section 2.1.4). We use the notations of Section 2.3.1, in particular the coordinate chart ψ_q.

We set $Y_i = (\psi_q)_* X_i$ and $X_i^\varepsilon = \varepsilon \delta_\varepsilon^* Y_i$ which is a vector field on $\mathbb{R}^n$. Recall that

$$X_i^\varepsilon = \widehat{X}_i^q + \varepsilon R_i^\varepsilon$$

where R_i^ε depends smoothly on ε for the C^∞ topology. Therefore, using the homogeneity of $\widehat{X}_i^q$, we get, for any $\varepsilon > 0$,

$$Y_i = \frac{1}{\varepsilon}(\delta_\varepsilon)_* X_i^\varepsilon = \frac{1}{\varepsilon}(\delta_\varepsilon)_*(\widehat{X}_i^q + \varepsilon R_i^\varepsilon) = \widehat{X}_i^q + (\delta_\varepsilon)_* R_i^\varepsilon. \tag{2.27}$$

The vector field $(\delta_\varepsilon)_* R_i^\varepsilon(x)$ does not depend on ε and has a size which tends uniformly to 0 as $x \to 0 \in \widehat{M}^q \simeq \mathbb{R}^n$. Recall that the Hamiltonian $\widehat{H}$ associated to the vector fields $\widehat{X}_i^q$ is given by

$$\widehat{H} = \sum_{i=1}^m h_{\widehat{X}_i^q}^2.$$

Similarly, we set

$$H = \sum_{i=1}^m h_{Y_i}^2.$$

We note that (2.27) gives

$$h_{Y_i} = h_{\widehat{X}_i^q} + h_{(\delta_\varepsilon)_* R_i^\varepsilon}.$$

Hence

$$\vec{H} = 2\sum_{i=1}^m h_{Y_i} \vec{h}_{Y_i} = \vec{\widehat{H}} + \vec{\Theta}, \tag{2.28}$$

where $\vec{\Theta}$ is a smooth vector field on $T^*\mathbb{R}^n$ such that

$$\|(d\pi \circ \vec{\Theta})(x,\xi)\| \leqslant C\|x\| \tag{2.29}$$

when $\|x\| \to 0$ (independently of ξ) where $\pi : T^*\mathbb{R}^n \to \mathbb{R}^n$ is the canonical projection. This last point comes from the smooth dependence of R_i^ε on ε for the C^∞ topology (uniform convergence of all derivatives on compact subsets of $\mathbb{R}^n$).

Given the projection of an integral curve $c(\cdot)$ of $\vec{H}$, we denote by $\widehat{c}(\cdot)$ the projection of the integral curve of $\vec{\widehat{H}}$ with same initial covector. Combining (2.28) and (2.29), and using Gronwall's lemma, we obtain the following result:

Fix $T_0 > 0$. For any neighborhood V of 0 in $\mathbb{R}^n$, there exists another neighborhood V' of 0 such that if $c_{|[0,T_0]} \subset V'$, then $\widehat{c}_{|[0,T_0]} \subset V$.

Therefore, if the property (**P**) holds at $0 \in \mathbb{R}^n$ for $\widehat{X}_1^q, \dots, \widehat{X}_m^q$, then it holds also at $0 \in \mathbb{R}^n$ for the vector fields $Y_1, \dots, Y_m$.

Using that $X_i = \psi_q^* Y_i$, we can pull back the result to M and obtain that the property (**P**) holds at point q for $X_1, \dots, X_m$, which concludes the proof of Proposition 2.17. □

Thanks to Lemma 2.18, it is sufficient to prove the property (**P**) under the additional assumption that

$$M \subset \mathbb{R}^n \text{ and } \mathrm{Lie}(X_1, \dots, X_m) \text{ is nilpotent.} \tag{2.30}$$

In all the sequel, we assume that this is the case.

2.3.3 End of the proof of Proposition 2.17

Let us finish the proof of Proposition 2.17. Our ideas are inspired by [AG01, Section 6].

First step: reduction to the constant Goh matrix case. We consider an adapted frame $Y_1, \ldots, Y_n$ at q. We take exponential coordinates of the second kind at q: we consider the inverse ψ_q of the diffeomorphism

$$(x_1, \ldots, x_n) \mapsto \exp(x_1 Y_1) \ldots \exp(x_n Y_n)(q).$$

Then we write the Taylor expansion (2.26) of $X_1, \ldots, X_m$ in these coordinates. Thanks to Lemma 2.18, we can assume that all terms in these Taylor expansions have non-holonomic order -1. We denote by ξ_i the dual variable of x_i. We use the notations $n_1, n_2, \ldots$ introduced in Section 2.3.1, and we make a strong use of (2.25).

Claim 1. If a normal geodesic $(x(t), \xi(t))_{t\in\mathbb{R}}$ has initial momentum satisfying $\xi_k(0) = 0$ for any $k \geqslant n_2 + 1$, then $\dot{\xi}_k \equiv 0$ for any $k \geqslant n_1 + 1$, and in particular $\xi_k \equiv 0$ for any $k \geqslant n_2 + 1$.

Proof. We write

$$X_j(x) = \sum_{i=1}^{n} a_{ij}(x)\partial_{x_i}, \qquad j = 1, \ldots, m$$

where the a_{ij} are homogeneous polynomials. We have

$$g^*(x, \xi) = \sum_{j=1}^{m} \left(\sum_{i=1}^{n} a_{ij}(x)\xi_i \right)^2. \tag{2.31}$$

Let $k \geqslant n_2 + 1$, which means that x_k has non-holonomic order $\geqslant 3$. If $a_{ij}(x)$ depends on x_k, then necessarily $i \geqslant n_3 + 1$, since $a_{ij}(x)\partial_{x_i}$ has non-holonomic order -1. Thus, writing explicitly $\dot{\xi}_k = -\frac{\partial g^*}{\partial x_k}$ thanks to (2.31), there is in front of each term a ξ_i for some i which is in particular $\geqslant n_2 + 1$. By Cauchy uniqueness, we deduce that $\xi_k \equiv 0$ for any $k \geqslant n_2 + 1$.

Now, let $k \geqslant n_1 + 1$, which means that x_k has non-holonomic order $\geqslant 2$. If $a_{ij}(x)$ depends on x_k, then necessarily $i \geqslant n_2 + 1$, since $a_{ij}(x)\partial_{x_i}$ has non-holonomic order -1. Thus, writing explicitly $\dot{\xi}_k = -\frac{\partial g^*}{\partial x_k}$ thanks to (2.31), there is in front of each term a ξ_i for some i which is $\geqslant n_2 + 1$. It is null by the previous conclusion, hence $\dot{\xi}_k \equiv 0$. $\square$

The previous claim will help us reducing the complexity of the vector fields X_i once again (after the first reduction provided by Lemma 2.18). Let us consider, for any $1 \leqslant j \leqslant m$, the vector field

$$X_j^{\text{red}} = \sum_{i=1}^{n_2} a_{ij}(x)\partial_{x_i} \tag{2.32}$$

where the sum is taken only up to n_2. We also consider the reduced Hamiltonian on T^*M

$$g^*_{\text{red}} = \sum_{j=1}^{m} h^2_{X_j^{\text{red}}}.$$

Claim 2. If $X_1^{\text{red}}, \ldots, X_m^{\text{red}}$ satisfy Property ($\mathbf{P}$) at q, then $X_1, \ldots, X_m$ satisfy Property ($\mathbf{P}$) at q.

Proof. Let us assume that $X_1^{\mathrm{red}}, \dots, X_m^{\mathrm{red}}$ satisfy Property (**P**) at q. Let $T_0 > 0$ and let $(x^{\mathrm{red},\varepsilon}(0), \xi^{\mathrm{red},\varepsilon}(0))$ be initial data for the Hamiltonian system associated to g^*_{red} which yield speed 1 normal geodesics $(x^{\mathrm{red},\varepsilon}(t), \xi^{\mathrm{red},\varepsilon}(t))$ such that $x^{\mathrm{red},\varepsilon}(t) \to q$ uniformly over $(0, T_0)$ as $\varepsilon \to 0$.

We can assume without loss of generality that $\xi_i^{\mathrm{red},\varepsilon}(0) = 0$ for any $i \geqslant n_2 + 1$, since these momenta (preserved under the reduced Hamiltonian evolution) do not change the projection $x^{\mathrm{red},\varepsilon}(t)$ of the normal geodesic. We consider $(x^\varepsilon(0), \xi^\varepsilon(0)) = (x^{\mathrm{red},\varepsilon}(0), \xi^{\mathrm{red},\varepsilon}(0))$ as initial data for the (non-reduced) Hamiltonian evolution associated to g^*. Then we notice that $\xi_k^\varepsilon \equiv 0$ for $k \geqslant n_2 + 1$ thanks to Claim 1. It follows that when $i \leqslant n_2$, we have $x_i^\varepsilon(t) = x_i^{\mathrm{red},\varepsilon}(t)$, i.e., the coordinate x_i is the same for the reduced and the non-reduced Hamiltonian evolution.

Finally, we take k such that $n_2 + 1 \leqslant k \leqslant n_3$. Since g^* is given by (2.31), we have

$$\dot{x}_k^\varepsilon = \frac{\partial g^*}{\partial \xi_k} = 2 \sum_{j=1}^m a_{kj}(x^\varepsilon) \left(\sum_{i=1}^n a_{ij}(x^\varepsilon) \xi_i^\varepsilon \right). \tag{2.33}$$

But a_{kj} has necessarily non-holonomic order 2 since ∂_{x_k} has non-holonomic order -3. Thus, $a_{kj}(x)$ is a non-constant homogeneous polynomial in $x_1, \dots, x_{n_2}$. Since $x_1^\varepsilon, \dots, x_{n_2}^\varepsilon$ converge to q uniformly over $(0, T_0)$ as $\varepsilon \to 0$, it is also the case of x_k^ε according to (2.33), noticing that

$$\left| \sum_{i=1}^n a_{ij}(x^\varepsilon) \xi_i^\varepsilon \right| \leqslant (g^*)^{1/2} = 1/2$$

for any j. In other words, $x_{n_2+1}^\varepsilon, \dots, x_{n_3}^\varepsilon$ also converge to q uniformly over $(0, T_0)$ as $\varepsilon \to 0$.

We can repeat this argument successively for $k \in \{n_3 + 1, \dots, n_4\}$, $k \in \{n_4 + 1, \dots, n_5\}$, etc, and we finally obtain the result: for any $1 \leqslant k \leqslant n$, x_k^ε converges to q uniformly over $(0, T_0)$ as $\varepsilon \to 0$. □

Thanks to the previous claim, we are now reduced to prove Proposition 2.17 for the vector fields $X_1^{\mathrm{red}}, \dots, X_m^{\mathrm{red}}$. In order to keep notations as simple as possible, we simplify these notations into $X_1, \dots, X_m$, i.e., we drop the upper notation "red". Also, without loss of generality we assume that $q = 0$.

If we choose our normal geodesics so that $x(0) = 0$, then $x_i \equiv 0$ for any $i \geqslant n_2 + 1$ thanks to (2.32). In other words, we forget the coordinates $x_{n_2+1}, \dots, x_n$ in the sequel, since they all vanish.[2]

Second step: conclusion of the proof. Now, we write the normal extremal system in its "control" form. We refer the reader to [ABB19, Chapter 4]. We have

$$\dot{x}(t) = \sum_{i=1}^m u_i(t) X_i(x(t)), \tag{2.34}$$

where the u_i are the controls, explicitly given by (see [ABB19, Theorem 4.20])

$$u_i(t) = h_{X_i}(x(t), \xi(t)). \tag{2.35}$$

[2] Note that this is the case only because we are now working with the reduced Hamiltonian evolution; otherwise, under the original Hamiltonian evolution associated to (2.31), the x_i (for $i \geqslant n_2 + 1$) remain small according to Claim 2, but do not necessarily vanish.

Thanks to (2.32), we rewrite (2.34) as

$$\dot{x}(t) = F(x(t))u(t), \tag{2.36}$$

where $F = (a_{ij})$, which has size $n_2 \times m$, and $u = {}^t(u_1, \ldots, u_m)$. Differentiating (2.35), we have the complementary equation

$$\dot{u}(t) = G(x(t), \xi(t))u(t)$$

where G is the Goh matrix

$$G = (2\{h_{X_i}, h_{X_j}\})_{1 \leqslant i,j \leqslant m}$$

(it differs from the usual Gox matrix by a factor -2 due to the absence of factor $\frac{1}{2}$ in the Hamiltonian g^* in our notations).

Let us prove that $G(t)$ is constant in t. Fix $1 \leqslant j, j' \leqslant m$. We notice that in (2.32), a_{ij} is a constant (independent of x) as soon as $1 \leqslant i \leqslant n_1$ since ∂_{x_i} has weight -1. This implies that

$$[X_j, X_{j'}] \text{ is spanned by the vector fields } \partial_{x_{n_1+1}}, \partial_{x_{n_1+2}}, \ldots, \partial_{x_{n_2}}. \tag{2.37}$$

Putting this into the relation $\{h_{X_j}, h_{X_{j'}}\} = h_{[X_j, X_{j'}]}$, and using that the dual variables ξ_k for $n_1 + 1 \leqslant k \leqslant n_2$ are preserved under the Hamiltonian evolution (due to Claim 1), we get that $G(t) \equiv G$ is constant in t.

We know that $G \neq 0$ and that G is antisymmetric. The whole control space $\mathbb{R}^m$ is the direct sum of the image of G and the kernel of G, and G is nondegenerate on its image. We take u_0 in an invariant plane of G; in other words its projection on the kernel of G vanishes (see Remark 2.20). We denote by $\widetilde{G}$ the restriction of G to this invariant plane. We also assume that u_0, decomposed as $u_0 = (u_{01}, \ldots, u_{0m}) \in \mathbb{R}^m$, satisfies $\sum_{i=1}^m u_{0i}^2 = 1/4$. Then $u(t) = e^{t\widetilde{G}}u_0$ and since $e^{t\widetilde{G}}$ is an orthogonal matrix, we have $\|e^{t\widetilde{G}}u_0\| = \|u_0\|$. We have by integration by parts

$$\begin{aligned} x(t) &= \int_0^t F(x(s))e^{s\widetilde{G}}u_0\, ds \\ &= F(x(t))\widetilde{G}^{-1}(e^{t\widetilde{G}} - I)u_0 - \int_0^t \frac{d}{ds}(F(x(s))\widetilde{G}^{-1}(e^{s\widetilde{G}} - I)u_0\, ds. \end{aligned} \tag{2.38}$$

Let us now choose the initial data of our family of normal geodesics (indexed by ε). The starting point $x^\varepsilon(0) = 0$ is the same for any ε, we only have to specify the initial covectors $\xi^\varepsilon = \xi^\varepsilon(0) \in T_0^*\mathbb{R}^m$. For any $i = 1, \ldots, m$, we impose that

$$\langle \xi^\varepsilon, X_i \rangle = u_{0i}. \tag{2.39}$$

It follows that $g^*(x(0), \xi^\varepsilon(0)) = \sum_{i=1}^m u_{0i}^2 = 1/4$ for any $\varepsilon > 0$. Now, we notice that $\mathrm{Span}(X_1, \ldots, X_m)$ is in direct sum with the Span of the $[X_i, X_j]$ for i, j running over $1, \ldots, m$ (this follows from (2.37)). Fixing $G^0 \neq 0$ an antisymmetric matrix and $\widetilde{G}^0$ its restriction to an invariant plane, we can specify, simultaneously to (2.39), that

$$\langle \xi^\varepsilon, 2[X_j, X_i] \rangle = \varepsilon^{-1} G^0_{ij}.$$

Then $x^\varepsilon(t)$ is given by (2.38) applied with $\widetilde{G} = \varepsilon^{-1}\widetilde{G}^0$, which brings a factor ε in front of (2.38).

Recall finally that the coefficients a_{ij} which compose F have non-holonomic order 0 or 1, thus they are degree 1 (or constant) homogeneous polynomials in $x_1, \ldots, x_{n_1}$. Thus $\frac{d}{ds}(F(x(s))$ is a linear combination of $\dot{x}_i(s)$ which we can rewrite thanks to (2.36) as a combination with bounded coefficients (since $\sum_{i=1}^m u_i^2 = 1/4$) of the $x_i(s)$. Hence, applying the Gronwall lemma in (2.38), we get $\|x^\varepsilon(t)\| \leqslant C\varepsilon$, which concludes the proof.

Remark 2.19. The normal geodesics constructed above lose their optimality quickly, in the sense that their first conjugate point and their cut-point are close to q.

Remark 2.20. If we take u_0 in the kernel of G, then the corresponding trajectory is singular, see [ABB19, Chapter 4]. In this case, we can find normal geodesics which spiral around this singular curve, and do not remain close to their initial point over $(0, T_0)$ although their initial covector is "high in the cylinder bundle U^*M". For example, for the Hamiltonian $\xi_1^2+(\xi_2+x_1^2\xi_3)^2$ associated to the "Martinet" vector fields $X_1 = \partial_{x_1}$, $X_2 = \partial_{x_2} + x_1^2\partial_{x_3}$ in $\mathbb{R}^3$, there exist normal geodesics which spiral around the singular curve $(t, 0, 0)$.

2.4 Proofs

2.4.1 Proof of Theorem 2.2

In this section, we conclude the proof of Theorem 2.2.

Fix a point q in the interior of $M \setminus \omega$ and $1 \leqslant i, j \leqslant m$ such that $[X_i, X_j](q) \notin \mathcal{D}_q$. Fix also an open neighborhood V of q in M such that $V \subset M \backslash \omega$. Fix V' an open neighborhood of q in M such that $\overline{V'} \subset V$, and fix also $T_0 > 0$.

As already explained in Section 2.1.3, to conclude the proof of Theorem 2.2, we use Proposition 2.16 applied to the particular normal geodesics constructed in Proposition 2.17.

By Proposition 2.17, we know that there exists a normal geodesic $t \mapsto x(t)$ such that $x(t) \in V'$ for any $t \in (0, T_0)$. It is the projection of a bicharacteristic $(x(t), \xi(t))$ and since it is non-stationary and traveled at speed 1, there holds $g^*(x(t), \xi(t)) = 1/4$. We denote by $(u_k)_{k\in\mathbb{N}}$ a sequence of solutions of (2.12) as in Proposition 2.16 whose energy at time t concentrates on $x(t)$ for $t \in (0, T_0)$. Because of (2.22), we know that

$$\|(u_k(0), \partial_t u_k(0))\|_{\mathcal{H}\times L^2} \geqslant c > 0$$

uniformly in k.

Therefore, in order to establish Theorem 2.2, it is sufficient to show that

$$\int_0^{T_0} \int_\omega |\partial_t u_k(t, x)|^2 d\mu(x) dt \underset{k\to+\infty}{\to} 0. \tag{2.40}$$

Since $x(t) \in V'$ for any $t \in (0, T_0)$, we get that for V_t chosen sufficiently small for any $t \in (0, T_0)$, the inclusion $V_t \subset V$ holds (see Proposition 2.16 for the definition of V_t). Combining this last remark with (2.23), we get (2.40), which concludes the proof of Theorem 2.2.

2.4.2 Proof of Corollary 2.4

We endow the topological dual $\mathcal{H}(M)'$ with the norm $\|v\|_{\mathcal{H}(M)'} = \|(-\Delta)^{-1/2}v\|_{L^2(M)}$.

The following proposition is standard (see, e.g., [TW09], [LLTT17]).

Lemma 2.21. *Let $T_0 > 0$, and $\omega \subset M$ be a measurable set. Then the following two observability properties are equivalent:*

(P1): *There exists C_{T_0} such that for any $(v_0, v_1) \in D((-\Delta)^{\frac{1}{2}}) \times L^2(M)$, the solution $v \in C^0(0, T_0; D((-\Delta)^{\frac{1}{2}})) \cap C^1(0, T_0; L^2(M))$ of (2.1) satisfies*

$$\int_0^{T_0} \int_\omega |\partial_t v(t, q)|^2 d\mu(q) dt \geqslant C_{T_0} \|(v_0, v_1)\|_{\mathcal{H}(M)\times L^2(M)}. \tag{2.41}$$

(P2): *There exists C_{T_0} such that for any $(v_0, v_1) \in L^2(M) \times D((-\Delta)^{-\frac{1}{2}})$, the solution $v \in C^0(0, T_0; L^2(M)) \cap C^1(0, T_0; D((-\Delta)^{-\frac{1}{2}}))$ of (2.1) satisfies*

$$\int_0^{T_0} \int_\omega |v(t,q)|^2 d\mu(q) dt \geqslant C_{T_0} \|(v_0, v_1)\|^2_{L^2 \times \mathcal{H}(M)'}. \tag{2.42}$$

Proof. Let us assume that (P2) holds. Let u be a solution of (2.1) with initial conditions $(u_0, u_1) \in D((-\Delta)^{\frac{1}{2}}) \times L^2(M)$. We set $v = \partial_t u$, which is a solution of (2.1) with initial data $v_{|t=0} = u_1 \in L^2(M)$ and $\partial_t v_{|t=0} = \Delta u_0 \in D((-\Delta)^{-\frac{1}{2}})$. Since $\|(v_0, v_1)\|_{L^2 \times \mathcal{H}(M)'} = \|(u_1, \Delta u_0)\|_{L^2 \times \mathcal{H}(M)'} = \|(u_0, u_1)\|_{\mathcal{H}(M) \times L^2}$, applying the observability inequality (2.42) to $v = \partial_t u$, we obtain (2.41). The proof of the other implication is similar. □

Finally, using Theorem 2.2, Lemma 2.21 and the standard HUM method ([Lio88]), we get Corollary 2.4.

2.4.3 Proof of Theorem 2.11

We consider the space of functions $u \in C^\infty([0, T] \times M_H)$ such that $\int_{M_H} u(t, \cdot) d\mu = 0$ for any $t \in [0, T]$, and we denote by $\mathcal{H}_T$ its completion for the norm $\|\cdot\|_{\mathcal{H}_T}$ induced by the scalar product

$$(u, v)_{\mathcal{H}_T} = \int_0^T \int_{M_H} \left(\partial_t u \partial_t v + (\nabla^{\mathrm{sR}} u) \cdot (\nabla^{\mathrm{sR}} v)\right) d\mu(q) dt.$$

We consider also the topological dual $\mathcal{H}_0'$ of the space $\mathcal{H}_0$ (see Section 2.1.5).

Lemma 2.22. *The injections $\mathcal{H}_0 \hookrightarrow L^2(M_H)$, $L^2(M_H) \hookrightarrow \mathcal{H}_0'$ and $\mathcal{H}_T \hookrightarrow L^2((0, T) \times M_H)$ are compact.*

Proof. Let $(\varphi_k)_{k \in \mathbb{N}}$ be an orthonormal basis of eigenfunctions of $L^2(M_H)$, labeled with increasing eigenvalues $0 = \lambda_0 < \lambda_1 \leqslant \ldots \leqslant \lambda_k \to +\infty$, so that $-\Delta \varphi_k = \lambda_k \varphi_k$. The fact that $\lambda_1 > 0$, which will be used in the sequel, can be proved as follows: if $-\Delta \varphi = 0$ then $\int_{M_H} |\nabla^{\mathrm{sR}} \varphi|^2 d\mu = 0$ and, since $\varphi \in C^\infty(M_H)$ by hypoelliptic regularity, we get $X_1 \varphi(x) = X_2 \varphi(x) = 0$ for any $x \in M_H$. Hence, $[X_1, X_2]\varphi \equiv 0$, and alltogether, this proves that φ is constant, hence $\lambda_1 > 0$.

We prove the last injection. Let $u \in \mathcal{H}_T$. Writing $u(t, \cdot) = \sum_{k=1}^\infty a_k(t) \varphi_k(\cdot)$ (note that there is no 0-mode since $u(t, \cdot)$ has null average), we see that

$$\begin{aligned}\|u\|^2_{\mathcal{H}_T} \geqslant \|\nabla^{\mathrm{sR}} u\|^2_{L^2((0,T) \times M_H)} = \sum_{k=1}^\infty \lambda_k \|a_k\|^2_{L^2((0,T))} &\geqslant \lambda_1 \sum_{k=1}^\infty \|a_k\|^2_{L^2((0,T))} \\ &= \lambda_1 \|u\|^2_{L^2((0,T) \times M_H)},\end{aligned}$$

thus $\mathcal{H}_T$ imbeds continuously into $L^2((0, T) \times M_H)$. Then, using a classical subelliptic estimate (see [Hor67] and [RS76, Theorem 17]), we know that there exists $C > 0$ such that

$$\|u\|_{H^{\frac{1}{2}}((0,T) \times M_H)} \leqslant C(\|u\|_{L^2((0,T) \times M_H)} + \|u\|_{\mathcal{H}_T}).$$

Together with the previous estimate, we obtain that for any $u \in \mathcal{H}_T$, $\|u\|_{H^{\frac{1}{2}}((0,T) \times M_H)} \leqslant C \|u\|_{\mathcal{H}_T}$. Then, the result follows from the fact that the injection $H^{\frac{1}{2}}((0, T) \times M_H) \hookrightarrow L^2((0, T) \times M_H)$ is compact.

The proof of the compact injection $\mathcal{H}_0 \hookrightarrow L^2(M_H)$ is similar, and the compact injection $L^2(M_H) \hookrightarrow \mathcal{H}_0'$ follows by duality. □

Proof of Theorem 2.11. In this proof, we use the notation $P = \partial_{tt}^2 - \Delta_H$. For the sake of a contradiction, suppose that there exists a sequence $(u^k)_{k\in\mathbb{N}}$ of solutions of the wave equation such that $\|(u_0^k, u_1^k)\|_{\mathcal{H}\times L^2} = 1$ for any $k \in \mathbb{N}$ and

$$\|(u_0^k, u_1^k)\|_{L^2\times\mathcal{H}_0'} \to 0, \qquad \int_0^T |(\mathrm{Op}(a)\partial_t u^k, \partial_t u^k)_{L^2(M_H,\mu)}|dt \to 0 \tag{2.43}$$

as $k \to +\infty$. Following the strategy of [Tar90] and [Ger91b], our goal is to associate a defect measure to the sequence $(u^k)_{k\in\mathbb{N}}$. Since the functional spaces involved in our result are unusual, we give the argument in detail.

First, up to extraction of a subsequence which we omit, (u_0^k, u_1^k) converges weakly in $\mathcal{H}_0 \times L^2(M_H)$ and, using the first convergence in (2.43) and the compact embedding $\mathcal{H}_0 \times L^2(M_H) \hookrightarrow L^2(M_H) \times \mathcal{H}_0'$, we get that $(u_0^k, u_1^k) \rightharpoonup 0$ in $\mathcal{H}_0 \times L_0^2$. Using the continuity of the solution with respect to the initial data, we obtain that $u^k \rightharpoonup 0$ weakly in $\mathcal{H}_T$. Using Lemma 2.22, we obtain $u^k \to 0$ strongly in $L^2((0,T)\times M_H)$.

Fix $B \in \Psi_{\mathrm{phg}}^0((0,T)\times M_H)$. We have

$$\begin{aligned}(Bu^k, u^k)_{\mathcal{H}_T} &= \int_0^T \int_{M_H} \left(\partial_t(Bu^k)\partial_t u^k + \left(\nabla^{\mathrm{sR}}(Bu^k)\right)\cdot\left(\nabla^{\mathrm{sR}}u^k\right)\right) d\mu(q)dt \\ &= \int_0^T \int_{M_H} \left(([\partial_t, B]u^k)\partial_t u^k + \left([\nabla^{\mathrm{sR}}, B]u^k\right)\cdot\left(\nabla^{\mathrm{sR}}u^k\right)\right) d\mu(q)dt \\ &\quad + \int_0^T \int_{M_H} \left(\left(B\partial_t u^k\right)\left(\partial_t u^k\right) + \left(B\nabla^{\mathrm{sR}}u^k\right)\cdot\left(\nabla^{\mathrm{sR}}u^k\right)\right) d\mu(q)dt \end{aligned} \tag{2.44}$$

Since $[\partial_t, B] \in \Psi_{\mathrm{phg}}^0((0,T)\times M_H)$, $[\nabla^{\mathrm{sR}}, B] \in \Psi_{\mathrm{phg}}^0((0,T)\times M_H)$ and $u^k \to 0$ strongly in $L^2((0,T)\times M_H)$, the first one of the two lines in (2.44) converges to 0 as $k \to +\infty$. Moreover, the last line is bounded uniformly in k since $B \in \Psi_{\mathrm{phg}}^0((0,T)\times M_H)$. Hence $(Bu^k, u^k)_{\mathcal{H}_T}$ is uniformly bounded. By a standard diagonal extraction argument (see [Ger91b] for example), there exists a subsequence, which we still denote by $(u^k)_{k\in\mathbb{N}}$ such that (Bu^k, u^k) converges for any B of principal symbol b in a countable dense subset of $C_c^\infty((0,T)\times M_H)$. Moreover, the limit only depends on the principal symbol b, and not on the full symbol.

Let us now prove that

$$\liminf_{k\to+\infty} (Bu^k, u^k)_{\mathcal{H}_T} \geqslant 0 \tag{2.45}$$

when $b \geqslant 0$. With a bracket argument as in (2.44), we see that it is equivalent to proving that the liminf as $k \to +\infty$ of the quantity

$$Q_k(B) = (B\partial_t u^k, \partial_t u^k)_{L^2} + (B\nabla^{\mathrm{sR}}u^k, \nabla^{\mathrm{sR}}u^k)_{L^2} \tag{2.46}$$

is $\geqslant 0$. But there exists $B' \in \Psi_{\mathrm{phg}}^0((0,T)\times M_H)$ such that $B' - B \in \Psi_{\mathrm{phg}}^{-1}((0,T)\times M_H)$ and B' is positive (this is the so-called Friedrichs quantization, see for example [Tay74, Chapter VII]). Then, $\liminf_{k\to+\infty} Q_k(B') \geqslant 0$, and $Q_k(B'-B) \to 0$ since $(B'-B)\partial_t \in \Psi_{\mathrm{phg}}^0((0,T)\times M_H)$ and $u^k \to 0$ strongly in $L^2((0,T)\times M_H)$. It immediately implies that (2.45) holds.

Therefore, setting $p = \sigma_p(P)$ and denoting by $\mathcal{C}(p)$ the characteristic manifold $\mathcal{C}(p) = \{p = 0\}$, there exists a non-negative Radon measure ν on $S^*(\mathcal{C}(p)) = \mathcal{C}(p)/(0,+\infty)$ such that

$$(\mathrm{Op}(b)u^k, u^k)_{\mathcal{H}_T} \to \int_{S^*(\mathcal{C}(p))} b d\nu$$

for any $b \in S_{\mathrm{phg}}^0((0,T)\times M_H)$.

Let $C \in \Psi^{-1}_{\rm phg}((0,T)\times M_H)$ of principal symbol c. We have $\vec{H}_p c = \{p,c\} \in S^0_{\rm phg}((0,T)\times M_H)$ and, for any $k \in \mathbb{N}$,

$$((CP-PC)u^k,u^k)_{\mathcal{H}_T} = (CPu^k,u^k)_{\mathcal{H}_T} - (Cu^k,Pu^k)_{\mathcal{H}_T} = 0 \tag{2.47}$$

since $Pu^k = 0$. To be fully rigorous, the identity of the previous line, which holds for any solution $u \in \mathcal{H}_T$ of the wave equation, is first proved for smooth initial data since $Pu \notin \mathcal{H}_T$ in general, and then extended to general solutions $u \in \mathcal{H}_T$. Taking principal symbols in (2.47), we get $\langle \nu, \vec{H}_p c\rangle = 0$.

Therefore, denoting by $(\psi_s)_{s\in\mathbb{R}}$ the maximal solutions of

$$\frac{d}{ds}\psi_s(\rho) = \vec{H}_p(\psi_s(\rho)), \qquad \rho \in T^*(\mathbb{R}\times M_H)$$

(see (2.7)), we get that, for any $s \in (0,T)$,

$$0 = \langle \nu, \vec{H}_p c\circ\psi_s\rangle = \langle \nu, \frac{d}{ds} c\circ\psi_s\rangle = \frac{d}{ds}\langle \nu, c\circ\psi_s\rangle$$

and hence

$$\langle \nu, c\rangle = \langle \nu, c\circ\psi_s\rangle. \tag{2.48}$$

We note here that the precise homogeneity of c (namely $c \in S^{-1}_{\rm phg}((0,T)\times M_H)$) does not matter since ν is a measure on the sphere bundle $S^*(\mathcal{C}(p))$. The identity (2.48) means that ν is invariant under the flow $\vec{H}_p$.

From the second convergence in (2.43), we can deduce that

$$\nu = 0 \text{ in } S^*(\mathcal{C}(p)) \cap T^*((0,T)\times \mathrm{Supp}(a)). \tag{2.49}$$

The proof of this fact, which is standard (see for example [BG02, Section 6.2]), is given in Section 2-A.2.

Let us prove that any normal geodesic of M_H with momentum $\xi \in V_\varepsilon^c$ enters ω in time at most $\kappa\varepsilon^{-1}$ for some $\kappa > 0$ which does not depend on ε. Indeed, the solutions of the bicharacteristic equations (2.10) with $g^* = 1/4$ and $\xi_3 \neq 0$ are given by

$$x_1(t) = \frac{1}{2\xi_3}\cos(2\xi_3 t+\phi) + \frac{\xi_2}{\xi_3}, \qquad x_2(t) = B - \frac{1}{2\xi_3}\sin(2\xi_3 t+\phi)$$
$$x_3(t) = C + \frac{t}{4\xi_3} + \frac{1}{16\xi_3^2}\sin(2(2\xi_3 t+\phi)) + \frac{\xi_2}{2\xi_3^2}\sin(2\xi_3 t+\phi)$$

where B, C, ξ_2, ξ_3 are constants. Since $\xi \in V_\varepsilon^c$ and $g^* = 1/4$, there holds $\frac{1}{4|\xi_3|} \geqslant \frac{\varepsilon}{2}$. Hence, we can conclude using the expression for x_3 (whose derivative is roughly $(4|\xi_3|)^{-1}$) and the fact that $\omega = M_H\backslash B$ contains a horizontal strip. Note that if $\xi_3 = 0$, the expressions of $x_1(t), x_2(t), x_3(t)$ are much simpler and we can conclude similarly.

Hence, together with (2.49), the propagation property (2.48) implies that $\nu \equiv 0$. It follows that $\|u^k\|_{\mathcal{H}_T} \to 0$. By conservation of energy, it is a contradiction with the normalization $\|(u_0^k,u_1^k)\|_{\mathcal{H}\times L^2} = 1$. Hence, (2.11) holds. $\square$

2-A Supplementary material

2-A.1 Proof of Proposition 2.12

In this Section, we give a second proof of Proposition 2.12 written in a more elementary form than the one of Section 2.2.1. Let us first prove the result when $M \subset \mathbb{R}^n$, following the proof of [Ral82]. The general case is addressed at the end of this section.

As in the proof of Section 2.2.1, we suppress the time variable t. Thus we use $x = (x_0, x_1, \ldots, x_n)$ where $x_0 = t$. Similarly, $\xi = (\xi_0, \xi_1, \ldots, \xi_n)$ where $\xi_0 = \tau$ previously. Let Γ be the curve given by $x(s) \in \mathbb{R}^{n+1}$. We insist on the fact that in the proof the bicharacteristics are parametrized by s, as in (2.7). We consider functions of the form

$$v_k(x) = k^{\frac{n}{4}-1} a_0(x) e^{ik\psi(x)}.$$

We would like to choose $\psi(x)$ such that for all $s \in \mathbb{R}$, $\psi(x(s))$ is real-valued and $\mathrm{Im} \frac{\partial^2 \psi}{\partial x_i \partial x_j}(x(s))$ is positive definite on vectors orthogonal to $\dot{x}(s)$. Roughly speaking, $|e^{ik\psi(x)}|$ will then look like a Gaussian distribution on planes perpendicular to Γ in $\mathbb{R}^{n+1}$.

We first observe that $\partial_{tt}^2 v_k - \Delta v_k$ can be decomposed as

$$\partial_{tt}^2 v_k - \Delta v_k = (k^{\frac{n}{4}+1} A_1 + k^{\frac{n}{4}} A_2 + k^{\frac{n}{4}-1} A_3) e^{ik\psi} \tag{2.50}$$

with

$$\begin{aligned} A_1(x) &= p_2\left(x, \nabla\psi(x)\right) a_0(x) \\ A_2(x) &= L a_0(x) \\ A_3(x) &= \partial_{tt}^2 a_0(x) - \Delta a_0(x). \end{aligned}$$

Here we have set

$$L a_0 = \frac{1}{i} \sum_{j=0}^{n} \frac{\partial p_2}{\partial \xi_j}\left(x, \nabla\psi(x)\right) \frac{\partial a_0}{\partial x_j} + \frac{1}{2i} \left(\sum_{j,k=0}^{n} \frac{\partial^2 p_2}{\partial \xi_j \partial \xi_k}\left(x, \nabla\psi(x)\right) \frac{\partial^2 \psi}{\partial x_j \partial x_k} \right) a_0 \tag{2.51}$$

(For general strictly hyperbolic operators, L contains a term with the sub-principal symbol of the operator, but here it is null, see Appendix A.)

In what follows, we construct a_0 and ψ so that $A_1(x)$ vanishes at order 2 along Γ and $A_2(x)$ vanishes at order 0 along the same curve. We will then be able to use Lemma 2.14 with $S = 3$ and $S = 1$ respectively.

Analysis of $A_1(x)$. Our goal is to show that, if we choose ψ adequately, we can make the quantity

$$f(x) = p_2\left(x, \nabla\psi(x)\right) \tag{2.52}$$

vanish at order 2 on Γ. For the vanishing at order 0, we prescribe that ψ satisfies $\nabla\psi(x(s)) = \xi(s)$, and then $f(x(s)) = 0$ since $(x(s), \xi(s))$ is a null-bicharacteristic. Note that this is possible since $x(s) \neq x(s')$ for any $s \neq s'$, due to $\dot{x}_0(s) = 1$ (bicharacteristics are traveled at speed 1, see Section 2.1.4). For the vanishing at order 1, using (2.52) and (2.7), we remark that for any

$0 \leqslant j \leqslant n$,

$$\begin{aligned}\frac{\partial f}{\partial x_j}(x(s)) &= \frac{\partial p_2}{\partial x_j}(x(s)) + \sum_{k=0}^{n} \frac{\partial p_2}{\partial \xi_k}(x(s)) \frac{\partial \psi}{\partial x_j \partial x_k}(x(s)) \\ &= -\dot{\xi}_j(s) + \sum_{k=0}^{n} \dot{x}_k(s) \frac{\partial \psi}{\partial x_j \partial x_k}(x(s)) \\ &= -\frac{d}{ds}\left(\frac{\partial \psi}{\partial x_j}(x(s))\right) + \sum_{k=0}^{n} \dot{x}_k(s) \frac{\partial \psi}{\partial x_j \partial x_k}(x(s)) \\ &= 0.\end{aligned} \tag{2.53}$$

Therefore, f vanishes automatically at order 1 along Γ (without making any particular choice for ψ): it just follows from (2.52) and the bicharacteristic equations (2.7). But for $f(x)$ to vanish at order 2 along Γ, it is required to choose a particular ψ. In the end, we will find that if ψ is given by the formula (2.59) below, with M being a solution of (2.54), then f vanishes at order 2 along Γ. Let us explain why.

Using the Einstein summation notation, we want that for any $0 \leqslant i, j \leqslant n$, there holds

$$\begin{aligned}0 &= \frac{\partial^2 f}{\partial x_j \partial x_i} \\ &= \frac{\partial^2 p_2}{\partial x_j \partial x_i} + \frac{\partial^2 p_2}{\partial \xi_k \partial x_i} \frac{\partial^2 \psi}{\partial x_j \partial x_k} + \frac{\partial^2 p_2}{\partial x_j \partial \xi_k} \frac{\partial^2 \psi}{\partial x_i \partial x_k} + \frac{\partial^2 p_2}{\partial \xi_l \partial \xi_k} \frac{\partial^2 \psi}{\partial x_i \partial x_k} \frac{\partial^2 \psi}{\partial x_j \partial x_l} + \frac{\partial p_2}{\partial \xi_k} \frac{\partial^3 \psi}{\partial x_j \partial x_k \partial x_i}\end{aligned}$$

along Γ. Introducing the matrices

$$(M(s))_{ij} = \frac{\partial^2 \psi}{\partial x_i \partial x_j}(x(s)), \qquad (A(s))_{ij} = \frac{\partial^2 p_2}{\partial x_i \partial x_j}(x(s), \xi(s)),$$

$$(B(s))_{ij} = \frac{\partial^2 p_2}{\partial \xi_i \partial x_j}(x(s), \xi(s)), \qquad (C(s))_{ij} = \frac{\partial^2 p_2}{\partial \xi_i \partial \xi_j}(x(s), \xi(s))$$

this amounts to solving the matricial Riccati equation

$$\frac{dM}{ds} + MCM + B^T M + MB + A = 0 \tag{2.54}$$

on a finite-length time-interval. While solving (2.54), we also require $M(s)$ to be symmetric, $\mathrm{Im}(M(s))$ to be positive definite on the orthogonal complement of $\dot{x}(s)$, and $M(s)\dot{x}(s) = \dot{\xi}(s)$ to hold for all s due to (2.53).

Let M_0 be a symmetric $(n+1) \times (n+1)$ matrix with $\mathrm{Im}(M_0) > 0$ on the orthogonal complement of $\dot{x}(0)$ and $M_0\dot{x}(0) = \dot{\xi}(0)$ (in particular $\mathrm{Im}(M_0)\dot{x}(0) = 0$). It is shown in [Ral82] that there exists a global solution $M(s)$ on $[0, T]$ of (2.54) which satisfies all the above conditions and such that $M(0) = M_0$. The proof just requires that A, C are symmetric, but does not need anything special about p_2 (in particular, it applies to our sub-Riemannian case where p_2 is degenerate). For the sake of completeness, we recall the proof here.

We consider $(Y(s), N(s))$ the matrix solution with initial data $(Y(0), N(0)) = (\mathrm{Id}, M_0)$ (where Id is the $(n+1) \times (n+1)$ identity matrix) to the linear system

$$\begin{cases} \dot{Y} = BY + CN \\ \dot{N} = -AY - B^T N. \end{cases} \tag{2.55}$$

We note that $(Y(s)\dot{x}(0), N(s)\dot{x}(0))$ then also solves (2.55), with Y and N being this time vectorial. One can check that $(\dot{x}(s), \dot{\xi}(s))$ is the solution of the same linear system with same initial data, and therefore, for any $s \in \mathbb{R}$,

$$\dot{x}(s) = Y(s)\dot{x}(0), \qquad \dot{\xi}(s) = N(s)\dot{x}(0). \tag{2.56}$$

All the coefficients in (2.55) are real and A and C are symmetric, and it follows that the flow defined by (2.55) *on vectors* preserves both the real symplectic form acting on pairs $(y, \eta) \in (\mathbb{R}^{n+1})^2$ and $(y', \eta') \in (\mathbb{R}^{n+1})^2$ given by

$$\sigma((y, \eta), (y', \eta')) = y \cdot \eta' - \eta \cdot y'$$

and the complexified form $\sigma_{\mathbb{C}}((y, \eta), (y', \eta')) = \sigma((y, \eta), (\overline{y'}, \overline{\eta'}))$ for $(y, \eta) \in (\mathbb{C}^{n+1})^2$ and $(y', \eta') \in (\mathbb{C}^{n+1})^2$. When we say that $\sigma_{\mathbb{C}}$ is invariant under (2.55), it means that we allow complex vectorial initial data in (2.55).

Let us prove that $Y(s)$ is invertible for any s. Let $v \in \mathbb{C}^{n+1}$ and $s_0 \in \mathbb{R}$ be such that $Y(s_0)v = 0$. We set $y(s_0) = Y(s_0)v$ and $\eta(s_0) = N(s_0)v$ and consider $\chi(s_0) = (y(s_0), \eta(s_0))$. From the conservation of $\sigma_{\mathbb{C}}$, we get

$$0 = \sigma_{\mathbb{C}}(\chi(s_0), \chi(s_0)) = \sigma_{\mathbb{C}}(\chi(0), \chi(0)) = v \cdot \overline{M_0 v} - \overline{v} \cdot M_0 v = -2i\overline{v} \cdot (\mathrm{Im}(M_0))v.$$

Since $\mathrm{Im}(M_0)$ is positive definite on the orthogonal complement to $\dot{x}(0)$, there holds $v = \lambda \dot{x}(0)$ for some $\lambda \in \mathbb{C}$. Hence

$$0 = Y(s_0)v = \lambda Y(s_0)\dot{x}(0) = \lambda \dot{x}(s_0)$$

where the last equality comes from (2.56). Since $\dot{x}_0(s_0) = \frac{\partial p_2}{\partial \xi_0}(s_0) = -2\xi_0(s_0) = 1$, there holds $\dot{x}(s_0) \neq 0$, hence $\lambda = 0$. It follows that $v = 0$ and $Y(s_0)$ is invertible.

Now, for any $s \in \mathbb{R}$, we set

$$M(s) = N(s)Y(s)^{-1}$$

which is a solution of (2.54) with $M(0) = M_0$. It verifies $M(s)\dot{x}(s) = \dot{\xi}(s)$ thanks to (2.56). Moreover, it is symmetric: if we denote by $y^i(s)$ and $\eta^i(s)$ the column vectors of Y and N, by preservation of σ, for any $0 \leqslant i, j \leqslant n$, the quantity

$$\sigma((y^i(s), \eta^i(s)), (y^j(s), \eta^j(s)) = y^i(s) \cdot M(s)y^j(s) - y^j(s) \cdot M(s)y^i(s)$$

is equal to the same quantity at $s = 0$, which is equal to 0 since M_0 is symmetric.

Let us finally prove that for any $s \in \mathbb{R}$, $\mathrm{Im}(M(s))$ is positive definite on the orthogonal complement of $\dot{x}(s)$. Let $y(s_0) \in \mathbb{C}^{n+1}$ be in the orthogonal complement of $\dot{x}(s_0)$. We decompose $y(s_0)$ on the column vectors of $Y(s_0)$:

$$y(s_0) = \sum_{i=0}^{n} b_i y^i(s_0), \qquad b_i \in \mathbb{C}.$$

For $s \in \mathbb{R}$, we consider $y(s) = \sum_{i=0}^{n} b_i y^i(s)$ and we set $\chi(s) = \sum_{i=0}^{n} b_i (y^i(s), \eta^i(s))$. Then,

$$\sigma_{\mathbb{C}}(\chi(s), \chi(s)) = -2i\overline{y(s)} \cdot \mathrm{Im}(M(s))y(s). \tag{2.57}$$

By preservation of $\sigma_{\mathbb{C}}$ and using (2.57), we get that

$$\overline{y(s_0)} \cdot \mathrm{Im}(M(s_0))y(s_0) = \overline{y(0)} \cdot \mathrm{Im}(M_0)y(0). \tag{2.58}$$

But $y(0)$ cannot be proportional to $\dot{x}(0)$ otherwise, using (2.56), we would get that $y(s_0)$ is proportional to $\dot{x}(s_0)$. Hence, the right hand side in (2.58) is > 0, which implies that $\mathrm{Im}(M(s_0))$ is positive definite on the orthogonal complement to $\dot{x}(s_0)$.

Therefore, we found a choice for the second order derivatives of ψ along Γ which meets all our conditions. For $x = (t, x') \in \mathbb{R} \times \mathbb{R}^n$ and s such that $t = t(s)$, we set

$$\psi(x) = \xi'(s) \cdot (x' - x'(s)) + \frac{1}{2}(x' - x'(s)) \cdot M(s)(x' - x'(s)), \tag{2.59}$$

and for this choice of ψ, f vanishes at order 2 along Γ.

To sum up, as in the Riemannian (or "strictly hyperbolic") case handled by Ralston in [Ral82], the key observation is that the invariance of σ and $\sigma_{\mathbb{C}}$ prevents the solutions of (2.54) with positive imaginary part on the orthogonal complement of $\dot{x}(0)$ to blowup.

Analysis of $A_2(x)$. We note that A_2 vanishes along Γ if and only if $La_0(x(s)) = 0$. According to (2.51), this turns out to be a linear transport equation on $a_0(x(s))$. Moreover, the coefficient of the first-order term, namely $\nabla_\xi p_2(x(s), \xi(s))$, is different from 0. Therefore, given $a_0 \neq 0$ at $(t = 0, x = x(0))$, this transport equation has a solution $a_0(x(s))$ with initial datum a_0, and, by Cauchy uniqueness, $a_0(x(s)) \neq 0$ for any s. Note that we have prescribed a_0 only along Γ, and we may choose a_0 in a smooth (and arbitrary) way outside Γ. We choose it to vanish outside a small neighborhood of Γ.

Proof of (2.13). We use (2.50) and we apply Lemma 2.14 to $S = 3$, $c = A_1$ and to $S = 1$, $c = A_2$, and we get

$$\|\partial_{tt}^2 v_k - \Delta v_k\|_{L^1(0,T;L^2(M))} \leqslant C(k^{-\frac{1}{2}} + k^{-\frac{1}{2}} + k^{-1}),$$

which implies (2.13).

Proof of (2.14). We first observe that since $\mathrm{Im}(M(s))$ is positive definite on the orthogonal complement of $\dot{x}(s)$ and continuous as a function of s, there exist $\alpha, C > 0$ such that for any $t(s) \in [0, T]$ and any $x' \in M$,

$$|\partial_t v_k(t(s), x')|^2 + |\nabla^{\mathrm{sR}} v_k(t(s), x')|^2 \geqslant \left(C|a_0(t(s), x')|^2 k^{\frac{n}{2}} + O(k^{2(\frac{n}{2}-1)})\right) e^{-\alpha k d(x', x'(s))^2}$$

where $d(\cdot, \cdot)$ denotes the Euclidean distance in $\mathbb{R}^n$. We denote by ℓ_n the Lebesgue measure on $\mathbb{R}^n$. Using the observation that for any function f,

$$\int_M f(x') e^{-\alpha k d(x', x'(s))^2} d\mu(x') \sim \frac{\pi^{n/2}}{k^{n/2}\sqrt{\alpha}} f(x'(s)) \frac{d\mu}{d\ell_n}(x'(s)) \tag{2.60}$$

as $k \to +\infty$, and the fact that $a_0(x(s)) \neq 0$, we obtain (2.14).

Proof of (2.15). We observe that since $\mathrm{Im}(M(s))$ is positive definite (uniformy in s) on the orthogonal complement of $\dot{x}(s)$, there exist $C, \alpha' > 0$ such that for any $t \in [0, T]$, for any $x' \in M$, $|\partial_t v_k(t(s), x')|$ and $|\nabla^{\mathrm{sR}} v_k(t(s), x')|$ are both bounded above by $Ck^{\frac{n}{4}} e^{-\alpha' k d(x', x'(s))^2}$. Therefore

$$\begin{aligned}\int_{M \setminus V_{t(s)}} &\left(|\partial_t v_k(t(s), x')|^2 + |\nabla^{\mathrm{sR}} v_k(t(s), x')|^2\right) d\mu(x') \\ &\leqslant Ck^{n/2} \int_{M \setminus V_{t(s)}} e^{-2\alpha' k d(x', x'(s))^2} d\mu(x') \\ &\leqslant Ck^{n/2} \int_{M \setminus V_{t(s)}} e^{-2\alpha' k d(x', x'(s))^2} d\ell_n(x') + \mathrm{o}(1) \end{aligned} \tag{2.61}$$

where, in the last line, we used the fact that $|d\mu/d\ell_n| \leqslant C$ in a fixed compact subset of M (since μ is a smooth volume), and the o(1) comes from the eventual blowup of μ at the boundary of M.

Now, $M \subset \mathbb{R}^n$, and there exists $r > 0$ such that $B_d(x(s), r) \subset V_{t(s)}$ for any s such that $t(s) \in (0, T)$, where $d(\cdot, \cdot)$ still denotes the Euclidean distance in $\mathbb{R}^n$. Therefore, we bound above the integral in (2.61) by

$$Ck^{n/2} \int_{\mathbb{R}^n \setminus B_d(x(s),r)} e^{-2\alpha' k d(x',x'(s))^2} d\ell_n(x') \tag{2.62}$$

Making the change of variables $y = k^{-1/2}(y - x(s))$, we bound above (2.62) by

$$C \int_{\mathbb{R}^n \setminus B_d(0,rk^{1/2})} e^{-2\alpha' \|y\|^2} d\ell_n(y)$$

with $\|\cdot\|$ the Euclidean norm. This last expression is bounded above by

$$Ce^{-\alpha' r^2 k} \int_{\mathbb{R}^n} e^{-\alpha' \|y\|^2} d\ell_n(y)$$

which implies (2.15).

Extension of the result to any manifold M. In the case of a general manifold M, not necessarily included in $\mathbb{R}^n$, we use charts together with the above construction. We cover M by a set of charts $(U_\alpha, \varphi_\alpha)$, where (U_α) is a family of open sets of M covering M and $\varphi_\alpha : U_\alpha \to \mathbb{R}^n$ is an homeomorphism U_α onto an open subset of $\mathbb{R}^n$. Take a solution $(x(t), \xi(t))_{t\in[0,T]}$ of (2.8). It visits a finite number of charts in the order $U_{\alpha_1}, U_{\alpha_2}, \ldots$, and we choose the charts and a_0 so that $v_k(t, \cdot)$ is supported in a unique chart at each time t. The above construction shows how to construct a_0 and ψ as long as $x(t)$ remains in the same chart. For any $l \geqslant 1$, we choose t_l so that $x(t_l) \in U_{\alpha_l} \cap U_{\alpha_{l+1}}$ and $a_0(t_l, \cdot)$ is supported in $U_{\alpha_l} \cap U_{\alpha_{l+1}}$. Since there is a (local) solution v_k for any choice of initial $a_0(t_l, x(t_l))$ and $\mathrm{Im}\left(\frac{\partial^2 \psi}{\partial x_i \partial x_j}\right)(t_l, x(t_l))$ in Proposition 2.12, we see that v_k may be continued from the chart U_{α_l} to the chart $U_{\alpha_{l+1}}$. This continuation is smooth since the two solutions coincide as long as $a_0(t, \cdot)$ is supported in $U_{\alpha_l} \cap U_{\alpha_{l+1}}$. Patching all solutions on the time intervals $[t_l, t_{l+1}]$ together, it yields a global in time solution v_k, as desired.

2-A.2 Proof of (2.49)

Because of the second convergence in (2.43) and the non-negativity of a, it amounts to proving that

$$\int_0^T |(\nabla^{\mathrm{sR}} \mathrm{Op}(a) u^k, \nabla^{\mathrm{sR}} u^k)_{L^2(M_H,\mu)}| dt \to 0.$$

Now, we notice that for any $B \in \Psi^0_{\mathrm{phg}}((0,T) \times M_H)$, there holds

$$(Bu^k, \nabla^{\mathrm{sR}} u^k)_{L^2((0,T)\times M_H)} \underset{k\to+\infty}{\longrightarrow} 0 \qquad \text{and} \qquad (Bu^k, \partial_t u^k)_{L^2((0,T)\times M_H)} \underset{k\to+\infty}{\longrightarrow} 0 \tag{2.63}$$

since $u^k \to 0$ strongly in $L^2((0,T) \times M_H)$ and both $\nabla^{\mathrm{sR}} u^k$ and $\partial_t u^k$ are bounded in $L^2((0,T) \times M_H)$. We apply this to $B = [\nabla^{\mathrm{sR}}, \mathrm{Op}(a)]$, and then, also using (2.63), we see that we can replace $\mathrm{Op}(a)$ by its Friedrichs quantization $\mathrm{Op}^F(a)$, which is positive (see [Tay74, Chapter VII]). In other words, we are reduced to prove

$$(\mathrm{Op}^F(a) \nabla^{\mathrm{sR}} u^k, \nabla^{\mathrm{sR}} u^k)_{L^2((0,T)\times M_H)} \underset{k\to+\infty}{\longrightarrow} 0. \tag{2.64}$$

Let $\delta > 0$ and $\widetilde{a} \in S^0_{\rm phg}((-\delta, T+\delta) \times M_H)$, $0 \leqslant \widetilde{a} \leqslant \sup(a)$ and such that $\widetilde{a}(t,\cdot) = a(\cdot)$ for $0 \leqslant t \leqslant T$. Making repeated use of (2.63) and of integrations by parts (since $\widetilde{a}$ is compactly supported in time), we have

$$\begin{aligned}
(\mathrm{Op}^F(\widetilde{a})\nabla^{\rm sR}u^k, \nabla^{\rm sR}u^k)_{L^2((0,T)\times M_H)} &= (\nabla^{\rm sR}\mathrm{Op}^F(\widetilde{a})u^k, \nabla^{\rm sR}u^k)_{L^2((0,T)\times M_H)} + o(1) \\
&= -(\mathrm{Op}^F(\widetilde{a})u^k, \Delta u^k)_{L^2((0,T)\times M_H)} + \mathrm{o}(1) \\
&= -(\mathrm{Op}^F(\widetilde{a})u^k, \partial_t^2 u^k)_{L^2((0,T)\times M_H)} + \mathrm{o}(1) \\
&= (\partial_t \mathrm{Op}^F(\widetilde{a})u^k, \partial_t u^k)_{L^2((0,T)\times M_H)} + \mathrm{o}(1) \\
&= (\mathrm{Op}^F(\widetilde{a})\partial_t u^k, \partial_t u^k)_{L^2((0,T)\times M_H)} + \mathrm{o}(1).
\end{aligned}$$

Finally we note that since Op^F is a positive quantization, we have

$$\begin{aligned}
(\mathrm{Op}^F(a)\nabla^{\rm sR}u^k, \nabla^{\rm sR}u^k)_{L^2((0,T)\times M_H)} &\leqslant (\mathrm{Op}^F(\widetilde{a})\nabla^{\rm sR}u^k, \nabla^{\rm sR}u^k)_{L^2((0,T)\times M_H)} \\
&= (\mathrm{Op}^F(\widetilde{a})\partial_t u^k, \partial_t u^k)_{L^2((0,T)\times M_H)} + \mathrm{o}(1) \\
&\leqslant C\delta + (\mathrm{Op}^F(a)\partial_t u^k, \partial_t u^k)_{L^2((0,T)\times M_H)} + \mathrm{o}(1) \\
&\leqslant C\delta + o(1)
\end{aligned}$$

where C does not depend on δ. Making $\delta \to 0$, it concludes the proof of (2.64), and consequently (2.49) holds.

Chapter 3

Observability of Baouendi-Grushin-type equations

"Science sans conscience n'est que ruine de l'âme."
François Rabelais.
"Conscience sans science n'est qu'un vilain gros mot."
Pierre Dac.

This chapter is adapted from [LS20]. Among other things, we prove Theorem 2 (restated as Theorem 3.4).

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3.1 Introduction and main results

3.1.1 Motivation

In this chapter, we will mainly use resolvent estimates to establish observability inequalities. Resolvent estimates consist in a quantitative measurement of how much approximate solutions (also named quasimodes) of an operator can concentrate away from an open set ω, and in particular resolvent estimates do not involve the time variable, at least in this context. See for example [BZ04] and [Mil12] for detailed studies about the link between observability and resolvent estimates.

Since the study of the controllability/observability properties of evolution equations driven by sub-Laplacians in full generality seems out of reach (except for wave equations, see the previous chapter), in this chapter we focus on a particular family of models, which we now describe.

Let

$$M = (-1, 1)_x \times \mathbb{T},$$

where $\mathbb{T}$ is the 1D torus in the y-variable and let $\gamma \geqslant 0$. We consider the Baouendi-Grushin-type sub-Laplacian

$$\Delta_\gamma = \partial_x^2 + |x|^{2\gamma}\partial_y^2,$$

together with the domain

$$D(\Delta_\gamma) = \{u \in \mathcal{D}'(M) : \partial_x^2 u, |x|^{2\gamma}\partial_y^2 u \in L^2(M) \text{ and } u_{|\partial M} = 0\}$$

(see Example 1.17). By Hörmander's theorem, in the case where $\gamma \in \mathbb{N}$, Δ_γ is subelliptic, since ∂_y can be obtained by taking γ times the bracket of ∂_x with $x^\gamma \partial_y$.

The observation region ω that we consider is assumed to contain a horizontal strip $(-1, 1)_x \times I_y$ where $I \subset \mathbb{T}$ is a non-empty open interval of the 1D-torus. This choice for ω is natural if one is interested in understanding the specific features of propagation in the subelliptic directions (here, the vertical y-axis), see Section 3.1.3 below; this choice for ω has already been made in different but related subelliptic frameworks, see for example [Koe17], [BS19], [FL21]. We note, and we will come back to this point later in our analysis, that ω does not satisfy the Geometric Control Condition, which is known to be equivalent to observability of elliptic waves (see [BLR92]) and to imply the observability of the elliptic Schrödinger equation in any time (see [Leb92b]). Several other choices for ω could have been made (see [BCG14] for example).

3.1.2 Main results.

Our first main result is a resolvent estimate in the case $\gamma \geqslant 1$, which reads as follows:

Theorem 3.1. *Let $\gamma \in \mathbb{R}$, $\gamma \geqslant 1$ and let ω contain a horizontal strip $(-1, 1) \times I$. There exist $C, h_0 > 0$ such that for any $v \in D(\Delta_\gamma)$ and any $0 < h \leqslant h_0$, there holds*

$$\|v\|_{L^2(M)} \leqslant C(\|v\|_{L^2(\omega)} + h^{-(\gamma+1)}\|(h^2\Delta_\gamma + 1)v\|_{L^2(M)}). \tag{3.1}$$

Remark 3.2. In [LL21] (see Corollary 1.9), a resolvent estimate with an exponential cost (replacing the above polynomial cost $h^{-(\gamma+1)}$) was proved for any sub-Riemannian manifold of step k and for any of its subsets ω of positive Lebesgue measure. It was shown to be sharp for the Baouendi-Grushin-type sub-Laplacian Δ_γ (with $\gamma + 1 = k$) and for any open set ω whose closure does not touch the line $\{x = 0\}$. Our resolvent estimate is much stronger, but heavily relies on the particular geometric situation under study.

Remark 3.3. From the proof of (2) of Theorem 3.4, the resolvent estimate (3.1) is sharp in the sense that there exists a sequence of quasi-modes v_h which saturates the inequality. Indeed, a better resolvent estimate than (3.1), together with [BZ04, Theorem 4], would contradict the lack of observability for short times in Point (2) of Theorem 3.4 (see the argument after Theorem 3.21 in Section 3.3.1).

Furthermore, the conclusion of Theorem 3.1 does not apply to the case $\gamma < 1$, at least if we remove the boundary. For example, when $\gamma = 0$ and Δ_0 is the usual Laplace operator on the torus $\mathbb{T}^2$, it follows from [BLR92] that the resolvent estimate (3.1) with order $O(h^{-1})$ cannot hold if ω does not satisfy the geometric control condition with respect to the geodesic flow.

In this chapter, we will explore the consequences of this resolvent estimate for the observability of evolution equations driven by Δ_γ.

Let us consider the Schrödinger-type equation with Dirichlet boundary conditions

$$\begin{cases} i\partial_t u - (-\Delta_\gamma)^s u = 0 \\ u_{|t=0} = u_0 \in L^2(M) \\ u_{|x=\pm 1} = 0 \end{cases} \tag{3.2}$$

where $s \in \mathbb{N}$ is a fixed integer and $\gamma \geqslant 0$, $\gamma \in \mathbb{R}$. Here $(-\Delta_\gamma)^s$ is defined "spectrally" by its action on eigenspaces of the operator Δ_γ associated with Dirichlet boundary conditions. In other words, by classical embedding theorems (recalled in Lemma 3.22), $(\Delta_\gamma, D(\Delta_\gamma))$ has a compact resolvent, and thus there exists an orthonormal Hilbert basis of eigenfunctions $(\varphi_j)_{j\in\mathbb{N}}$ such that $-\Delta_\gamma \varphi_j = \lambda_j^2 \varphi_j$, with the λ_j sorted in increasing order. The domain of $(-\Delta_\gamma)^s$ is given by

$$D((-\Delta_\gamma)^s) = \{u \in L^2(M) : \sum_{j\in\mathbb{N}} \lambda_j^{4s} |(u, \varphi_j)_{L^2(M)}|^2 < \infty\}. \tag{3.3}$$

Note that a function u in $D((-\Delta_\gamma)^s)$ verifies the boundary conditions

$$(-\Delta_\gamma)^k u|_{\partial M} = 0, \qquad \text{for any } 0 \leqslant k < s - \frac{1}{4}. \tag{3.4}$$

In Section 3-A.1, we prove this fact and we also show that (3.2) is well-posed in $L^2(M)$. Of course, the solution of (3.2) does not live in general in the energy space given by the form domain of $(-\Delta_\gamma)^s$, but only in $L^2(M)$.

Given an open subset $\widetilde{\omega} \subset M$, we say that (3.2) is observable in time $T_0 > 0$ in $\widetilde{\omega}$ if there exists $C > 0$ such that for any $u_0 \in L^2(M)$, there holds

$$\|u_0\|^2_{L^2(M)} \leqslant C \int_0^{T_0} \|e^{-it(-\Delta_\gamma)^s} u_0\|^2_{L^2(\widetilde{\omega})} dt. \tag{3.5}$$

Our second main result, which is a reformulation of Theorem 2, roughly says that observability holds if and only if the subellipticity, measured by the step $\gamma + 1$, is not too strong compared to s:

Theorem 3.4. *Assume that $\gamma \in \mathbb{R}$, $\gamma \geqslant 1$. Let $I \subsetneq \mathbb{T}_y$ be a strict open subset, and let $\omega = (-1, 1)_x \times I$. Then, for $s \in \mathbb{N}$, we have:*

1. *If $\frac{1}{2}(\gamma + 1) < s$, (3.2) is observable in ω for any $T_0 > 0$;*
2. *If $\frac{1}{2}(\gamma + 1) = s$, there exists $T_{\inf} > 0$ such that (3.2) is observable in ω for T_0 if and only if $T_0 \geqslant T_{\inf}$;*
3. *If $\frac{1}{2}(\gamma + 1) > s$, for any $T_0 > 0$, (3.2) is not observable in ω.*

Indeed, Points (1) and (2) hold under the weaker assumption that ω contains a horizontal band of the form $(-1,1)_x \times I$; and Point (3) holds under the weaker assumption that $M \setminus \omega$ contains an open neighborhood of some point $(x,y) \in M$ with $x = 0$.

Let us make several comments about this result:

- In the case $\frac{1}{2}(\gamma+1) = s$, our proof only provides a lower bound on T_{inf} (see Remark 3.26). The exact value of T_{inf} was explicitly computed in [BS19] in the case $\gamma = s = 1$. It is an interesting problem to compute this exact value for s, γ satisfying $s = \frac{1}{2}(\gamma+1)$, and more importantly, to give a geometric interpretation for this exact constant in a more general subelliptic setting.

- The number $\frac{1}{2}(\gamma+1)$ appearing in Theorem 3.4 is already known to play a key role in many other problems. Recall that the step of the manifold (defined as the least number of brackets required to generate the whole tangent space) is equal to $\gamma+1$ (when $\gamma \in \mathbb{N}$). Then, $2/(\gamma+1)$ is the exponent known as the gain of Sobolev derivatives in subelliptic estimates. Note that $\frac{1}{2}(\gamma+1)$ is also the threshold found in the work [BCG14] which deals with observability of the heat equation with sub-Laplacian Δ_γ, and that it is related to the growth of eigenvalues for the operator $-\partial_x^2 + x^{2\gamma}$, see for example Section 2.3 in [BCG14].

- In the statement of Theorem 3.4, we took $s \in \mathbb{N}$ in order to avoid technical issues of non-local effects due to the fractional Laplacian. We expect that the statements in Theorem 3.4 are also true for all $s > 0$.

- The assumption that $\gamma \geqslant 1$ for Points (1) and (2) is mainly due to the technical issue that the Hamiltonian flow associated with the symbol $\partial_x^2 + |x|^{2\gamma}\partial_y^2$ may not be unique if $0 < \gamma < 1$ (see Section 3.2.3). Dealing with this case, and more generally addressing the question of propagation of singularities for metrics with lower regularity, is an open problem.

We now derive from Theorems 3.1 and 3.4 two consequences. First, Theorem 3.4 implies the following result about observability of heat-type equations associated to Δ_γ (which are well-posed, as proved in Section 3-A.1):

Corollary 3.5. *Assume that $\gamma \in \mathbb{R}$, $\gamma \geqslant 1$ and ω contains a horizontal strip $(-1,1)_x \times I$. For any $s \in \mathbb{N}$, $s > \frac{1}{2}(\gamma+1)$ and any $T_0 > 0$, final observability for the heat equation with Dirichlet boundary conditions*

$$\begin{cases} \partial_t u + (-\Delta_\gamma)^s u = 0 \\ u_{|t=0} = u_0 \in L^2(M) \\ u_{|x=\pm 1} = 0 \end{cases} \tag{3.6}$$

holds in time T_0. In other words, there exists $C > 0$ such that for any $u_0 \in L^2(M)$, there holds

$$\|e^{-T_0(-\Delta_\gamma)^s} u_0\|^2_{L^2(M)} \leqslant \int_0^{T_0} \|e^{-t(-\Delta_\gamma)^s} u_0\|^2_{L^2(\omega)} dt.$$

This is a direct consequence of Corollary 2 in [DM12] and Point (2) of Theorem 3.4. Note also that observability for (3.6) fails for any time if $\gamma = s = 1$ (see [Koe17]), so that we cannot expect that an analogue of Point (2) of Theorem 3.4 holds for heat-type equations. This last fact - observability of a Schrödinger semigroup while the associated heat semigroup is not observable - gives an illustration of Proposition 3 of [DM12] (which states that the same phenomenon occurs for the harmonic oscillator on the real line observed in a set of the form $(-\infty, x_0)$, $x_0 \in \mathbb{R}$).

Finally, Theorem 3.1 also implies a decay rate for the damped wave equation associated to Δ_γ. To state it, we introduce the following adapted Sobolev spaces: for $k = 1, 2$,

$$H^k_\gamma(M) = \{v \in \mathcal{D}'(M),\ (-\Delta_\gamma + 1)^{k/2} v \in L^2(M)\}, \qquad \|v\|_{H^k_\gamma(M)} = \|(-\Delta_\gamma + 1)^{k/2} v\|_{L^2(M)}$$

and $H^1_{\gamma,0}(M)$ is the completion of $C^\infty_c(M)$ for the norm $\|\cdot\|_{H^1_\gamma(M)}$.

Let $b \in L^\infty(M), b \geqslant 0$ such that $\inf_{q \in \overline{\omega}} b(q) > 0$. On the space $\mathcal{H} := H^1_{\gamma,0}(M) \times L^2(M)$, the operator

$$\mathcal{A} = \begin{pmatrix} 0 & 1 \\ \Delta_\gamma & -b \end{pmatrix}$$

with domain $D(\mathcal{A}) = (H^2_\gamma(M) \cap H^1_{\gamma,0}(M)) \times H^1_{\gamma,0}(M)$ generates a bounded semigroup (from the Hille-Yosida theorem) and the damped wave equation

$$(\partial_t^2 - \Delta_\gamma + b\partial_t)u = 0 \tag{3.7}$$

with Dirichlet boundary conditions and given initial datum $(u_0, u_1) \in \mathcal{H}$ admits a unique solution $u \in C^0(\mathbb{R}^+; H^1_{\gamma,0}(M)) \cap C^1(\mathbb{R}^+; L^2(M))$, see Section 3-A.1.

Corollary 3.6. *Assume $\gamma \in \mathbb{R}$, $\gamma \geqslant 1$ and ω contains a horizontal strip $(-1,1)_x \times I$. There exists $C > 0$ such that, for any $(u_0, u_1) \in D(\mathcal{A})$, the solution $u(t)$ of (3.7) with initial conditions $(u, \partial_t u)_{|t=0} = (u_0, u_1)$ satisfies*

$$E(u(t), \partial_t u(t))^{\frac{1}{2}} \leqslant \frac{C}{t^{\frac{1}{2\gamma}}} E(\mathcal{A}(u_0, u_1))^{\frac{1}{2}} \tag{3.8}$$

for any $t \geqslant 1$, where

$$E(v, w) = \|\partial_x v\|^2_{L^2(M)} + \||x|^\gamma \partial_y v\|^2_{L^2(M)} + \|w\|^2_{L^2(M)}.$$

In particular, $E(u(t), \partial_t u(t)) \to 0$ as $t \to +\infty$.

Remark 3.7. As usual for the damped wave equation, one cannot replace $E(\mathcal{A}(u_0, u_1))^{\frac{1}{2}}$ in the r.h.s. of (3.8) by $E(u_0, u_1)^{\frac{1}{2}}$, otherwise the rate $t^{-\frac{1}{2\gamma}}$ could be improved to an exponential decay.

The proof of this corollary from Theorem 3.1 is essentially contained in Proposition 2.4 of [AL14]. To be self-contained, we prove Corollary 3.6 in Section 3-A.2. Note that the decay rate $t^{-\frac{1}{2}}$ when $\gamma = 1$ is not new. This special case is a direct consequence of the Schrödinger observability proved in [BS19] and an abstract result (Theorem 2.3) in [AL14], linking the Schrödinger observability and the decay rate of the associated damped wave equation. However, when $\gamma > 1$, the Schrödinger equation is not observable ((3) of Theorem 3.4), and we have to apply Theorem 3.1. Also, we do not address here the question of the optimality of the decay rate given by Corollary 3.6. See [AL14, Section 2C] for other open questions related to decay rates of damped waves.

3.1.3 Comments and sketch of proof

Let us describe in a few words the intuition underlying our results, notably Theorem 3.4. For that, we start with the case $s = 1/2$ (corresponding to wave equations) which, although not covered by Theorem 3.4, is of interest. Whereas elliptic wave equations are observable in finite time under a condition of geometric control ([BLR92]), it is known that for (strictly) subelliptic wave equations, observability fails in any time (see Chapter 2). This is due to the fact that in (co)-directions where the sub-Laplacian is not elliptic, the propagation of waves, and more generally of any evolution equation built with sub-Laplacians, is slowed down. On the other side, large s correspond to a quicker propagation along all directions. Therefore, Theorem 3.4 characterizes the threshold for the ratio of γ and s to get an exact balance between subelliptic effects (measured by the step $\gamma+1$) and elliptic phenomena (measured by s), and thus "finite speed of propagation" along subelliptic directions.

This same analysis underlies the result on the Baouendi-Grushin-Schrödinger equation [BS19], which was the starting point of our analysis: indeed, [BS19] deals with the critical case $\frac{1}{2}(\gamma+1) = s = 1$. Although the elliptic Schrödinger equation propagates at infinite speed, in subelliptic geometries, observability may hold only for sufficiently large time or even fail in any time if the degeneracy measured by γ is sufficiently strong. To our knowledge, the paper [BGX00], which exhibited a family of travelling waves solutions of the Schrödinger equation (3.2) for $\gamma = 1$, moving at speeds proportional to $n \in \mathbb{N}$, was the first result showing the slowdown of propagation in degenerate directions.

The chapter is organized as follows.

In Section 3.2, we prove Theorem 3.1, roughly following the same lines as in [BS19]. Due to the absence of the time-variable in our resolvent estimate, our proof is however slightly simpler, but as a counterpart, our method does not allow us to compute explicitly the minimal time $T_{\inf}$ of observability in Point (2) of Theorem 3.4. After having spectrally localized the sub-Laplacian Δ_γ around h^{-2}, our proof relies on a careful analysis of several regimes of comparison between $|D_y|$ and Δ_γ, which roughly correspond to different types of trajectories for the geodesics in M: we split the function v appearing in (3.1) according to Fourier modes in y and then we establish estimates for different "spectrally localized" parts of v of the form $\psi(h^2\Delta_\gamma)\chi_h(D_y)v$. Here, χ_h localizes D_y in some subinterval of $\mathbb{R}$ which depends on h. Fixing a small constant $b_0 \ll 1$, the three different regimes which we distinguish are:

- the degenerate regime in Section 3.2.2 ($|D_y| \geqslant b_0^{-1}h^{-1}$), for which we use a positive commutator method (also known as "energy method", and used for example to prove propagation of singularities in the literature, see [Hor71a, Section 3.5]);
- the regime of the geometric control condition in Section 3.2.3 ($b_0^{-1}h^{-1} \geqslant |D_y| \geqslant b_0 h^{-1}$), handled with semi-classical defect measures;
- the regime of horizontal propagation ($|D_y| \leqslant b_0 h^{-1}$) in Sections 3.2.4 and 3.2.5, for which we use a positive commutator argument, and then a normal form method.

In Section 3.3, using the link between resolvent estimates and observability of Schrödinger-type semigroups established in [BZ04], we deduce Points (1) and (2) of Theorem 3.4 from Theorem 3.1. Indeed, we first establish a *spectrally localized* observability inequality, from which we deduce the full observability using a classical procedure described for example in [BZ12].

In Section 3.4, we prove Point (3) of Theorem 3.4. For that, we construct a sequence of approximate solutions of (3.2) whose energy concentrates on a point $(x, y) \in (-1, 1) \times \mathbb{T}$ with $x = 0$ and $y \notin I$. The existence of such a sequence contradicts the observability inequality (3.5) and is possible only when $\frac{1}{2}(\gamma+1) > s$. For constructing the sequence of initial data, we add in a careful way the ground states of the operators $-\partial_x^2 + |x|^{2\gamma}\eta^2$ for different η's (the Fourier variable of y). These initial data propagate at nearly null speed along the vertical axis $x = 0$.

Finally, Section 3-A is devoted to the proof of basic results which were postponed to the end of the chapter. In Section 3-A.1, we prove the well-posedness of the Schrödinger-type equation (3.2), the heat-type equation (3.6) and the damped wave equation (3.7), using standard techniques such as the Hille-Yosida theorem. In Section 3-A.2, we prove Corollary 3.6. Using results of [BT10], it is sufficient to estimate the size of $(i\lambda\mathrm{Id} - \mathcal{A})^{-1}$ for large $\lambda \in \mathbb{R}$ (and in appropriate functional spaces). This is done mainly thanks to a priori estimates on the system $(i\lambda \mathrm{Id} - \mathcal{A})U = F$, and using the resolvent estimate of Theorem 3.1.

3.2 Proof of Theorem 3.1

This section is devoted to the proof of Theorem 3.1. In all the sequel, $\gamma \geqslant 1$ is fixed. It is sufficient to deal with the case where $\omega = (-1,1)_x \times I$ where I is a simple interval, since if Theorem 3.1 holds for some $\omega = \omega_1$, then it holds for any $\omega_2 \supset \omega_1$. Hence, in all the sequel, we assume that I is a simple interval (a_1, a_2). Also, we use the notations $D_x = \frac{1}{i}\partial_x$ and $D_y = \frac{1}{i}\partial_y$.

We will argue by contradiction. Assume that there exists a sequence $(v_h)_{h>0}$ such that

$$\|v_h\|_{L^2(M)} = 1, \quad \|v_h\|_{L^2(\omega)} = o(1), \quad \|f_h\|_{L^2(M)} = o(h^{\gamma+1}) \tag{3.9}$$

where $f_h = (h^2\Delta_\gamma + 1)v_h$, and we seek for a contradiction, which would prove Theorem 3.1. Let us show that we can furthermore assume that v_h has localized spectrum: for that, we consider an even cutoff $\psi \in C_c^\infty(\mathbb{R})$, such that $\psi \equiv 1$ near ± 1 and $\psi = 0$ outside $(-2, -\frac{1}{2}) \cup (\frac{1}{2}, 2)$. We set $w_h = (1 - \psi(h^2\Delta_\gamma))v_h$. Then $(h^2\Delta_\gamma + 1)w_h = (1 - \psi(h^2\Delta_\gamma))f_h$ has L^2 norm which is $o(h^{\gamma+1})$. Moreover, we also deduce that $w_h = (h^2\Delta_\gamma + 1)^{-1}(1 - \psi(h^2\Delta_\gamma))f_h$ and since $(h^2\Delta_\gamma + 1)^{-1}(1 - \psi(h^2\Delta_\gamma))$ is elliptic and thus bounded from $L^2(M)$ to $L^2(M)$, we obtain $\|w_h\|_{L^2(M)} = o(1)$. Hence, considering $v_h - w_h$ instead of v_h, we can furthermore assume that $v_h = \psi(h^2\Delta_\gamma)v_h$.

In the next subsections, we use a decomposition of v_h as $v_h = v_h^1 + v_h^2 + v_h^3 + v_h^4$ where

$$\begin{aligned} v_h^1 &= (1 - \chi_0(b_0 h D_y))v_h, \qquad & v_h^2 &= (\chi_0(b_0 h D_y) - \chi_0(b_0^{-1} h D_y))v_h \\ v_h^3 &= (\chi_0(b_0^{-1} h D_y) - \chi_0(h^\epsilon D_y))v_h, \qquad & v_h^4 &= \chi_0(h^\epsilon D_y)v_h, \end{aligned}$$

where $0 < \epsilon \ll 1, 0 < b_0 \ll 1$ are small parameters which will be fixed throughout the article and will be specified later (respectively in Proposition 3.14 and in Lemma 3.10). This is a decomposition according to the dual Fourier variable of y and defined by functional calculus. The cut-off $\chi_0 \in C_c^\infty(\mathbb{R})$ will be defined later (see (3.10)). We prove that $v_h^j = o(1)$ for $j = 1, 2, 3, 4$, which contradicts (3.9). The methods used for each j are quite different, and roughly correspond to the different behaviours of geodesics according to their momentum $\eta \sim D_y$.

3.2.1 A priori estimate and elliptic regularity

We start with the following coercivity estimate:

Lemma 3.8. *There exists $C_1 > 0$ such that for any u, the following inequality holds:*

$$\| |D_y|^{\frac{2}{\gamma+1}} u\|_{L^2(M)} \leqslant C_1 \|\Delta_\gamma u\|_{L^2(M)}$$

Proof of Lemma 3.8. We write a Fourier expansion in y: for $\eta \in \mathbb{Z}$, we set $\widehat{u}_\eta(\cdot) := \mathcal{F}_y(u)(\cdot, \eta)$. Then, we have

$$\mathcal{F}_y(-\Delta_\gamma u)(x, \eta) = (D_x^2 + |x|^{2\gamma}\eta^2)\widehat{u}_\eta(x).$$

We make the change of variables $z = |\eta|^{\frac{1}{\gamma+1}}x$, and we set $f(z, \eta) = \mathcal{F}_y(-\Delta_\gamma u)(x, \eta)$ and $\widehat{v}_\eta(z) = \widehat{u}_\eta(x)$. Then we obtain

$$f(z, \eta) = |\eta|^{\frac{2}{\gamma+1}}(D_z^2 + |z|^{2\gamma})\widehat{v}_\eta(z),$$

and thus, using that $D_z^2 + |z|^{2\gamma}$ is elliptic (since its spectrum is strictly above 0), we get

$$|\eta|^{\frac{2}{\gamma+1}} \|\widehat{v}_\eta\|_{L_z^2} \leqslant C \|f(\cdot,\eta)\|_{L_z^2}$$

for some constant $C > 0$ (independent of η). Coming back to the x variable and summing over η, we obtain

$$\begin{aligned}\||D_y|^{\frac{2}{\gamma+1}} u\|_{L^2(M)}^2 &= \sum_{\eta\in\mathbb{Z}} |\eta|^{\frac{4}{\gamma+1}} \|\widehat{u}_\eta\|_{L_x^2}^2 \\ &\leqslant C_1 \sum_{\eta\in\mathbb{Z}} \|\mathcal{F}_y(-\Delta_\gamma u)(\cdot,\eta)\|_{L_x^2}^2 \\ &= C_1 \|\Delta_\gamma u\|_{L^2(M)}^2\end{aligned}$$

thanks to Plancherel formula, which finishes the proof. □

Let $\chi_0 \in C_c^\infty(\mathbb{R};[0,1])$ such that

$$\chi_0(\zeta) \equiv 1, \text{ if } |\zeta| \leqslant (4C_1)^{\frac{\gamma+1}{2}} \text{ and } \chi_0(\zeta) \equiv 0 \text{ if } |\zeta| > (8C_1)^{\frac{\gamma+1}{2}}. \tag{3.10}$$

Corollary 3.9. *For $0 < h < 1$, there holds*

$$\psi(h^2\Delta_\gamma)(1 - \chi_0(h^{\gamma+1}D_y)) = 0.$$

Proof. For $n \in \mathbb{Z}$, we consider an Hilbert basis of eigenfunctions $\varphi_{m,n}$ of L_x^2 satisfying

$$(D_x^2 + |x|^{2\gamma}n^2)\varphi_{m,n} = \lambda_{m,n}^2 \varphi_{m,n}, \quad \|\varphi_{m,n}(x)\|_{L^2((-1,1))} = 1, \tag{3.11}$$

so that $\varphi_{m,n}e^{iny}$ is an eigenfunction of Δ_γ with associated eigenvalue $-\lambda_{m,n}^2$. Let $f \in D(\Delta_\gamma)$, and consider $f_h = \psi(h^2\Delta_\gamma)(1 - \chi_0(h^{\gamma+1}D_y))f$. We write

$$f_h = \sum_{m,n} a_{m,n}\psi(-h^2\lambda_{m,n}^2)(1 - \chi_0(h^{\gamma+1}n))\varphi_{m,n}e^{iny}.$$

We use Plancherel formula, apply Lemma 3.8 to f_h and we obtain

$$\sum_{m,n} |n|^{\frac{4}{\gamma+1}} |a_{m,n}|^2 \psi(-h^2\lambda_{m,n}^2)^2 (1-\chi_0(h^{\gamma+1}n))^2 \leqslant C_1^2 \sum_{m,n} \lambda_{m,n}^4 |a_{m,n}|^2 \psi(-h^2\lambda_{m,n}^2)^2 (1-\chi_0(h^{\gamma+1}n))^2 \tag{3.12}$$

On the support of $\theta(h,m,n) := \psi(-h^2\lambda_{m,n}^2)^2(1 - \chi_0(h^{\gamma+1}n))^2$, there holds $|n|^{\frac{4}{\gamma+1}} \geqslant 16C_1^2h^{-4} > C_1^2\lambda_{m,n}^4$. Indeed, for the first inequality, we used the support properties of χ_0, and for the second the support of ψ. This contradicts (3.12), except if all $a_{m,n}$ vanish, i.e., $f_h \equiv 0$. □

Corollary 3.9 implies that

$$v_h = \psi(h^2\Delta_\gamma)\chi_0(h^{\gamma+1}D_y)v_h. \tag{3.13}$$

The next lemma shows that in the regime $|D_y|\gamma h^{-1}$, the energy of v_h concentrates in the region $|x| \ll 1$. Let $\chi \in C_c^\infty(\mathbb{R})$ such that $\chi(\zeta) \equiv 1$ for $|\zeta| \leqslant 2^{\frac{1}{\gamma}}$. Also, possibly taking a larger C_1 in Lemma 3.8, we can assume that $C_1 \geqslant 1$: in particular, $\chi_0(\zeta) \equiv 1$ for $|\zeta| \leqslant 1$.

Lemma 3.10 (Elliptic regularity)**.** *There exist small constants* $0 < h_0 \ll 1$ *and* $0 < b_0 \ll 1$ *such that for all* $0 < h < h_0$*, there holds*

$$\begin{aligned}&\big\|\big(1-\chi(b_0^{-\frac{1}{\gamma}}x)\big)(1-\chi_0(b_0hD_y))v_h\big\|_{L^2(M)}+\big\|\big(1-\chi(b_0^{-\frac{1}{\gamma}}x)\big)(1-\chi_0(b_0hD_y))h\partial_x v_h\big\|_{L^2(M)}\\ \leqslant& C_Nh^N\Big(\|v_h\|_{L^2(M)}+\|h\nabla_\gamma v_h\|_{L^2(M)}\Big),\end{aligned}$$

for any $N\in\mathbb{N}$*.*

Proof. As in the previous lemma, we write the eigenfunction expansion of v_h as

$$(1-\chi_0(b_0hD_y))v_h=\sum_{\substack{m,n:|n|\geqslant b_0^{-1}h^{-1}\\ \frac{1}{\sqrt{2}}h^{-1}\leqslant\lambda_{m,n}\leqslant\sqrt{2}h^{-1}}}a_{m,n}\mathrm{e}^{iny}\varphi_{m,n}(x)$$

since $\chi_0(\zeta)\equiv 1$ for $|\zeta|\leqslant 1$ and $v_h=\psi(h^2\Delta_\gamma)v_h$.

We claim that it suffices to prove:

$$\|(1-\chi(b_0^{-\frac{1}{\gamma}}x))\varphi_{m,n}\|_{L^2}+\|(1-\chi(b_0^{-\frac{1}{\gamma}}x))h\partial_x\varphi_{m,n}\|_{L^2}\leqslant C_Nh^N \tag{3.14}$$

for all $N\in\mathbb{N}$ and m,n such that $\frac{1}{\sqrt{2}}h^{-1}\leqslant\lambda_{m,n}\leqslant\sqrt{2}h^{-1}, |n|\geqslant b_0^{-1}h^{-1}$. Indeed, Cauchy-Schwarz and (3.14) together imply

$$\begin{aligned}&\big\|\big(1-\chi(b_0^{-\frac{1}{\gamma}}x)\big)(1-\chi_0(b_0hD_y))v_h\big\|_{L^2(M)}\\ \leqslant&\sum_{\substack{m,n:b_0^{-1}h^{-1}\leqslant|n|\leqslant Ch^{-(\gamma+1)}\\ \frac{1}{\sqrt{2}}h^{-1}\lambda_{m,n}\leqslant\sqrt{2}h^{-1}}}|a_{m,n}|\big\|(1-\chi(b_0^{-\frac{1}{\gamma}}x))\varphi_{m,n}\big\|_{L^2}\\ \leqslant& C_Nh^N\|v_h\|_{L^2}(\#\{(m,n):b_0^{-1}h^{-1}\leqslant|n|\leqslant Ch^{-(\gamma+1)},\frac{1}{\sqrt{2}}h^{-1}\leqslant\lambda_{m,n}\leqslant\sqrt{2}h^{-1}\})^{1/2}.\end{aligned}$$

Since $\lambda_{m,n}=|n|^{\frac{2}{\gamma+1}}\mu_{m,n}$ where $\mu_{m,n}$ is the m-th eigenvalue of the operator $D_z^2+|z|^{2\gamma}$ on $L^2(|z|\leqslant|n|^{\frac{1}{\gamma+1}})$ with Dirichlet boundary condition, we deduce from Weyl's law that

$$\#\{(m,n):b_0^{-1}h^{-1}\leqslant|n|\leqslant Ch^{-(\gamma+1)},\frac{1}{\sqrt{2}}h^{-1}\leqslant\lambda_{m,n}\leqslant\sqrt{2}h^{-1}\}\leqslant Ch^{-N_0}$$

for some $N_0\in\mathbb{N}$.[1] Therefore, it is sufficient to establish (3.14), which roughly says that in the regime we consider, the energy of eigenfunctions concentrates near $x=0$.

Multiplying (3.11) by $(1-\chi(b_0^{-\frac{1}{\gamma}}x))^2\overline{\varphi}_{m,n}$ and integrating over $x\in(-1,1)$, we obtain that

$$\int_{-1}^1(1-\chi(b_0^{-\frac{1}{\gamma}}x))^2\lambda_{m,n}^2|\varphi_{m,n}(x)|^2dx=\int_{-1}^1(1-\chi(b_0^{-\frac{1}{\gamma}}x))^2\overline{\varphi}_{m,n}(x)\cdot(-\partial_x^2+|x|^{2\gamma}n^2)\varphi_{m,n}dx.$$

Doing integration by part for the r.h.s., and using the fact that $n^2|x|^{2\gamma}\geqslant\frac{4}{h^2}$ on the support of $1-\chi(b_0^{-\frac{1}{\gamma}}x)$ when $|n|\geqslant b_0^{-1}h^{-1}$, we deduce that the r.h.s. can be bounded from below by

$$\begin{aligned}&\frac{4}{h^2}\int_{-1}^1(1-\chi(b_0^{-\frac{1}{\gamma}}x))^2|\varphi_{m,n}(x)|^2dx+\int_{-1}^1(1-\chi(b_0^{-\frac{1}{\gamma}}x))^2|\partial_x\varphi_{m,n}(x)|^2dx\\ &-\int_{-1}^1 2b_0^{-\frac{1}{\gamma}}\chi'(b_0^{-\frac{1}{\gamma}}x)(1-\chi(b_0^{-\frac{1}{\gamma}}x))\overline{\varphi}_{m,n}(x)\partial_x\varphi_{m,n}(x)dx.\end{aligned}$$

[1]To obtain this rough estimate, it suffices to apply Weyl's law for each fixed n and count the number of n.

Using the fact that $\frac{4}{h^2} - \lambda_{m,n}^2 \geqslant \frac{2}{h^2}$, we obtain that

$$\begin{aligned}&2h^{-2}\|(1-\chi(b_0^{-\frac{1}{\gamma}}x))\varphi_{m,n}\|_{L^2}^2 + \|(1-\chi(b_0^{-\frac{1}{\gamma}}x))\partial_x\varphi_{m,n}\|_{L^2}^2\\ \leqslant &Cb_0^{-\frac{1}{\gamma}}\|\chi'(b_0^{-\frac{1}{\gamma}}x)\varphi_{m,n}\|_{L^2}\|(1-\chi(b_0^{-\frac{1}{\gamma}}x))\partial_x\varphi_{m,n}\|_{L^2}.\end{aligned} \tag{3.15}$$

Using Young's inequality in the r.h.s., this implies

$$\|(1-\chi(b_0^{-\frac{1}{\gamma}}x))\varphi_{m,n}\|_{L^2} + \|(1-\chi(b_0^{-\frac{1}{\gamma}}x))h\partial_x\varphi_{m,n}\| \leqslant Cb_0^{-\frac{1}{\gamma}}h.$$

To prove a better estimate, i.e. with an h^N in the r.h.s. instead of h, we observe that

$$\|\chi'(b_0^{-\frac{1}{\gamma}}x)\varphi_{m,n}\|_{L^2} \leqslant C\|(1-\widetilde{\chi}(b_0^{-\frac{1}{\gamma}}x))\varphi_{m,n}\|_{L^2}$$

for another cutoff $\widetilde{\chi}$ such that $\widetilde{\chi}\chi = \widetilde{\chi}$. Therefore, we choose cutoffs $\chi_{(1)}, \chi_{(2)}, \cdots, \chi_{(N)} \in C_c^\infty(\mathbb{R})$ such that $\chi_{(1)} = \chi$ and $\chi_{(k)}\chi_{(k+1)} = \chi_{(k+1)}$ for all $1 \leqslant k \leqslant N$ and such that (3.15) holds by replacing χ by $\chi_{(k)}$ and

$$\|\chi'_{(k)}(b_0^{-\frac{1}{\gamma}}x)\varphi_{m,n}\|_{L^2} \leqslant C_k\|(1-\chi_{(k+1)}(b_0^{-\frac{1}{\gamma}}x))\varphi_{m,n})\|_{L^2}, \quad k = 1, 2, \cdots, N-1.$$

Now since for $\chi_{(N)}$,

$$\|(1-\chi_{(N)}(b_0^{-\frac{1}{\gamma}}x))\varphi_{m,n}\|_{L^2} + \|(1-\chi_{(N)}(b_0^{-\frac{1}{\gamma}}x))h\partial_x\varphi_{m,n}\| \leqslant Cb_0^{-\frac{1}{\gamma}}h,$$

we deduce by induction (in the reverse order) that

$$\|(1-\chi_{(1)}(b_0^{-\frac{1}{\gamma}}x))\varphi_{m,n}\|_{L^2} + \|(1-\chi_{(1)}(b_0^{-\frac{1}{\gamma}}x))h\partial_x\varphi_{m,n}\| \leqslant Cb_0^{-\frac{N}{\gamma}}h^N.$$

This completes the proof of Lemma 3.10. $\square$

3.2.2 Degenerate regime

For $0 < h < 1$ and b_0 fixed once for all thanks to Lemma 3.10, we define the semiclassical spectral projector

$$\Pi_h^{b_0h} := \psi(h^2\Delta_\gamma)(\chi_0(h^{\gamma+1}D_y) - \chi_0(b_0hD_y)).$$

In this subsection, we will show that

$$\|\Pi_h^{b_0h}v_h\|_{L^2(M)} = o(1), \quad h \to 0. \tag{3.16}$$

We prove it by contradiction. If not, we must have $\|w_h\|_{L^2(M)} \gtrsim 1$ where $w_h = \Pi_h^{b_0h}v_h$. We set $\widetilde{f} = \Pi_h^{b_0h}f$ so that

$$(h^2\Delta_\gamma + 1)w_h = \widetilde{f}_h.$$

Let us notice that

$$\left|\|h\nabla_\gamma w_h\|_{L^2(M)}^2 - \|w_h\|_{L^2(M)}^2\right| \leqslant \|w_h\|_{L^2(M)}\|(h^2\Delta_\gamma + 1)w_h\|_{L^2(M)}$$

where $\nabla_\gamma = (\partial_x, x^\gamma\partial_y)$ is the horizontal gradient. This follows from integration by part in the integral $\int w_h(h^2\Delta_\gamma + 1)w_h$. We deduce

$$\|h\nabla_\gamma w_h\|_{L^2(M)} = \|w_h\|_{L^2(M)}^2 + o(1). \tag{3.17}$$

The proof of (3.16) is mainly based on the following commutator relation:

$$[\Delta_\gamma, x\partial_x + (\gamma+1)y\partial_y] = 2\Delta_\gamma.$$

This is an illustration for the positive commutator method, which we shall use again in other parts of the proof. This method dates back at least to [Hor71a, Section 3.5] and has been widely used, for example for proving propagation of singularities for the wave equation.

Note that $y\partial_y$ is not defined globally on $\mathbb{T}_y$. This is why we introduce the following cut-off procedure. Let $\phi \in C^\infty(\mathbb{T})$ such that $\phi \equiv 1$ on $\mathbb{T} \setminus (a_1, a_2)$, $\mathrm{supp}(\phi') \subset (a_1, a_2)$ and $\phi \equiv 0$ on a strict sub-interval of $I = (a_1, a_2)$. Then, considering $\phi(y)y\partial_y$ on the interval $[\frac{a_1+a_2}{2}, \frac{a_1+a_2}{2} + 2\pi]$ and then periodizing, we obtain an objet globally defined on $\mathbb{T}$.

We also set $\chi_{b_0}(x) = \chi(b_0^{-\frac{1}{\gamma}} x)$ (see Lemma 3.10). We compute the inner product

$$C_\gamma := ([h^2\Delta_\gamma + 1, \chi_{b_0}(x)\phi(y)(x\partial_x + (\gamma+1)y\partial_y)]w_h, w_h)_{L^2(M)}$$

in two ways. The first way is to expand the bracket and use the self-adjointness of Δ_γ:

$$C_\gamma = (\chi_{b_0}(x)\phi(y)(x\partial_x v_h + (\gamma+1)y\partial_y w_h), \widetilde{f}_h)_{L^2(M)} - (\chi_{b_0}(x)\phi(y)(x\partial_x \widetilde{f}_h + (\gamma+1)y\partial_y \widetilde{f}_h), w_h)_{L^2(M)}.$$

The second way is to use the computation

$$\begin{aligned}
&[h^2\Delta_\gamma + 1, \chi_{b_0}(x)\phi(y)(x\partial_x + (\gamma+1)y\partial_y)] \\
=&2h^2\chi_{b_0}(x)\phi(y)\Delta_\gamma + h^2(\gamma+1)|x|^{2\gamma}(\phi''(y)y + 2\phi'(y))\chi_{b_0}(x)\partial_y \\
&+h^2\phi''(y)|x|^{2\gamma}x\chi_{b_0}(x)\partial_x + 2h^2(\gamma+1)|x|^{2\gamma}y\phi'(y)\chi_{b_0}(x)\partial_y^2 + 2h^2|x|^{2\gamma}x\chi_{b_0}(x)\phi'(y)\partial_{xy}^2 \\
&+h^2\phi(y)(\chi''_{b_0}(x) + 2\chi'_\epsilon(x)\partial_x)(x\partial_x + (\gamma+1)y\partial_y).
\end{aligned}$$

From the elliptic regularity (Lemma 3.10) and (3.17), on the supports of $1-\chi_{b_0}(x), \chi'_{b_0}(x), \chi''_{b_0}(x)$, the L^2 norm of w_h and $\nabla_\gamma w_h$ is of order $O(h^N)\|w_h\|_{L^2}$ for any $N \in \mathbb{N}$. Then, using integration by part and Young's inequality, we obtain

$$C_\gamma = (2\phi(y)h^2\Delta_\gamma w_h, w_h)_{L^2(M)} + O(h)\|h\nabla_\gamma w_h\|^2_{L^2(M)} + O(h)\|w_h\|^2_{L^2(M)} + O(1)\|h\nabla_\gamma w_h\|^2_{L^2(\mathrm{supp}(\phi'))}.$$

Equating the two ways of computing C_γ and using integration by parts, we obtain

$$\begin{aligned}
\|\phi(y)^{1/2}h\nabla_\gamma w_h\|^2_{L^2(M)} \leqslant\ & O(h)\|h\nabla_\gamma w_h\|^2_{L^2(M)} + O(h)\|w_h\|^2_{L^2(M)} + O(1)\|h\nabla_\gamma w_h\|^2_{L^2(\mathrm{supp}(\phi'))} \\
&+ O(1)\|\widetilde{f}_h\|_{L^2(M)}(\|\partial_x w_h\|_{L^2(M)} + \|\partial_y w_h\|_{L^2(M)}).
\end{aligned}$$

First, we notice that we can replace the left hand side simply by $\|h\nabla_\gamma w_h\|^2_{L^2(M)}$ (which is $\gtrsim 1$ thanks to (3.17)) and the above inequality remains true: this is due to the presence of $O(1)\|h\nabla_\gamma w_h\|^2_{L^2(\mathrm{supp}(\phi'))}$ in the right hand side. Then, for h sufficiently small, we absorb the $O(h)\|h\nabla_\gamma w\|^2_{L^2(M)}$ and the $O(h)\|w_h\|^2_{L^2(M)}$ terms in the left hand side. Finally, we use

$$\begin{aligned}
\|\partial_x w_h\|_{L^2(M)} &\leqslant h^{-1}\|h\nabla_\gamma w_h\|_{L^2(M)} \lesssim h^{-1} \\
\|\partial_y w_h\|_{L^2(M)} &\leqslant h^{-(\gamma+1)}\|w_h\|_{L^2(M)} \\
\|\widetilde{f}_h\|_{L^2(M)} &\leqslant \|f_h\|_{L^2(M)} = o(h^{\gamma+1})
\end{aligned}$$

where the first line comes from (3.17), the second line from Corollary 3.9 together with $w_h = \psi(h^2\Delta_\gamma)w_h$, and the third line from Plancherel formula and (3.9). We obtain

$$1 \lesssim \|h\nabla_\gamma w_h\|^2_{L^2(\mathrm{supp}(\phi'))}. \tag{3.18}$$

Let us prove that this contradicts (3.9). Let $\phi_1 \in C^\infty(\mathbb{T}_y)$ such that $\phi_1 = 1$ on $\mathrm{supp}(\phi')$ and $\phi_1 = 0$ on $\mathbb{T}_y \setminus I$. In particular, together with (3.18), this implies

$$1 \lesssim \|\phi_1(y) h \nabla_\gamma w_h\|^2_{L^2(M)}.$$

By integration by parts, there holds

$$\begin{aligned}\|\phi_1(y) h \nabla_\gamma w_h\|^2_{L^2(M)} &= -h^2 \int_M w_h (\nabla_\gamma(\phi_1^2) \cdot \nabla_\gamma w_h) dxdy - h^2 \int_M \phi_1^2 w_h \Delta_\gamma w_h dxdy \\ &= -h^2 \int_M w_h (\nabla_\gamma(\phi_1^2) \cdot \nabla_\gamma w_h) dxdy + \int_M \phi_1^2 w_h (w_h - \widetilde{f}_h) dxdy\end{aligned}$$

where in the last line we used the equation of w_h. Using (3.9), (3.17) and Cauchy-Schwarz inequality, we see that the first term in the last line is $O(h)$. For the second term, we write

$$\left| \int_M \phi_1^2 w_h (w_h - f_h) dxdy \right| = \|\phi_1 w_h\|^2_{L^2(M)} + o(1),$$

and we note that

$$\|\phi_1 w_h\|_{L^2(M)} \leqslant \|[\phi_1, \Pi_h^{b_0 h}] v_h\|_{L^2(M)} + \|\Pi_h^{b_0 h}(\phi_1 v_h)\|_{L^2(M)} \leqslant O(h) + \|v_h\|_{L^2(\omega)} = o(1)$$

as $h \to 0$, by assumption. All in all, we obtain $\|\phi_1(y) h \nabla_\gamma w_h\|^2_{L^2(M)} = o(1)$, which is a contradiction. This concludes the proof of (3.16).

3.2.3 Regime of the geometric control condition

Let

$$\Pi_{h,b_0} = \psi(h^2 \Delta_G) \chi_0(b_0 h D_y)(1 - \chi_0(b_0^{-1} h D_y))$$

and $z_h = \Pi_{h,b_0} v_h$. In this subsection, we will show that

$$\|z_h\|_{L^2(M)} = o(1), \quad h \to 0. \tag{3.19}$$

We will use a defect-measure based argument as in [BS19, Section 5]. It consists in showing that the semi-classical defect measure associated with a subsequence of $(z_h)_{h>0}$ is invariant along the Melrose-Sjöstrand flow (corresponding to the principal symbol $p = \xi^2 + |x|^{2\gamma}\eta^2$). Then to obtain a contradiction, we just need to check the geometric control condition: there exists $T_0 > 0$ such that any trajectory of the Melrose-Sjöstrand flow enters ω within time T_0; but we recall that only trajectories corresponding to $|\eta| \in (b_0, b_0^{-1})$ are considered here. We omit the standard steps of constructing the semi-classical measure and proving the invariance of the measure[2], and only proceed to check the geometric control condition.

For the principal symbol

$$p(x, y; \xi, \eta) = \xi^2 + |x|^{2\gamma}\eta^2, \quad \gamma > 1,$$

the Hamiltonian flow is given by the ODE

$$\begin{cases} \dot{x} = \partial_\xi p = 2\xi \\ \dot{\xi} = -\partial_x p = -2\gamma |x|^{2(\gamma-1)} x \eta^2 \\ \dot{y} = 2|x|^{2\gamma}\eta \\ \dot{\eta} = 0. \end{cases} \tag{3.20}$$

[2]The argument is the same as in the Baouendi-Grushin-context $\gamma = 1$ handled in [BS19, Section 5].

Thanks to the integrability of (3.20), we can define the Melrose-Sjöstrand flow associated with the symbol p on the compressed cotangent bundle ${}^bT^*\overline{M}$.[3] We will denote by $\varphi_s(\cdot)$ this flow.

Remark 3.11. The assumption that $\gamma \geqslant 1$ is used here, since otherwise the coefficients of (3.20) are not Lipschitz and the Cauchy-Lipschitz theorem does not allow us to conclude that its solutions are unique.

Lemma 3.12. *Assume that $\gamma \geqslant 1$ and $\omega \subset (-1,1) \times \mathbb{T}$ is a horizontal strip. There exist $T_0 > 0, c_0 > 0$, such that for all $\rho_0 = (x_0, y_0; \xi_0, \eta_0)$ with $|\eta_0| \in \left(b_0, b_0^{-1}\right)$ and $p(x_0, y_0; \xi_0, \eta_0) = p_0 \in \left(\frac{1}{2}, 2\right)$, there holds*

$$\frac{1}{T_0}\int_0^{T_0} \mathbf{1}_\omega(\varphi_s(\rho_0))ds \geqslant c_0 > 0.$$

In particular, the geometric control condition (GCC) holds for ω.

Proof. It suffices to show that any trajectory $\varphi_s(\rho_0)$ satisfying

$$p(\rho_0) = p_0 \in \left(\frac{1}{2}, 2\right), \quad |\eta_0| \in \left(b_0, b_0^{-1}\right)$$

will enter the interior of ω before some uniform time $T_0 > 0$. By shifting the y variable we may assume that $y_0 = 0$. Without loss of generality we can also assume that $\eta_0 > 0$. Let $\varphi_s(\rho_0) = (x(s), y(s); \xi(s), \eta(s))$. Note that $\eta(s) = \eta_0 \neq 0$, so that $x(\cdot)$ is periodic. Moreover, we have the first integrals

$$p_0 = \frac{1}{4}|\dot{x}(s)|^2 + |x(s)|^{2\gamma}\eta_0^2, \quad y(s) = 2\eta_0 \int_0^s |x(s')|^{2\gamma} ds' \pmod{2\pi} \tag{3.21}$$

In a nutshell, to show that the flow reaches ω, we first notice that $y(\cdot)$ evolves in a monotone way in $\mathbb{T}$, and that the larger $|x|$ is, the more y varies. Now, if $|x|$ remains too small, then (3.21) gives that $|\dot{x}| \sim 2\sqrt{p_0}$, which implies that $|x|$ cannot remain too small, thus a contradiction.

To put it into a rigorous form, consider the interval $J_\delta = (-\delta, \delta)$ (for the x variable) for $0 < \delta \ll 1$. For $\delta > 0$ sufficiently small (not depending on $|\eta_0| \in (b_0, b_0^{-1})$) and if $x(s) \in J_\delta$, using (3.21), we have $|\dot{x}(s)| \geqslant \sqrt{p_0}$. Therefore, following the flow, it takes a time at most $\tau_0 := \frac{2\delta}{\sqrt{p_0}}$ to leave the regionl $J_\delta \times \mathbb{T}$.
Let us fix s_0 such that $x(s_0) \in J_\delta$ (if it does not exist, we are done thanks to the second relation in (3.21)). We know that there exists $s_0 \leqslant s_1 \leqslant s_0 + \tau_0$ such that $|x(s_1)| = \delta$. We consider the minimal time $s_2 \geqslant s_1$ such that $|x(s_2)| = \frac{\delta}{2}$. Since $\|\dot{x}\|_\infty \leqslant 2\sqrt{p_0}$ (thanks to (3.21)), we know that $s_2 \geqslant s_3 := s_1 + \frac{\delta}{4\sqrt{p_0}}$. Finally,

$$y(s_3) - y(s_0) = 2\eta_0 \int_{s_0}^{s_3} |x(s')|^{2\gamma} ds' \geqslant 2b_0 \int_{s_1}^{s_3} \left(\frac{\delta}{2}\right)^{2\gamma} ds' = \frac{b_0\delta}{2\sqrt{p_0}} \left(\frac{\delta}{2}\right)^{2\gamma}.$$

In other words, in any case, y increases of at least $\frac{b_0\delta}{2\sqrt{p_0}}\left(\frac{\delta}{2}\right)^{2\gamma}$ within any time period of length $\tau_0 + \frac{\delta}{4\sqrt{p_0}} \leqslant \frac{3\delta}{\sqrt{p_0}}$. Hence, the result holds for some T_0 of order $\delta^{-2\gamma}$. □

[3] In our specific example, the flow in the interior is defined via (3.20); when it reaches the boundary, the flow is continued directly at diffractive points and by reflection at hyperbolic points. There is no higher order contact in this simple geometry, see [BS19, Section 5].

3.2.4 Horizontal propagation regime I

Now we treat the regime $|D_y| \leqslant b_0 h^{-1}$. We set $\kappa_h := \psi(h^2\Delta_\gamma)\chi_0(b_0^{-1}hD_y)v_h$. To finish the proof of Theorem 3.1, it remains to show that

$$\|\kappa_h\|_{L^2(M)} = o(1), \quad h \to 0. \tag{3.22}$$

Let μ be a semi-classical measure associated to a subsequence of $(\kappa_h)_{h>0}$. Since it is invariant along the Hamiltonian flow associated with $p = \xi^2 + |x|^{2\gamma}\eta^2$ subject to the reflection and diffraction at the boundary. Since $\mu\mathbf{1}_\omega = 0$ and ω is a horizontal strip (or union of horizontal strips), we deduce that the only possible place where the defect measure concentrates is the set $\{\eta = 0\}$ on which the trajectories are horizontal. To exclude this possibility, we need to decompose $|D_y|$ in a finer way. For some small parameter $\epsilon > 0$ to be chosen later, we let

$$\kappa_h^\epsilon = (1 - \chi_0(h^\epsilon D_y))\kappa_h, \quad \kappa_{h,\epsilon} = \chi_0(h^\epsilon D_y)\kappa_h.$$

Our goal of this subsection is to show that

$$\|\kappa_h^\epsilon\|_{L^2(M)} = o(1), \quad h \to 0 \tag{3.23}$$

We use the positive commutator method (already used in Section 3.2.2) with the relation

$$\begin{aligned}[h^2\Delta_\gamma + 1, \phi(y)y\partial_y] =& 2\phi(y)|x|^{2\gamma}(h\partial_y)^2 + 2y\phi'(y)|x|^{2\gamma}(h\partial_y)^2 \\ &+ h^2\phi''(y)y|x|^{2\gamma}\partial_y + 2h^2\phi'(y)|x|^{2\gamma}\partial_y,\end{aligned}$$

where ϕ has been introduced in Section 3.2.2. As in Section 3.2.2, we compute the inner product $([h^2\Delta_\gamma + 1, \phi(y)y\partial_y]\kappa_h^\epsilon, \kappa_h^\epsilon)_{L^2(M)}$ in two ways, and using Cauchy-Schwarz, it gives

$$\begin{aligned}\|\phi(y)^{1/2}h|x|^\gamma\partial_y\kappa_h^\epsilon\|_{L^2(M)}^2 \leqslant& Ch\||x|^\gamma h\partial_y\kappa_\epsilon^h\|_{L^2(M)}\|\kappa_h^\epsilon\|_{L^2(M)} + Ch^2\|\kappa_h^\epsilon\|_{L^2(M)}^2 \\ &+ C\|\phi'(y)^{1/2}|x|^\gamma h\partial_y\kappa_h^\epsilon\|_{L^2(M)}^2 + Ch^{-1}\|f_h\|_{L^2(M)}\|h\partial_y\kappa_h^\epsilon\|_{L^2(M)}.\end{aligned}$$

Using Young's inequality, we deduce that for any $\delta > 0$, for any sufficiently small $h > 0$,

$$\begin{aligned}\|\phi(y)^{1/2}|x|^\gamma h\partial_y\kappa_h^\epsilon\|_{L^2(M)}^2 \leqslant& \delta\|h\partial_y\kappa_h^\epsilon\|_{L^2(M)}^2 + C\||x|^\gamma h\partial_y\kappa_h^\epsilon\|_{L^2(\mathrm{supp}(\phi'))}^2 \\ &+ C(\delta)h^{-2}\|f_h\|_{L^2(M)}^2 + C(\delta)h^2\|\kappa_h^\epsilon\|_{L^2(M)}^2\end{aligned}$$

and therefore, using the $\||x|^\gamma h\partial_y\kappa_h^\epsilon\|_{L^2(\mathrm{supp}(\phi'))}$ term in the right hand side, we obtain

$$\begin{aligned}\||x|^\gamma h\partial_y\kappa_h^\epsilon\|_{L^2(M)}^2 \leqslant& \delta\|h\partial_y\kappa_h^\epsilon\|_{L^2(M)}^2 + C\||x|^\gamma h\partial_y\kappa_h^\epsilon\|_{L^2(\mathrm{supp}(\phi'))}^2 \\ &+ C(\delta)h^{-2}\|f_h\|_{L^2(M)}^2 + C(\delta)h^2\|\kappa_h^\epsilon\|_{L^2(M)}^2.\end{aligned} \tag{3.24}$$

We need the following lemma, which roughly states that in the horizontal regime, the mass cannot concentrate on $x = 0$:

Lemma 3.13. *We have*

$$\|\partial_y\kappa_h^\epsilon\|_{L^2(M)} \leqslant C\||x|^\gamma\partial_y\kappa_h^\epsilon\|_{L^2(M)} + o(1),$$

as $h \to 0$.

Let us postpone the proof of Lemma 3.13 for the moment and proceed to finish the proof of (3.23). Thanks to (3.24) and Lemma 3.13, by choosing δ small enough, we have

$$\||x|^\gamma\partial_y\kappa_h^\epsilon\|_{L^2(M)}^2 \leqslant C\||x|^\gamma\partial_y\kappa_h^\epsilon\|_{L^2(\mathrm{supp}(\phi'))}^2 + C(\delta)o(h^{2(\gamma-1)}) + C(\delta)\|\kappa_h^\epsilon\|_{L^2(M)}^2 + o(1).$$

Applying Lemma 3.13 again and plugging into the inequality above, we have

$$\|\partial_y\kappa_h^\epsilon\|_{L^2(M)}^2 + \||x|^\gamma\partial_y\kappa_h^\epsilon\|_{L^2(M)}^2 \leqslant C\||x|^\gamma\partial_y\kappa_h^\epsilon\|_{L^2(\mathrm{supp}(\phi'))}^2 + C(\delta)\|\kappa_h^\epsilon\|_{L^2(M)}^2 + o(1).$$

Now since $\mathcal{F}_y(\kappa_h^\epsilon)(x,n) = 0$, for all $|n| \leqslant h^{-\epsilon}$, by definition of κ_h^ϵ, we have

$$\|\kappa_h^\epsilon\|_{L^2(M)}^2 \leqslant h^{2\epsilon}\|\partial_y\kappa_h^\epsilon\|_{L^2(M)}^2,$$

hence, we have

$$\|\partial_y\kappa_h^\epsilon\|_{L^2(M)}^2 + \||x|^\gamma\partial_y\kappa_h^\epsilon\|_{L^2(M)}^2 \leqslant C\||x|^\gamma\partial_y\kappa_h^\epsilon\|_{L^2(\mathrm{supp}(\phi'))}^2 + o(1).$$

Now, we proceed as in Section 3.2.2: we insert a smooth cutoff $\phi_1(y)$ such that $\mathrm{supp}(\phi_1) \subset \omega$ and $\phi_1(y) \equiv 1$ on $\mathrm{supp}(\phi')$. Hence $\||x|^\gamma\partial_y\kappa_h^\epsilon\|_{L^2(\mathrm{supp}(\phi'))}^2 \leqslant \||x|^\gamma\phi_1(y)\partial_y\kappa_h^\epsilon\|^2$. Then we can write

$$|x|^\gamma\phi_1(y)h\partial_y\kappa_h^\epsilon = |x|^\gamma h\partial_y\psi(h^2\Delta_\gamma)\chi_0(b_0^{-1}hD_y)(\phi_1(y)v_h) + O_{L^2(M)}(h),$$

where the second term on the r.h.s. comes from the commutator. Therefore,

$$\|\kappa_h^\epsilon\|_{L^2(M)}^2 \leqslant h^{2\epsilon}\|\partial_y\kappa_h^\epsilon\|_{L^2(M)}^2 \leqslant o(h^{2\epsilon}) + Ch^{2\epsilon}\|\phi_1(y)v_h\|_{L^2(M)}^2 + O(h^{2\epsilon}).$$

Since $\mathrm{supp}(\phi_1) \subset \omega$, there holds $\|\phi_1(y)v_h\|_{L^2(M)} = o(1)$. The proof of (3.23) is complete.

It remains to prove Lemma 3.13:

Proof of Lemma 3.13. This is a variant of horizontal propagation estimates in the spirit of Lemma 6.2 in [BS19]. However, due to the absent of the time variable, here we need a slightly different argument. The main idea is to use propagation arguments in the horizontal direction in order to “get out” from the singular region $x = 0$.

Let $z_h = \partial_y\kappa_h^\epsilon$. Since ∂_y commutes with $h^2\Delta_\gamma + 1$, z_h satisfies the equation

$$(h^2\Delta_\gamma + 1)z_h = g_h = o_{L^2}(h^\gamma),$$

where $g_h = (1 - \chi_0(h^\epsilon D_y))\chi_0(b_0^{-1}hD_y)\partial_y f_h$. Let us show that for some $r_0 \in (0, \frac{1}{2})$,

$$\|z_h\|_{L^2(|x|\leqslant 2r_0)} \leqslant C(r_0)\|z_h\|_{L^2(r_0<|x|<1)} + o(1)$$

as $h \to 0$, which is sufficient for proving Lemma 3.13. We choose $\psi^\pm \in C_c^\infty(\mathbb{R})$ such that

$$\psi^\pm(\xi) = \begin{cases} 1, & \text{if } \ \frac{3}{4}\sqrt{\frac{1}{2} - (8C_1)^{\gamma+1}b_0^2} \leqslant \pm\xi \leqslant 2\sqrt{2}; \\ 0, & \text{if } \ |\xi| > 3 \text{ or } |\xi| < \frac{1}{2}\sqrt{\frac{1}{2} - (8C_1)^{\gamma+1}b_0^2}. \end{cases}$$

Let $\chi \in C_c^\infty((0,1))$ such that $\chi(x) = 1$ if $|x| \leqslant r_0$ and $\chi(x) = 0$ if $|x| > 3r_0/2$. From the localization property of z_h, we know that

$$\begin{aligned} \mathrm{WF}_h(z_h) &\subset \big\{(x,y;\xi,\eta) : p = \xi^2 + |x|^{2\gamma}\eta^2 \in (\frac{1}{2}, 2), |\eta| \leqslant (8C_1)^{\frac{\gamma+1}{2}}b_0\big\} \\ &\subset \{(x,y,\xi,\eta) : \xi \in \mathrm{supp}(\psi^+) \cup \mathrm{supp}(\psi^-)\}, \end{aligned}$$

thus it suffices to estimate $\|\chi(x)\psi^{\pm}(hD_x)z_h\|_{L^2(M)}$ and by symmetry we only need to estimate $\|\chi(x)\psi^{+}(hD_x)z_h\|_{L^2(M)}$. Moreover, by our choice of $\psi^{\pm}$,

$$\mathrm{WF}_h(z_h)\cap\{(x,y;\xi,\eta):\xi\in\mathrm{supp}((\psi^{\pm})')\}=\emptyset.$$

Note that for any (semi-classical) pseudo-differential operator $\mathrm{Op}_h(a)$, compactly supported in the interior of M, we have

$$\frac{1}{ih}\big([\mathrm{Op}_h(a),h^2\Delta_\gamma+1]z_h,z_h\big)_{L^2(M)}=o(h^{\gamma-1})=o(1),\tag{3.25}$$

thanks to the equation of z_h. Now we consider a specific pseudo-differential operator $\mathrm{Op}_h(a^{\pm})$ with principal symbol $\chi^2(x)\sin\big(\frac{\pi x}{4r_0}\big)(\psi^{\pm}(\xi))^2$. By symbolic calculus,

$$\frac{1}{ih}[\mathrm{Op}_h(a^+),h^2\Delta_\gamma+1]=\mathrm{Op}_h(\{\xi^2+|x|^{2\gamma}\eta^2,\chi^2(x)\sin(\frac{\pi x}{4r_0})(\psi^{\pm}(\xi))^2\})+\mathcal{O}_{L^2\to L^2}(h).$$

We compute

$$\begin{aligned}&\{\xi^2+|x|^{2\gamma}\eta^2,\chi^2(x)\sin(\frac{\pi x}{4r_0})(\psi^{+}(\xi))^2\}\\=&2\xi(\psi^+(\xi))^2\cdot\frac{\pi}{4r_0}\chi^2(x)\cos\big(\frac{\pi x}{4r_0}\big)+4\xi(\psi^+(\xi))^2\chi(x)\chi'(x)\sin\big(\frac{\pi x}{4r_0}\big)\\&-4\gamma|x|^{2\gamma-2}x\eta^2\psi^+(\xi)(\psi^+)'(\xi)\chi^2(x)\sin\big(\frac{\pi x}{4r_0}\big).\end{aligned}$$

Let

$$a_1=2\xi(\psi^+(\xi))^2\cdot\frac{\pi}{4r_0}\chi^2(x)\cos\big(\frac{\pi x}{4r_0}\big),\quad a_2=4\xi(\psi^+(\xi))^2\chi(x)\chi'(x)\sin\big(\frac{\pi x}{4r_0}\big)$$

and $a_3=-4\gamma|x|^{2\gamma-2}x\eta^2\psi^+(\xi)(\psi^+)'(\xi)\chi^2(x)\sin\big(\frac{\pi x}{4r_0}\big)$. From the property of $\mathrm{WF}_h(z_h)$, we have $(\mathrm{Op}_h(a_3)z_h,z_h)_{L^2(M)}=O(h^N)$, for any $N\in\mathbb{N}$. From the support property of a_2, we have

$$|(\mathrm{Op}_h(a_2)z_h,z_h)_{L^2(M)}|\leqslant C\|z_h\|^2_{L^2(r_0<|x|<1)}.$$

Thus from (3.25), we have

$$(\mathrm{Op}_h(a_1)z_h,z_h)_{L^2(M)}\leqslant o(1)+C\|z_h\|^2_{L^2(r_0<|x|<1)}.\tag{3.26}$$

Since $a_1\geqslant c_0$ for some uniform constant $c_0>0$ on $|x|\leqslant 3r_0/2$, we can decompose $a_1=a_1^{(0)}+a_1^{(1)}$ where $a_1^{(0)}\geqslant c_0\chi(x)^2(\psi^+(\xi))^2$ and $\mathrm{supp}(a_1^{(1)})\subset\{|x|>\frac{3r_0}{2}\}$. Using the sharp Gårding inequality, we have

$$(\mathrm{Op}_h(a_1^{(0)})z_h,z_h)_{L^2(M)}\geqslant c_0(\mathrm{Op}_h(\chi(x)^2(\psi^+(\xi))^2)z_h,z_h)_{L^2(M)}-Ch\|z_h\|^2_{L^2(M)}.$$

Together with (3.26), this yields

$$\|\chi_0(x)\psi(hD_x)^+z_h\|^2_{L^2}\leqslant o(1)+C\|z_h\|_{L^2(r_0<|x|<1)}.$$

The proof of Lemma 3.13 is now complete. $\square$

3.2.5 Horizontal propagation regime II

To finish the proof of (3.22) (and hence that of Theorem 3.1), it remains to show that

$$\|\kappa_{h,\epsilon}\|_{L^2(M)} = o(1), \quad h \to 0, \tag{3.27}$$

where $\kappa_{h,\epsilon} = \chi_0(h^\epsilon D_y)\kappa_h = \psi(h^2\Delta_\gamma)\chi_0(h^\epsilon D_y)\kappa_h$ with small parameter $\epsilon > 0$ to be fixed later. In this subsection, we prove the following result which, combined with (3.9), directly yields (3.27):

Proposition 3.14. *There exist $C > 0$, $h_0 > 0$, $\varepsilon_0 > 0$ such that for all $0 < h < h_0$ and $0 < \epsilon < \varepsilon_0$, we have*

$$\|\kappa_{h,\epsilon}\|_{L^2(M)} \leqslant C\|\kappa_{h,\epsilon}\|_{L^2(\omega)} + Ch^{-2}\|(h^2\Delta_\gamma + 1)\kappa_{h,\epsilon}\|_{L^2(M)} + Ch^{1-2\epsilon}\|v_h\|_{L^2(M)}.$$

We follow the normal form method as in [BS19, Section 7], originally inspired by the work [BZ04]. The key point is to search for a microlocal transformation

$$w = (1 + hQD_y^2)v$$

for some suitable semi-classical pseudo-differential operator $Q = q(x, hD_x)$, such that the conjugated equation (satisfed by w) is

$$h^2\partial_x^2 w + h^2 M\partial_y^2 w + w = \text{ errors},$$

where

$$M = \frac{1}{2}\int_{-1}^{1} |x|^{2\gamma} dx$$

is the mean value of $|x|^{2\gamma}$. Roughly speaking, this normal form method puts into a rigorous form the intuition that in the horizontal propagation regime, the vector field $|x|^\gamma\partial_y$ acts as if it were averaged along horizontal trajectories.

Then we will be able to use the following theorem:

Proposition 3.15 ([AL14],[BZ04],[AM14])**.** *Let $\Delta_M = \partial_x^2 + M\partial_y^2$. Then for any non-empty open set $\omega_0 \subset \mathbb{T}^2$, we have*

$$\|u\|_{L^2(\mathbb{T}^2)} \leqslant C\|u\|_{L^2(\omega_0)} + Ch^{-2}\|(h^2\Delta_M + 1)u\|_{L^2(\mathbb{T}^2)}.$$

However, dealing with Dirichlet boundary value problem induces difficulties and consequently, we prefered to extend the analysis to the periodic setting. First we introduce several notations. Let

$$\widetilde{\mathbb{T}} := [-1, 3]/\{-1, 3\} \quad \text{and} \quad \widetilde{\mathbb{T}}^2 := \widetilde{\mathbb{T}}_x \times \mathbb{T}_y.$$

Define

$$a(x) = |x|^\gamma, \text{ if } |x| \leqslant 1 \text{ and } a(x) = |2 - x|^\gamma, \text{ if } 1 \leqslant x \leqslant 3,$$

and the operator

$$P_a := \partial_x^2 + a(x)^2\partial_y^2.$$

Note that $a(x)$ and $a(x)^2$ are Lipschitz functions on $\widetilde{\mathbb{T}}$. Let

$$H_a^k(\widetilde{\mathbb{T}}^2) := \{f \in \mathcal{D}'(\widetilde{\mathbb{T}}^2) : P_a^j f \in L^2(\widetilde{\mathbb{T}}^2), \forall 0 \leqslant j \leqslant k\}$$

the associated function spaces and the domain of P_a is $D(P_a) = H_a^2(\widetilde{\mathbb{T}}^2)$. Recall that $D(\Delta_\gamma) = H_{\gamma,0}^1(M) \cap H_\gamma^2(M)$. Consider the extension map:

$$\iota_1 : D(\Delta_\gamma) \to D(P_a), \quad f \mapsto \widetilde{f},$$

with

$$\widetilde{f}(x,y) = f(x,y), \text{ if } |x| \leqslant 1, \text{ and } \widetilde{f}(x,y) = -f(2-x,y), \text{ if } 1 \leqslant x \leqslant 3.$$

The mapping ι_1 is the odd extension with respect to $x = 1$. Note that for $f \in C^\infty(\overline{M})$, we have

$$\partial_x f|_{x=1-} = \partial_x(\iota_1 f)|_{x=1+}.$$

Recall the following lemmas from [BS19, Section 7]:

Lemma 3.16 ([BS19])**.** *The extension map $\iota_1 : D(\Delta_\gamma) \to D(P_a)$ is continuous. Moreover, for all $f \in D(\Delta_\gamma)$, $\|\iota_1 f\|_{L^2(\widetilde{\mathbb{T}}^2)} = \sqrt{2}\|f\|_{L^2(\Omega)}$.*

Note that this result was only proved for $\gamma = 1$ in [BS19, Section 7], but the proof given there works without any modification for general $\gamma \geqslant 1$.

Lemma 3.17 ([BS19])**.** *Let S_1, S_2 be two self-ajoint operators on Banach spaces E_1, E_2 with domains $D(S_1), D(S_2)$ respectively. Assume that $j : D(S_1) \to D(S_2)$ is a continuous embedding and that there holds $j \circ S_1 = S_2 \circ j$. Then, for any Schwartz function $g \in \mathcal{S}(\mathbb{R})$, we have*

$$j \circ g(S_1) = g(S_2) \circ j$$

Lemma 3.17 ensures the preservation of the spectral localization property by odd extension procedure. We deduce from Lemma 3.17 that for any Schwartz function $g : \mathbb{R} \to \mathbb{C}$,

$$\iota_1 \circ g(h^2\Delta_\gamma) = g(h^2 P_a) \circ \iota_1.$$

Consequently, we have the following lemma, reducing the proof of Proposition 3.14 to the observability of the extended solutions:

Lemma 3.18. *Let $\phi_1(y)$ be a smooth function which is supported in ω. Assume that there exist $h_0, \varepsilon_0 > 0$ such that for any $0 < h < h_0, 0 < \epsilon < \varepsilon_0$, the following observability holds for all $\widetilde{v} \in L^2(\widetilde{\mathbb{T}}^2)$:*

$$\begin{aligned} \|\psi(h^2P_a)\chi_0(h^\epsilon D_y)\widetilde{v}\|_{L^2(\widetilde{\mathbb{T}}^2)}^2 \leqslant & C\|(h^2P_a+1)\psi(h^2P_a)\chi_0(h^\epsilon D_y)\widetilde{v}\|_{L^2(\widetilde{\mathbb{T}}^2)}^2 \\ & + C\|\phi_1(y)\psi(h^2P_a)\chi_0(h^\epsilon D_y)\widetilde{v}(t)\|_{L^2(\widetilde{\mathbb{T}}^2)}^2 dt + Ch\|\widetilde{v}\|_{L^2(\widetilde{\mathbb{T}}^2)}^2. \end{aligned} \tag{3.28}$$

Then Proposition 3.14 is true. More precisely, with the same constant $C > 0$, for all $0 < h < h_0$, $0 < \epsilon < \varepsilon_0$, the resolvent estimate

$$\begin{aligned} \|\psi(h^2\Delta_\gamma)\chi_0(h^\epsilon D_y)v\|_{L^2(M)}^2 \leqslant & C\|(h^2\Delta_\gamma+1)v\|_{L^2(M)}^2 \\ & + C\|\phi_1(y)\psi(h^2\Delta_\gamma)\chi_0(h^\epsilon D_y)v\|_{L^2(M)}^2 + Ch\|v\|_{L^2(M)}^2 \end{aligned}$$

holds for all $v \in L^2(M)$.

The proof of Lemma 3.18 is straightforward and we omit the detail.

Remark 3.19. Since the extension operation is done for the x-variable, we keep the notation $\phi_1(y)$ for the extension of this function.

Before proving (3.28), we need a lemma which, modulo errors, allows us to replace the operator $\psi(h^2P_a)\chi_0(h^\epsilon D_y)$ by $\psi_1(hD_x)\chi_0(h^\epsilon D_y)$.

Lemma 3.20. *Let $\psi_1 \in C_c^\infty(\frac{1}{4} < |\xi| < 4)$ such that $\psi_1 = 1$ on supp(ψ). Then, as a bounded operator on $L^2(\widetilde{\mathbb{T}}^2)$, we have*

$$(1-\psi_1(hD_x))\,\psi(h^2P_a)\chi_0(h^\epsilon D_y) = O_{L^2\to L^2}(h^{2-2\epsilon}).$$

Proof. As D_y commutes with D_x and P_a, by Plancherel, it suffices to show that, uniformly in $|n| \leqslant (8C_1)^{\frac{1}{\gamma+1}}h^{-\epsilon}$,

$$(1-\psi_1(hD_x))\psi(h^2\mathcal{L}_n) = O_{L^2\to L^2}(h^{2(1-\epsilon)}),$$

where $\mathcal{L}_n = -\partial_x^2 + n^2a(x)^2$. The key point here is that $(1-\psi_1(\xi))\psi(\xi) = 0$. We will make use of the Helffer-Sjöstrand formula (see [DS99] and [BGT04]) :

$$\psi(h^2\mathcal{L}_n) = \frac{1}{2\pi i}\int_{\mathbb{C}} \overline{\partial}\widetilde{\psi}(z)(z-h^2\mathcal{L}_n)^{-1}dz\wedge d\overline{z},$$

where $\widetilde{\psi}(z)$ is an almost analytic extension of ψ, for example

$$\widetilde{\psi}(z) := \chi(\mathrm{Im}z)\cdot\sum_{n=0}^{N+1}\frac{\psi^{(n)}(\mathrm{Re}z)}{n!}(i\mathrm{Im}z)^n, \quad N \geqslant 2.$$

Note that as an operator-valued meromorphic function, we have

$$(z-h^2\mathcal{L}_n)^{-1} = (z-h^2D_x^2)^{-1} + h^2n^2(z-h^2D_x^2)^{-1}a(x)^2(z-h^2\mathcal{L}_n)^{-1},$$

we obtain that

$$(1-\psi_1(hD_x))\psi(h^2\mathcal{L}_n) = \frac{h^2n^2}{2\pi i}(1-\psi_1(hD_x))\int_{\mathbb{C}}\overline{\partial}\widetilde{\psi}(z)(z-h^2D_x)^{-1}a(x)^2(z-h^2\mathcal{L}_n)^{-1}dz\wedge d\overline{z},$$

where we used the Cauchy integral formula

$$\psi(h^2D_x) = \frac{1}{2\pi i}\int_{\mathbb{C}}\overline{\partial}\widetilde{\psi}(z)(z-h^2D_x^2)^{-1}dz\wedge d\overline{z}$$

and $(1-\psi_1(hD_x))\psi(hD_x) = 0$. Using the fact that $|\overline{\partial}\widetilde{\psi}(z)| \leqslant C_N|\mathrm{Im}z|^N\chi(\mathrm{Im}z)$ and $\|(z-P)^{-1}\| \leqslant |\mathrm{Im}z|^{-1}$ for any self-adjoint operator P, we deduce that

$$\|(1-\psi_1(hD_x))\psi(h^2\mathcal{L}_n)\| \leqslant Ch^{2(1-\epsilon)}.$$

This completes the proof of Lemma 3.20. □

Proof of Proposition 3.14. From Lemma 3.18, it is sufficient to prove (3.28). With a little abuse of notation, we denote by $v_0 = \widetilde{\kappa}_{h,\epsilon}$ the extension of $\kappa_{h,\epsilon}$, which verifies $v_0 = \psi(h^2P_a)\chi_0(h^\epsilon D_y)v_0$. We are now in the periodic setting. Yet, we should pay an extra attention to the fact that $P_a = \partial_x^2 + a(x)^2\partial_y^2$ is a hypoelliptic operator with only Lipschitz coefficient. More precisely, $a \in \mathrm{Lip}(\widetilde{\mathbb{T}}^2)$ which is not C^1 when $\gamma = 1$ at $x = 1$.

Modulo an error $O_{L^2}(h^{2-2\epsilon})\|v_0\|_{L^2(\widetilde{\mathbb{T}}^2)}$, we may assume that $v_0 = \psi_1(hD_x)\chi_0(h^\epsilon D_y)v_0$. Note that

$$(h^2P_a+1)v_0 = f_0 := \iota((h^2\Delta_\gamma+1)\kappa_{h,\epsilon}).$$

Now we search for the function

$$w_0 = (1-hQ\partial_y^2)v_0$$

with an h-pseudo-differential operator Q acting only on x, to be chosen later. Let $M = \frac{1}{4}\int_{-1}^{3} a(x)^2 dx = \frac{1}{2}\int_{-1}^{1}|x|^{2\gamma}dx$ be the average of $a(x)^2$ along the horizontal trajectory $y = \text{const}$. Using the equation $(h^2P_a+1)v_0 = f_0$, we have

$$\begin{aligned}(h^2\partial_x^2 + Mh^2\partial_y^2)w_0 + w_0 =&(1-hQ\partial_y^2)(h^2\Delta_\gamma+1)v_0 + (1-hQ\partial_y^2)(M-a(x)^2)h^2\partial_y^2 v_0\\ &-\frac{1}{h}[h^2\partial_x^2, Q]h^2\partial_y^2 v_0\\ =&(1-hQ\partial_y^2)f_0 + \big(M-a(x)^2-\frac{1}{h}[h^2\partial_x^2,Q]\big)h^2\partial_y^2 v_0\\ &-hQ\partial_y^2(M-a(x)^2)h^2\partial_y^2 v_0\end{aligned}$$

Take $\psi_2 \in C_c^\infty(1/8 \leqslant |\xi| \leqslant 8)$, such that $\psi_2\psi_1 = \psi_1$. We define the operator

$$Q = \frac{1}{2i}\left(\int_{-1}^{x}(M-a(z)^2)dz\right)(hD_x)^{-1}\psi_2(hD_x),$$

and set $b(x) = \frac{1}{2i}\int_{-1}^{x}(M-a(z)^2)dz$, $m(hD_x) = (hD_x)^{-1}\psi_2(hD_x)$. Since $a(x)^2 - M$ has zero average, the function b is well-defined as a periodic function in the space $C^1(\widetilde{\mathbb{T}}) \cap W^{2,\infty}(\widetilde{\mathbb{T}})$. From a direct calculation, we have

$$h[\partial_x^2, Q] = 2ib'(x)m(hD_x)hD_x + i[hD_x, b'(x)]m(hD_x).$$

Note that $[hD_x, b'(x)] = -ihb''(x)$, and $b'' \in L^\infty(\widetilde{\mathbb{T}})$, thus

$$\|\big(M-a(x)^2-\frac{1}{h}[h^2\partial_x^2,Q]\big)h^2\partial_y^2 v_0\|_{L^2(\widetilde{\mathbb{T}}^2)} = O(h^{3-2\epsilon})\|v_0\|_{L^2(\widetilde{\mathbb{T}}^2)}.$$

Therefore,

$$\|(h^2\Delta_M+1)w_0\|_{L^2(\widetilde{\mathbb{T}}^2)} \leqslant C\|f_0\|_{L^2(\widetilde{\mathbb{T}}^2)} + O(h^{3-4\epsilon})\|v_0\|_{L^2(\widetilde{\mathbb{T}}^2)}.$$

where $\Delta_M = \partial_x^2 + M\partial_y^2$. Applying Proposition 3.15, we obtain that

$$\|w_0\|_{L^2(\widetilde{\mathbb{T}}^2)} \leqslant C\|\phi_1(y)w_0\|_{L^2(\widetilde{\mathbb{T}}^2)} + Ch^{-2}\|f_0\|_{L^2(\widetilde{\mathbb{T}}^2)} + Ch^{1-4\epsilon}\|v_0\|_{L^2(\widetilde{T}^2)}.$$

Since $w_0 = v_0 + O_{L^2(\widetilde{\mathbb{T}}^2)}(h^{1-2\epsilon})\|v_0\|_{L^2(\widetilde{\mathbb{T}}^2)}$ and $\text{supp}(\phi_1) \subset \omega$, the proof of Proposition 3.14 is now complete. □

End of the proof of Theorem 3.1. We choose ε as in Proposition 3.14. Combining (3.13), (3.16), (3.19), (3.22) and (3.27), we obtain $\|v_h\|_{L^2(M)} = o(1)$, which contradicts (3.9) and proves Theorem 3.1. □

3.3 Theorem 3.4: proofs of observability

3.3.1 Localized observability

In this section, we prove Point (1) and one part of Point (2) of Theorem 3.4, namely that observability holds for sufficiently large time in case $s = \frac{\gamma+1}{2}$. The proofs of these two results are both based on the resolvent estimate given by Theorem 3.1.

In general, we have the following abstract theorem:

Theorem 3.21 ([BZ04]). *Let $P(h)$ be self-adjoint on some Hilbert space $\mathcal{H}$ with densely defined domain $\mathcal{D}$ and $A(h) : \mathcal{D} \to \mathcal{H}$ be bounded. Fix $\chi_0 \in C_c^\infty((-b,-a))$. Assume that uniformly for $\tau \in I = [-b,-a] \subset \mathbb{R}$, we have the following resolvent inequality*

$$\|u\|_{\mathcal{H}} \leqslant \frac{G(h)}{h}\|(P(h)+\tau)u\|_{\mathcal{H}} + g(h)\|A(h)u\|_{\mathcal{H}}$$

for some $1 \leqslant G(h) \leqslant O(h^{-N_0})$. Then there exist constants $C_0, c_0, h_0 > 0$, such that for every $T(h)$ satisfying

$$\frac{G(h)}{T(h)} < c_0,$$

we have, $\forall 0 < h < h_0$

$$\|\chi_0(P(h))u\|_{\mathcal{H}}^2 \leqslant C_0 \frac{g(h)^2}{T(h)} \int_0^{T(h)} \|A(h)e^{-\frac{itP(h)}{h}}\chi_0(P(h))u\|_{\mathcal{H}}^2 dt,$$

where $\psi \in C_c^\infty((a,b))$.

Let us prove Points (1) and (2) of Theorem 3.4, using Theorems 3.1 and 3.21. For $s \in \mathbb{N}$, $s \geqslant 1$, there holds:

$$(-h^2\Delta_\gamma)^s - 1 = Q_{h,\gamma}(-h^2\Delta_\gamma - 1)$$

where

$$Q_{h,\gamma} = (-h^2\Delta_\gamma)^{s-1} + \ldots + 1$$

which is an elliptic operator, such that $Q_{h,\gamma}^{-1}$ is bounded from $L^2(M)$ to $L^2(M)$ (independently on h). Hence if

$$(-h^2\Delta_\gamma)^s u_h - u_h = g_h,$$

then

$$-h^2\Delta_\gamma u_h - u_h = Q_{h,\gamma}^{-1} g_h$$

and, applying Theorem 3.1 and using the $L^2(M)$ boundedness of $Q_{h,\gamma}^{-1}$, we get

$$\|u_h\|_{L^2(M)} \leqslant O(1)\|u_h\|_{L^2(\omega)} + O(h^{-(\gamma+1)})\|g_h\|_{L^2(M)}.$$

Let us now prove a *spectrally localized* observability inequality thanks to a rescaling argument. We first assume $s > \frac{\gamma+1}{2}$. The previous estimate gives

$$\|u_h\|_{L^2(M)} \leqslant O(1)\|u_h\|_{L^2(\omega)} + \frac{G(h)}{h}\|g_h\|_{L^2(M)}$$

with $G(h) = o(h^{1-2s})$. Applying Theorem 3.21 to $g(h) = 1, A(h) = \mathbf{1}_\omega$, $P(h) = (-h^2\Delta_\gamma)^s$, by denoting $u_h = \chi_0((-h^2\Delta_\gamma)^s)u_0$ where $\chi_0 \in C_c^\infty((1/2,2))$, we have

$$\|u_h\|_{L^2(M)}^2 \leqslant \frac{C_0}{T(h)} \int_0^{T(h)} \|e^{-\frac{it(-h^2\Delta_\gamma)^s}{h}} u_h\|_{L^2(\omega)}^2 dt$$

for $T(h) = C_1 G(h)$ with $C_1 = \frac{2}{c_0}$. By changing variables $t' = h^{2s-1}t$, we have

$$\|u_h\|_{L^2(M)}^2 \leqslant \frac{C_0}{C_1 G(h) h^{2s-1}} \int_0^{h^{2s-1}C_1G(h)} \|e^{-it'(-\Delta_\gamma)^s} u_h\|_{L^2(\omega)}^2 dt'.$$

Fix $T > 0$. Since $h^{2s-1}G(h) = o(1)$ as $h \to 0$, we can apply the inequality above $\sim \frac{T}{C_1 h^{2s-1}G(h)}$ times, together with the conservation of the $L^2(M)$ norm along the flow $e^{-it'(-\Delta_\gamma)^s}$. This yields

$$\|u_h\|^2_{L^2(M)} \leqslant C \int_0^T \|e^{-it'(-\Delta_\gamma)^s} u_h\|^2_{L^2(\omega)} dt'. \tag{3.29}$$

In the case of Point (2) where $\frac{1}{2}(\gamma+1) = s$, doing the same argument with $G(h) = O(h^{1-2s})$, we obtain that (3.29) holds only for T sufficiently large, that is, $T \geqslant T_{\inf}$.

It remains to show how to deduce Point (1) (or Point (2)) from the localized observability inequality (3.29). This procedure is standard (see [BZ12, Section 4]), but we recall it here briefly for the sake of completeness.

3.3.2 From the localized observability to the full observability

In the next lemma, $H^{-1}_\gamma(M)$ denotes the dual of $H^1_{\gamma,0}(M)$ (defined in Section 3.1.2).

Lemma 3.22. *The embeddings $H^1_{\gamma,0}(M) \hookrightarrow L^2(M)$ and $L^2(M) \hookrightarrow H^{-1}_\gamma(M)$ are compact.*

Proof. By duality, we only need to prove that $H^1_{\gamma,0}(M) \hookrightarrow L^2(M)$ is compact. Since $H^1_{\gamma,0}(M) \hookrightarrow H^1_\gamma(M)$, it suffices to show that $H^1_\gamma(M) \hookrightarrow L^2(M)$ is compact. For $s_1 \in \mathbb{N}, s_2 \geqslant 0$, denote by $H^{s_1,s_2}(M)$ be the Sobolev space with respect to the norm

$$\|f\|^2_{H^{s_1,s_2}(M)} := \|f\|^2_{L^2(M)} + \|\partial_x^{s_1} f\|^2_{L^2(M)} + \||D_y|^{s_2} f\|^2_{L^2(M)}.$$

Note that $H^1_\gamma(M) = [L^2(M), H^2_\gamma(M)]_{\frac{1}{2}}$ and $H^{0,\frac{1}{\gamma+1}}(M) = [L^2(M), H^{0,\frac{2}{\gamma+1}}(M)]_{\frac{1}{2}}$, here $[\mathcal{X},\mathcal{Y}]_\theta$ is the standard notation of interpolation spaces (see Chapter 4 of [Tay11]). By Lemma 3.8, we know that $H^2_\gamma(M) \hookrightarrow H^{0,\frac{2}{\gamma+1}}(M)$. Interpolating with the trivial embedding[4] $L^2(M) \hookrightarrow L^2(M)$, we obtain that $H^1_\gamma(M) \hookrightarrow H^{0,\frac{1}{\gamma+1}}(M)$ is continuous. Moreover, since $\|\partial_x u\|_{L^2(M)} \leqslant \|u\|_{H^1_{\gamma,0}(M)}$, we deduce that $H^1_\gamma(M) \hookrightarrow H^{1,\frac{1}{\gamma+1}}(M)$ is continuous. Thus from the compactness of the embedding $H^{1,\frac{1}{\gamma+1}}(M) \hookrightarrow L^2(M)$, we deduce that $H^1_{\gamma,0}(M) \hookrightarrow L^2(M)$ is compact. □

Proof of Point (1) and (2) of Theorem 3.4. Let $\psi(\rho) := \chi_0((-\rho)^s)$, hence $u_h = \psi(h^2\Delta_\gamma)u_0$. From (3.29), we deduce that for sufficiently small $h_0 > 0$, $0 < h < h_0$ and any $T > T_{\inf}$ (for Point (1) we say that $T_{\inf} = 0$), there holds

$$\|\psi(h^2\Delta_\gamma)u_0\|^2_{L^2(M)} \leqslant C_T \int_0^T \|\phi_1 e^{-it(-\Delta_\gamma)^s} \psi(h^2\Delta_\gamma)u_0\|^2_{L^2(M)} dt, \tag{3.30}$$

where $\mathrm{supp}(\phi_1) \subset \omega$. Taking $h = 2^{-j}$ and summing over the inequality above for $j \geqslant j_0 = \lfloor \log_2 (h_0^{-1}) \rfloor + 1$, by decreasing h_0 if necessary, we will get

$$\|u_0\|^2_{L^2(M)} \leqslant C_T \int_0^T \|e^{-it(-\Delta_\gamma)^s} u_0\|^2_{L^2(\omega)} dt + C_T \|\psi_0(2^{-2j_0}\Delta_\gamma)u_0\|^2_{L^2(M)}, \tag{3.31}$$

for some $\psi_0 \in C_c^\infty(\mathbb{R})$. To see this, we first take $\psi_0 \in C_c^\infty(\mathbb{R})$, equaling to 1 on $(-\frac{1}{2}, 0]$. By the almost orthogonality, we have

$$\|u_0\|^2_{L^2(M)} \leqslant \|\psi_0(2^{-2j_0}\Delta_\gamma)u_0\|^2_{L^2(M)} + C \sum_{j=j_0}^{\infty} \|\psi(2^{-2j}\Delta_\gamma)u_0\|^2_{L^2(M)} \leqslant C\|u_0\|^2_{L^2(M)}.$$

[4]Here we use the complex interpolation theorem, see for example [LP64].

Applying (3.30), we have for each $j \geqslant j_0$,

$$
\begin{aligned}
&\|\psi(2^{-2j}\Delta_\gamma)u_0\|^2_{L^2(M)}\\
\leqslant& C_T\int_0^T \|\psi(2^{-2j}\Delta_\gamma)(\phi_1 e^{-it(-\Delta_\gamma)^s}u_0)\|^2_{L^2(M)}dt + C_T\int_0^T \|[\psi(2^{-2j}\Delta_\gamma),\phi_1]e^{-it(-\Delta_\gamma)^s}u_0\|^2_{L^2(M)}dt\\
\leqslant& C_T\int_0^T \|\psi(2^{-2j}\Delta_\gamma)(\phi_1 e^{-it(-\Delta_\gamma)^s}u_0)\|^2_{L^2(M)}dt + C_T 2^{-2j}\|u_0\|^2_{L^2(M)},
\end{aligned}
$$

where for the last step, we used the symbolic calculus for the commutator $[\psi(2^{-2j}\Delta_\gamma),\phi_1]$ and the fact that $e^{-it(-\Delta_\gamma)^s}$ is unitary on $L^2(M)$. Summing the above inequality over $j \geqslant j_0$, we obtain (3.31), provided that $h_0 > 0$ is chosen smaller so that $C_T h_0^2 = C_T 2^{-2j_0} < \frac{1}{2}$. Note that the second term on the right side of (3.31) can be controlled by $\|u_0\|^2_{H^{-1}_\gamma(M)}$.

To conclude, we follow the approach of Bardos-Lebeau-Rauch [BLR92]. For $T' > 0$, defining the set

$$
\mathcal{N}_{T'} := \left\{u_0 \in L^2(M) : e^{-it(-\Delta_\gamma)^s}u_0|_{[0,T']\times\omega} = 0\right\}
$$

Let $T' \in (T_{\inf}, T)$. (3.31) implies that any function $u_0 \in \mathcal{N}_{T'}$ satisfies

$$
\|u_0\|_{L^2(M)} \leqslant C_T\|u_0\|_{H^{-1}_\gamma(M)}.
$$

Thanks to Lemma 3.22, we deduce that $\dim(\mathcal{N}_{T'}) < \infty$. Note that for any $T_1 < T_2$, $\mathcal{N}_{T_2} \subset \mathcal{N}_{T_1}$. Consider the mapping $S(\delta) := \delta^{-1}\left(e^{-i\delta(-\Delta_\gamma)^s} - \mathrm{Id}\right) : \mathcal{N}_{T'} \to \mathcal{N}_{T'-\delta}$. For fixed $T' \in (T_{\inf}, T)$, when $\delta < T' - T_{\inf}$, $\dim\mathcal{N}_{T'-\delta} < \infty$. Since the dimension is an integer, up to a slight diminution of T', there exists $\delta_0 > 0$, such that for all $0 < \delta < \delta_0$, $\mathcal{N}_{T'-\delta} = \mathcal{N}_{T'}$. Therefore, $S(\delta)$ is a linear map on $\mathcal{N}_{T'}$. Let $\delta \to 0$, we obtain that

$$
-i(-\Delta_\gamma)^s|_{\mathcal{N}_{T'}} : \mathcal{N}_{T'} \to \mathcal{N}_{T'}
$$

is a well-defined linear operator. Take any eigenvalue $\lambda \in \mathbb{C}$ of this operator, and assume that $u_* \in \mathcal{N}_{T'}$ is a corresponding eigenfunction (if it exists). There holds

$$
(-\Delta_\gamma)^s u_* = i\lambda u_*.
$$

This implies that u_* is an eigenfunction of $(-\Delta_\gamma)^s$ (thus u_* is also an eigenfunction of $-\Delta_\gamma$). However, $u_*|_\omega \equiv 0$, hence $u_* \equiv 0$ by the unique continuation property for Δ_γ (see [Gar93]). Therefore, $\mathcal{N}_{T'} = \{0\}$.

Now we choose $T_0 = T'$ as above. By contradiction, assume that Point (1) or Point (2) of Theorem (3.4) is untrue. Then there exists a sequence $(u_{k,0})_{k\in\mathbb{N}}$, such that

$$
\|u_{k,0}\|_{L^2(M)} = 1, \quad \lim_{k\to\infty}\int_0^{T_0}\|e^{-it(-\Delta_\gamma)^s}u_{k,0}\|^2_{L^2(M)}dt = 0.
$$

Up to extraction of a subsequence, we may assume that $u_{k,0} \rightharpoonup u_0$ in $L^2(M)$. Thus from Lemma 3.22, $u_{k,0} \to u_0$, strongly in $H^{-1}_\gamma(M)$. Since $e^{-it(-\Delta_\gamma)^s}u_{k,0} \to 0$ in $L^2([0,T_0]\times\omega)$, we know that $u_0 \in \mathcal{N}_{T_0} = \{0\}$. Besides, letting $k \to \infty$ in (3.31), we obtain $\|u_0\|_{H^{-1}_\gamma(M)} > 0$. This is a contradiction, which concludes the proof of Points (1) and (2) of Theorem 3.4. $\square$

3.4 Theorem 3.4: proofs of non-observability

In this section, we prove the second part of Point (2) of Theorem 3.4, namely that observability fails for small times in case $s = \frac{\gamma+1}{2}$, and Point (3) also follows from this analysis. The proof

is totally different from the proofs of observability presented in Section 3.3. Let us assume $\frac{1}{2}(\gamma+1) = s$.
We note that if $\gamma = 1$, then necessarily $s = 1$, and the result was proved in [BS19]. Therefore, in the sequel, we assume $\gamma > 1$: this will be useful for establishing precise asymptotics of eigenfunctions, see Proposition 3.25.

The non-observability part of Point (2) immediately follows from:

Proposition 3.23. *There exist $T_0 > 0$ and a sequence of solutions $(u_n)_{n\in\mathbb{N}}$ of (3.2) with initial data $(u_n^0)_{n\in\mathbb{N}}$ such that $\|u_n^0\|_{L^2(M)} = 1$ and*

$$\int_0^{T_0} \int_\omega |u_n(t,x,y)|^2 dxdydt \underset{n\to+\infty}{\longrightarrow} 0. \tag{3.32}$$

The goal of this section is to prove Proposition 3.23. In all the sequel, using the invariance by y-translation, we assume without lost of generality that $\mathbb{T}_y \setminus I$ contains a neighborhood of 0.

Here is a sketch of the proof, which borrows ideas from [BS19, Section 9]:

- We can reduce the analysis to the construction of solutions of $i\partial_t u - (-\Delta_\gamma)^s u = 0$ in $\mathbb{R}\times\mathbb{T}$: then, using an appropriate cut-off, we transplant it into solutions of (3.2) (thus in $(-1,1)\times\mathbb{T}$).
- In $\mathbb{R}\times\mathbb{T}$ and for $\eta\in\mathbb{R}$, we consider as initial datum the normalized ground state $|\eta|^{\frac{1}{2(\gamma+1)}}\phi_\gamma(|\eta|^{\frac{1}{\gamma+1}}x)$ of the operator $D_x^2 + |\eta|^2|x|^{2\gamma}$, mutiplied by $e^{iy\eta}$. The associated solution of $i\partial_t u - (-\Delta_\gamma)^s u = 0$ is obtained by mutiplication by a phase, and the intuition is that this solution has all its energy concentrated near $x = 0$ when η is large: it is analoguous to the "degenerate regime" of Section 3.2.2. Taking linear combinations of these solutions for large η's (this is the role of the multiplication by $\psi(h_n k)$ in (3.34)), we obtain a solution which travels along the y-axis at finite speed.

Let us now start the proof. The normalized ground state of the operator $P_{\gamma,w} = -\partial_x^2 + |x|^{2\gamma}w^2$ on $\mathbb{R}_x$ is denoted by $p_\gamma(w,\cdot)$ and the associated eigenvalue is $\lambda_\gamma(w)$. We set $z = |w|^{\frac{1}{\gamma+1}}x$, and we are then left to study the operator $Q_\gamma = -\partial_z^2 + |z|^{2\gamma}$ on $\mathbb{R}_z$. Recall that its normalized ground state is ϕ_γ which satisfies

$$Q_\gamma\phi_\gamma = \mu_0\phi_\gamma$$

on $\mathbb{R}_z$. In particular, we have $\lambda_\gamma(w) = \mu_0|w|^{\frac{2}{\gamma+1}}$ and

$$p_\gamma(w,x) = |w|^{\frac{1}{2(\gamma+1)}}\phi_\gamma(|w|^{\frac{1}{\gamma+1}}x).$$

Definition 3.24. *We write $f(x) = \widetilde{O}(g(x))$ as $x\to+\infty$ if for any $k\in\mathbb{N}$,*

$$|f^{(k)}(x)| = O(|g^{(k)}(x)|), \quad x\to\infty.$$

We need the following estimate, which is specific to the case $\gamma > 1$:

Proposition 3.25. *We consider the ground state*

$$-\phi_\gamma'' + |z|^{2\gamma}\phi_\gamma = \mu_0\phi_\gamma, \quad \phi_\gamma(x) > 0, \quad \phi_\gamma \text{ even}, \quad \|\phi_\gamma\|_{L^2(\mathbb{R})} = 1.$$

Then, for some constant $c_\gamma\in\mathbb{R}$ we have the asymptotic behavior

$$\phi_\gamma(x) \sim \frac{c_\gamma}{x^{\frac{\gamma}{2}}} e^{-\frac{x^{\gamma+1}}{\gamma+1}}, \quad x\to\infty, \tag{3.33}$$

and $\phi_\gamma = \widetilde{O}(x^{-\frac{\gamma}{2}}e^{-\frac{x^{\gamma+1}}{\gamma+1}})$.

Proposition 3.25 is proved in Section 3.4.4 below, but let us first explain how to deduce Proposition 3.23 from these estimates.

3.4.1 Estimate of the source term

We set $h_n = 2^{-n}$ and we consider

$$v_n(t,x,y) = \sum_{k\in\mathbb{Z}} \psi(h_n k) e^{iyk - it\mu_0^s |k|^{\frac{2s}{\gamma+1}}} |k|^{\frac{1}{2(\gamma+1)}} \phi_\gamma(|k|^{\frac{1}{\gamma+1}} x), \tag{3.34}$$

where $\psi \in C_c^\infty(\frac{1}{2} \leqslant \eta \leqslant 1)$, and ϕ_γ is the first normalized eigenfunction of the operator $-\partial_x^2 + |x|^{2\gamma}$ on $L^2(\mathbb{R}_x)$ with the eigenvalue $\mu_0 > 0$. Then v_n satisfies

$$i\partial_t v_n - (-\Delta_\gamma)^s v_n = 0$$

on $\mathbb{R}_x \times \mathbb{T}_y$.

We consider a cut-off $\chi \in C_c^\infty(\mathbb{R}_x)$ with $\chi = 1$ for $|x| \leqslant 1/4$ and $\chi(x) = 0$ for $|x| \geqslant 1/2$. Let $u_n = \chi v_n$, then

$$i\partial_t u_n - (-\Delta_\gamma)^s u_n = -[(-\Delta_\gamma)^s, \chi] v_n.$$

Our first goal is to show that

$$f_n := [(-\Delta_\gamma)^s, \chi] v_n \underset{n\to+\infty}{\longrightarrow} 0, \tag{3.35}$$

in $L^2_{t,x,y}$ as $n \to +\infty$, uniformly in $t \in [0, T_0]$.

We write

$$[(-\Delta_\gamma)^s, \chi] = \sum_{j=0}^{s-1} (-\Delta_\gamma)^j [-\Delta_\gamma, \chi] (-\Delta_\gamma)^{s-j-1} \tag{3.36}$$

and we note that

$$[-\Delta_\gamma, \chi] = -2\chi' \partial_x - \chi''.$$

Let us fix $0 \leqslant j \leqslant s-1$ and focus on the term indexed by j in (3.36). We know that

$$[-\Delta_\gamma, \chi](-\Delta_\gamma)^{s-j-1} v_n(t,x,y) = \sum_{k\in\mathbb{Z}} \left(-2|k|^{\frac{1}{\gamma+1}} \phi_\gamma'(|k|^{\frac{1}{\gamma+1}} x) \chi'(x) - \phi_\gamma(|k|^{\frac{1}{\gamma+1}} x) \chi''(x) \right) \theta_n(t,y,k), \tag{3.37}$$

with

$$\theta_n(t,n,k) = \psi(h_n k) e^{iky - it\mu_0^s |k|^{\frac{2s}{\gamma+1}}} (\mu_0 |k|^{\frac{2}{\gamma+1}})^{s-j-1} |k|^{\frac{1}{2(\gamma+1)}}.$$

Now we have to take j times $(-\Delta_\gamma)$ on the left of the above expression. For that, we determine the size of the new factors brought by any new ∂_x or $|x|^\gamma \partial_y$ derivative. Indeed, we see that $(-\Delta_\gamma)^j [-\Delta_\gamma, \chi](-\Delta_\gamma)^{s-j-1} v_n$ is a sum of terms of the form

$$I^{n,j}_{j_1,j_2,j_3,j_4}(t,x,y) := \sum_{k\in\mathbb{Z}} |k|^{j_1} \phi_\gamma^{(j_2)}(|k|^{\frac{1}{\gamma+1}} x) \chi^{(j_3)}(x) (|x|^\gamma \partial_y)^{j_4} \theta_n(t,y,k)$$

with $j_1, j_2, j_3, j_4 \geqslant 0$ bounded above by a constant which only depends on s. We also notice that necessarily $j_3 \geqslant 1$, so that, with the properties of χ, $I^{n,j}_{j_1,j_2,j_3,j_4}(t,x,y) = 0$ for $|x| < 1/4$. Therefore, we can assume $|x| \geqslant 1/4$. Because of the term $\psi(h_n k)$ in $\theta_n(t,y,k)$, the sum in the definition of $I^{n,j}_{j_1,j_2,j_3,j_4}$ can be taken only over $k \in (h_n^{-1}/2, h_n^{-1})$. Now, using the profile of $\phi_\gamma^{(j_2)}$ given by Proposition 3.25, we see that $I^{n,j}_{j_1,j_2,j_3,j_4} = o(1)$ as $n \to +\infty$. Therefore, (3.35) holds.

By Duhamel's formula, we then have, for fixed $T_0 > 0$,

$$\| u_n(t) - e^{-it(-\Delta_\gamma)^s} (\chi v_n(0)) \|_{L^2_{x,y}} \underset{n\to+\infty}{\longrightarrow} 0$$

uniformly in $t \in [0, T_0]$. Therefore, now considering u_n as a function on $(-1,1)_x \times \mathbb{T}_y$, we see that (3.32) holds if and only if

$$\int_0^{T_0} \int_{\mathbb{R}\times I} |v_n(t,x,y)|^2 dxdydt \underset{n\to+\infty}{\longrightarrow} 0. \tag{3.38}$$

3.4.2 Proof of (3.38)

Recall that $\mathbb{T}_y \setminus I$ is assumed to contain a neighborhood of 0. We prove that for any $c > 0$, there exists $T_0 > 0$ such that $\|v_n \mathbf{1}_{|y|\geqslant c}\|_{L^2((0,T_0)\times\mathbb{R}_x\times\mathbb{T}_y)} \underset{n\to+\infty}{\longrightarrow} 0$, which implies (3.38). We consider the phase

$$\Phi_m(t,y,w) = wy - \lambda_\gamma(w)^s t - 2\pi m w.$$

By the Poisson formula,

$$v_n(t,x,y) = \sum_{m\in\mathbb{Z}} \widehat{K^{(n)}_{t,x,y}}(2\pi m)$$

where

$$\widehat{K^{(n)}_{t,x,y}}(2\pi m) = \int_{\mathbb{R}} \psi(h_n w) p_\gamma(w,x) e^{i\Phi_m(t,y,w)} dw \tag{3.39}$$

The goal is to prove that for $|y| \geqslant c$, each $\widehat{K^{(n)}_{t,x,y}}(2\pi m)$ is small; therefore v_n is also small for y outside a neighborhood of 0.

We do the usual integration by part argument, writing

$$e^{i\Phi_m(w)} = \frac{1}{i\partial_w \Phi_m} \frac{\partial}{\partial w} e^{i\Phi_m}. \tag{3.40}$$

Here, using $\lambda_\gamma(w) = \mu_0 |w|^{\frac{2}{\gamma+1}}$ and $s = \frac{\gamma+1}{2}$, we find

$$\partial_w \Phi_m(t,y,w) = y - 2\pi m - t\mu_0^s,$$

(for $w > 0$) and in particular $\partial_w^2 \Phi_m = 0$ (for $w > 0$). Using (3.40), we integrate by parts three times in (3.39):

$$\widehat{K^{(n)}_{t,x,y}}(2\pi m) = \frac{1}{i} \int_{\mathbb{R}} \frac{\partial_w^3(\psi(h_n w) p_\gamma(w,x))}{|\partial_w \Phi_m(t,y,w)|^3} e^{i\Phi_m} dw. \tag{3.41}$$

There is a $\partial_w \Phi_m$ at the denominator, for which we need an estimate. We assume without loss of generality that $I \subsetneq \mathbb{T}_y$ is an interval, which we denote by (a,b), with $0 < a < b < \pi$. Let us fix $T_0 < a/\mu_0^s$. We see that $|\partial_w \Phi_m(t,y,w)|$ is bounded away from 0 uniformly in $y \in I$ and $0 \leqslant t \leqslant T_0$. Moreover $|m| \lesssim |\partial_w \Phi_m(t,y,w)|$ when $m \to +\infty$, uniformly in $w \in \mathbb{R}$, $y \in I$ and $0 \leqslant t \leqslant T_0$. We write $|\partial_w \Phi_m(w)| \gtrsim |m - c_0|$ for some $0 < c_0 < 1$ which does not depend on m.

Let us now analyze (3.41). Since on the support of $\psi(h_n w)$, $|w| \sim h_n^{-1}$, the main contribution of $\partial_w^3(\psi(h_n w) p_\gamma(w,x))$ comes from the situation where every derivative falls on the factor $\phi_\gamma(|w|^{\frac{1}{\gamma+1}} x)$, thus bounded by

$$O(|w|^{\frac{1}{2(\gamma+1)} + \frac{3}{\gamma+1} - 3}) = O(h_n^{\frac{3\gamma}{\gamma+1} - \frac{1}{2(\gamma+1)}}), \quad |w| \sim h_n^{-1}.$$

Therefore, we obtain that

$$\sup_{(t,x,y)\in(0,T)\times\omega} |\widehat{K^{(n)}_{t,x,y}}(2\pi m)| \leqslant \frac{C h_n^{\frac{3\gamma}{\gamma+1} - \frac{1}{2(\gamma+1)} - 1}}{|m - c_0|^3}.$$

Hence, the sum over m of the $|\widehat{K^{(n)}_{t,x,y}}(2\pi m)|$ is $O(h_n^{\frac{4\gamma-3}{2(\gamma+1)}})$[5]. It gives (3.38), since $\gamma \geqslant 1$.

Remark 3.26. Note that this proof provides the lower bound $T_{\inf} \geqslant a/\mu_0^s$.

[5] Writing similar relations as (3.40), but at higher order, and then integrating by part sufficiently many times in (3.41), we can obtain better bounds $O(h_n^N)$ for any $N \in \mathbb{N}$.

3.4.3 End of the proof of Proposition 3.23

We finally need to estimate the size of the initial data.

Lemma 3.27. *There exists $c > 0$ such that $\|v_{n,0}\|_{L^2(M)} \geqslant c$ for any $n \in \mathbb{N}$.*

Proof of Lemma 3.27. By Plancherel (used for fixed $x \in \mathbb{R}$), we have

$$\begin{aligned}\|v_{n,0}\|_{L^2}^2 &= \sum_{k\in\mathbb{Z}} \int_{\mathbb{R}} |\psi(h_n k)|^2 |k|^{\frac{1}{\gamma+1}} |\phi_\gamma(|k|^{\frac{1}{\gamma+1}} x)|^2 dx \\ &= \sum_{k\in\mathbb{Z}} |\psi(h_n k)|^2 \gtrsim 1,\end{aligned}$$

hence the conclusion. □

Combining Lemma 3.27 and (3.38), we get Proposition 3.23, and the non-observability part of Point (2) of Theorem 3.1 follows. Point (3) then follows immediately from the abstract result [Mil12, Corollary 3.9]: if (3.2) was observable for some $T > 0$ and some $s < \frac{\gamma+1}{2}$, then it would be observable in any time for $s = \frac{\gamma+1}{2}$, which is not the case thanks to the non-observability part of Point (2).

Remark 3.28. Note that it would be possible to obtain Point (3) by a similar construction as the one of Section 3.4.2: if $s < \frac{\gamma+1}{2}$, the phase $\partial_w \Phi_m$ verifies an estimate of the form $\partial_w \Phi_m = y - 2\pi m + O(h_n^{1-\frac{2s}{\gamma+1}} T_0)$, and, since $h_n^{1-\frac{2s}{\gamma+1}} T_0$ tends to 0 in any case as $n \to +\infty$, an analysis similar to the above one shows that observability fails for any $T_0 > 0$.

Remark 3.29. The proof of Proposition 3.23 is adapted from the vertical Gaussian-beam like construction of [BS19] and this strategy was inspired by [RS20] for the controllability of the Kadomtsev-Petviashvili equation. Since s is a natural number, our construction here simplifies the analysis of Section 9 in [BS19], without appealing to the properties of first eigenfunctions of the semi-classical generalized harmonic oscillators $-\partial_x^2 + n^2|x|^{2\gamma}$ with Dirichlet boundary conditions. When s is fractional, we do not have the nice formulas (3.36) and (3.37), due to the non-local feature, and the analysis will be considerably more involved. Nevertheless, we believe that it is possible to handle a more precise analysis as in Section 9 of [BS19] to prove Point (3) for general $s > 0$, not necessarily in $\mathbb{N}$.

Remark 3.30. It might be possible to generalize Proposition 3.23 to a more general setting thanks to a normal form procedure. By normal form, we mean that a complicated sub-Laplacian can sometimes be (micro)-locally conjugated (by a Fourier Integral Operator) to a simpler one, see [CHT18, Theorem 5.2] for the example of 3D contact sub-Laplacians. Since in the above proof of Point (3) the constructed sequence of solutions stays localized around a single fixed point of the manifold, we could hope to disprove observability for equations involving sub-Laplacians which are microlocally conjugated to $-\Delta_\gamma$.

3.4.4 Proof of Proposition 3.25

Our proof is inspired by [Si70, Appendix IV]. Note that we are only interested in the region $x\gamma 1$. Let $Y = \binom{\psi}{\psi'}$, and

$$A = \begin{pmatrix} 0 & 1 \\ |x|^{2\gamma} - \mu_0 & 0 \end{pmatrix},$$

hence $Q_\gamma \psi = \mu_0 \psi$ is equivalent to $Y' = AY$. We set

$$\phi_-(x) = x^{-\gamma/2} e^{-\frac{x^{\gamma+1}}{\gamma+1}}, \quad \phi_+(x) = x^{-\gamma/2} e^{\frac{x^{\gamma+1}}{\gamma+1}}.$$

We compute

$$\phi_-'(x) = -(x^{\frac{\gamma}{2}} + \frac{\gamma}{2} x^{-\frac{\gamma}{2}-1}) e^{-\frac{x^{\gamma+1}}{\gamma+1}}, \quad \phi_+'(x) = (x^{\frac{\gamma}{2}} - \frac{\gamma}{2} x^{-\frac{\gamma}{2}-1}) e^{\frac{x^{\gamma+1}}{\gamma+1}},$$

$$\phi_-''(x) = (x^{\frac{3\gamma}{2}} + \frac{\gamma}{2}(\frac{\gamma}{2}+1) x^{-\frac{\gamma}{2}-2}) e^{-\frac{x^{\gamma+1}}{\gamma+1}}, \quad \phi_+''(x) = (x^{\frac{3\gamma}{2}} + \frac{\gamma}{2}(\frac{\gamma}{2}+1) x^{-\frac{\gamma}{2}-2}) e^{\frac{x^{\gamma+1}}{\gamma+1}}.$$

These two functions can be viewed as approximate solutions, as $x \to +\infty$, to

$$L\psi := -\psi'' + (x^{2\gamma} - \mu_0)\psi = 0$$

and we will give an expression of ϕ_γ in terms of ϕ_- and ϕ_+, which will imply (3.33). Let

$$U = \begin{pmatrix} \phi_- & \phi_+ \\ \phi_-' & \phi_+' \end{pmatrix}$$

and $a = \binom{a_-}{a_+} := U^{-1} Y$, or equivalently,

$$\psi(x) = a_-(x)\phi_-(x) + a_+(x)\phi_+(x), \quad \psi'(x) = a_-(x)\phi_-'(x) + a_+(x)\phi_+'(x).$$

We remark that the inverse of U exists since $\det(U) = \phi_+'\phi_- - \phi_-'\phi_+ = 2$ and is given by

$$U^{-1} = \frac{1}{\det(U)} \begin{pmatrix} \phi_+' & -\phi_+ \\ -\phi_-' & \phi_- \end{pmatrix}.$$

We set the ansatz $Y = Ua$, hence $L\psi = 0$ is equivalent to

$$a' = -Ra,$$

where

$$R = U^{-1}(U'U^{-1} - A)U = U^{-1} \begin{pmatrix} 0 & 0 \\ \mu_0 + \frac{\gamma}{2}(\frac{\gamma}{2}+1)x^{-2} & 0 \end{pmatrix} U$$

i.e.,

$$R = \frac{\mu_0 + \frac{\gamma}{2}(\frac{\gamma}{2}+1)x^{-2}}{x^\gamma} \begin{pmatrix} -1 & -e^{\frac{2x^{\gamma+1}}{\gamma+1}} \\ e^{-\frac{2x^{\gamma+1}}{\gamma+1}} & 1 \end{pmatrix}.$$

To solve $a' = -Ra$, we expand the Neumann series as

$$a(x) = \sum_{n=0}^{\infty} a_n(x), \quad a_n = \begin{pmatrix} a_{n,-}(x) \\ a_{n,+}(x) \end{pmatrix}.$$

where

$$a_{n+1}(x) = \int_x^\infty R(z) a_n(z) dz,$$

provided that the series and the integration converge. In order to avoid the divergence at $x = +\infty$, we initially choose

$$a_0(x) = \begin{pmatrix} a_{0,-} \\ 0 \end{pmatrix},$$

where we can set $a_{0,-} = 1$ is a constant. It turns out that the Neumann series $a = \sum_{n=0}^{\infty} a_n$ converges to a smooth function a. Hence $Y = Ua$ is the solution of $Y' = AY$ which tends to 0 as $x \to +\infty$.

Lemma 3.31. *There holds*

$$a_-(x) - 1 = e^{\frac{2x^{\gamma+1}}{\gamma+1}} \widetilde{O}(\frac{1}{x^{\gamma-1}} e^{-\frac{2x^{\gamma+1}}{\gamma+1}}), \quad a_+(x) = \widetilde{O}(\frac{1}{x^{\gamma-1}} e^{-\frac{2x^{\gamma+1}}{\gamma+1}}).$$

Proof. It follows from a simple recurrence that there exist $C > 0$ and some (large) $x_0 > 0$ such that for any $n \in \mathbb{N}$ and any $x \geqslant x_0$, we have

$$|a_{n,-}(x)| \leqslant \frac{C\mu_0^n}{x^{n(\gamma-1)}}, \qquad |a_{n,+}(x)| \leqslant \frac{C\mu_0^n}{x^{n(\gamma-1)}} e^{-2\frac{x^{\gamma+1}}{\gamma+1}}$$

It follows that $a_-(x) - 1 = O(1/x^{\gamma-1})$ and $a_+(x) = O(e^{-\frac{2x^{\gamma+1}}{\gamma+1}}/x^{\gamma-1})$. Then, the estimates on the derivatives of a_- and a_+ follow from a recurrence using the relation $a' = -Ra$. □

Thus we have constructed an explicit solution

$$\psi_\infty(x) := a_-(x)\phi_-(x) + a_+(x)\phi_+(x)$$

with the asymptotic behavior

$$\psi_\infty(x) \sim x^{-\frac{\gamma}{2}} e^{-\frac{x^{\gamma+1}}{\gamma+1}}, \quad x \to +\infty$$

and $\psi_\infty = \widetilde{O}(x^{-\frac{\gamma}{2}} e^{-\frac{x^{\gamma+1}}{\gamma+1}})$.

Note that the Wronskian of the equation $L\psi = 0$ is constant (so we can choose it to be 1), so we find another independent solution (with some $x_0 \gamma 1$ fixed)

$$\psi_{-\infty}(x) := \psi_\infty(x) \int_{x_0}^x \frac{dz}{(\psi_\infty(z))^2} \sim x^{-\frac{\gamma}{2}} e^{\frac{x^{\gamma+1}}{\gamma+1}}.$$

Now the fundamental solution $\phi_\gamma(x)$ should be a linear combination of $\psi_\infty, \psi_{-\infty}$, namely, there exist constants $a, b \in \mathbb{R}$ such that

$$\phi_\gamma(x) = a\psi_\infty(x) + b\psi_{-\infty}(x)$$

for all large $x > x_0$ (this identity is only valid for large $x > 0$). Since $\phi_\gamma(x) \to 0$ as $x \to +\infty$, we must have $b = 0$, which finishes the proof.

3-A Supplementary material

3-A.1 Proof of the well-posedness

We intend to prove the well-posedness of (3.2), (3.6) and (3.7).

Schrödinger equation

The equation (3.2) can be solved by spectral theory. Expanding the initial datum $u_0(x, y)$ as

$$u_0(x, y) = \sum_{j\in\mathbb{N}} a_j \varphi_j(x, y), \qquad \text{with } -\Delta_\gamma \varphi_j = \lambda_j^2 \varphi_j, \tag{3.42}$$

the solution of (3.2) is given by

$$(e^{-it(-\Delta_\gamma)^s} u_0)(t, x, y) = \sum_{j\in\mathbb{N}} a_j e^{-it\lambda_j^{2s}} \varphi_j(x, y),$$

which belongs to $L^2(M)$ for any $t \in \mathbb{R}$.

Let us now prove (3.4). For each N, we set

$$u_N = \sum_{j \leqslant N} (u, \varphi_j)\varphi_j.$$

Then, $(-\Delta_\gamma)^k u_N|_{\partial M} = 0$ for all $k \geqslant 0$. When $k \leqslant s$,

$$(-\Delta_\gamma)^k u_N = \sum_{j \leqslant N} \lambda_j^{2k} (u, \varphi_j)\varphi_j$$

converges uniformly in $H_\gamma^{2(s-k)}(M)$ to u. When $k < s - \frac{1}{4}$, since this is equivalent to $2(s-k) > \frac{1}{2}$, $(-\Delta_\gamma)^k u_N|_{\partial M}$ converges in $L^2(\partial M)$ by trace theorem[6]. In particular, we have $(-\Delta_\gamma)^k u|_{\partial M} = 0$. Note that when $s = \frac{k_0}{2} \in \frac{1}{2}\mathbb{N}$, $0 \leqslant k < s - \frac{1}{4}$ is equivalent to $0 \leqslant k \leqslant \lfloor \frac{k_0 - 1}{2} \rfloor$.

Heat equation

To prove the well-posedness in $L^2(M)$, we will apply the Hille-Yosida theorem with generator $\widetilde{\mathcal{A}} = -(-\Delta_\gamma)^s$. The domain $D(\widetilde{\mathcal{A}})$ is given by (3.3), and it is dense in $L^2(M)$. For $u_0 \in D(\widetilde{\mathcal{A}})$, written as in (3.42), there holds

$$\operatorname{Re}(\langle \widetilde{\mathcal{A}} u_0, u_0 \rangle_{L^2(M)}) = -\sum_{j \in \mathbb{N}} |a_j|^2 \lambda_j^{2s} \|\varphi_j\|_{L^2(M)}^2 \leqslant 0,$$

hence $\widetilde{\mathcal{A}}$ is dissipative. Let us show that it is maximally dissipative, i.e., $\mathrm{Id} - \mu\widetilde{\mathcal{A}}$ is surjective for any $\mu > 0$. Let u_0 as in (3.42) and $\mu > 0$. We consider

$$u = \sum_{j \in \mathbb{N}} \frac{a_j}{1 + \mu \lambda_j^{2s}} \varphi_j.$$

Then $u \in L^2(M)$ and $(\mathrm{Id} - \mu\widetilde{\mathcal{A}})u = u_0$. Therefore, by the Hille-Yosida theorem, $\widetilde{\mathcal{A}}$ generates a strongly continuous semigroup of contraction, and in particular (3.6) is well-posed.

Damped wave equation

Consider the damped wave equation

$$\partial_t^2 u - \Delta_\gamma u + b\partial_t u = 0$$

where $b \in L^\infty(M)$ and $b \geqslant 0$. For its well-posedness in the energy space $\mathcal{H} = H^1_{\gamma,0}(M) \times L^2(M)$, we will apply the Hille-Yosida theorem to prove the existence and uniqueness of the semi-group $e^{t\mathcal{A}}$ with generator

$$\mathcal{A} = \begin{pmatrix} 0 & 1 \\ \Delta_\gamma & -b \end{pmatrix}.$$

We need to check the condition that $\mathcal{A}$ is maximally dissipative, which we formulate this time under the form

(a) $(0, \infty) \subset \rho(\mathcal{A})$;

(b) $\|(\mu \mathrm{Id} - \mathcal{A})^{-1}\|_{\mathcal{L}(\mathcal{H})} \leqslant \mu^{-1}$, for any $\mu > 0$.

[6] Though H_γ^s is not the usual Sobolev space, the usual trace theorem applies since near the boundary, $-\Delta_\gamma$ is uniformly elliptic.

Indeed, (a) is proved in the beginning of the proof of Corollary 3.33. We only need to check (b). Let $U=(u,v)^t$ and $F=(u,v)^t$ such that $(\mu-\mathcal{A})U=F$. Equipped with the inner product

$$\big((u_1,v_1),(u_2,v_2)\big)_{\mathcal{H}}:=(\nabla_\gamma u_1,\nabla_\gamma u_2)_{L^2(M)}+(v_1,v_2)_{L^2(M)},$$

we verify directly that

$$\mathrm{Re}\big(\mathcal{A}U,U\big)_{\mathcal{H}}=-(bv,v)_{L^2(M)}\leqslant 0.$$

Therefore,

$$\mu\|U\|_{\mathcal{H}}^2\leqslant\mu(U,U)_{\mathcal{H}}-\mathrm{Re}(\mathcal{A}U,U)_{\mathcal{H}}=\mathrm{Re}((\mu\mathrm{Id}-\mathcal{A})U,U)_{\mathcal{H}}\leqslant\|U\|_{\mathcal{H}}\|(\mu\mathrm{Id}-\mathcal{A})U\|_{\mathcal{H}}.$$

This means that $\mu\|(\mu\mathrm{Id}-\mathcal{A})^{-1}F\|_{\mathcal{H}}\leqslant\|F\|_{\mathcal{H}}$. Therefore, (b) is verified. The proof of well-posedness for the damped wave equation is then complete.

3-A.2 Proof of Corollary 3.6

Recall that $\gamma\geqslant 1$ is fixed. Given $b\in L^\infty(M)$, $b\geqslant 0$, consider the damped wave equation

$$\partial_t^2u-\Delta_\gamma u+b\partial_t u=0$$

which can be written as $\partial_t U=\mathcal{A}U$ with $U=(u,\partial_t u)^t$ and

$$\mathcal{A}=\begin{pmatrix}0 & 1\\ \Delta_\gamma & -b\end{pmatrix}.$$

Let $\mathcal{H}:=H_{0,\gamma}^1(M)\times L^2(M)$ and H_γ^{-1} be the dual of $H_{0,\gamma}^1(M)$. When $b=\mathbf{1}_\omega$, we have a stronger version of Theorem 3.1:

Proposition 3.32. *There exist $C,h_0>0$, such that for all $0<h\leqslant h_0$, and any solution v of*

$$(h^2\Delta_\gamma+1)v=g_1+g_2,$$

with $g_1\in L^2(M),g_2\in H_\gamma^{-1}$, we have

$$\|h\nabla_\gamma v\|_{L^2(M)}+\|v\|_{L^2(M)}\leqslant C\|v\mathbf{1}_\omega\|_{L^2(M)}+\frac{C}{h^{\gamma+1}}\|g_1\|_{L^2(M)}+\frac{C}{h^{\gamma+2}}\|g_2\|_{H_\gamma^{-1}(M)}.$$

Proof. Let $P_h=-h^2\Delta_\gamma-1+ih^{\gamma+1}$. We first show that P_h is invertible. Note that for $v\in D(\Delta_\gamma)$, we have

$$(P_hv,v)_{L^2(M)}=\|h\nabla_\gamma v\|_{L^2(M)}^2-\|v\|_{L^2(M)}^2+ih^{\gamma+1}(bv,v)_{L^2(M)}.$$

Taking the imaginary part of the identity above, we have (using $b^2=b$)

$$\|bv\|_{L^2(M)}^2\leqslant h^{-(\gamma+1)}|\mathrm{Im}(P_hv,v)_{L^2(M)}|. \tag{3.43}$$

Taking the real part of the identity and inserting Theorem 3.1, we have

$$\begin{aligned}\|h\nabla_\gamma v\|_{L^2(M)}^2+\|v\|_{L^2(M)}^2&\leqslant 2\|v\|_{L^2(M)}^2+|\mathrm{Re}(P_hv,v)_{L^2(M)}|\\&\leqslant C\|bv\|_{L^2(M)}^2+Ch^{-2(\gamma+1)}\|P_hv\|_{L^2(M)}^2+\|P_hv\|_{L^2(M)}\|v\|_{L^2(M)}.\end{aligned}$$

Applying Young's inequality and (3.43), we have

$$\|h\nabla_\gamma v\|_{L^2(M)}^2+\|v\|_{L^2(M)}^2\leqslant Ch^{-2(\gamma+1)}\|P_hv\|_{L^2(M)}^2.$$

This implies that P_h is invertible and

$$P_h^{-1} = O(h^{-(\gamma+1)}) : L^2(M) \to L^2(M), \qquad P_h^{-1} = O(h^{-(\gamma+2)}) : L^2(M) \to H^1_{\gamma,0}(M).$$

Now if $(h^2\Delta_\gamma + 1)v = g_1 + g_2$, for any $w \in L^2(M)$, let $z = P_h^{-1}w$, and we have

$$\begin{aligned}(v, w)_{L^2(M)} =&(v, P_h z)_{L^2(M)} = (P_h v, z)_{L^2(M)} = (ih^{\gamma+1}b - g_1 - g_2, z)_{L^2(M)} \\ \leqslant&\|ih^{\gamma+1}b - g_1\|_{L^2(M)}\|z\|_{L^2(M)} + \|g_2\|_{H_\gamma^{-1}}\|z\|_{H^1_{\gamma,0}} \\ \leqslant&Ch^{-(\gamma+1)}\|ih^{\gamma+1}b - g_1\|_{L^2(M)}\|w\|_{L^2(M)} + Ch^{-(\gamma+2)}\|g_2\|_{H_\gamma^{-1}}\|w\|_{L^2(M)}.\end{aligned}$$

Since $w \in L^2(M)$ is arbitrary, by duality, we complete the proof of Proposition 3.32 □

Consequently, the following resolvent estimate for the damped wave equation holds:

Corollary 3.33. *We have $i\mathbb{R} \subset \rho(\mathcal{A})$ and there exists $\lambda_0 \geqslant 1$, such that for every $\lambda \in \mathbb{R}$, $|\lambda| \geqslant \lambda_0$,*

$$\|(i\lambda \mathrm{Id} - \mathcal{A})^{-1}\|_{\mathcal{L}(\mathcal{H})} \leqslant C|\lambda|^{2\gamma}. \tag{3.44}$$

Proof of Corollary 3.33 from Theorem 3.32. We show that $i\mathbb{R} \subset \rho(\mathcal{A})$. This consists of two steps. First, we prove that $\mu \in \rho(\mathcal{A})$ for all $\mu > 0$. Let $U = (u, v)^t$ and $F = (f, g)^t$, then

$$(\mu \mathrm{Id} - \mathcal{A})U = F$$

is equivalent to

$$\begin{cases} \mu u - v = f \\ -\Delta_\gamma u + \mu v + bv = g, \end{cases} \tag{3.45}$$

hence u satisfies the equation

$$-\Delta_\gamma u + (\mu b + \mu^2)u = g + (b + \mu)f. \tag{3.46}$$

Consider the bilinear form on $H^1_{0,\gamma}$:

$$B_\mu[u, v] := \mathrm{Re}(-\Delta_\gamma u + (\mu b + \mu^2)u, v)_{L^2(M)} = \mathrm{Re}\left((\nabla_\gamma u, \nabla_\gamma v)_{L^2(M)} + \mu^2(u, v)_{L^2(M)} + \mu(bu, v)_{L^2(M)}\right)$$

which is coercive for all $\mu > 0$. By Lax-Milgram, given $(f, g) \in \mathcal{H}$, (3.46) posseses a unique solution $u \in H^1_{0,\gamma}$, and setting $v = \mu u - f$, we obtain a solution $(u, v) \in \mathcal{H}$ of (3.45). Hence $\mu \in \rho(\mathcal{A})$. Moreover, we claim that $(\mathrm{Id} - \mathcal{A})^{-1}$ is compact. Indeed, from the equation of u, we deduce that $u \in H^2_\gamma(M)$. Since $v = \mu u - f$, we then deduce that $v \in H^1_{\gamma,0}(M)$. Now the compactness of $(\mathrm{Id} - \mathcal{A})^{-1}$ comes from the fact that the embedding $H^{k+1}_\gamma(M) \hookrightarrow H^k_\gamma(M)$ is compact (which we only need for $k = 0, 1$).

Now for any $z \in \mathbb{C}$, we write

$$z - \mathcal{A} = (\mathrm{Id} + (1 - z)(\mathcal{A} - \mathrm{Id})^{-1})(\mathrm{Id} - \mathcal{A}),$$

since $\mathrm{Id} + (1 - z)(\mathcal{A} - \mathrm{Id})^{-1}$ is Fredholm with index 0, we deduce that $z - \mathcal{A}$ is invertible (i.e. $z \in \rho(\mathcal{A})$) if and only if it is injective. To prove that $i\lambda - \mathcal{A}$ is injective for all $\lambda \in \mathbb{R}$, it suffices to show that any solution u of

$$-\Delta_\gamma u - \lambda^2 u + i\lambda bu = 0$$

is zero. Multiplying by $\overline{u}$, doing the integration by part and taking the imaginary part, we have

$$(bu,u)_{L^2}=0.$$

Since $b=\mathbf{1}_\omega$, we have $bu=0$ a.e., hence we deduce that u is an eigenfunction of $-\Delta_\gamma$ which vanishes on ω. By the unique continuation property of $-\Delta_\gamma$ (see [Gar93]), we deduce that $u\equiv 0$. This proves that $i\mathbb{R}\subset\rho(\mathcal{A})$.

It remains to prove (3.44) for large λ. Without loss of generality, we assume that $\lambda\geqslant 1$. Let $U=(u,v)^t\in\mathcal{H}$ and $F=(f,g)^t\in\mathcal{H}$ such that $(i\lambda-\mathcal{A})U=F$. Equivalently, with $h=\lambda^{-1}$,

$$\begin{cases} u=-ih(v+f),\\ (h^2\Delta_\gamma+1)v=ihbv-ihg-h^2\Delta_\gamma f.\end{cases}$$

Applying Theorem 3.32 to v and $g_1=ihg+ihbv, g_2=h^2\Delta_\gamma f$, we have

$$\begin{aligned}\|v\|_{L^2}\leqslant& C\|b^{\frac12}v\|_{L^2}+Ch^{-(\gamma+1)}\|ihbv-ihg\|_{L^2}+Ch^{-(\gamma+2)}\|h^2\Delta_\gamma f\|_{H_\gamma^{-1}}\\ \leqslant& Ch^{-\gamma}\|b^{\frac12}v\|_{L^2}+Ch^{-\gamma}\|g\|_{L^2}+Ch^{-\gamma}\|f\|_{H_\gamma^1}.\end{aligned}\tag{3.47}$$

We need to estimate $\|b^{\frac12}v\|_{L^2}$. Multiplying the equation $(h^2\Delta_\gamma+1)v=ihbv-ihg-h^2\Delta_\gamma f$ by $\overline{v}$, integrating it and taking the imaginary part, we have

$$\begin{aligned}(bv,v)_{L^2}\leqslant&|(g,v)_{L^2}|+h^{-1}|(h^2\Delta_\gamma f,v)_{L^2}|\leqslant\|g\|_{L^2}\|v\|_{L^2}+h\|\Delta_\gamma f\|_{H_\gamma^{-1}}\|v\|_{H_\gamma^1}\\ \leqslant&\|g\|_{L^2}\|v\|_{L^2}+h\|f\|_{H_\gamma^1}\|ih^{-1}u-f\|_{H_\gamma^1}\leqslant\|g\|_{L^2}\|v\|_{L^2}+h\|f\|_{H_\gamma^1}^2+\|f\|_{H_\gamma^1}\|u\|_{H_\gamma^1}.\end{aligned}$$

Plugging into (3.47) and using the fact that $\|b^{\frac12}v\|_{L^2}^2=(bv,v)_{L^2}$ since $b\gtrsim\mathbf{1}_\omega$, we obtain that

$$\|v\|_{L^2}\leqslant Ch^{-\gamma}\|g\|_{L^2}^{1/2}\|v\|_{L^2}^{1/2}+Ch^{-\gamma}\|f\|_{H_\gamma^1}^{1/2}\|u\|_{H_\gamma^1}^{1/2}+Ch^{-\gamma}\|g\|_{L^2}+Ch^{-\gamma}\|f\|_{H_\gamma^1}.\tag{3.48}$$

It remains to estimate $\|u\|_{H_\gamma^1}$. From the equation $u=-ihv-ihf$, we have

$$\|u\|_{H_\gamma^1}\leqslant h\|v\|_{H_\gamma^1}+h\|f\|_{H_\gamma^1}.$$

Next, multiplying the equation $(h^2\Delta_\gamma+1)v=ihbv-ihg-h^2\Delta_\gamma f$ by $\overline{v}$, integrating it and taking the real part, we have

$$\begin{aligned}\|h\nabla_\gamma v\|_{L^2}^2\leqslant&\|v\|_{L^2}^2+h|(g,v)_{L^2}|+|(h^2\Delta_\gamma f,v)_{L^2}|\\ \leqslant&\|v\|_{L^2}^2+Ch\|g\|_{L^2}^2+\frac12 h\|v\|_{L^2}^2+Ch^2\|\Delta_\gamma f\|_{H_\gamma^{-1}}^2+\frac12 h^2\|v\|_{H_\gamma^1}^2,\end{aligned}$$

hence $\|hv\|_{H_\gamma^1}\leqslant Ch^{1/2}\|g\|_{L^2}+Ch\|f\|_{H_\gamma^1}+\|v\|_{L^2}$, and $\|u\|_{H_\gamma^1}\leqslant\|v\|_{L^2}+Ch\|f\|_{H_\gamma^1}+Ch^{1/2}\|g\|_{L^2}$. Plugging into (3.48) and using Young's inequality, we have

$$\|u\|_{H_\gamma^1}+\|v\|_{L^2}\leqslant Ch^{-2\gamma}\|g\|_{L^2}+Ch^{-2\gamma}\|f\|_{H_\gamma^1}.$$

This completes the proof of Corollary 3.33. □

Now, using [BT10, Theorem 2.4], we obtain Corollary 3.6.

Chapter 4

Observability in quotients of groups of Heisenberg type

"On a beau intervertir l'ordre des facteurs, le courrier n'arrive pas plus vite."
Pierre Dac.

This chapter is adapted from [FL21]. Its main result is Theorem 3 (restated as Theorem 4.2).

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4.1 Introduction

In this chapter, we consider on a compact manifold M a sub-Laplacian $\Delta_M = \sum_{j=1}^{2d} V_j^2$ which is the sum of the squares of $2d$ vector fields $(V_j)_{1\leqslant j\leqslant 2d}$ that satisfy the Hörmander condition: together with their commutators, they generate the tangent bundle to M. We give necessary and sufficient conditions for the controllability and observability of the Schrödinger equation associated with $\frac{1}{2}\Delta_M + \mathbb{V}$ where the potential $\mathbb{V}$ is analytic.

More precisely, the manifold $M = \widetilde{\Gamma}\backslash G$ is obtained by taking the quotient of a group of Heisenberg type (or H-type) G by one of its discrete cocompact sub-groups $\widetilde{\Gamma}$. The Lie group G, as a differential manifold, is diffeomorphic to $\mathbb{R}^{2d+p}$, where p is the dimension of the center of the group, and it is an important example of stratified Lie group of step 2. We study the controllability and the observability of the Schrödinger equation on M thanks to the Harmonic analysis properties of the group G, and of M. Contrarily to what happens for the usual elliptic Schrödinger equation for example on flat tori or on negatively curved manifolds, there exists a minimal time of observability. The main tools used in the proofs are (operator-valued) semi-classical measures constructed by use of representation theory and a notion of semi-classical wave packets that we introduce here in the context of groups of Heisenberg type. The concrete example given in Section 4.1.5, which is constructed in Heisenberg groups, will probably help the reader to follow the notations in the present introduction.

4.1.1 The quotient-manifold M and the Schrödinger equation

We consider an H-type group G, i.e., a connected and simply connected Lie group whose Lie algebra is an H-type algebra, denoted by $\mathfrak{g}$. This means that:

- $\mathfrak{g}$ is a step 2 stratified Lie algebra: it is equipped with a vector space decomposition
$$\mathfrak{g} = \mathfrak{v} \oplus \mathfrak{z}\,,$$
such that $[\mathfrak{v}, \mathfrak{v}] = \mathfrak{z} \neq \{0\}$ and $\mathfrak{z}$ is the center of $\mathfrak{g}$.
- $\mathfrak{g}$ is endowed with a scalar product $\langle\cdot,\cdot\rangle$ such that, for all $\lambda \in \mathfrak{z}^*$, the skew-symmetric map
$$J_\lambda : \mathfrak{v} \to \mathfrak{v}$$
defined by
$$\langle J_\lambda(U), V\rangle = \lambda([U, V]) \quad \forall U, V \in \mathfrak{v} \tag{4.1}$$
satisfies $J_\lambda^2 = -|\lambda|^2\mathrm{Id}$. In other words, J_λ is an orthogonal map as soon as $|\lambda| = 1$. Here, to define $|\lambda|$, we first identify $\mathfrak{z}^*$ to $\mathfrak{z}$ thanks to $\langle\cdot,\cdot\rangle$, then we define $|\lambda|$ as the norm (deriving from $\langle\cdot,\cdot\rangle$) of the image of λ through this identification.

The Heisenberg group in any (odd) dimension is an example of H-type group, as will be recalled below. H-type groups were introduced in [Kap80], the main motivation being that the sub-Laplacians in these groups admit explicit fundamental solutions of an elementary form.

Via the exponential map
$$\mathrm{Exp} : \mathfrak{g} \to G$$
which is a diffeomorphism from $\mathfrak{g}$ to G, one identifies G and $\mathfrak{g}$ as a set and a manifold. We may identify $\mathfrak{g}$ with the space of left-invariant vector fields via
$$Xf(x) = \frac{d}{dt} f(x\mathrm{Exp}(tX))\Big|_{t=0}\,, \tag{4.2}$$

which acts on functions of $x \in G$ and on functions of $x \in M$. Choosing an orthonormal basis V_j of $\mathfrak{v}$ and identifying $\mathfrak{g}$ with the Lie algebra of left-invariant vector fields on G, one defines the sub-Laplacian

$$\Delta_M = \sum_{j=1}^{2d} V_j^2,$$

on M, where $\dim \mathfrak{v} = 2d$. Note that this makes sense since the V_j are left-invariant, and thus pass to the quotient. We consider the hypoelliptic second order equation (see [Hor67])

$$i\partial_t \psi + \frac{1}{2}\Delta_M \psi + \mathbb{V}\psi = 0 \tag{4.3}$$

on M, where $\mathbb{V}$ is an analytic function defined on M (the latter assumption could be relaxed, see Remark 4.17 below).

4.1.2 Controllability and observability

One says that the Schrödinger equation (4.3) is *controllable* in time T on the measurable set $U \subset M$ if for any $u_0, u_1 \in L^2(M)$, there exists $f \in L^2((0,T) \times M)$ such that the solution $\psi \in L^2((0,T) \times M)$ of

$$i\partial_t \psi + \frac{1}{2}\Delta_M \psi + \mathbb{V}\psi = f\mathbf{1}_U$$

(where $\mathbf{1}_U$ denotes the characteristic function of U) with initial condition $\psi(0,x) = u_0(x)$ satisfies $\psi(T,x) = u_1(x)$. By the Hilbert Uniqueness Method (see [Lio88]), it is well-known that controllability is equivalent to an observability inequality.

The Schrödinger equation (4.3) is said to be *observable* in time T on the measurable set U if there exists a constant $C_{T,U} > 0$ such that

$$\forall u_0 \in L^2(M), \quad \|u_0\|^2_{L^2(M)} \leqslant C_{T,U} \int_0^T \left\| \mathrm{e}^{it(\frac{1}{2}\Delta_M + \mathbb{V})} u_0 \right\|^2_{L^2(U)} dt. \tag{4.4}$$

For the usual (Riemannian) Schrödinger equation, it is known that if the so-called *Geometric Control Condition* is satisfied in some time T' (which means that any ray of geometric optics enters U within time T'), then observability, and thus controllability, hold in any time $T > 0$ (see [Leb92b]). Much less is known about the converse implication, due to curvature effects.

4.1.3 Main result

Our main result gives a similar condition, replacing the rays of geometric optics by the curves of the flow map on $M \times \mathfrak{z}^*$:

$$\Phi_0^s : (x, \lambda) \mapsto (\mathrm{Exp}(sd\mathcal{Z}^{(\lambda)}/2)x, \lambda),$$

where, for $\lambda \in \mathfrak{z}^*$, $\mathcal{Z}^{(\lambda)}$ is the element of $\mathfrak{z}$ defined by $\lambda(\mathcal{Z}^{(\lambda)}) = |\lambda|$. Note that the integral curves of this flow are transverse to the space spanned by the V_j's. We introduce the following H-type geometric control condition.

(H-GCC) The measurable set U satisfies **H-type GCC** in time T if

$$\forall (x,\lambda) \in M \times (\mathfrak{z}^* \setminus \{0\}), \ \ \exists s \in (0,T), \ \ \Phi_0^s((x,\lambda)) \in U \times \mathfrak{z}^*.$$

Definition 4.1. *We denote by $T_{\mathrm{GCC}}(U)$ the infimum of all $T > 0$ such that H-type GCC holds in time T (and we set $T_{\mathrm{GCC}}(U) = +\infty$ if H-type GCC does not hold in any time).*

In the sequel, we will also consider an additional assumption (A). In geometric terms, it consists in saying that all the normal geodesics reach U in finite time. To give a rigorous statement, we fix an orthonormal basis $V = (V_1, \ldots, V_{2d})$ of $\mathfrak{v}$ to write the coordinates $v = (v_1, \ldots, v_{2d})$ of a vector

$$V = v_1 V_1 + \ldots + v_{2d} V_{2d} \in \mathfrak{v}.$$

Given $\omega \in \mathfrak{v}^*$, we write ω_j for the coordinates of ω in the dual basis of V, and we write $|\omega| = 1$ when $\sum_{j=1}^{2d} \omega_j^2 = 1$.

(A) For any $(x, \omega) \in M \times \mathfrak{v}^*$ such that $|\omega| = 1$, there exists $s \in \mathbb{R}$ such that

$$\operatorname{Exp}(s \sum_{j=1}^{2d} \omega_j V_j) x \in U.$$

Theorem 4.2. *Assume that the potential $\mathbb{V}$ in* (1.20) *is analytic. Let $U \subset M$ be open and denote by $\overline{U}$ its closure.*

1. *Assume that U satisfies* **(A)** *and that $T > T_{\mathrm{GCC}}(U)$, then the observability inequality* (4.4) *holds, i.e. the Schrödinger equation* (4.3) *is observable in time T on U and thus* (4.3) *is controllable in time T on U.*

2. *Assume $T \leqslant T_{\mathrm{GCC}}(\overline{U})$, then the observability inequality* (4.4) *fails, and thus the controllability in time T also fails on U.*

Although this will be commented more thoroughly in Remark 4.15, let us already say that we conjecture that the observability inequality (4.4) holds in U at time T under the only condition that $T > T_{\mathrm{GCC}}(U)$ (and thus one could avoid using Assumption (A)). We also point out Remark 4.17 about the assumption that the potential is analytic. Finally, we notice that in general $T_{\mathrm{GCC}}(U) \neq T_{\mathrm{GCC}}(\overline{U})$. This is due to the possible existence of "grazing rays", see Remark 4.25 for more comments on this issue.

The existence of a minimal time of control in Theorem 4.2 contrasts strongly with the observability in arbitrary small time, under Geometric Control Condition, of the usual elliptic Schrödinger equation (see [Leb92b]), which is related to its "infinite speed of propagation". In the subelliptic setting which we consider here (meaning that Δ_M is subelliptic but not elliptic), in the directions defined by $\mathfrak{z}$, the Schrödinger operator has a very different behaviour, possessing for example a family of travelling waves moving at speeds proportional to $n \in \mathbb{N}$, as was first noticed in [BGX00, Section 1] (see also [FF21, Theorem 2.10]).

More recently, in [BS19], it was shown that the Grushin Schrödinger equation $i\partial_t u - \partial_x^2 u - x^2 \partial_y^2 u = 0$ in $(-1, 1)_x \times \mathbb{T}_y$ is observable on a set of horizontal strips if and only the time T of observation is sufficiently large. With related ideas, it is shown in [LS20] (see Chapter 3) that the observability of the Grushin-type Schrödinger equation $i\partial_t u + (-\partial_x^2 - |x|^{2\gamma} \partial_y^2)^s u = 0$ in $(-1, 1)_x \times \mathbb{T}_y$ (with observation on the same horizontal strips as in [BS19]) depends on the value of the ratio $(\gamma + 1)/s$: observability may hold in arbitrarily small time, or only for sufficiently large times, or even never hold if $(\gamma+1)/s$ is large enough. These results share many similarities with ours, although their proofs use totally different techniques. The existence of a minimal time of observability for hypoelliptic PDEs was first shown in the context of the heat equation: for instance the case of the heat equation with Heisenberg sub-Laplacian was investigated in [BC17]. Finally, in contrast with the usual "finite time of observability" of elliptic waves (under GCC), it was shown in [Let20b] (see Chapter 2) that subelliptic waves are never observable. We can roughly summarize all these results by saying that *the subellipticity of the sub-Laplacian slows down the propagation of evolution equations in the directions needing brackets to be generated.*

The proof of Theorem 4.2 is based on adapting standard semi-classical approach to prove observability for a class of Schrödinger equations with *subelliptic* Laplacian, through the use of the operator-valued semi-classical measures of [FF21] which are adapted to this stratified setting. The proof also uses the introduction of wave packets playing in this non-commutative setting a role similar to the ones introduced in [CR12] and [Hag80] in the Euclidean case. To say it differently, we follow the usual scheme for proving or disproving observability inequalities, but with all the analytic tools (i.e., pseudodifferential operators, semiclassical measures and wave packets) adapted to our subelliptic setting: we do not use, for instance, classical pseudodifferential operators.

4.1.4 Strategy of the proof

The theorem consists in two parts: firstly that the condition **(A)** guarantees that the observability holds when $T > T_{\mathrm{GCC}}(U)$ and, secondly, that the observability fails when $T \leqslant T_{\mathrm{GCC}}(\overline{U})$. Beginning with the first part, it is standard (see [Leb92b]) to start with a *localized observability* result as stated in the next lemma.

Lemma 4.3 (Localized observability)**.** *Assume the set U satisfies assumption* **(A)** *and that* **(H-GCC)** *holds in time T for U. Let $h > 0$ and $\chi \in C_c^\infty((1/2, 2), [0, 1])$. Using functional calculus, we set*

$$\mathcal{P}_h f = \chi\left(-h^2\left(\frac{1}{2}\Delta_M + \mathbb{V}\right)\right) f, \quad f \in L^2(M). \tag{4.5}$$

Then, there exists a constant $C_0 > 0$ such that for any sufficiently small $h > 0$ and any $u_0 \in L^2(M)$,

$$\|\mathcal{P}_h u_0\|^2_{L^2(M)} \leqslant C_0 \int_0^T \left\| \mathrm{e}^{it(\frac{1}{2}\Delta_M + \mathbb{V})} \mathcal{P}_h u_0 \right\|^2_{L^2(U)} dt. \tag{4.6}$$

Remark 4.4. By conservation of mass in the LHS (and invariance of H-type GCC by translation in time), this inequality also holds when the integral in the RHS is taken over an arbitrary time interval (T_1, T_2) such that $T_2 - T_1 \geqslant T$.

The proof of the localized observablity is done in Section 4.3.1 below. The argument is by contradiction (as in [BZ12] or [AM14, Section 7]) and it uses the semi-classical setting based on representation theory and developed in [FF19, FF21] that we extend to the setting of quotient manifolds in Section 4.2. In particular, this argument relies in a strong way on the operator-valued semi-classical measures constructed in Sections 4.2.3 and 4.2.4.

The role of semiclassical measures in the context of observability estimates was first noticed by Gilles Lebeau [Leb96] and has been widely used since then [Mac10, AM14, AFM15, MR18], with all the developments of semi-classical measures, especially two-scale (also called two-microlocal) semi-classical measures that allow to analyze more precisely the concentration of families on submanifolds. These two-scale measures introduced in the end of the 90-s (see [Fer00, Fer05, FG02, Nie96, Mil96]) have known since then a noticeable development in control theory (see the survey [Mac15]) and in a large range of problems from conical intersections in quantum chemistry [LT05, FL08] to effective mass equations [CFM19, CFM20]. The semi-classical measures that we consider here have common features with the two-scales ones in the sense that they are operator-valued. This operator-valued feature arises from the inhomogeneity of the nilmanifolds, in parallel with the homogeneity introduced by a second scale of concentration as in the references above. However, the operator-valued feature is more fundamental here since it is due to non-commutativity of nilmanifolds and is a direct consequence of the original features of Fourier analysis on nilpotent groups: it is thus intrinsic to the structure of the problem.

The second step of the proof of the first part of Theorem 4.2 consists in passing from the localized observability to observability itself. Standard arguments (see [BZ12]) that we describe in Section 4.3.2 allow to derive from Lemma 4.3, a *weak observability* inequality in time T on the domain U: there exists a constant $C_1 > 0$ such that

$$\forall u_0 \in L^2(M), \ \ \|u_0\|^2_{L^2(M)} \leqslant C_1 \int_0^T \left\| \mathrm{e}^{it(\frac{1}{2}\Delta_M + \mathbb{V})} u_0 \right\|^2_{L^2(U)} dt + C_1 \|(\mathrm{Id} - \Delta_M)^{-1} u_0\|^2_{L^2(M)}. \quad (4.7)$$

Note that compared to (4.4), the latter inequality has an added term in its RHS which controls the low frequencies. This weak observability inequality (4.7) implies (4.4) via a Unique continuation principle for $\frac{1}{2}\Delta_M + \mathbb{V}$ (see [Bon69] and [LL20]), as we describe in Section 4.3.3. It is then non surprising that the result of Theorem 4.2 holds as soon as a Unique continuation principle is known for $\frac{1}{2}\Delta_M + \mathbb{V}$, without further assumption of analyticity on $\mathbb{V}$ (see Remark 4.17).

For proving the second part of Theorem 4.2 – the necessity of the condition **(H-GCC)** – we construct a family of initial data (u_0^ε) for which the solution $(\psi^\varepsilon(t))$ of the Schrödinger equation (4.3) concentrates on the curve $\Phi_0^t(x_0, \lambda_0)$, for any choice of $(x_0, \lambda_0) \in M \times \mathfrak{z} \setminus \{0\}$. As mentioned above, this set of initial data is the non-commutative counterpart to the wave packets (also called coherent states) in the Euclidean setting [CR12, Hag80]. These aspects are the subject of Section 4.4. Our proof relies on a statement of propagation of semiclassical measures which was proved in [FF21] when $\mathbb{V} = 0$ and that we adapt to our setting. A second proof consists in using the results of Section 4-A.3, which are of independent interest: we prove that, if the initial datum is a wave packet, the solution of (4.3) is also (approximated by) a wave packet.

Our approach could be developed in general graded Lie groups through the generalization of the tools we use: for semi-classical measures in graded groups, see Remarks 3.3 and 4.4 in [FF19], and for an extension of non-commutative wave packets to a more general setting, see Sections 6.3 and 6.4 in [FF] (based on [Ped94]).

4.1.5 An example

Before closing this introduction, let us describe an example of a quotient manifold M to which our result applies. It is known (see [BLU07, Theorem 18.2.1], and also [BFG16]) that any H-type group is isomorphic to one of the "prototype H-type groups", which are defined as follows: let $P^{(1)}, \dots, P^{(p)}$ be p linearly independent $2d \times 2d$ orthogonal skew-symmetric matrices satisfying the property

$$P^{(r)} P^{(s)} + P^{(s)} P^{(r)} = 0, \ \ \forall r, s \in \{1, ..., p\}\,, \ \ r \neq s.$$

Let us denote by $(z, s) = (z_1, \cdots, z_{2d}, s_1, \cdots, s_p)$ the points of $\mathbb{R}^{2d+p}$, that is endowed with the group law

$$(z, s) \cdot (z', s') := \begin{pmatrix} z + z' \\ s_j + s'_j + \frac{1}{2}\langle z, P^{(j)} z' \rangle, \ \ j = 1, ..., p \end{pmatrix}$$

This defines a Lie group with a Lie algebra of left invariant vector fields spanned by the following vector fields: for j running from 1 to $2d$ and k from 1 to p,

$$X_j := \partial_{z_j} + \frac{1}{2} \sum_{k=1}^{p} \sum_{l=1}^{2d} z_l \, P^{(k)}_{l,j} \partial_{s_k}, \qquad \text{and} \qquad \partial_{s_k}.$$

For more explicit examples of H-type groups, see [BLU07, Section 18.1] (e.g., Example 18.1.3). It includes the Heisenberg group $\mathbf{H}^d$ (of dimension $2d+1$), but also groups with a center of dimension $p>1$.

In this representation, the Heisenberg group $\mathbf{H}^d$ corresponds to $p=1$ and the choice of

$$P^{(1)} = \begin{pmatrix} 0 & \mathbf{1}_{\mathbb{R}^d} \\ -\mathbf{1}_{\mathbb{R}^d} & 0 \end{pmatrix}.$$

The group law then is

$$(x,y,s)\cdot(x',y',s') := \begin{pmatrix} x+x' \\ y+y' \\ s+s'+\frac{1}{2}\sum_{j=1}^d (x_j y_j' - x_j' y_j) \end{pmatrix}$$

where $x,y,x',y' \in \mathbb{R}^d$ and $s,s'\in\mathbb{R}$. We define the scalar product on $\mathfrak{v}$ by saying that the $2d$ vector fields

$$X_j = \partial_{x_j} - \frac{y_j}{2}\partial_s, \quad Y_j = \partial_{y_j} + \frac{x_j}{2}\partial_s, \qquad j=1,\dots,d$$

form an orthonormal basis, and we define the scalar product on $\mathfrak{z}$ by saying that ∂_s has norm 1 (and $\mathfrak{v}$ and $\mathfrak{z}$ are orthogonal for the scalar product on $\mathfrak{g}$). Then we obtain

$$J_\lambda\left(\sum_{j=1}^d (a_j X_j + b_j Y_j)\right) = \lambda \sum_{j=1}^d (-b_j X_j + a_j Y_j).$$

where J_λ has been introduced in (4.1). An example of discrete cocompact subgroup of the Heisenberg group $\mathbf{H}^d$ is

$$\widetilde{\Gamma} = (\sqrt{2\pi}\mathbb{Z})^{2d} \times \pi\mathbb{Z}, \tag{4.8}$$

and the associated quotient manifold is the left quotient $M = \widetilde{\Gamma}\backslash\mathbf{H}^d$. For more general examples of discrete cocompact subgroups in H-type groups, see [CG04, Chapter 5].

A typical open set $U \subset \widetilde{\Gamma}\backslash\mathbf{H}^d$ of control which one may consider is the periodization (through the multiplication on the left by elements of $\widetilde{\Gamma}$) of the set

$$A = \{(x,y,s),\ (x,y)\in[0,\sqrt{2\pi})^{2d},\ s\in I\}$$

where I is a strict open subset of $[0,\pi)$. One can verify that both Assumption (A) and (H-GCC) (in sufficiently large time, which depends on I) are satisfied.

4.2 Semi-classical analysis on quotient manifolds

Semi-classical analysis is based on the analysis of the scales of oscillations of functions. It uses a microlocal approach, meaning that one understands functions in the phase space, i.e. the space of position/impulsion of quantum mechanics. As the impulsion variable is the dual variable of the position variable via the Fourier transform, microlocal analysis crucially relies on the Fourier representation of functions, and on the underlying harmonic analysis.

Recall that, in the usual Euclidean setting, the algebra of pseudodifferential operators contains those of multiplications by functions together with Fourier multipliers. These operators are defined by their symbols via the Fourier inversion formula and are used for analyzing families of functions in the phase space. Indeed, their boundedness in L^2 for adequate classes of symbols allows to build a linear map on the set of symbols, the weak limits of which are characterized by non-negative Radon measures. These measures give phase space information on the obstruction to strong convergence of bounded families in $L^2(\mathbb{R}^d)$. In a context where no specific scale is specified, they are called microlocal defect measures, or H-measures and were first introduced independently in [Ger91a, Tar90]. When a specific scale of oscillations is prescribed, this scale is called the semi-classical parameter and they are called semi-classical (or Wigner) measures (see [HMR87, Ger91a, GL93, LP93, GMMP00]). If these functions are moreover solutions of some equation, the semi-classical measures may have additional properties such as invariance under a flow.

In the next sections, we follow the same steps, adapted to the context of quotients of H-type groups, which are non-commutative: following the theory of non-commutative harmonic analysis (see [CG04, Tay11] and some elements given in Section 4-A.1), we define the (operator-valued) Fourier transform (4.13), based on the unitary irreducible representations of the group, recalled in (4.12), which form an analog to the usual frequency space. Then, we use the Fourier inversion formula (4.14) to define in (4.16) a class of symbols and the associated semi-classical pseudodifferential operators in (4.18). From this, Proposition 4.11 guarantees the existence of semi-classical measures, whose additional invariance properties for solutions of the Schrödinger equation are listed in Proposition 4.13.

4.2.1 Harmonic analysis on quotient manifolds

Let G be a stratified nilpotent Lie group of H-type and $\widetilde{\Gamma}$ be a discrete cocompact subgroup of G. We consider the left quotient $M = \widetilde{\Gamma}\backslash G$ and we denote by π the canonical projection

$$\pi : G \to M$$

which associates to $x \in G$ its class modulo $\widetilde{\Gamma}$.

For each $\lambda \in \mathfrak{z}^* \setminus \{0\}$, one associates with λ the canonical skew-symmetric form $B(\lambda)$ defined on $\mathfrak{v}$ by

$$B(\lambda)(U, V) = \lambda([U, V]).$$

The map $J_\lambda : \mathfrak{v} \to \mathfrak{v}$ of Section 4.1 is the natural endomorphism associated with $B(\lambda)$. In H-type groups, the symmetric form $-J_\lambda^2$ is the scalar map $|\lambda|^2\mathrm{Id}$ (note that $-J_\lambda^2$ is always a non-negative symmetric form). Therefore, one can find a λ-dependent orthonormal basis

$$\left(P_1^{(\lambda)}, \dots, P_d^{(\lambda)}, Q_1^{(\lambda)}, \dots, Q_d^{(\lambda)}\right)$$

of $\mathfrak{v}$ where J_λ is represented by

$$\langle J_\lambda(U), V\rangle = B(\lambda)(U, V) = |\lambda| U^t J V \quad \text{with} \quad J = \begin{pmatrix} 0 & \mathrm{Id} \\ -\mathrm{Id} & 0 \end{pmatrix},$$

the vectors $U, V \in \mathfrak{v}$ being written in the $\left(P_1^{(\lambda)}, \dots, P_d^{(\lambda)}, Q_1^{(\lambda)}, \dots, Q_d^{(\lambda)}\right)$-basis. We then decompose $\mathfrak{v}$ in a λ-depending way as $\mathfrak{v} = \mathfrak{p}_\lambda + \mathfrak{q}_\lambda$ with

$$\mathfrak{p} := \mathfrak{p}_\lambda := \mathrm{Span}\left(P_1^{(\lambda)}, \dots, P_d^{(\lambda)}\right), \quad \mathfrak{q} := \mathfrak{q}_\lambda := \mathrm{Span}\left(Q_1^{(\lambda)}, \dots, Q_d^{(\lambda)}\right).$$

Denoting by $z = (z_1, \cdots, z_p)$ the coordinates of Z in a fixed orthonormal basis $(Z_1, \cdots, Z_p)$ of $\mathfrak{z}$, and once given $\lambda \in \mathfrak{z}^* \setminus \{0\}$, we will often use the writing of an element $x \in G$ or $X \in \mathfrak{g}$ as

$$x = \mathrm{Exp}(X), \qquad X = p_1 P_1^{(\lambda)} + \ldots + p_d P_d^{(\lambda)} + q_1 Q_1^{(\lambda)} + \ldots + q_d Q_d^{(\lambda)} + z_1 Z_1 + \ldots + z_p Z_p, \tag{4.9}$$

where $X = P + Q + Z$, $p = (p_1, \cdots, p_d)$ are the λ-dependent coordinates of P on the vector basis $(P_1^{(\lambda)}, \cdots, P_d^{(\lambda)})$, $q = (q_1, \cdots, q_d)$ those of Q on $(Q_1^{(\lambda)}, \cdots, Q_d^{(\lambda)})$, and $z = (z_1, \cdots, z_p)$ of Z are independent of λ.

Example 4.5. In the Heisenberg group $\mathbf{H}^d$, if $\lambda = \alpha dz$ with $\alpha \in \mathbb{R}$, we have $P_j^{(\lambda)} = X_j$, $Q_j^{(\lambda)} = Y_j$ for $\alpha > 0$, and $P_j^{(\lambda)} = Y_j$, $Q_j^{(\lambda)} = X_j$ for $\alpha < 0$. Therefore, the above (p, q, z) coordinates are not the usual coordinates in Heisenberg groups (see [Tay86, Chapter 1]). This choice is due to the fact that there is no canonical choice of coordinates in general H-type groups. As a consequence, the formula for irreducible representations (4.11) is not the same as the usual one in Heisenberg groups [Tay86, Equation (2.23) in Chapter 1].

As already mentioned in Section 4.1.3, we also fix an orthonormal basis $(V_1, \ldots, V_{2d})$ of $\mathfrak{v}$ to write the coordinates $v = (v_1, \ldots, v_{2d})$ of a vector

$$V = v_1 V_1 + \ldots + v_{2d} V_{2d} \in \mathfrak{v};$$

both this orthonormal basis and the coordinates are independent of λ. With these coordinates, we define a quasi-norm by setting

$$|x| = \left(|v_1|^4 + \cdots + |v_{2d}|^4 + |z_1|^2 + \cdots + |z_p|^2\right)^{1/4}, \quad x = \mathrm{Exp}(V + Z) \in G. \tag{4.10}$$

We recall that it satisfies a triangle inequality up to a constant.

Functional spaces

We shall say that a function f on G is $\widetilde{\Gamma}$-leftperiodic if we have

$$\forall x \in G, \ \ \forall \gamma \in \widetilde{\Gamma}, \ \ f(\gamma x) = f(x).$$

With a function f defined on M, we associate the $\widetilde{\Gamma}$-leftperiodic function $f \circ \pi$ defined on G. Conversely, a $\widetilde{\Gamma}$-leftperiodic function f naturally defines a function on M. Thus the set of functions on M is in a one-to-one relation with the set of $\widetilde{\Gamma}$-left periodic functions on G.

The inner products on $\mathfrak{v}$ and $\mathfrak{z}$ allow us to consider the Lebesgue measure $dv\, dz$ on $\mathfrak{g} = \mathfrak{v} \oplus \mathfrak{z}$. Via the identification of G with $\mathfrak{g}$ by the exponential map, this induces a Haar measure dx on G and on M. This measure is invariant under left and right translations:

$$\forall f \in L^1(M), \quad \forall x \in M, \quad \int_M f(y)dy = \int_M f(xy)dy = \int_M f(yx)dy.$$

The convolution of two functions f and g on M is given by

$$f * g(x) = \int_M f(xy^{-1})g(y)dy = \int_M f(y)g(y^{-1}x)dy.$$

Using the bijection of the set of functions on M with the set of $\widetilde{\Gamma}$-leftperiodic functions on G, we deduce that $f * g$ is well-defined as a function on M. Finally, we define Lebesgue spaces by

$$\|f\|_{L^q(M)} := \left(\int_M |f(y)|^q \, dy\right)^{\frac{1}{q}},$$

for $q \in [1, \infty)$, with the standard modification when $q = \infty$.

Homogeneous dimension

Since G is stratified, there is a natural family of dilations on $\mathfrak{g}$ defined for $t>0$ as follows: if X belongs to $\mathfrak{g}$, we decompose X as $X=V+Z$ with $V\in\mathfrak{v}$ and $Z\in\mathfrak{z}$ and we set

$$\delta_t X := tV + t^2 Z\,.$$

The dilation is defined on G via the identification by the exponential map as the map $\mathrm{Exp}\circ\delta_t\circ\mathrm{Exp}^{-1}$ that we still denote by δ_t. The dilations δ_t, $t>0$, on $\mathfrak{g}$ and G form a one-parameter group of automorphisms of the Lie algebra $\mathfrak{g}$ and of the group G. The Jacobian of the dilation δ_t is t^Q where

$$Q := \dim\mathfrak{v} + 2\dim\mathfrak{z} = 2d+2p$$

is called the homogeneous dimension of G. A differential operator T on G (and more generally any operator T defined on $C_c^\infty(G)$ and valued in the distributions of $G\sim\mathbb{R}^{2d+p}$) is said to be homogeneous of degree ν (or ν-homogeneous) when $T(f\circ\delta_t)=t^\nu(Tf)\circ\delta_t$. We recall that the quasi-norm introduced in (4.10) satisfies $|\delta_r x|=r|x|$ for all $r>0$ and $x\in G$. It is a homogeneous quasi-norm and we recall that any homogeneous quasi-norm is equivalent to it.

Irreducible representations and Fourier transform

For the sake of completeness, many details about the results of this section, which are standard in non-commutative harmonic analysis, are given in Section 4-A.1.

The infinite dimensional irreducible representations of G are parametrized by $\mathfrak{z}^*\setminus\{0\}$: for $\lambda\in\mathfrak{z}^*\setminus\{0\}$, one defines $\pi^\lambda_\cdot$ on $L^2(\mathfrak{p}_\lambda)\sim L^2(\mathbb{R}^d)$ by

$$\pi^\lambda_x\Phi(\xi) = \mathrm{e}^{i\lambda(z)+\frac{i}{2}|\lambda|\,p\cdot q+i\sqrt{|\lambda|}\,\xi\cdot q}\,\Phi\left(\xi+\sqrt{|\lambda|}p\right), \tag{4.11}$$

where x has been written as in (4.9). The representations π^λ, $\lambda\in\mathfrak{z}^*\setminus\{0\}$, are infinite dimensional. The other unitary irreducible representations of G are given by the characters of the first stratum in the following way: for every $\omega\in\mathfrak{v}^*$, we set

$$\pi^{0,\omega}_x = \mathrm{e}^{i\omega(V)},\quad x=\mathrm{Exp}(V+Z)\in G,\quad \text{with } V\in\mathfrak{v} \text{ and } Z\in\mathfrak{z}.$$

The 0 in the notation $(0,\omega)$ is here to differentiate $\pi^{(0,\omega)}$ from π^λ. It is natural since we think of $\mathfrak{v}^*$ as "horizontal" and $\mathfrak{z}^*$ as "vertical". The set $\widehat{G}$ of all unitary irreducible representations modulo unitary equivalence is then parametrized by $(\mathfrak{z}^*\setminus\{0\})\sqcup\mathfrak{v}^*$:

$$\widehat{G} = \{\text{class of } \pi^\lambda \ :\ \lambda\in\mathfrak{z}^*\setminus\{0\}\}\sqcup\{\text{class of } \pi^{0,\omega}\ :\ \omega\in\mathfrak{v}^*\}. \tag{4.12}$$

We will identify each representation π^λ with its equivalence class. Note that the trivial representation $1_{\widehat{G}}$ corresponds to the class of $\pi^{(0,\omega)}$ with $\omega=0$, i.e. $1_{\widehat{G}}:=\pi^{(0,0)}$.

The set $G\times\widehat{G}$ will be interpreted in our analysis as the phase space of G, and $M\times\widehat{G}$ as the phase space of M, in analogy with the fact that $\mathbb{R}^d\times\mathbb{R}^d$ and $\mathbb{T}^d\times\mathbb{R}^d$ are respectively the phase space of the Euclidean space $\mathbb{R}^d$ and of the torus $\mathbb{T}^d$.

The Fourier transform is defined on $\widehat{G}$ and is valued in the space of bounded operators on $L^2(\mathfrak{p}_\lambda)$: for any $\lambda\in\mathfrak{z}^*$, $\lambda\neq 0$,

$$\mathcal{F}f(\lambda) := \int_G f(x)\left(\pi^\lambda_x\right)^*\,dx\,, \tag{4.13}$$

Besides, above finite dimensional representations, the Fourier transform is defined for $\omega \in \mathfrak{v}^*$ by

$$\widehat{f}(0,\omega) = \mathcal{F}f(0,\omega) := \int_G f(x)(\pi_x^{(0,\omega)})^* dx = \int_{\mathfrak{v}\times\mathfrak{z}} f(\mathrm{Exp}(V+Z))e^{-i\omega(V)}dVdZ.$$

Functions f of $L^1(G)$ have a Fourier transform $(\mathcal{F}(f)(\lambda))_{\lambda\in\mathfrak{z}^*}$ which is a bounded family of bounded operators on $L^2(\mathfrak{p}_\lambda)$ with uniform bound:

$$\|\mathcal{F}f(\lambda)\|_{\mathcal{L}(L^2(\mathfrak{p}_\lambda))} \leqslant \int_G |f(x)| \|(\pi_x^\lambda)^*\|_{\mathcal{L}(L^2(\mathfrak{p}_\lambda))} dx = \|f\|_{L^1(G)}.$$

since the unitarity of π^λ implies $\|(\pi_x^\lambda)^*\|_{\mathcal{L}(L^2(\mathfrak{p}_\lambda))} = 1$.

The Fourier transform can be extended to an isometry from $L^2(G)$ onto the Hilbert space of measurable families $A = \{A(\lambda)\}_{\lambda\in\mathfrak{z}^*\setminus\{0\}}$ of operators on $L^2(\mathfrak{p}_\lambda)$ which are Hilbert-Schmidt for almost every $\lambda \in \mathfrak{z}^* \setminus \{0\}$, with norm

$$\|A\| := \left(\int_{\mathfrak{z}^*\setminus\{0\}} \|A(\lambda)\|^2_{HS(L^2(\mathfrak{p}_\lambda))} |\lambda|^d \, d\lambda\right)^{\frac{1}{2}} < \infty\,.$$

We have the Fourier-Plancherel formula:

$$\int_G |f(x)|^2 \, dx = c_0 \int_{\mathfrak{z}^*\setminus\{0\}} \|\mathcal{F}f(\lambda)\|^2_{HS(L^2(\mathfrak{p}_\lambda))} |\lambda|^d \, d\lambda\,,$$

where $c_0 > 0$ is a computable constant. The Plancherel measure is $c_0|\lambda|^d d\lambda$, and is supported in the subset $\{\text{class of } \pi^\lambda \ : \ \lambda \in \mathfrak{z}^* \setminus \{0\}\}$ of $\widehat{G}$. Besides, an inversion formula for $f \in \mathcal{S}(G)$ and $x \in G$ writes:

$$f(x) = c_0 \int_{\mathfrak{z}^*\setminus\{0\}} \mathrm{Tr}\left(\pi_x^\lambda \mathcal{F}f(\lambda)\right) |\lambda|^d \, d\lambda\,, \tag{4.14}$$

where Tr denotes the trace of operators of $\mathcal{L}(L^2(\mathfrak{p}_\lambda))$ (see [Tay86, Chapter 1, Theorem 2.7]). This formula makes sense since for Schwartz functions $f \in \mathcal{S}(G)$, the operators $\mathcal{F}f(\lambda)$, $\lambda \in \mathfrak{z}^* \setminus \{0\}$, are trace-class, with enough regularity in λ so that $\int_{\mathfrak{z}^*\setminus\{0\}} \mathrm{Tr}\left|\mathcal{F}f(\lambda)\right| |\lambda|^d \, d\lambda$ is finite.

To conclude this section, it is important to notice that the differential operators have a Fourier resolution that allows to think them as Fourier multipliers. In particular, the resolution of the sub-Laplacian $-\Delta_G$ is well-understood

$$\forall f \in \mathcal{S}(G),\ \ \mathcal{F}(-\Delta_G f)(\lambda) = H(\lambda)\mathcal{F}(f)(\lambda).$$

At $\pi^{(0,\omega)}$, $\omega \in \mathfrak{v}^*$, it is the number $\mathcal{F}(-\Delta_G)(0,\omega) = |\omega|^2$, and at π^λ, $\lambda \in \mathfrak{z}^* \setminus \{0\}$, it is the unbounded operator

$$H(\lambda) = |\lambda| \sum_{j=1}^d \left(-\partial_{\xi_j}^2 + \xi_j^2\right), \tag{4.15}$$

where we have used the identification $\mathfrak{p}_\lambda \sim \mathbb{R}^d$.

4.2.2 Semi-classical pseudodifferential operators on quotient manifolds

As observables of quantum mechanics are functions on the phase space, the symbols of pseudodifferential operators on M are functions defined on $M \times \widehat{G}$. In this non-commutative framework, they have the same properties as the Fourier transform and they are operator-valued symbols.

Following [FF21, FF19], we consider the class of symbols $\mathcal{A}_0$ of fields of operators defined on $M \times \widehat{G}$ by

$$\sigma(x,\lambda) \in \mathcal{L}(L^2(\mathfrak{p}_\lambda)), \quad (x,\lambda) \in M \times \widehat{G},$$

that are smooth in the variable x and Fourier transforms of functions of the set $\mathcal{S}(G)$ of Schwartz functions on G in the variable λ: for all $(x,\lambda) \in M \times \widehat{G}$,

$$\sigma(x,\lambda) = \mathcal{F}\kappa_x(\lambda), \quad \kappa_x \in \mathcal{C}^\infty(M, \mathcal{S}(G)). \tag{4.16}$$

Note that a similar class of symbols in the Euclidean context was introduced in [LP93, Section 3]. Note that we kept in (4.16) the notation λ also for the parameters $(0,\omega)$, $\omega \in \mathfrak{v}^*$. In this case, the operator $\mathcal{F}\kappa_x((0,\omega)) = \sigma(x,(0,\omega))$ reduces to a complex number since the associated Hilbert space is $\mathbb{C}$.

If $\varepsilon > 0$, we associate with κ_x (and thus with $\sigma(x,\lambda)$) the function κ_x^ε defined on G by

$$\kappa_x^\varepsilon(z) = \varepsilon^{-Q}\kappa_x(\delta_{\varepsilon^{-1}}(z)), \tag{4.17}$$

We then define the semi-classical pseudodifferential operator $\mathrm{Op}_\varepsilon(\sigma)$ via the identification of functions f on M with $\widetilde{\Gamma}$-leftperiodic functions on G:

$$\mathrm{Op}_\varepsilon(\sigma)f(x) = \int_G \kappa_x^\varepsilon(y^{-1}x)f(y)dy. \tag{4.18}$$

When $\varepsilon = 1$, we omit the index ε and just write Op instead of Op_ε.

Remark 4.6. The formulas (4.18), (4.17) and (4.16) may be compared to the formulas of the semiclassical (standard) quantization on the torus $\mathbb{T}^n = (\mathbb{R}/2\pi\mathbb{Z})^n$, namely, for $\sigma(x,\xi), x \in \mathbb{T}^n, \xi \in \mathbb{R}^n$ and f a $(2\pi\mathbb{Z})^n$-periodic function,

$$\begin{aligned}
\mathrm{Op}_\varepsilon^{\mathbb{T}^n}(\sigma)f(x) &= \int_{\mathbb{R}^n} K^\varepsilon\left(x, x-y\right) f(y)dy \\
\text{where} \quad K^\varepsilon(x,z) &= \varepsilon^{-n}K(x,\varepsilon^{-1}z), \\
K(x,w) &= \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} e^{iw\cdot\xi}\sigma(x,\xi)d\xi \in C^\infty(\mathbb{T}^n, \mathcal{S}(\mathbb{R}^n)), \\
\text{i.e.,} \quad \sigma(x,\xi) &= \mathcal{F}_w^{\mathbb{R}^n}K(x,\xi).
\end{aligned}$$

We observe the following facts:

1. The operator $\mathrm{Op}_\varepsilon(\sigma)$ is well-defined as an operator on M. Indeed,

$$\begin{aligned}
\mathrm{Op}_\varepsilon(\sigma)f(\gamma x) &= \int_G \kappa_{\gamma x}^\varepsilon(y^{-1}\gamma x)f(y)dy \\
&= \int_G \kappa_x^\varepsilon(y^{-1}x)f(\gamma y)dy \\
&= \mathrm{Op}_\varepsilon(\sigma)f(x).
\end{aligned}$$

Here we have used a change of variable and the relations $\kappa_{\gamma x}(\cdot) = \kappa_x(\cdot)$ and $f(\gamma y) = f(y)$.

2. Using (4.14) and (4.16), we have the useful identity

$$\mathrm{Op}_\varepsilon(\sigma)f(x) = \varepsilon^{-Q}\int_G \kappa_x(\delta_\varepsilon(y^{-1}x))f(y)dy = \int_{G\times\widehat{G}} \mathrm{Tr}(\pi^\lambda_{y^{-1}x}\sigma(x,\varepsilon^2\lambda))f(y)|\lambda|^d d\lambda dy.$$

Note that the rescaling $\sigma(x,\varepsilon^2\lambda)$ is considered here only for $\lambda \in \mathfrak{z}^* \setminus \{0\}$ due to (4.14).

3. The kernel of $\mathrm{Op}_\varepsilon(\sigma)$ is given by

$$k_\varepsilon(x,y)=\sum_{\gamma\in\widetilde{\Gamma}}\kappa_x^\varepsilon(\gamma y^{-1}x)$$

4. The family of operators $(\mathrm{Op}_\varepsilon(\sigma))_{\varepsilon>0}$ is uniformly bounded in $\mathcal{L}(L^2(M))$:

$$\|\mathrm{Op}_\varepsilon(\sigma)\|_{\mathcal{L}(L^2(M))}\leqslant\int_G\sup_{x\in M}|\kappa_x(y)|dy. \tag{4.19}$$

5. Semi-classical pseudodifferential operators act locally: let $\sigma\in\mathcal{A}_0$ be compactly supported in an open set U such that $\overline{U}$ is strictly included in a unit cell $\mathcal{B}$ of $\widetilde{\Gamma}$ and $\chi\in\mathcal{C}_c^\infty(\mathcal{B})$ such that $\chi\sigma=\sigma$. Then, for all $N\in\mathbb{N}$, there exists a constant c_N such that, for any $\varepsilon>0$,

$$\|\mathrm{Op}_\varepsilon(\sigma)-\chi\mathrm{Op}_\varepsilon(\sigma)\chi\|_{\mathcal{L}(L^2(M))}=\|\mathrm{Op}_\varepsilon(\sigma)-\mathrm{Op}_\varepsilon(\sigma)\chi\|_{\mathcal{L}(L^2(M))}\leqslant c_N\,\varepsilon^N. \tag{4.20}$$

Remark 4.7. The last property is crucial for our analysis since it allows to transfer results obtained in the nilpotent group G for functions in $L^2_{\mathrm{loc}}(G)$ to the case of square-integrable functions of the homogeneous manifold M. Indeed, if $f\in L^2(M)$, then f can be identified to a $\widetilde{\Gamma}$-leftperiodic function on $L^2_{loc}(G)$. In particular, we have $\chi f\in L^2(G)$ and $\mathrm{Op}_\varepsilon(\sigma)\chi f=\chi\mathrm{Op}_\varepsilon(\sigma)\chi f$ coincides with the standard definition of [FF19, FF21]. This correspondance between computations in M and in G will be further developed at the beginning of Section 4.4.1, notably through the periodization operator $\mathbb{P}$.

Properties (3), (4) and (5) are discussed more in details in Section 4-A.2.

We deduce from the latter observation (5) the next two properties. For stating them, we introduce the difference operators, acting on $\mathcal{L}(L^2(\mathfrak{p}_\lambda))$:

$$\Delta_{p_j}^\lambda=|\lambda|^{-1/2}[\xi_j,\cdot],\qquad \Delta_{q_j}=|\lambda|^{-1/2}[i\partial_{\xi_j},\cdot],\quad 1\leqslant j\leqslant d.$$

(6) The following symbolic calculus result holds:

Proposition 4.8. *Let $\sigma\in\mathcal{A}_0$. Then, in $\mathcal{L}(L^2(M))$,*

$$\mathrm{Op}_\varepsilon(\sigma)^*=\mathrm{Op}_\varepsilon(\sigma^*)-\varepsilon\,\mathrm{Op}_\varepsilon(P\cdot\Delta_p^\lambda\sigma^*+Q\cdot\Delta_q^\lambda\sigma^*)+O(\varepsilon^2).$$

Let $\sigma_1,\sigma_2\in\mathcal{A}_0$. Then in $\mathcal{L}(L^2(M))$,

$$\mathrm{Op}_\varepsilon(\sigma_1)\circ\mathrm{Op}_\varepsilon(\sigma_2)=\mathrm{Op}_\varepsilon(\sigma_1\,\sigma_2)-\varepsilon\,\mathrm{Op}_\varepsilon\left(\Delta_p^\lambda\sigma_1\cdot P\,\sigma_2+\Delta_q^\lambda\sigma_1\cdot Q\,\sigma_2\right)+O(\varepsilon^2),$$

Proof. We take $f,g\in L^2(M)$. By using a partition of unity, we reduce to the case of σ and χ as in Point (5) above. Thanks to Proposition 3.6 of [FF19], we observe

$$\begin{aligned}(\mathrm{Op}_\varepsilon(\sigma)^*f,g)_{L^2(M)}&=(f,\mathrm{Op}_\varepsilon(\sigma)g)_{L^2(G)}\\&=(\chi f,\mathrm{Op}_\varepsilon(\sigma)\chi g)_{L^2(G)}=(\mathrm{Op}_\varepsilon(\sigma)^*\chi f,\chi g)_{L^2(G)}\\&=(\mathrm{Op}_\varepsilon(\sigma^*)\chi f,\chi g)_{L^2(G)}-\varepsilon(\mathrm{Op}_\varepsilon(P\cdot\Delta_p^\lambda\sigma^*+Q\cdot\Delta_q^\lambda\sigma^*)\chi f,\chi g)_{L^2(G)}\\&\qquad+O(\varepsilon^2\|\chi f\|_{L^2(G)}\|\chi g\|_{L^2(G)})\\&=(\mathrm{Op}_\varepsilon(\sigma^*)f,g)_{L^2(M)}-\varepsilon(\mathrm{Op}_\varepsilon(P\cdot\Delta_p^\lambda\sigma^*+Q\cdot\Delta_q^\lambda\sigma^*)f,g)_{L^2(M)}\\&\qquad+O(\varepsilon^2\|f\|_{L^2(M)}\|g\|_{L^2(M)}).\end{aligned}$$

Indeed, if $\chi\sigma = \sigma$ in $\mathcal{B}$, we also have $\chi\Delta_p^\lambda\sigma = \Delta_p^\lambda\sigma$ and $\chi\Delta_q^\lambda\sigma = \Delta_q^\lambda\sigma$ whence $\chi\sigma^* = \sigma^*$ in $\mathcal{B}$. The proof follows similarly for the product by considering

$$(\mathrm{Op}_\varepsilon(\sigma_1) \circ \mathrm{Op}_\varepsilon(\sigma_2)f, g)_{L^2(M)},$$

passing in $L^2(G)$, using Point (5) and the observations above. □

(7) The main contribution of the function $(x,z) \mapsto \kappa_x(z)$ to the operator $\mathrm{Op}_\varepsilon(\sigma)$, $\sigma(x,\lambda) = \mathcal{F}(\kappa_x)(\lambda)$ is due to its values close to $z = \mathbf{1}_G$.

Proposition 4.9. *Let $\chi_0 \in \mathcal{C}^\infty(G)$ be compactly supported close to 1_G. Let $\sigma = \mathcal{F}(\kappa_x)(\lambda)$ and*

$$\sigma_\varepsilon = \mathcal{F}(\kappa_x \chi_0 \circ \delta_\varepsilon).$$

Then, in $L^2(M)$, for all $N \in \mathbb{N}$,

$$\mathrm{Op}_\varepsilon(\sigma) = \mathrm{Op}_\varepsilon(\sigma_\varepsilon) + O(\varepsilon^{NQ}).$$

Proof. Here again, we reduce by using a partition of unity to the case of σ as in (5) above and introduce the associated function $\chi \in \mathcal{C}_c^\infty(\mathcal{B})$ such that $\chi\sigma = \sigma$. We observe that $\chi\sigma_\varepsilon = \sigma_\varepsilon$ and we use Proposition 3.4 of [FF19] to write for $f, g \in L^2(M)$,

$$\begin{aligned}(\mathrm{Op}_\varepsilon(\sigma)f, g)_{L^2(M)} &= (\mathrm{Op}_\varepsilon(\sigma)\chi f, \chi g)_{L^2(G)}\\ &= (\mathrm{Op}_\varepsilon(\sigma_\varepsilon)\chi f, \chi g)_{L^2(G)} + O(\varepsilon^{NQ}\|\chi f\|_{L^2(G)}\|\chi g\|_{L^2(G)})\\ &= (\mathrm{Op}_\varepsilon(\sigma_\varepsilon)f, g)_{L^2(M)} + O(\varepsilon^{NQ}\|f\|_{L^2(M)}\|g\|_{L^2(G)})\end{aligned}$$

which concludes the proof. □

4.2.3 Semi-classical measures

When given a bounded sequence $(f^\varepsilon)_{\varepsilon>0}$ in $L^2(M)$, one defines an observation $\ell_\varepsilon(\sigma)$ in analogy with quantum mechanics as the action of observables on this family, i.e. the families

$$\ell_\varepsilon(\sigma) = (\mathrm{Op}_\varepsilon(\sigma)f^\varepsilon, f^\varepsilon), \quad \sigma \in \mathcal{A}_0.$$

Since these quantities are bounded sequences of real numbers, it is then natural to study the asymptotic $\varepsilon \to 0$. The families $(\ell_\varepsilon(\sigma))_{\varepsilon>0}$ have weak limits that depend linearly on σ and enjoy additional properties. We call semi-classical measure of $(f^\varepsilon)_{\varepsilon>0}$ any of these linear forms.

For describing the properties of semi-classical measures, we need to introduce a few notations. If Z is a locally compact Hausdorff set, we denote by $\mathcal{M}(Z)$ the set of finite Radon measures on Z and by $\mathcal{M}^+(Z)$ the subset of its positive elements. Considering the metric space $M \times \widehat{G}$ endowed with the field of complex Hilbert spaces $L^2(\mathfrak{p}_\lambda)$ defined above elements $(x,\lambda) \in M \times \widehat{G}$, we denote by $\widetilde{\mathcal{M}}_{ov}(M \times \widehat{G})$ the set of pairs (γ, Γ) where γ is a positive Radon measure on $M \times \widehat{G}$ and $\Gamma = \{\Gamma(x,\lambda) \in \mathcal{L}(L^2(\mathfrak{p}_\lambda)) : \lambda \in \widehat{G}\}$ is a measurable field of trace-class operators such that

$$\|\Gamma d\gamma\|_{\mathcal{M}} := \int_{M\times\widehat{G}} \mathrm{Tr}(|\Gamma(x,\lambda)|) d\gamma(x,\lambda) < \infty.$$

Here, as usual, $|\Gamma| := \sqrt{\Gamma\Gamma^*}$. Note that $\Gamma(x,\lambda)$ is defined as a linear operator on the space $L^2(\mathfrak{p}_\lambda)$ which does not depend on x but which depends on λ. Considering that two pairs

(γ, Γ) and (γ', Γ') in $\widetilde{\mathcal{M}}_{ov}(M \times \widehat{G})$ are equivalent when there exists a measurable function $f : M \times \widehat{G} \to \mathbb{C} \setminus \{0\}$ such that

$$d\gamma'(x, \lambda) = f(x, \lambda) d\gamma(x, \lambda) \text{ and } \Gamma'(x, \lambda) = \frac{1}{f(x, \lambda)} \Gamma(x, \lambda)$$

for γ-almost every $(x, \lambda) \in M \times \widehat{G}$, we define the equivalence class of (γ, Γ) by $\Gamma d\gamma$, and the resulting quotient by $\mathcal{M}_{ov}(M \times \widehat{G})$. One checks readily that $\mathcal{M}_{ov}(M \times \widehat{G})$ equipped with the norm $\| \cdot \|_{\mathcal{M}}$ is a Banach space.

Finally, we say that a pair (γ, Γ) in $\widetilde{\mathcal{M}}_{ov}(M \times \widehat{G})$ is positive when $\Gamma(x, \lambda) \geqslant 0$ for γ-almost all $(x, \lambda) \in M \times \widehat{G}$. In this case, we write $(\gamma, \Gamma) \in \widetilde{\mathcal{M}}^+_{ov}(M \times \widehat{G})$, and $\Gamma d\gamma \geqslant 0$ for $\Gamma d\gamma \in \mathcal{M}^+_{ov}(M \times \widehat{G})$.

With these notations in mind, one can mimic the proofs of [FF21], considering the C^*-algebra $\mathcal{A}$ obtained as the closure of $\mathcal{A}_0$ for the norm $\sup_{(x,\lambda) \in M \times \widehat{G}} \|\sigma(x, \lambda)\|_{\mathcal{L}(L^2(\mathfrak{p}_\lambda))}$. Indeed, the properties of this algebra depend on those of $\widehat{G}$ and the analysis of the set and of [FF19, FF21] also applies in this context. Then, arguing as in [FF19, FF21], one can define semi-classical measures as follows.

Theorem 4.10. *Let $(f^\varepsilon)_{\varepsilon>0}$ be a bounded family in $L^2(M)$. There exist a sequence $(\varepsilon_k) \in (\mathbb{R}^*_+)^{\mathbb{N}}$ with $\varepsilon_k \underset{k \to +\infty}{\longrightarrow} 0$, and $\Gamma d\gamma \in \mathcal{M}^+_{ov}(M \times \widehat{G})$ such that for all $\sigma \in \mathcal{A}$,*

$$(\mathrm{Op}_{\varepsilon_k}(\sigma) f^{\varepsilon_k}, f^{\varepsilon_k})_{L^2(M)} \underset{k \to +\infty}{\longrightarrow} \int_{M \times \widehat{G}} \mathrm{Tr}(\sigma(x, \lambda) \Gamma(x, \lambda)) d\gamma(x, \lambda).$$

Given the sequence $(\varepsilon_k)_{k \in \mathbb{N}}$, the measure $\Gamma d\gamma$ is unique up to equivalence. Besides,

$$\int_{M \times \widehat{G}} \mathrm{Tr}(\Gamma(x, \lambda)) d\gamma(x, \lambda) \leqslant \limsup_{\varepsilon \to 0} \|f^\varepsilon\|^2_{L^2(M)}.$$

We emphasize on the operator-valued nature of $\Gamma(x, \lambda) \mathbf{1}_{\lambda \in \mathfrak{z}^*}(\lambda)$ in opposition to the fact that $\Gamma(x, \lambda) \mathbf{1}_{\lambda \in \mathfrak{v}^*}(\lambda) \in \mathbb{R}^+$ (since finite dimensional representations are scalar operators).

The link of semi-classical measures with the limit of energy densities $|f^\varepsilon(x)|^2 dx$ will be discussed below, it is solved thanks to the notion of ε-oscillating families (see Section 4.2.4).

4.2.4 Time-averaged semi-classical measures

The local observability inequality takes into account time-averaged quadratic quantities of the solution of Schrödinger equation. Physically, it corresponds to an observation, i.e. the measurement of an observable during a certain time. For example, when $\mathbb{V} = 0$, the right-hand side of inequality (4.6) can be expressed with the set of observables introduced in the previous section using the symbol $\sigma(x, \lambda) = \mathbf{1}_{x \in M} \chi(H(\lambda))$ (see (4.15) for a definition of $H(\lambda)$). Therefore, when considering time-dependent families, as solutions to the Schrödinger equation (4.3), we are interested in the limits of time-averaged quantities: let $(u^\varepsilon)_{\varepsilon>0}$ be a bounded family in $L^\infty(\mathbb{R}, L^2(M))$, $\theta \in L^1(\mathbb{R})$ and $\sigma \in \mathcal{A}_0$, we define

$$\ell_\varepsilon(\theta, \sigma) = \int_{\mathbb{R}} \theta(t) \left(\mathrm{Op}_\varepsilon(\sigma) u^\varepsilon(t), u^\varepsilon(t) \right)_{L^2(M)} dt$$

and we are interested in the limit as ε goes to 0 of these quantities.

When introduced, semi-classical measures were first used for systems with a semi-classical time scaling, i.e. involving $\varepsilon \partial_t$ derivatives, which is not the case here when multiplying the equation (4.3) by ε^2. It is then difficult to derive results for the semi-classical measures at each time t. However, one can deduce results for the time-averaged semi-classical measures that hold almost everywhere in time. Indeed, these measures satisfy important geometric properties that can lead to their identification (for example in Zoll manifolds). This was first remarked by [Mac10] and lead to important results in control [AM14, AFM15, MR19], but also for example in the analysis of dispersion effects of operators arising in solid state physics [CFM19, CFM20]. This approach has been extended to H-type groups in [FF21] and, arguing in the same manner as for the proof of Theorem 2.8 therein, we obtain the next result on the nilmanifold M.

Proposition 4.11. *Let $(u^\varepsilon)_{\varepsilon>0}$ be a bounded family in $L^\infty(\mathbb{R}, L^2(M))$. There exist a sequence $(\varepsilon_k) \in (\mathbb{R}_+^*)^{\mathbb{N}}$ with $\varepsilon_k \underset{k\to+\infty}{\longrightarrow} 0$ and a map $t \mapsto \Gamma_t d\gamma_t$ in $L^\infty(\mathbb{R}, \mathcal{M}_{ov}^+(M \times \widehat{G}))$ such that we have for all $\theta \in L^1(\mathbb{R})$ and $\sigma \in \mathcal{A}$,*

$$\int_{\mathbb{R}} \theta(t)(\mathrm{Op}_{\varepsilon_k}(\sigma)u^{\varepsilon_k}(t), u^{\varepsilon_k}(t))_{L^2(M)}dt \underset{k\to+\infty}{\longrightarrow} \int_{\mathbb{R}\times M\times\widehat{G}} \theta(t)\mathrm{Tr}(\sigma(x,\lambda)\Gamma_t(x,\lambda))d\gamma_t(x,\lambda)dt.$$

Given the sequence $(\varepsilon_k)_{k\in\mathbb{N}}$, the map $t \mapsto \Gamma_t d\gamma_t$ is unique up to equivalence. Besides,

$$\int_{\mathbb{R}}\int_{M\times\widehat{G}} \mathrm{Tr}(\Gamma_t(x,\lambda))d\gamma_t(x,\lambda)dt \leqslant \limsup_{\varepsilon\to 0} \|u^\varepsilon\|^2_{L^\infty(\mathbb{R},L^2(M))}.$$

ε-oscillating families

The link between semi-classical measures and the weak limits of time-averaged energy densities is solved thanks to the notion of ε-oscillation. Let $(u^\varepsilon)_{\varepsilon>0}$ be a bounded family in $L^\infty(\mathbb{R}, L^2(M))$. We say that the family $(u^\varepsilon)_{\varepsilon>0}$ is *uniformly ε-oscillating* when we have for all $T > 0$,

$$\limsup_{\varepsilon\to 0} \sup_{t\in[-T,T]} \left\|\mathbf{1}_{-\varepsilon^2\Delta_M>R}u^\varepsilon(t)\right\|_{L^2(M)} \underset{R\to+\infty}{\longrightarrow} 0.$$

Proposition 4.12. *[[FF21]Proposition 5.3] Let $(u^\varepsilon) \in L^\infty(\mathbb{R}, L^2(M))$ be a uniformly ε-oscillating family admitting a time-averaged semi-classical measure $t \mapsto \Gamma_t d\gamma_t$ for the sequence $(\varepsilon_k)_{k\in\mathbb{N}}$. Then for all $\phi \in \mathcal{C}^\infty(M)$ and $\theta \in L^1(\mathbb{R})$,*

$$\lim_{k\to+\infty} \int_{\mathbb{R}\times M} \theta(t)\phi(x)|u^{\varepsilon_k}(t,x)|^2 dxdt = \int_{\mathbb{R}} \theta(t) \int_{M\times\widehat{G}} \phi(x)\mathrm{Tr}\left(\Gamma_t(x,\lambda)\right) d\gamma_t(x,\lambda)\, dt,$$

Semi-classical measures for families of Schrödinger equations

Families of solutions to the Schrödinger equation (4.3) have special features. We recall that in the (non compact) group G, the operator

$$H(\lambda) = |\lambda| \sum_{j=1}^{d} \left(-\partial_{\xi_j}^2 + \xi_j^2\right)$$

introduced in (4.15) is the Fourier resolution of the sub-Laplacian $-\Delta_G$ above $\lambda \in \mathfrak{z}^* \setminus \{0\}$. Up to a constant, this is a quantum harmonic oscillator with discrete spectrum $\{|\lambda|(2n+d), n \in \mathbb{N}\}$ and finite dimensional eigenspaces. For each eigenvalue $|\lambda|(2n + d)$, we denote by $\Pi_n^{(\lambda)}$ and $\mathcal{V}_n^{(\lambda)}$ the corresponding spectral orthogonal projection and eigenspace. Even though the spectral resolution of $-\Delta_G$ and $-\Delta_M$ are quite different, we shall use the operator $H(\lambda)$ as one uses the function $\xi \mapsto |\xi|^2$ on the phase space of the torus $\mathbb{T}^d$, when studying the operator $-\Delta_{\mathbb{T}^d}$.

Proposition 4.13. *Assume $\Gamma_t d\gamma_t$ is associated with a family of solutions to* (4.3)*.*

1. *For* $(x,\lambda) \in M \times \mathfrak{z}^*$

$$\Gamma_t(x,\lambda) = \sum_{n\in\mathbb{N}} \Gamma_{n,t}(x,\lambda) \;\; with \;\; \Gamma_{n,t}(x,\lambda) := \Pi_n^{(\lambda)} \Gamma_t(x,\lambda) \Pi_n^{(\lambda)}. \tag{4.21}$$

Moreover, the map $(t,x,\lambda) \mapsto \Gamma_{n,t}(x,\lambda) d\gamma_t(x,\lambda)$ *defines a continuous function from* $\mathbb{R}$ *into the set of distributions on* $M \times (\mathfrak{z}^* \setminus \{0\})$ *valued in the finite dimensional space* $\mathcal{L}(\mathcal{V}_n^{(\lambda)})$ *which satisfies*

$$\left(\partial_t - (n + \frac{d}{2}) \mathcal{Z}^{(\lambda)}\right) (\Gamma_{n,t}(x,\lambda) d\gamma_t(x,\lambda)) = 0 \tag{4.22}$$

2. *For* $(x,(0,\omega)) \in M \times \mathfrak{v}^*$*, the scalar measure* $\Gamma_t d\gamma_t$ *is invariant under the flow*

$$\Xi^s : (x,\omega) \mapsto (x \mathrm{Exp}(s\omega \cdot V), \omega).$$

Here, $\omega \cdot V = \sum_{j=1}^{2d} \omega_j V_j$ *where* ω_j *denote the coordinates of* ω *in the dual basis of* V*.*

The proof of this proposition follows ideas from [FF21] that we adapt to our situation. We give some elements on the proof of this Proposition in Section 4-A.2, in particular we explain the continuity of the map $t \mapsto \Gamma_t d\gamma_t$.

We have now all the tools that we shall use for proving Theorem 4.2 in the next two sections.

4.3 Proof of the sufficiency of the geometric conditions

We prove here the first part of Theorem 4.2, that if U satisfies condition **(A)**, $T_{\mathrm{GCC}}(U) < +\infty$ and $T > T_{\mathrm{GCC}}(U)$, then the Schrödinger equation (4.3) is observable on U in time T.

4.3.1 Proof of localized observability.

We argue by contradiction. If (4.6) is false, then there exist $(u_0^k)_{k\in\mathbb{N}}$ and $(h_k)_{k\in\mathbb{N}}$ such that $u_0^k = \mathcal{P}_{h_k} u_0^k$,

$$\|u_0^k\|_{L^2(M)} = 1 \;\text{ and }\; \int_0^T \left\| \mathrm{e}^{it(\frac{1}{2}\Delta_M + \mathbb{V})} \mathcal{P}_{h_k} u_0^k \right\|_{L^2(U)}^2 dt \underset{k\to+\infty}{\longrightarrow} 0. \tag{4.23}$$

Because $u_0^k = \mathcal{P}_{h_k} u_0^k$ with χ compactly supported in an annulus (see (4.5)) and $\mathbb{V}$ is bounded, the family u_0^k is h_k-oscillating in the sense of Section 4.2.4 and so it is for

$$\psi_k(t) = \mathrm{e}^{it(\frac{1}{2}\Delta_M + \mathbb{V})} \mathcal{P}_{h_k} u_0^k.$$

We consider (after extraction of a subsequence if necessary), the semi-classical measure $\Gamma_t d\gamma_t$ of $\psi_k(t)$ given by Proposition 4.11 and satisfying the properties listed in Proposition 4.13.

Proposition 4.14. *We have the following facts:*

1. *There holds*

$$\int_0^T \int_{U \times \widehat{G}} \mathrm{Tr}(\Gamma_t(x,\lambda)) d\gamma_t(x,\lambda) dt = 0. \tag{4.24}$$

2. γ_t *is supported above* $\mathfrak{z}^* \setminus \{0\}$ *for almost every* $t \in \mathbb{R}$*.*

Proof of Proposition 4.14. To prove (1), let us recall that for $\theta \in L^1(\mathbb{R})$ and $\sigma \in \mathcal{A}_0$,

$$\int_{\mathbb{R}} \theta(t)(\mathrm{Op}_{h_k}(\sigma)\psi_k(t), \psi_k(t))_{L^2(M)}dt \underset{k\to+\infty}{\longrightarrow} \int_{\mathbb{R}\times M\times\widehat{G}} \theta(t)\mathrm{Tr}(\sigma(x,\lambda)\Gamma_t(x,\lambda))d\gamma_t(x,\lambda)dt. \quad (4.25)$$

We take $\varphi_j(x)$ a sequence of smooth non-negative functions converging to $\mathbf{1}_U(x)$ and bounded above by 1, and $\alpha \in C_c^\infty((-1,1))$ non-negative with $\alpha = 1$ in a neighborhood of 0. Since $\psi_k(t)$ is uniformly ε-oscillating for $\varepsilon = h_k$, we have

$$\int_0^T \int_{\mathbb{R}\times M\times\widehat{G}} \mathrm{Tr}(\varphi_j(x)\Gamma_t(x,\lambda))d\gamma_t(x,\lambda)dt = \lim_{R\to+\infty}\lim_{k\to+\infty}\int_0^T \left(\mathrm{Op}_{h_k}(\varphi_j(x)\alpha(R^{-1}H(\lambda)))\psi_k(t), \psi_k(t)\right)_{L^2(M)} dt.$$

Besides, $\mathrm{Op}_{h_k}(\varphi_j(x)\alpha(R^{-1}H(\lambda))) = \varphi_j(x)\alpha(-h_k^2R^{-1}\Delta_M)$, thus

$$\|\mathrm{Op}_{h_k}(\varphi_j(x)\alpha(R^{-1}H(\lambda)))\|_{\mathcal{L}(M)} \leqslant 1$$

and

$$\left|\int_0^T \left(\mathrm{Op}_{h_k}(\varphi_j(x)\alpha(R^{-1}H(\lambda)))\psi_k(t), \psi_k(t)\right)_{L^2(M)}\right| \leqslant \int_0^T \|\psi_k(t)\|^2_{L^2(U)}dt.$$

We deduce from (4.23) that

$$\int_0^T \int_{\mathbb{R}\times M\times\widehat{G}} \mathrm{Tr}(\varphi_j(x)\Gamma_t(x,\lambda))d\gamma_t(x,\lambda)dt = 0.$$

Taking the limit $j \to +\infty$ and using Lebesgue's dominated convergence theorem (since $\Gamma_t d\gamma_t \geqslant 0$), we get (4.24).

Point (2) follows from Point (1), the positivity of $\Gamma_t d\gamma_t$, Assumption (A) and Point (2) of Proposition 4.13. □

Set

$$\gamma_{n,t}(x,\lambda) = \mathrm{Tr}\left(\Gamma_{n,t}(x,\lambda)\right)\gamma_t(x,\lambda).$$

We have obtained

$$0 = \sum_{n\in\mathbb{N}}\int_0^T\int_{U\times\widehat{G}} \mathrm{Tr}(\Gamma_{n,t}(x,\lambda))d\gamma_t(x,\lambda)dt = \sum_{n\in\mathbb{N}}\int_0^T\int_{U\times\widehat{G}} d\gamma_{n,t}(x,\lambda)dt$$

whence, the positivity of Γ_t (and thus of $\gamma_{n,t}$) yields

$$\int_{U\times\mathfrak{z}^*} d\gamma_{n,t}(x,\lambda) = 0, \text{ for almost every } t\in[0,T], \ \forall n\in\mathbb{N},$$

where we have also used that the support of $d\gamma_{n,t}$ is above $\mathfrak{z}^*$.

We now use transport equation (4.22). For $n \in \mathbb{N}$ and $\lambda \in \mathfrak{z}^* \setminus \{0\}$, we set

$$Z_n(\lambda) = (n + \frac{d}{2})\mathcal{Z}^{(\lambda)}$$

and we have

$$|Z_n(\lambda)| = n + \frac{d}{2}.$$

We introduce the map Φ_n^s defined for $s \in \mathbb{R}$ and $n \in \mathbb{N}$ as an application from $M \times (\mathfrak{z}^* \setminus \{0\})$ to itself by

$$\Phi_n^s : (x, \lambda) \mapsto (\mathrm{Exp}[sZ_n(\lambda)]x, \lambda) \, .$$

The flows Φ_n^s and Φ_0^s are related by

$$\Phi_n^s(x,\lambda) = \Phi_0^{s'}(x,\lambda), \qquad s' = \left(\frac{2n}{d} + 1\right) s.$$

The transport equation (4.22) implies that for any interval I and any $\Lambda \subset M \times (\mathfrak{z}^* \setminus \{0\})$,

$$\frac{d}{ds}\left(\int_{(I+s)\times\Phi_n^s(\Lambda)} d\gamma_{n,t} dt\right) = 0,$$

which means

$$\int_{(I+s)\times\Phi_n^s(\Lambda)} d\gamma_{n,t} dt = \int_{I\times\Lambda} d\gamma_{n,t} dt. \tag{4.26}$$

Since $T > T_{\mathrm{GCC}}(U)$, we may choose T' such that $T_{\mathrm{GCC}}(U) < T' < T$ and (H-GCC) holds in time T'. Assume that there exists τ with $0 < \tau < T - T'$ such that

$$\int_0^\tau \int_{M\times\mathfrak{z}^*} d\gamma_t dt > 0. \tag{4.27}$$

We seek for a contradiction.

Writing $\gamma_t = \sum_{n=0}^\infty \gamma_{n,t}$, with all $\gamma_{n,t}$ being non-negative Radon measures on $M \times (\mathfrak{z}^* \setminus \{0\})$ (since Point 2 of Proposition 4.14 ensures that it has no mass on the trivial representation), we see that there exists $n_0 \in \mathbb{N}$ and a bounded open subset $\Lambda \subset M \times (\mathfrak{z}^* \setminus \{0\})$ such that

$$\int_0^\tau \int_\Lambda d\gamma_{n_0,t} dt > 0.$$

Fix $(x, \lambda) \in \Lambda$ and $s \in (0, T')$ such that $\Phi_0^s((x, \lambda)) \in U \times \mathfrak{z}^*$. Note that, making Λ smaller if necessary, by continuity of the flow and using that U is open, $\Phi_0^s((x', \lambda')) \in U \times \mathfrak{z}^*$ for any $(x', \lambda') \in \Lambda$. Therefore $\Phi_{n_0}^{s(n_0)}((x', \lambda')) \in U \times \mathfrak{z}^*$ for any $(x', \lambda') \in \Lambda$, where $s(n_0) = \frac{sd}{2n_0+d}$ (with a slight abuse of notation).

From (4.24), we get

$$\gamma_{n_0,t}(\Phi_{n_0}^{s(n_0)}(\Lambda)) = 0, \;\; a.e. \; t \in (0, T),$$

and in particular

$$\int_{s(n_0)}^T \int_{\Phi_{n_0}^{s(n_0)}(\Lambda)} d\gamma_{n_0,t} dt = 0.$$

Therefore, by (4.26),

$$\int_0^{T-s(n_0)} \int_\Lambda d\gamma_{n,t} dt = 0.$$

Since $\tau < T - T' < T - s(n_0)$, we get

$$\int_0^\tau \int_\Lambda d\gamma_{n,t} dt = 0$$

which is a contradiction. Therefore

$$\int_0^\tau \int_{M\times\mathfrak{z}^*} d\gamma_t dt = 0.$$

This implies $\gamma_t = 0$ for almost every $t \in (0, \tau)$. In turn, this contradicts the fact that $\|\psi_k(t)\|_{L^2} = 1$. Therefore (4.6) holds.

Remark 4.15. Assumption (A) exactly corresponds to the usual Geometric Control Condition which is known to be a sufficient condition for the control/observation of the Riemannian Schrödinger equation (see [Leb92b]), but this time it is assumed to hold for the sub-Riemannian geodesic flow. It is well known that, in the Riemaninan setting, this condition is not always necessary : it is not for the Euclidean torus (see [Jaf90, AM14, BZ12]) while it is for Zoll manifolds [Mac11] (these manifolds have geodesics that are all periodic); so, it depends on the manifold. Thus, in the case of general subelliptic Schrödinger equations, it is likely that an Assumption such as (A) has to be required in some cases for proving the observability of the Schrödinger equation. However, as already mentioned in the introduction, we tend to think that in the particular case considered in this chapter (quotients of H-type groups), Theorem 4.2 still holds without this assumption. Assumption (A) has been used in the proof of Point (2) of Proposition 4.14, and it is the only place of the chapter where we use it. By analogy with the results of [AM14, AFM15, BS19], it is likely that as in [BS19, Section 7], a key argument should be a reduction to a problem on the Euclidian torus, as those studied in [AFM15] for example. Then, the semiclassical analysis of this reduced problem would show that the part of the measure γ_t located above $M \times \mathfrak{v}^*$ vanishes. That would prove that H-type GCC alone is enough and would avoid the use of Assumption (A).

4.3.2 Proof of weak observability

We prove here (4.6) $\Longrightarrow$ (4.7).

Consider a partition of unity over the positive real half-line $\mathbb{R}^+$:

$$\forall x \in \mathbb{R}^+, \quad 1 = \chi_0(x)^2 + \sum_{j=1}^{\infty} \chi_j(x)^2 \tag{4.28}$$

where, for $j \geqslant 1$, $\chi_j(x) = \chi(2^{-j}x)$ with $\chi \in C_c^\infty((1/2, 2), [0,1])$. To construct such a partition of unity, consider $\psi \in C_c^\infty((-2,2),[0,1])$ such that $\psi \equiv 1$ on a neighborhood of $[-1,1]$, and set $\chi(x) = \sqrt{\psi(x) - \psi(2x)}$ for $x \geqslant 0$, which is smooth for well-chosen ψ. Finally, define $\chi_0(x)$ for $x \geqslant 0$ by $\chi_0(x)^2 = 1 - \sum_{j=1}^\infty \chi_j(x)^2$, so that $\chi_0(x) = 0$ for $x \geqslant 2$. Then (4.28) holds.

We follow the proof of [BZ12, Proposition 4.1]. Set $h_j = 2^{\frac{-j}{2}}$ for $j \geqslant 1$, and note that $\mathcal{P}_{h_j} = \chi_j(-(\frac{1}{2}\Delta_M + \mathbb{V}))$. We choose K so that $h_K \leqslant h_0$, where h_0 is taken so that (4.6) holds for $0 < h \leqslant h_0$. We take $\varepsilon > 0$ such that $T' + 2\varepsilon < T$ and $\psi \in C_c^\infty((0,T),[0,1])$ with $\psi = 1$ on a neighborhood of $[\varepsilon, T' + 2\varepsilon]$. Then

$$\begin{aligned}
\|u_0\|^2_{L^2(M)} &= \sum_{j=0}^{\infty} \left\| \chi_j \left(-\frac{1}{2}\Delta_M + \mathbb{V} \right) u_0 \right\|^2_{L^2(M)} \\
&= \sum_{j=0}^{K} \|\mathcal{P}_{h_j} u_0\|^2_{L^2(M)} + \sum_{j=K+1}^{\infty} \|\mathcal{P}_{h_j} u_0\|^2_{L^2(M)} \\
&\leqslant C \left\| \left(\mathrm{Id} - (\frac{1}{2}\Delta_M + \mathbb{V}) \right)^{-1} u_0 \right\|^2_{L^2(M)} + \sum_{j=K+1}^{\infty} \|\mathcal{P}_{h_j} u_0\|^2_{L^2(M)} \\
&\leqslant C \|(\mathrm{Id} - \Delta_M)^{-1} u_0\|^2_{L^2(M)} + C \sum_{j=K+1}^{\infty} \left\| \psi(t) e^{it(\frac{1}{2}\Delta_M + \mathbb{V})} \mathcal{P}_{h_j} u_0 \right\|^2_{L^2((0,T)\times U)}
\end{aligned}$$

where in the third line we bounded above the low frequencies with a constant $C = C_K$, and in the last line we used (4.6) (with the term on U being integrated for $t \in (\varepsilon, T' + 2\varepsilon)$, which is of length $> T'$, see Remark 4.4). Note that we also used the fact that $\mathbb{V}$ is analytic and thus bounded, and therefore the resolvents of the operators $\frac{1}{2}\Delta_M + \mathbb{V}$ and Δ_M are comparable in L^2 norm. Using equation (4.3), we may change $\mathcal{P}_{h_j} = \chi_j(-(\frac{1}{2}\Delta_M + \mathbb{V}))$ into $\chi_j(-D_t)$ where $D_t = \partial_t/i$. We get

$$\|u_0\|^2_{L^2(M)} \leqslant C\|(\mathrm{Id} - \Delta_M)^{-1}u_0\|^2_{L^2(M)} + C\sum_{j=K+1}^{\infty}\left\|\psi(t)\chi_j(-D_t)\mathrm{e}^{it(\frac{1}{2}\Delta_M+\mathbb{V})}u_0\right\|^2_{L^2((0,T)\times U)} \tag{4.29}$$

If $\widetilde{\psi} \in C_c^\infty((0,T),[0,1])$ satisfies $\widetilde{\psi} = 1$ on $\mathrm{supp}(\psi)$, we note that

$$\begin{aligned}\psi(t)\chi_j(-D_t) &= \psi(t)\chi_j(-D(t))\widetilde{\psi}(t) + \psi(t)[\widetilde{\psi}(t), \chi_j(-D_t)]\\ &= \psi(t)\chi_j(-D(t))\widetilde{\psi}(t) + E_j(t, D_t)\end{aligned} \tag{4.30}$$

where E_j is smoothing, i.e.,

$$\partial^\alpha E_j = O(\langle t\rangle^{-N}\langle\tau\rangle^{-N}2^{-Nj})$$

for any $\alpha \in \mathbb{N}$, any $N \in \mathbb{N}$ and uniformly in j. This fact follows from the remark that, on the support of ψ, $\widetilde{\psi}$ is constant and therefore the bracket vanishes.

Therefore, integrating by parts in the time variable in the second term of the right-hand side and absorbing the error terms $E_j(t, D_t)$ in $\|(\mathrm{Id} - \Delta_M)^{-1}u_0\|^2_{L^2}$, we get

$$\begin{aligned}
\|u_0\|^2_{L^2(M)} &\leqslant C\|(\mathrm{Id} - \Delta_M)^{-1}u_0\|^2_{L^2(M)} + C\sum_{j=K+1}^{\infty}\|\psi(t)\chi_j(-D_t)\widetilde{\psi}(t)\mathrm{e}^{it(\frac{1}{2}\Delta_M+\mathbb{V})}u_0\|^2_{L^2((0,T)\times U)}\\
&\leqslant C\|(\mathrm{Id} - \Delta_M)^{-1}u_0\|^2_{L^2(M)} + C\sum_{j=K+1}^{\infty}\|\chi_j(-D_t)\widetilde{\psi}(t)\mathrm{e}^{it(\frac{1}{2}\Delta_M+\mathbb{V})}u_0\|^2_{L^2((0,T)\times U)}\\
&= C\|(\mathrm{Id} - \Delta_M)^{-1}u_0\|^2_{L^2(M)} + C\sum_{j=K+1}^{\infty}\left(\chi_j(-D_t)^2\widetilde{\psi}(t)\mathrm{e}^{it(\frac{1}{2}\Delta_M+\mathbb{V})}u_0\,,\,\widetilde{\psi}(t)\mathrm{e}^{it(\frac{1}{2}\Delta_M+\mathbb{V})}u_0\right)_{L^2((0,T)\times U)}\\
&\leqslant C\|(\mathrm{Id} - \Delta_M)^{-1}u_0\|^2_{L^2(M)} + C\left(\sum_{j=0}^{\infty}\chi_j(-D_t)^2\widetilde{\psi}(t)\mathrm{e}^{it(\frac{1}{2}\Delta_M+\mathbb{V})}u_0\,,\,\widetilde{\psi}(t)\mathrm{e}^{it(\frac{1}{2}\Delta_M+\mathbb{V})}u_0\right)_{L^2((0,T)\times U)}\\
&\leqslant C\|(\mathrm{Id} - \Delta_M)^{-1}u_0\|^2_{L^2(M)} + C\|\mathrm{e}^{it(\frac{1}{2}\Delta_M+\mathbb{V})}u_0\|^2_{L^2((0,T)\times U)}
\end{aligned}$$

where we used (4.28) in the last line. This concludes the proof of (4.7).

4.3.3 Proof of observability

We prove here (4.7) $\Longrightarrow$ (4.4), which concludes the proof of the sufficiency of the geometric condition **H-type GCC**. We follow the classical Bardos-Lebeau-Rauch argument, see for example [BZ12].

For $\delta \geqslant 0$, we set

$$\mathcal{N}_\delta = \{u_0 \in L^2(M) \ \mid\ e^{it(\frac{1}{2}\Delta_M+\mathbb{V})}u_0 \equiv 0 \text{ on } (0, T-\delta)\times U\}.$$

Lemma 4.16. *There holds $\mathcal{N}_0 = \{0\}$.*

Proof. Let $u_0 \in \mathcal{N}_0$. We define

$$v_{\epsilon,0} = \frac{1}{\epsilon}\left(e^{i\epsilon(\frac{1}{2}\Delta_M + \mathbb{V})} - \mathrm{Id}\right) u_0. \tag{4.31}$$

If $\epsilon \leqslant \delta$, then $e^{it(\frac{1}{2}\Delta_M+\mathbb{V})}v_{\epsilon,0} = 0$ on $(0, T-\delta) \times U$. We write u_0 in terms of orthonormal eigenvectors f_λ of $\frac{1}{2}\Delta_M + \mathbb{V}$ (associated with $\lambda \in \mathrm{Sp}$, the spectrum of $\frac{1}{2}\Delta_M + \mathbb{V}$):

$$u_0 = \sum_{\lambda \in \mathrm{Sp}} u_{0,\lambda} f_\lambda$$

For small enough α, β, applying (4.7) with a slightly smaller T, we have

$$\begin{aligned}
\|v_{\alpha,0} - v_{\beta,0}\|_{L^2}^2 &\leqslant C\|(\mathrm{Id} - (\frac{1}{2}\Delta_M + \mathbb{V}))^{-1}(v_{\alpha,0} - v_{\beta,0})\|_{L^2}^2 \\
&\leqslant C \sum_{\lambda \in \mathrm{Sp}} \left| \frac{e^{i\alpha\lambda} - 1}{\alpha} - \frac{e^{i\beta\lambda} - 1}{\beta} \right|^2 (1+\lambda)^{-2}|u_{0,\lambda}|^2 \\
&\leqslant C \sum_{\lambda \in \mathrm{Sp}} \lambda^2 |\alpha - \beta|^2 (1+\lambda)^{-2}|u_{0,\lambda}|^2 \\
&\leqslant C|\alpha - \beta|^2.
\end{aligned}$$

Hence there exists $v_0 \in L^2(M)$ such that $v_0 = \lim_{\alpha \to 0} v_{\alpha,0}$ where the limit is taken in $L^2(M)$. This limit is necessarily in $\mathcal{N}_\delta$ for all $\delta > 0$, hence in $\mathcal{N}_0$. Moreover, thanks to (4.31), there holds in the sense of distributions

$$e^{it(\frac{1}{2}\Delta_M+\mathbb{V})}v_0 = \partial_t e^{it(\frac{1}{2}\Delta_M+\mathbb{V})}u_0$$

and therefore

$$v_0 = i(\frac{1}{2}\Delta_M + \mathbb{V})u_0.$$

Therefore $\Delta_M : \mathcal{N}_0 \to \mathcal{N}_0$ is a well-defined operator. Moreover, according to (4.7), on $\mathcal{N}_0$, we have

$$\|(\mathrm{Id} - \Delta_M) \cdot \|_{L^2(M)} \leqslant C\| \cdot \|_{L^2(M)}$$

and, by compact embedding (see Lemma 4.18 below), the unit ball of $\mathcal{N}_0 \subset L^2(M)$ is compact. Hence $\mathcal{N}_0$ is finite dimensional and there exists an eigenfunction $w \in \mathcal{N}_0$ of $\frac{1}{2}\Delta_M + \mathbb{V} : \mathcal{N}_0 \to \mathcal{N}_0$, i.e.,

$$(\frac{1}{2}\Delta_M + \mathbb{V})w = \mu w, \quad w_{|U} = 0$$

By a standard unique continuation principle (see [Bon69] and [LL20, Theorem 1.12]), since $\mathbb{V}$ and Δ_M are analytic (see [BLU07, Section 5.10] for example), we conclude that $w = 0$, hence $\mathcal{N}_0 = \{0\}$. □

Remark 4.17. To our knowledge, the unique continuation principle used in the above proof is only known when $\mathbb{V}$ is analytic. In C^∞ regularity, counterexamples to the unique continuation principle exist, see [Ba86]. However, the result of Theorem 4.2 holds as soon as a unique continuation principle holds for $\frac{1}{2}\Delta_M + \mathbb{V}$.

Lemma 4.18. *Set*

$$\mathcal{H}(M) = \{u \in L^2(M) \mid (\mathrm{Id} - \Delta_M)u \in L^2(M)\}.$$

Then $\mathcal{H}(M) \hookrightarrow L^2(M)$ with compact embedding.

Proof. By [LL20, Corollary B.1], we have $\|u\|_{H^1(M)} \leqslant \|(\mathrm{Id}-\Delta_M)u\|_{L^2(M)}$ since G is step 2. Therefore, $\mathcal{H}(M) \hookrightarrow H^1(M)$ continuously. The result then follows by the Rellich-Kondrachov (compact embedding) theorem. □

Assume that (4.4) does not hold. Then there exists a sequence $(u_0^k)_{k\in\mathbb{N}}$ such that

$$\|u_0^k\|_{L^2(M)} = 1 \ \text{ and } \ \int_0^T \left\| \mathrm{e}^{it(\frac{1}{2}\Delta_M+\mathbb{V})} u_0^k \right\|_{L^2(U)}^2 dt \underset{k\to+\infty}{\longrightarrow} 0. \tag{4.32}$$

Since $(u_0^k)_{k\in\mathbb{N}}$ is bounded in $L^2(M)$, we can extract from $(u_0^k)_{k\in\mathbb{N}}$ a subsequence which converges weakly to some u^∞ in $L^2(M)$. By Lemma 4.18, we then have $(\mathrm{Id}-\Delta_M)^{-1}u_0^k \to (\mathrm{Id}-\Delta_M)^{-1}u^\infty$ strongly in $L^2(M)$. Moreover, the second convergence in (4.32) gives $u^\infty \in \mathcal{N}_0$. Thanks to (4.7), we know that

$$\|u_0^k\|_{L^2(M)}^2 \leqslant C_1 \int_0^T \left\| \mathrm{e}^{it(\frac{1}{2}\Delta_M+\mathbb{V})} u_0^k \right\|_{L^2(U)}^2 dt + C_1 \left\| (\mathrm{Id}-\Delta_M)^{-1} u_0^k \right\|_{L^2(M)}^2 .$$

Therefore, taking the limit $k \to +\infty$, we get

$$1 \leqslant C_1 \|(\mathrm{Id}-\Delta_M)^{-1}u^\infty\|_{L^2(M)}^2.$$

Therefore $u^\infty \neq 0$, which contradicts Lemma 4.16 since $u^\infty \in \mathcal{N}_0$. Hence, (4.4) holds.

4.4 Non-commutative wave packets and the necessity of the geometric control

In this section, we conclude the proof of Theorem 4.2 and prove the necessity of the condition **(H-GCC)** (for $\overline{U}$). We use special data that we call non-commutative wave packets that we first introduce, together with their properties, on which we also elaborate in Section 4-A.3. Then, we conclude to the necessity of the H-type GCC.

4.4.1 Non-commutative wave packets

Let us first briefly recall basic facts about classical (Euclidean) wave packets. Given $(x_0,\xi_0) \in \mathbb{R}^d \times \mathbb{R}^d$ and $a \in \mathcal{S}(\mathbb{R}^d)$, we consider the family (indexed by ε) of functions

$$u^\varepsilon_{\mathrm{eucl}}(x) = \varepsilon^{-d/4} a\left(\frac{x-x_0}{\sqrt{\varepsilon}}\right) \mathrm{e}^{\frac{i}{\varepsilon}\xi_0\cdot(x-x_0)}, \ \ x \in \mathbb{R}^d. \tag{4.33}$$

Such a family is called a (Euclidean) wave packet.

The oscillation along ξ_0 is forced by the term $\mathrm{e}^{\frac{i}{\varepsilon}\xi_0\cdot(x-x_0)}$ and the concentration on x_0 is performed at the scale $\sqrt{\varepsilon}$ for symmetry reasons : the ε-Fourier transform of $u^\varepsilon_{\mathrm{eucl}}$, $\varepsilon^{-d/2}\widehat{u}^\varepsilon_{\mathrm{eucl}}(\xi/\varepsilon)$ presents a concentration on ξ_0 at the scale $\sqrt{\varepsilon}$. The regularity of the wave packets makes them a flexible tool. Besides, taking a compactly supported in the interior of a unit cell for the torus, one can generalize their definition to the case of the torus by extending them by periodicity. For example, let us consider the torus $\mathbb{T}^d = \mathbb{R}^d/(2\pi\mathbb{Z})^d$, we choose $a \in \mathcal{C}_c^\infty((-\pi,\pi)^d)$ and we define $a_\varepsilon(x)$ as

$$a_\varepsilon(x) = a\left(\frac{x-x_0}{\sqrt{\varepsilon}}\right).$$

We consider the periodisation operator $\mathbb{P}$ which associates with a function φ compactly supported inside a set of the form $x_0 + (-\pi,\pi)^d$ the periodic function defined on the sets $k + x_0 + (-\pi,\pi)^d$

for $k \in (2\pi\mathbb{Z})^d$ by $\mathbb{P}\varphi(x) = \varphi(x-k)$. Then, the definition of a wave packet extends to functions on the torus by setting

$$u^\varepsilon_{\rm torus}(x) = \varepsilon^{d/4}\mathbb{P}a_\varepsilon(x)\mathrm{e}^{\frac{i}{\varepsilon}\xi_0\cdot(x-x_0)}.$$

We introduce here a generalization of these wave packets to the non-commutative setting of Lie groups and nilmanifolds, in the context of H-type groups, which is strongly inspired by [FF19]. For $x \in G$, we write

$$x = \mathrm{Exp}(V+Z) = x_{\mathfrak z}x_{\mathfrak v} = x_{\mathfrak v}x_{\mathfrak z} \text{ with } V \in \mathfrak v,\;\; Z \in \mathfrak z,$$

where

$$x_{\mathfrak z} = \mathrm{e}^Z \in G_{\mathfrak z} := \mathrm{Exp}(\mathfrak z) \text{ and } x_{\mathfrak v} = \mathrm{e}^V \in G_{\mathfrak v} := G/G_{\mathfrak z}.$$

The concentration is performed by use of dilations: with $a \in \mathcal{C}^\infty_c(G)$, we associate

$$a_\varepsilon(x) = a\left(\delta_{\varepsilon^{-1/2}}(x)\right).$$

The oscillations are forced by using coefficients of the representations, in the spirit of [Ped94]: with $\lambda_0 \in \mathfrak z^*$, $\Phi_1,\ \Phi_2$ smooth vectors in the space of representations, i.e. in $\mathcal{S}(\mathbb{R}^d)$, we associate the oscillating term

$$e_\varepsilon(x) = \left(\pi^{\lambda_\varepsilon}_x\Phi_1, \Phi_2\right),\;\; \lambda_\varepsilon = \frac{\lambda_0}{\varepsilon^2}.$$

We restrict to $\varepsilon \in (0,1)$ and define the periodisation operator $\mathbb{P}$ in analogy with the case of the torus described above, using the multiplication on the left by elements of $\widetilde\Gamma$. We consider a subset $\mathcal{B}$ of G which is a neighborhood of 1_G and such that $\cup_{\gamma\in\widetilde\Gamma}(\gamma\mathcal{B}) = G$ and we choose functions a that are in $\mathcal{C}^\infty_c(\mathcal{B})$ (in other words, their support is a subset of the interior of $\mathcal{B}$).

Proposition 4.19. *Let $\Phi_1, \Phi_2 \in \mathcal{S}(\mathbb{R}^d)$, $a \in C^\infty_c(\mathcal{B})$, $x_0 \in M$, $\lambda_0 \in \mathfrak z^* \setminus \{0\}$. Then, there exists $\varepsilon_0 > 0$ such that the family $(v^\varepsilon)_{\varepsilon\in(0,\varepsilon_0)}$ defined by*

$$v^\varepsilon(x) = |\lambda_\varepsilon|^{d/2}\,\varepsilon^{-p/2}\,\mathbb{P}(e_\varepsilon a_\varepsilon)(x_0^{-1}x),$$

is a bounded ε-oscillating family in $L^2(M)$ with bounded ε-derivatives and momenta:

$$\forall k \in \mathbb{N},\;\; \exists C_k > 0,\;\; \forall \varepsilon > 0,\;\; \|(-\varepsilon^2\Delta_M)^{k/2}v^\varepsilon\|_{L^2(M)} \leqslant C_k. \tag{4.34}$$

Moreover, $(v^\varepsilon)_{\varepsilon\in(0,\varepsilon_0)}$ has only one semi-classical measure $\Gamma d\gamma$ where

$$\gamma = c_a\,\delta(x-x_0)\otimes\delta(\lambda-\lambda_0),\;\; c_a = \|\Phi_2\|^2\int_{G_{\mathfrak z}}|a(x_{\mathfrak z})|^2dx_{\mathfrak z}, \tag{4.35}$$

and Γ is the operator defined by

$$\Gamma\Phi = \frac{(\Phi,\Phi_1)}{\|\Phi_1\|^2}\Phi_1,\;\; \forall\Phi \in L^2(\mathbb{R}^d).$$

In the following, we shall say that the family v^ε is a wave packet on M with cores (x_0,λ_0), profile a and harmonics (Φ_1,Φ_2), and write

$$v^\varepsilon = WP^\varepsilon_{x_0,\lambda_0}(a,\Phi_1,\Phi_2) = |\lambda_\varepsilon|^{d/2}\,\varepsilon^{-p/2}\,\mathbb{P}(e_\varepsilon a_\varepsilon)(x_0^{-1}x).$$

Remark 4.20. 1. Note that ε_0 is chosen small enough so that for $\varepsilon \in (0,\varepsilon_0)$, the function $G \ni x \mapsto a_\varepsilon(x)$ has support included in a unit cell of G for $\widetilde\Gamma$ and thus $x \mapsto (e_\varepsilon a_\varepsilon)(x_0^{-1}x)$ can be extended by periodicity on G, which defines a function of M.

2. Omitting the periodisation operator $\mathbb{P}$, we construct wave packets on G that also satisfy estimates in momenta
$$\forall k \in \mathbb{N},\ \ \exists C_k > 0,\ \ \forall \varepsilon > 0,\qquad \sum_{1\leqslant p+q\leqslant k} \| |x|^p (-\varepsilon^2 \Delta_G)^{q/2} v^\varepsilon \|_{L^2(G)} \leqslant C_k.$$

3. The coefficient $|\lambda_\varepsilon|^{d/2}\varepsilon^{-p/2}$ guarantees the boundedness in $L^2(M)$ of the family $(v^\varepsilon)_{\varepsilon>0}$.

4. *Characterization of wave packets.* Let $x \in M$ be identified to a point of G and let us fix Φ_1, Φ_2, x_0 and λ_0. Then, v^ε is a wave packet on M if there exist $x_0 \in M$, $\lambda_0 \in \mathfrak{z}^* \setminus \{0\}$, $a \in C_c^\infty(\mathcal{B})$ and $\Phi_1, \Phi_2 \in \mathcal{S}(\mathbb{R}^d)$, such that
$$\begin{aligned} \varepsilon^{Q/4} v^\varepsilon(x_0 \delta_{\sqrt{\varepsilon}}(x)) &= |\lambda_\varepsilon|^{d/2} \varepsilon^{Q/4-p/2} a(x) (\Phi_1, (\pi^{\lambda_0}_{\delta_{\varepsilon^{-1/2}}(x)})^* \Phi_2) \\ &= |\lambda_0|^{d/2} \varepsilon^{-d/2} a(x) (\Phi_1, (\pi^{\lambda_0}_{\delta_{\varepsilon^{-1/2}}(x)})^* \Phi_2). \end{aligned} \tag{4.36}$$

5. *Generalization.* The construction we make here extends to more general Lie groups following ideas from Section 6.4 in [FF19] and [Ped94].

4.4.2 Proof of Proposition 4.19

The proof of Proposition 4.19 is relatively long, and we decompose it into several steps.

The norm of wave packets

By the definition of the periodisation operator $\mathbb{P}$,
$$\int_M |v^\varepsilon(x)|^2 dx = |\lambda_\varepsilon|^d \varepsilon^{-p} \int_G |a_\varepsilon(x_0^{-1}x)|^2 |e_\varepsilon(x_0^{-1}x)|^2 dx.$$
We then use (4.36) and we write
$$\begin{aligned} \|v^\varepsilon\|^2_{L^2(G)} &= |\lambda_0|^d \varepsilon^{-d} \int_G |a(x)|^2 (\pi^{\lambda_0}_{\delta_{\varepsilon^{-1/2}} x} \Phi_1, \Phi_2)|^2 dx \\ &= |\lambda_0|^d \int_G |a(\delta_{\sqrt{\varepsilon}}(x_{\mathfrak{v}}) x_{\mathfrak{z}})|^2 (\pi^{\lambda_0}_{x_{\mathfrak{v}}} \Phi_1, \Phi_2)|^2 dx_{\mathfrak{v}} dx_{\mathfrak{z}} \\ &\leqslant \left(\int_{G_{\mathfrak{z}}} \sup_{y_{\mathfrak{v}} \in G_{\mathfrak{v}}} |a(y_{\mathfrak{v}} x_{\mathfrak{z}})|^2 dx_{\mathfrak{z}} \right) \left(|\lambda_0|^d \int_{G_{\mathfrak{v}}} |(\pi^{\lambda_0}_{x_{\mathfrak{v}}} \Phi_1, \Phi_2)|^2 dx_{\mathfrak{v}} \right). \end{aligned}$$

Let us note that the following relation holds for any $\Phi, \widetilde{\Phi}, \Psi, \widetilde{\Psi} \in \mathcal{S}(\mathbb{R}^d)$:
$$|\lambda_0|^d \int_{G_{\mathfrak{v}}} (\pi^{\lambda_0}_{x_{\mathfrak{v}}} \Phi, \Psi) \overline{(\pi^{\lambda_0}_{x_{\mathfrak{v}}} \widetilde{\Phi}, \widetilde{\Psi})} dx_{\mathfrak{v}} = (\Phi, \widetilde{\Phi}) \overline{(\Psi, \widetilde{\Psi})}. \tag{4.37}$$

Therefore,
$$|\lambda_0|^d \int_{G_{\mathfrak{v}}} |(\pi^{\lambda_0}_{x_{\mathfrak{v}}} \Phi_1, \Phi_2)|^2 dx_{\mathfrak{v}} = \|\Phi_1\|^2 \|\Phi_2\|^2.$$
We deduce that v^ε is uniformly bounded in $L^2(G)$.

The ε-oscillation and the regularity of wave packets.

Straightforward computations give that if $\lambda \in \mathfrak{z}^* \setminus \{0\}$, $\Phi_1, \Phi_2 \in \mathcal{S}(\mathbb{R}^d)$, $x_{\mathfrak{v}} = \mathrm{Exp}[P+Q]$, $x = x_{\mathfrak{v}} x_{\mathfrak{z}}$

$$P = \sum_{j=1}^d p_j P_j^{(\lambda)} \ \text{ and } \ Q = \sum_{j=1}^d q_j Q_j^{(\lambda)},$$

then, for $1 \leqslant j \leqslant d$,

$$\sqrt{|\lambda|}\, q_j \left(\pi_x^\lambda \Phi_1, \Phi_2\right) = \left([\pi_x^\lambda, i\partial_{\xi_j}]\Phi_1, \Phi_2\right), \quad \sqrt{|\lambda|}\, p_j \left(\pi_x^\lambda \Phi_1, \Phi_2\right) = \left([\pi_x^\lambda, \xi_j]\Phi_1, \Phi_2\right). \qquad (4.38)$$

Besides,

$$P_j^{(\lambda)} \left(\pi_x^\lambda \Phi_1, \Phi_2\right) = \sqrt{|\lambda|} \left(\partial_{\xi_j} \pi_x^\lambda \Phi_1, \Phi_2\right) \ \text{ and } \ Q_j^{(\lambda)} \left(\pi_x^\lambda \Phi_1, \Phi_2\right) = i\sqrt{|\lambda|} \left(\xi_j \pi_x^\lambda \Phi_1, \Phi_2\right). \qquad (4.39)$$

For proving this formula for $P_j^{(\lambda)}$, we use (4.2) and we observe

$$\mathrm{Exp}(tP_j^{(\lambda)})\mathrm{Exp}(P+Q+Z) = \mathrm{Exp}(tP_j^{(\lambda)} + P + Q + Z + \frac{t}{2}[P_j^{(\lambda)}, P+Q]).$$

Since $[P_j^{(\lambda)}, Q_j^{(\lambda)}] = \mathcal{Z}^{(\lambda)}$ and for $k \neq j$, $[P_j^{(\lambda)}, P_k^{(\lambda)}] = [P_j^{(\lambda)}, Q_k^{(\lambda)}] = 0$, we deduce

$$\mathrm{Exp}(tP_j^{(\lambda)})\mathrm{Exp}(P+Q+Z) = \mathrm{Exp}(tP_j^{(\lambda)} + P + Q + Z + \frac{t}{2} q_j \mathcal{Z}^{(\lambda)}).$$

Therefore, using $\lambda(\mathcal{Z}^{(\lambda)}) = |\lambda|$, we obtain for $\Phi \in \mathcal{S}(\mathbb{R}^d)$ and $\xi \in \mathbb{R}^d$,

$$\frac{d}{dt}\left(\pi^\lambda_{\mathrm{Exp}(tP_j^{(\lambda)})x} \Phi(\xi)\right)\Big|_{t=0} = \sqrt{|\lambda|}\pi_x^\lambda \partial_{\xi_j}\Phi(\xi) + i|\lambda| q_j \pi_x^\lambda \Phi(\xi) = \sqrt{|\lambda|}\partial_{\xi_j}\pi_x^\lambda \Phi(\xi).$$

The proof for $Q_j^{(\lambda)}$ is similar. We deduce (4.34) and that the family (v^ε) is uniformly ε-oscillating by the Sobolev criteria of Proposition 4.6 in [FF19].

Action of pseudodifferential operators on wave packets.

For studying their semi-classical measure, it is convenient to analyze first the action of pseudodifferential operators on wave packets.

Lemma 4.21. *Let $\Phi_1,\ \Phi_2 \in \mathcal{S}(\mathbb{R}^d)$, $(x_0, \lambda_0) \in G \times \mathfrak{z}^*$, $a \in \mathcal{C}_c^\infty(\mathcal{B})$. Let $\sigma \in \mathcal{A}_0$ compactly supported in an open set U such that $\overline{U}$ is strictly included in a unit cell $\mathcal{B}$ of $\widetilde{\Gamma}$. Then there exist $\varepsilon_1 > 0$ and $c_1 > 0$ such that for all $\varepsilon \in (0, \varepsilon_1)$,*

$$\|\mathrm{Op}_\varepsilon(\sigma) WP^\varepsilon_{x_0,\lambda_0}(a, \Phi_1, \Phi_2) - WP^\varepsilon_{x_0,\lambda_0}(a, \sigma(x_0,\lambda_0)\Phi_1, \Phi_2)\|_{L^2(M)} \leqslant c_1 \sqrt{\varepsilon}.$$

Remark 4.22. The proof we perform below shows that there exist sequences of profiles $(a_j)_{j\in\mathbb{N}}$ and of harmonics $(\Phi_1^{(j)}, \Phi_2^{(j)})_{j\in\mathbb{N}}$ such that for all $N \in \mathbb{N}$,

$$\|\mathrm{Op}_\varepsilon(\sigma) WP^\varepsilon_{x_0,\lambda_0}(a, \Phi_1, \Phi_2) - \sum_{j=0}^N \varepsilon^{\frac{j}{2}} WP^\varepsilon_{x_0,\lambda_0}(a_j, \Phi_1^{(j)}, \Phi_2^{(j)})\|_{L^2(M)} \leqslant c_1 (\sqrt{\varepsilon})^{N+1}.$$

Moreover, by commuting the operator $(-\varepsilon^2\Delta_G)^{s/2}$ with the pseudodifferential operators, one can extend this result in Sobolev spaces. Note also that the same type of expansion holds in G, in refined functional spaces where momenta are controlled:

$$\|\mathrm{Op}_\varepsilon(\sigma)WP^\varepsilon_{x_0,\lambda_0}(a,\Phi_1,\Phi_2)-\sum_{j=0}^N \varepsilon^{\frac{j}{2}}WP^\varepsilon_{x_0,\lambda_0}(a_j,\Phi_1^{(j)},\Phi_2^{(j)})\|_{\Sigma^k_\varepsilon(G)}\leqslant c_1\,\varepsilon^{\frac{N+1}{2}}$$

where Σ^k_ε is the vector space of functions $f\in L^2(G)$ for which the semi-norms

$$\|f\|_{\Sigma^k_\varepsilon}:=\sum_{\ell=0}^k\Big(\||x|^\ell f\|_{L^2(G)}+\|(-\varepsilon^2\Delta_G)^{\ell/2}f\|_{L^2(G)}\Big) \tag{4.40}$$

are finite.

Proof. We first observe that, in view of Remark 4.7, it is enough to prove the result for wave packets in G. Indeed, consider $\chi\in\mathcal{C}_c^\infty(\overline{\mathcal{B}})$ with $\chi\sigma=\sigma$. Then for any function $f\in\mathcal{C}_c^\infty(\overline{\mathcal{B}})$ and $x\in M$ identified to the point x of $G\cap\mathcal{B}$, we have for all $N\in\mathbb{N}$, thanks to (4.20),

$$\begin{aligned}\mathrm{Op}_\varepsilon(\sigma)\mathbb{P}(f)(x)&=\mathrm{Op}_\varepsilon(\sigma)\chi\mathbb{P}(f)(x)+O(\varepsilon^N)\\&=\mathrm{Op}_\varepsilon(\sigma)\chi f(x)+O(\varepsilon^N)=\mathrm{Op}_\varepsilon(\sigma)f(x)+O(\varepsilon^N).\end{aligned}$$

Therefore, we are going to prove the result of Lemma 4.21 for wave packets and pseudodifferential operators in G. Besides, for simplicity, we assume that $\sigma(x,\cdot)$ is the Fourier transform of a compactly supported function. This technical assumption simplifies the proof which extends naturally to symbols that are Fourier transform of Schwartz class functions.

We write

$$\begin{aligned}\mathrm{Op}_\varepsilon(\sigma)v^\varepsilon(x)&=c_0|\lambda_\varepsilon|^{d/2}\varepsilon^{-p/2}\int_{G\times\widehat{G}}\mathrm{Tr}(\pi^\lambda_{y^{-1}x}\sigma(x,\varepsilon^2\lambda))a_\varepsilon(x_0^{-1}y)(\pi^{\lambda_\varepsilon}_{x_0^{-1}y}\Phi_1,\Phi_2)|\lambda|^d d\lambda dy\\&=c_0|\lambda_\varepsilon|^{d/2}\varepsilon^{-p/2}\int_{G\times\widehat{G}}\mathrm{Tr}(\pi^\lambda_{y^{-1}x_0^{-1}x}\sigma(x,\varepsilon^2\lambda))a_\varepsilon(y)(\pi^{\lambda_\varepsilon}_y\Phi_1,\Phi_2)|\lambda|^d d\lambda dy.\end{aligned}$$

where we have performed the change of variable $y\mapsto x_0y$. We now focus on $\varepsilon^{-Q/4}\mathrm{Op}_\varepsilon(\sigma)v^\varepsilon(x_0\delta_{\sqrt{\varepsilon}}x)$ in order to simplify the computations. Note that this quantity is uniformly bounded in $L^2(G)$.

$$\mathrm{Op}_\varepsilon(\sigma)v^\varepsilon(x_0\delta_{\sqrt{\varepsilon}}x)\ =c_0|\lambda_\varepsilon|^{d/2}\varepsilon^{-p/2}\int_{G\times\widehat{G}}\mathrm{Tr}(\pi^\lambda_{y^{-1}\delta_{\sqrt{\varepsilon}}x}\sigma(x_0\delta_{\sqrt{\varepsilon}}x,\varepsilon^2\lambda)a_\varepsilon(y)(\pi^{\lambda_\varepsilon}_y\Phi_1,\Phi_2)|\lambda|^d d\lambda dy.$$

We perform the change of variable $\widetilde{y}=\delta_{\varepsilon^{-1/2}}y$ and $\widetilde{\lambda}=\varepsilon^2\lambda$. We have

$$\pi^\lambda_{y^{-1}\delta_{\sqrt{\varepsilon}}x}=\pi^{\widetilde{\lambda}/\varepsilon^2}_{\delta_{\sqrt{\varepsilon}}(y^{-1}x)}=\pi^{\widetilde{\lambda}}_{\delta_{\varepsilon^{-1/2}}(\widetilde{y}^{-1}x)},\quad \pi^{\lambda_\varepsilon}_y=\pi^{\lambda_0/\varepsilon^2}_{\delta_{\sqrt{\varepsilon}}\widetilde{y}}=\pi^{\lambda_0}_{\delta_{\varepsilon^{-1/2}}(y)}$$

and

$$|\widetilde{\lambda}|^d d\widetilde{\lambda}d\widetilde{y}=\varepsilon^{2d}\varepsilon^{2p}\varepsilon^{-Q/2}|\lambda|^d d\lambda dy=\varepsilon^{Q/2}|\lambda|^d d\lambda dy.$$

We obtain

$$\begin{aligned}\mathrm{Op}_\varepsilon(\sigma)v^\varepsilon(x_0\delta_{\sqrt{\varepsilon}}x)&=c_0|\lambda_\varepsilon|^{d/2}\varepsilon^{-p/2}\varepsilon^{-Q/2}\\&\quad\times\int_{G\times\widehat{G}}\mathrm{Tr}(\pi^\lambda_{\delta_{\varepsilon^{-1/2}}(y^{-1}x)}\sigma(x_0\delta_{\sqrt{\varepsilon}}x,\lambda))a(y)(\pi^{\lambda_0}_{\delta_{\varepsilon^{-1/2}}(y)}\Phi_1,\Phi_2)|\lambda|^d d\lambda dy.\end{aligned}$$

The change of variables $w = \delta_{\varepsilon^{-1/2}}(y^{-1}x)$ (for which $dy = \varepsilon^{Q/2}dw$ and $y = x(\delta_{\sqrt{\varepsilon}}w)^{-1})$) gives

$$\begin{aligned}\mathrm{Op}_\varepsilon(\sigma)v^\varepsilon(x_0\delta_{\sqrt{\varepsilon}}x) &= c_0|\lambda_\varepsilon|^{d/2}\varepsilon^{-p/2}\\ &\quad\times\int_{G\times\widehat{G}}\mathrm{Tr}(\pi_w^\lambda\sigma(x_0\delta_{\sqrt{\varepsilon}}x,\lambda))a(x(\delta_{\sqrt{\varepsilon}}w)^{-1})(\pi^{\lambda_0}_{(\delta_{\varepsilon^{-1/2}}(x))w^{-1}}\Phi_1,\Phi_2)|\lambda|^d d\lambda dw\\ &= c_0|\lambda_\varepsilon|^{d/2}\varepsilon^{-p/2}\\ &\quad\times\int_{G\times\widehat{G}}\mathrm{Tr}(\pi_w^\lambda\sigma(x_0\delta_{\sqrt{\varepsilon}}x,\lambda))a(x(\delta_{\sqrt{\varepsilon}}w)^{-1})(\pi^{\lambda_0}_{w^{-1}}\Phi_1,(\pi^{\lambda_0}_{\delta_{\varepsilon^{-1/2}}(x)})^*\Phi_2)|\lambda|^d d\lambda dw.\end{aligned}$$

Computing the integral in λ thanks to the inverse Fourier transform formula (4.14) and denoting by κ_x the Schwartz function such that $\sigma(x,\cdot) = \mathcal{F}(\kappa_x)$ we have

$$\varepsilon^{Q/4}\mathrm{Op}_\varepsilon(\sigma)v^\varepsilon(x_0\delta_{\sqrt{\varepsilon}}x) = |\lambda_0|^{d/2}\varepsilon^{-d/2}\int_G \kappa_{x_0\delta_{\sqrt{\varepsilon}}x}(w)a(x(\delta_{\sqrt{\varepsilon}}w)^{-1})(\pi^{\lambda_0}_{w^{-1}}\Phi_1,(\pi^{\lambda_0}_{\delta_{\varepsilon^{-1/2}}(x)})^*\Phi_2)dw$$

that we can rewrite

$$\varepsilon^{Q/4}\mathrm{Op}_\varepsilon(\sigma)v^\varepsilon(x_0\delta_{\sqrt{\varepsilon}}x) = |\lambda_0|^{d/2}\varepsilon^{-d/2}\left(Q^\varepsilon(x)\Phi_1,(\pi^{\lambda_0}_{\delta_{\varepsilon^{-1/2}}(x)})^*\Phi_2\right)$$

with

$$Q^\varepsilon(x) = \int_G \kappa_{x_0\delta_{\sqrt{\varepsilon}}x}(w)a(x(\delta_{\sqrt{\varepsilon}}w)^{-1})\pi^{\lambda_0}_{w^{-1}}dw.$$

By performing a Taylor formula on the functions $x\mapsto\kappa_{x_0\delta_{\sqrt{\varepsilon}}x}(w)$ and $w\mapsto a(x(\delta_{\sqrt{\varepsilon}}w)^{-1})$, we see that the operator $Q^\varepsilon(x)$ admits a formal asymptotic expansion of the form

$$Q^\varepsilon(x) = Q_0(x)+\sqrt{\varepsilon}Q_1(x)+\cdots+\varepsilon^{\frac{j}{2}}Q_j(x)+\cdots \tag{4.41}$$

with

$$Q_0(x) = a(x)\int_G \kappa_{x_0}(w)\pi^{\lambda_0}_{w^{-1}}dw = a(x)\sigma(x_0,\lambda_0).$$

It remains to prove the convergence of this asymptotic expansion by examining the remainder term.

We examine the one-term expansion. We write

$$a(x(\delta_{\sqrt{\varepsilon}}w)^{-1}) = a(x)+A(x,\delta_{\sqrt{\varepsilon}}w) \tag{4.42}$$

with

$$|A(x,w)|\leqslant\sum_{j=1}^{2d}\sup_{|z|\leqslant|w|}|z_j||V_ja(xz)|\leqslant C_a|w|, \tag{4.43}$$

where for $z\in G$, $|z|$ denotes the homogeneous norm defined in (4.10). We obtain

$$\varepsilon^{Q/4}\mathrm{Op}_\varepsilon(\sigma)v^\varepsilon(x_0\delta_{\sqrt{\varepsilon}}x) = |\lambda_0|^{d/2}\varepsilon^{-d/2}\left(Q_0\Phi_1,(\pi^{\lambda_0}_{\delta_{\varepsilon^{-1/2}}(x)})^*\Phi_2\right)a(x)+\sqrt{\varepsilon}r_1^\varepsilon(x)+\sqrt{\varepsilon}r_2^\varepsilon(x) \tag{4.44}$$

with

$$r_1^\varepsilon(x) = |\lambda_0|^{d/2}\varepsilon^{-d/2}\left(R_1^\varepsilon(x)\Phi_1,(\pi^{\lambda_0}_{\delta_{\varepsilon^{-1/2}}(x)})^*\Phi_2\right),\quad R_1^\varepsilon(x) = \varepsilon^{-1/2}\int_G(\kappa_{x_0\delta_{\sqrt{\varepsilon}}x}(w)-\kappa_{x_0}(w))a(x)\pi^{\lambda_0}_{w^{-1}}dw$$

and

$$r_2^\varepsilon(x) = |\lambda_0|^{d/2}\varepsilon^{-d/2}\left(R_2^\varepsilon(x)\Phi_1,(\pi^{\lambda_0}_{\delta_{\varepsilon^{-1/2}}(x)})^*\Phi_2\right),\quad R_2^\varepsilon(x) = \varepsilon^{-1/2}\int_G \kappa_{x_0\delta_{\sqrt{\varepsilon}}x}(w)A(x,\delta_{\sqrt{\varepsilon}}w)\pi^{\lambda_0}_{w^{-1}}dw.$$

Lemma 4.23. *The families $(r_1^\varepsilon)_{\varepsilon>0}$ and $(r_2^\varepsilon)_{\varepsilon>0}$ are uniformly bounded in $L^2(G)$.*

Applying (4.36) to the first term in the right hand side of (4.44), we see that Lemma 4.23 implies Lemma 4.21. □

Proof of Lemma 4.23. The idea is that, for $j = 1, 2$, there holds $r_j^\varepsilon(x) = \varepsilon^{-d/2}\widetilde{r}_j^\varepsilon(\delta_{\varepsilon^{-1/2}}(x_\mathfrak{v}), x_\mathfrak{z}, x)$ with

$$y \mapsto \widetilde{r}_j^\varepsilon(y_\mathfrak{v}, y_\mathfrak{z}, x)$$

that is in $L^2(G)$, uniformly with respect to ε, with continuity of the map $x \mapsto \widetilde{r}_j^\varepsilon(\cdot, \cdot, x)$.

With this idea in mind, we write, for $j = 1, 2$,

$$\begin{aligned}\|r_j^\varepsilon\|^2_{L^2(G)} &= |\lambda_0|^d\varepsilon^{-d}\int_G \left|\left(R_j^\varepsilon(x)\Phi_1, (\pi^{\lambda_0}_{\delta_{\varepsilon^{-1/2}}(x)})^*\Phi_2\right)\right|^2 dx\\ &= |\lambda_0|^d\int_G \left|\left(R_j^\varepsilon(\delta_{\varepsilon^{1/2}}(x_\mathfrak{v})x_\mathfrak{z})\Phi_1, (\pi^{\lambda_0}_{x_\mathfrak{v}})^*\Phi_2\right)\right|^2 dx_\mathfrak{v}dx_\mathfrak{z}. \end{aligned}\tag{4.45}$$

Let us first deal with r_1^ε. Writing a Taylor formula, we notice that

$$\begin{aligned}R_1^\varepsilon(\delta_{\varepsilon^{1/2}}(x_\mathfrak{v})x_\mathfrak{z}) &= \varepsilon^{-1/2}\int_G(\kappa_{x_0\delta_\varepsilon(x_\mathfrak{v})\delta_{\sqrt{\varepsilon}}(x_\mathfrak{z})}(w) - \kappa_{x_0}(w))a(x)\pi^{\lambda_0}_{w^{-1}}dw\\ &= \sqrt{\varepsilon}\int_G B(x,w)a(x)\pi^{\lambda_0}_{w^{-1}}dw\end{aligned}$$

where $(x, w) \mapsto B(x, w)$ is continuous and compactly supported in w. Therefore $R_1^\varepsilon(\delta_{\varepsilon^{1/2}}(x_\mathfrak{v})x_\mathfrak{z})$ is a bounded operator for any $x \in G$. Since a is compactly supported, it implies that $(r_1^\varepsilon)_{\varepsilon>0}$ is uniformly bounded in $L^2(G)$.

Let us now deal with r_2^ε. We are going to use that for all multi-indexes $\alpha \in \mathbb{N}^{2d}$, the map

$$x \mapsto x_\mathfrak{v}^\alpha\left(R_2^\varepsilon(\delta_{\varepsilon^{1/2}}(x_\mathfrak{v})x_\mathfrak{z})\Phi_1, (\pi^{\lambda_0}_{x_\mathfrak{v}})^*\Phi_2\right)\tag{4.46}$$

is uniformly bounded and has compact support in $x_\mathfrak{z}$. Let us first prove these properties.

By assumption on the support of κ_x, we know that the w's contributing to the integral defining $R_2^\varepsilon(x)$ are contained in a compact set (independent of x). Then, using (4.42) and the fact that a has compact support, we obtain that R_2^ε has compact support. It follows that the map (4.46) has compact support in $x_\mathfrak{z}$, i.e., there exists $R_0 > 0$ such that $|x_\mathfrak{z}| \leqslant R_0$ for all x that are in the support of $R_2^\varepsilon(\delta_{\varepsilon^{1/2}}(x_\mathfrak{v})x_\mathfrak{z})$. Because of (4.43) and because the integral is compactly supported in w, $R_2^\varepsilon(x)$ is a bounded operator for all $x \in G$. Besides, the bound is uniform since x belongs to a compact set. Therefore, there exists a constant $C_0 > 0$ such that

$$\left|\left(R_2^\varepsilon(\delta_{\varepsilon^{1/2}}(x_\mathfrak{v})x_\mathfrak{z})\Phi_1, (\pi^{\lambda_0}_{x_\mathfrak{v}})^*\Phi_2\right)\right| \leqslant C_0\mathbf{1}_{x_\mathfrak{z}\leqslant R_0}(x).$$

One now wants to prove also decay at infinity in $x_\mathfrak{v}$. For this, we use the relations (4.38) and the fact that Φ_1 and Φ_2 are in the Schwartz class to absorb the factor $|x_\mathfrak{v}|$ in the right part of the scalar product. Therefore, for all $\alpha \in \mathbb{N}$, there exists C_α such that

$$|x_\mathfrak{v}|^\alpha\left|\left(R_2^\varepsilon(\delta_{\varepsilon^{1/2}}(x_\mathfrak{v})x_\mathfrak{z})\Phi_1, (\pi^{\lambda_0}_{x_\mathfrak{v}})^*\Phi_2\right)\right| \leqslant C_\alpha\mathbf{1}_{x_\mathfrak{z}\leqslant R_0}(x).$$

As a conclusion, there exists $C > 0$ such that

$$\int_G\left|\left(R_2^\varepsilon(\delta_{\varepsilon^{1/2}}(x_\mathfrak{v})x_\mathfrak{z})\Phi_1, (\pi^{\lambda_0}_{x_\mathfrak{v}})^*\Phi_2\right)\right|^2 dx_\mathfrak{v}dx_\mathfrak{z} \leqslant C\int \mathbf{1}_{|x_\mathfrak{z}|\leqslant R_0}(1 + |x_\mathfrak{v}|^2)^{-N}dx_\mathfrak{v}dx_\mathfrak{z} < +\infty$$

by choosing N large enough. This implies the uniform boundedness of the family (r_2^ε) in $L^2(G)$, which concludes the proof of Lemma 4.23. □

Let us now shortly discuss the generalization of this proof in order to obtain an asymptotic expansion at any order, as stated in Remark 4.22. The idea is to use a Taylor expansion at higher order (see Section 3.1.8 of [FR16]). The terms of the expansion (4.41) are of the form

$$Q_j(x) = x^\alpha a(x) \int_G w^\beta \kappa_{x_0}(w) \pi^{\lambda_0}_{w^{-1}} dw$$

where α and β are multi-indexes such that the sum of their homogeneous lengths is exactly j. Denoting by $\Delta_{w^\beta}\sigma(x,\lambda_0)$ the Fourier transform of $w \mapsto w^\beta \kappa_{x_0}(w)$, we obtain

$$Q_j(x) = x^\alpha a(x) \Delta_{w^\beta}\sigma(x,\lambda_0).$$

Observe that the operator Δ_{w^β} is a difference operator as defined in [FR16]. It order to justify Remark 4.22, one then needs to remark that the rest term produced by the Taylor expansion at order N is of the form

$$r^\varepsilon_N(x) = |\lambda_0|^{d/2} \varepsilon^{-d/2} \left(R^\varepsilon_N(x)\Phi_1, (\pi^{\lambda_0}_{\delta_{\varepsilon^{-1/2}}(x)})^* \Phi_2 \right)$$

and

$$R^\varepsilon_N(x) = \varepsilon^{-\frac{N+1}{2}} \int_G \kappa_{x_0 \delta_{\sqrt{\varepsilon}} x}(w) A_{N+1}(x, \delta_{\sqrt{\varepsilon}} w) \pi^{\lambda_0}_{w^{-1}} dw$$

where A_{N+1} satisfies convenient bounds so that an argument similar to the preceding one can be worked out. We do not develop the argument further because we do not need such a precise estimate for our purpose.

Semi-classical measure

We can now deduce (4.35) from Lemma 4.21 and the following lemma.

Lemma 4.24. *Let $(x_0,\lambda_0) \in G \times (\mathfrak{z}^* \setminus \{0\})$ $a, b \in \mathcal{C}^\infty_c(\mathcal{B})$ where $\mathcal{B}$ is a unit cell of M, and $\Phi_1, \Phi_2, \Psi_1, \Psi_2 \in \mathcal{S}(\mathbb{R}^p)$. Then*

$$\left(WP^\varepsilon_{x_0,\lambda_0}(a, \Phi_1, \Phi_2), WP^\varepsilon_{x_0,\lambda_0}(b, \Psi_1, \Psi_2) \right)_{L^2(M)} = (\Phi_1, \Psi_1)\overline{(\Phi_2, \Psi_2)} \int_{G_\mathfrak{z}} a(x_\mathfrak{z})\overline{b(x_\mathfrak{z})} dx_\mathfrak{z} + O(\sqrt{\varepsilon})$$

Proof. Define $u^\varepsilon = WP^\varepsilon_{x_0,\lambda_0}(a, \Phi_1, \Phi_2)$ and $v^\varepsilon = WP^\varepsilon_{x_0,\lambda_0}(b, \Psi_1, \Psi_2)$ the wave packets in G. We first use that

$$\left(WP^\varepsilon_{x_0,\lambda_0}(a, \Phi_1, \Phi_2), WP^\varepsilon_{x_0,\lambda_0}(b, \Psi_1, \Psi_2) \right)_{L^2(M)} = (u^\varepsilon, v^\varepsilon)_{L^2(G)}.$$

Besides,

$$\begin{aligned}(u^\varepsilon, v^\varepsilon)_{L^2(G)} &= |\lambda_\varepsilon|^d \varepsilon^{-p} \int_G a_\varepsilon(x_0^{-1}x)\bar{b}(x_0^{-1}x)(\pi^{\lambda_\varepsilon}_{x_0^{-1}x}\Phi_1, \Phi_2)\overline{(\pi^{\lambda_\varepsilon}_{x_0^{-1}x}\Psi_1, \Psi_2)} dx \\ &= |\lambda_0|^d \int_G a\left(\delta_{\sqrt{\varepsilon}}(x_\mathfrak{v})x_\mathfrak{z}\right) \bar{b}\left(\delta_{\sqrt{\varepsilon}}(x_\mathfrak{v})x_\mathfrak{z}\right) (\pi^{\lambda_0}_{x_\mathfrak{v}}\Phi_1, \Phi_2)\overline{(\pi^{\lambda_0}_{x_\mathfrak{v}}\Psi_1, \Psi_2)} dx_\mathfrak{v} dx_\mathfrak{z}.\end{aligned}$$

A Taylor expansion of the map $x \mapsto a(\delta_{\sqrt{\varepsilon}}(x_\mathfrak{v})x_\mathfrak{z})\overline{b(\delta_{\sqrt{\varepsilon}}(x_\mathfrak{v})x_\mathfrak{z})}$ gives

$$a(\delta_{\sqrt{\varepsilon}}(x_\mathfrak{v})x_\mathfrak{z})\overline{b(\delta_{\sqrt{\varepsilon}}(x_\mathfrak{v})x_\mathfrak{z})} = a(x_\mathfrak{z})\overline{b(x_\mathfrak{z})} + \sqrt{\varepsilon} \sum_{1 \leqslant j \leqslant 2d} v_j r_j(x_\mathfrak{z}, \delta_{\sqrt{\varepsilon}}(x_\mathfrak{v}))$$

where $x_{\mathfrak{v}} = \mathrm{Exp}(\sum_{1\leqslant j\leqslant 2d} v_j V_j)$ and with $|r_j(x,w)| \leqslant C_j$ for some constants C_j, $1 \leqslant j \leqslant 2d$. We deduce (using (4.38))

$$\begin{aligned}(u^\varepsilon, v^\varepsilon)_{L^2(G)} &= |\lambda_0|^d \int_{G_{\mathfrak{z}}} a(x_{\mathfrak{z}})\overline{b(x_{\mathfrak{z}})}dx_{\mathfrak{z}} \int_{G_{\mathfrak{v}}} (\pi^{\lambda_0}_{x_{\mathfrak{v}}}\Phi_1, \Phi_2)\overline{(\pi^{\lambda_0}_{x_{\mathfrak{v}}}\Psi_1, \Psi_2)}dx_{\mathfrak{v}} + O(\sqrt{\varepsilon})\\ &= (\Phi_1, \Psi_1)\overline{(\Phi_2, \Psi_2)} \int_{G_{\mathfrak{z}}} a(x_{\mathfrak{z}})\overline{b(x_{\mathfrak{z}})}dx_{\mathfrak{z}} + O(\sqrt{\varepsilon}),\end{aligned}$$

where the second line follows from (4.37). □

Here again, the reader will observe that the expansion can be pushed at any order.

It follows from Lemma 4.21 and Lemma 4.24 that

$$\begin{aligned}&(\mathrm{Op}_\varepsilon(\sigma)WP^\varepsilon_{x_0,\lambda_0}(a,\Phi_1,\Phi_2), WP^\varepsilon_{x_0,\lambda_0}(a,\Phi_1,\Phi_2))\\ &\qquad= (WP^\varepsilon_{x_0,\lambda_0}(a,\sigma(x_0,\lambda_0)\Phi_1,\Phi_2), WP^\varepsilon_{x_0,\lambda_0}(a,\Phi_1,\Phi_2)) + O(\sqrt{\varepsilon})\\ &\qquad= (\sigma(x_0,\lambda_0)\Phi_1,\Phi_1)\|\Phi_2\|^2 \int_{G_{\mathfrak{z}}} |a(x_{\mathfrak{z}})|^2 dx_{\mathfrak{z}} + O(\sqrt{\varepsilon})\end{aligned}$$

which concludes the proof of Proposition 4.19.

4.4.3 End of the proof of Theorem 4.2

By the results of Section 4.3, we only need to prove that if $T \leqslant T_{\mathrm{GCC}}(\overline{U})$, the observability inequality (4.4) does not hold.

We first note that if the observability inequality (4.4) is satisfied for some $T > 0$, then there exists $\delta > 0$ such that (4.4) also holds in time $T - \delta$. Indeed, if it were not the case, there would exist $u_0^n \in L^2(M)$ such that $\|u_0^n\|_{L^2(M)} = 1$ and

$$\begin{aligned}1 = \|u_0^n\|^2_{L^2(M)} &\geqslant n \int_0^{T-2^{-n}} \left\| \mathrm{e}^{it(\frac{1}{2}\Delta_M + \mathbb{V})} u_0^n \right\|^2_{L^2(U)} dt\\ &\geqslant n \int_0^{T} \left\| \mathrm{e}^{it(\frac{1}{2}\Delta_M + \mathbb{V})} u_0^n \right\|^2_{L^2(U)} dt - \frac{n}{2^n}.\end{aligned}$$

due to conservation of energy, and (4.4) would not hold in time T. Therefore, we shall assume in the sequel that $T < T_{\mathrm{GCC}}(\overline{U})$.

Let $T < T_{\mathrm{GCC}}(\overline{U})$ and $(x_0, \lambda_0) \in G \times (\mathfrak{z}^* \setminus \{0\})$ such that

$$\text{for all } s \in [0,T],\ \ \Phi_0^s(x_0,\lambda_0) \notin \overline{U} \times \mathfrak{z}^*. \tag{4.47}$$

Let us chose initial data u_0^ε in (4.3) which is a wave packet in M with harmonics given by the first Hermite function h_0:

$$u_0^\varepsilon = WP^\varepsilon_{x_0,\lambda_0}(a, h_0, h_0).$$

As a consequence, the semi-classical measure of (u_0^ε) is $\Gamma_0(x,\lambda)d\gamma_0$ with Γ_0 the orthogonal projector on h_0 (this is where we use the fact that h_0 is the first Hermite function) and

$$\gamma_0(x,\lambda) = c\,\delta(x - x_0) \otimes \delta(\lambda - \lambda_0)$$

where $c = \limsup \|u_0^\varepsilon\|_{L^2(M)} > 0$. Let us denote by $u^\varepsilon(t)$ the associated solution, $u^\varepsilon(t) = \mathrm{e}^{it(\frac{1}{2}\Delta_M + \mathbb{V})} u_0^\varepsilon$. By Proposition 4.13, any of its semi-classical measures $\Gamma_t d\gamma_t$ decomposes above $G \times \mathfrak{z}^*$ according to the eigenspaces of $H(\lambda)$ following (4.21). Moreover, by Proposition 4.13, the maps $(t,x,\lambda)\mapsto \Gamma_{n,t}(x,\lambda)d\gamma_t(x,\lambda)$ are continuous and satisfy the transport equation (4.22). We deduce that for $n\neq 0$, $\Gamma_{n,t}(x,\lambda)=0$,

$$\gamma_t(x,\lambda)=c\,\delta\left(x-\mathrm{Exp}\left(t\frac{d}{2}\mathcal{Z}^{(\lambda)}\right)x_0\right)\otimes\delta(\lambda-\lambda_0) \tag{4.48}$$

and Γ_0 is the orthogonal projector on h_0.

As a consequence of the conservation of the L^2-norm by the Schrödinger equation, $\|u^\varepsilon(t)\|_{L^2(M)}=\|u_0^\varepsilon\|_{L^2(M)}$. Besides, the ε-oscillation (see Proposition 4.12) gives that, for the subsequence defining $\Gamma_t d\gamma_t$,

$$\lim_{\varepsilon\to 0}\|u^\varepsilon(t)\|^2_{L^2(M)}=\int_{M\times\widehat{G}}\mathrm{Tr}(\Gamma_t(x,\lambda))d\gamma_t(x,\lambda),\ \ \forall t\in\mathbb{R}.$$

We deduce that we have, for any $t\in\mathbb{R}$,

$$\int_{M\times\widehat{G}}\mathrm{Tr}(\Gamma_t(x,\lambda))d\gamma_t(x,\lambda)=\int_{M\times\widehat{G}}\mathrm{Tr}(\Gamma_0(x,\lambda))d\gamma_0(x,\lambda).$$

On the other hand, the positivity of the measure $\mathrm{Tr}(\Gamma_t(x,\lambda))d\gamma_t(x,\lambda)$ combined with (4.48) gives

$$\begin{aligned}\int_{M\times\widehat{G}}\mathrm{Tr}(\Gamma_t(x,\lambda))d\gamma_t(x,\lambda)\geqslant\int_{M\times\mathfrak{z}^*}\mathrm{Tr}(\Gamma_t(x,\lambda))d\gamma_t(x,\lambda)&=\int_{M\times\mathfrak{z}^*}\mathrm{Tr}(\Gamma_0(x,\lambda))d\gamma_0(x,\lambda)\\&=\int_{M\times\widehat{G}}\mathrm{Tr}(\Gamma_0(x,\lambda))d\gamma_0(x,\lambda).\end{aligned}$$

We deduce that $\gamma_t\mathbf{1}_{\mathfrak{v}^*}=0$. Now, using (4.47), there exists a continuous function $\phi: M\to[0,1]$ such that $\phi(\Phi_0^s(x_0,\lambda_0))=0$ for any $s\in[0,T]$ and $\phi=1$ on $\overline{U}\times\mathfrak{z}^*$. Using Proposition 4.12 for the subsequence defining the semi-classical measure $\Gamma_t d\gamma_t$, we get

$$0\leqslant\int_0^T\int_U|u^\varepsilon(t,x)|^2dxdt\leqslant\int_0^T\int_M\phi(x)|u^\varepsilon(t,x)|^2dxdt\underset{\varepsilon\to 0}{\longrightarrow}\int_0^T\int_{M\times\mathfrak{z}^*}\phi(x)d\gamma_t(x,\lambda)dt=0.$$

Therefore, the observability inequality (4.4) cannot hold.

Remark 4.25. As already noticed in the introduction, it can happen that $T_{\mathrm{GCC}}(\overline{U})<T_{\mathrm{GCC}}(U)$, and in this case, Theorem 4.2 does not say anything about observability for times T such that $T_{\mathrm{GCC}}(\overline{U})<T\leqslant T_{\mathrm{GCC}}(U)$. This is due to the possible existence of grazing rays, which are rays which touch the boundary ∂U without entering the interior of U. This phenomenon already occurs in the context of the observability of Riemannian waves, as was shown for example in [Leb92a, Section VI.B]. The example given in this paper is the observation of the wave equation in the unit sphere $\mathbb{S}^2$ from its (open) northern hemisphere: although the GCC condition is violated by the geodesic following the equator, observability holds in time $T>\pi$. Intuitively, even wave packets following this geodesic have half of their energy located on the northern hemisphere.

4-A Supplementary material

4-A.1 Representations of H-type groups

In this section, we provide a proof of the description (4.12) of $\widehat{G}$. This material is standard in non-commutative Fourier analysis, see for example [CG04].

The orbits of $\mathfrak{g}$

As any group, a nilpotent connected, simply connected Lie group acts on itself by the inner automorphism $i_x : y \mapsto xyx^{-1}$. With this action, one derives the action of G on its Lie algebra $\mathfrak{g}$ called the *adjoint map*

$$\begin{array}{rccc} \mathrm{Ad}: & G & \to & \mathrm{Aut}(\mathfrak{g}) \\ & x & \mapsto & \mathrm{Ad}_x = d(i_x)_{|1_G}, \end{array}$$

and its action on $\mathfrak{g}^*$, the *co-adjoint map*

$$\begin{array}{rccc} \mathrm{Ad}^*: & G & \to & \mathrm{Aut}(\mathfrak{g}^*) \\ & x & \mapsto & \mathrm{Ad}^*_x \end{array}$$

defined by

$$\forall x \in G, \ \ \forall \ell \in \mathfrak{g}^*, \ \ \forall Y \in \mathfrak{g}, \ \ (\mathrm{Ad}^*_x \ell)(Y) = \ell(\mathrm{Ad}_x^{-1} Y).$$

It turns out that the orbits of this action play an important role in the representation theory of the group. Let us recall that the orbit of an element $\ell \in \mathfrak{g}^*$ is the set $\mathcal{O}_\ell$ defined by

$$\mathcal{O}_\ell = \{\mathrm{Ad}^*_x(\ell), \ \ x \in G\}.$$

The next proposition describes the orbits of H-type groups.

Proposition 4.26. *Let G be a H-type group, then there are only two types of orbits.*

(i) 0-th. dimensional orbits. If $\ell \in \mathfrak{v}^*$, then $\mathcal{O}_\ell = \{\ell\}$.

(ii) 2d-th. dimensional orbits. If $\ell = \omega + \lambda$ with $\omega \in \mathfrak{v}^*$ and $\lambda \in \mathfrak{z}^* \setminus \{0\}$, then $\mathcal{O}_\ell = \mathcal{O}_\lambda$ and

$$\mathcal{O}_\lambda = \{\omega' + \lambda, \ \ \omega' \in \mathfrak{v}^*\}.$$

Proof. Let $x = \mathrm{Exp}(V_x + Z_x) \in G$ and $y = \mathrm{Exp}(V_y + Z_y) \in G$. Then

$$\begin{aligned} i_x(y) = xyx^{-1} &= \mathrm{Exp}(V_x + Z_x)\mathrm{Exp}(V_y + Z_y)\mathrm{Exp}(-V_x - Z_x) \\ &= \mathrm{Exp}(V_y + Z_y + [V_x, V_y]). \end{aligned}$$

We deduce that if $Y = V_Y + Z_Y \in \mathfrak{g}$,

$$\mathrm{Ad}_x^{-1}(Y) = V_Y + Z_Y + [V_x, V_Y].$$

Therefore, if $\ell = \omega + \lambda$ with $\lambda \in \mathfrak{z}^*$ and $\omega \in \mathfrak{v}^*$,

$$\mathrm{Ad}^*_x \ell(Y) = \langle \ell, \mathrm{Ad}_x^{-1}(Y)\rangle = \langle \omega, V_Y\rangle + \langle \lambda, Z_Y + [V_x, V_Y]\rangle = \langle \omega + J_\lambda(V_x), V_Y\rangle + \langle \lambda, Z_Y\rangle$$

As a consequence, if $\lambda = 0$, $\mathrm{Ad}^*_x \ell(Y) = \ell(Y)$ for all $Y \in \mathfrak{g}$. We deduce $\mathrm{Ad}^*_x \ell = \ell$ for all $x \in G$, which gives the first type of orbits.
If now $\lambda \neq 0$ and if $\omega' \in \mathfrak{v}^*$, one can find $V_x \in \mathfrak{v}$ such that

$$\langle \omega', V\rangle = \langle \omega + J_\lambda(V_x), V\rangle, \ \ \forall V \in \mathfrak{v}.$$

One deduces that for all $Y \in \mathfrak{g}$, $\mathrm{Ad}^*_x \ell(Y) = \ell'(Y)$ with $\ell' = \omega' + \lambda$. We deduce that any of these ℓ' is in the orbit of ℓ, which concludes the proof. □

Let $\lambda \in \mathfrak{z}^* \setminus \{0\}$, the sets $\mathfrak{p}_\lambda \oplus \mathfrak{z}$ and $\mathfrak{q}_\lambda \oplus \mathfrak{z}$ are maximal isotropic sub-algebras of $\mathfrak{g}$ for the bilinear map $B(\lambda)$ (with associated endomorphism J_λ). Such an algebra is said to be a *polarizing algebra* of $\mathfrak{g}$. We shall use these algebras in the next section.

Unitary irreducible representations of G

The *unitary representations* of a locally compact group are homomorphisms π of G into the group of unitary operators on a Hilbert space that are continuous for the strong topology. The representations for which there is no proper closed $\pi(G)$-invariant subspaces in $\mathcal{H}_\pi$ are called *irreducible.* Arbitrary representations can be uniquely decomposed as sums of irreducible representations.

Kirillov theory establishes a one to one relation between the orbits $(\mathcal{O}_\ell)_{\ell\in\mathfrak{g}^*}$ and the irreducible unitary representations of G for any nilpotent Lie group which is connected and locally connected. We shall first explain how one associates to an orbit $\mathcal{O}_\ell$ a representation π_ℓ (which only depends on the class of the orbit $\mathcal{O}_\ell$). Then, in the next subsection, we shall explain how the Stone-Von Neumann Theorem implies that any representation can be associated with an orbit.

- Let $\omega \in \mathfrak{v}^*$, the map χ_ω defined below is a 1-dimensional representation of G.

$$\begin{array}{rccc} \chi_\omega : & G & \to & \mathbf{S}^1 \\ & \mathrm{Exp}(X) & \mapsto & \mathrm{e}^{i\omega(X)}. \end{array}$$

- Let $\lambda \in \mathfrak{z}^* \setminus \{0\}$. We consider the polarizing sub-algebra associated with λ

$$\mathfrak{m}_\lambda = \mathfrak{q}_\lambda \oplus \mathfrak{z}$$

and the subgroup of G defined by $M := \mathrm{Exp}(\mathfrak{m}_\lambda)$. Then, if $\ell \in \mathcal{O}_\lambda$, $\ell([\mathfrak{m}_\lambda, \mathfrak{m}_\lambda]) = 0$, and the map

$$\begin{array}{rccc} \chi_{\lambda,M} : & M & \to & \mathbf{S}^1 \\ & \mathrm{Exp}(Y) & \mapsto & \mathrm{e}^{i\lambda(Y)}. \end{array}$$

is a one-dimensional representation of M. This allows to construct an induced representation π_λ on G with Hilbert space $\mathfrak{p}_\lambda \sim L^2(\mathbb{R}^p)$ via the identification of $\mathrm{Exp}\left(\sum_{j=1}^d \xi_j P_j^{(\lambda)}\right) \in \mathrm{Exp}(\mathfrak{p}_\lambda)$ with $\xi = (\xi_1, \cdots, \xi_d) \in \mathbb{R}^d$. Indeed, let us take $\xi \in \mathfrak{p}_\lambda$ and $x = \mathrm{Exp}(X)$, with $X = P + Q + Z$ and $P \in \mathfrak{p}_\lambda$, $Q \in \mathfrak{q}_\lambda$ and $Z \in \mathfrak{z}$. We have, by the Baker-Campbell-Hausdorff formula,

$$\mathrm{Exp}(\xi)\mathrm{Exp}(X) = \mathrm{Exp}(Q + Z + [\xi, Q] + \frac{1}{2}[P, Q])\mathrm{Exp}(\xi + P),$$

with

$$Q + Z + [\xi, Q] + \frac{1}{2}[P, Q] \in \mathfrak{m}_\lambda \quad \text{and} \quad \xi + P \in \mathfrak{p}_\lambda.$$

Let us denote by $p, q \in \mathbb{R}^d$ the coordinates of P and Q in the bases $(P_j^{(\lambda)})_{1\leqslant j\leqslant d}$ and $(Q_j^{(\lambda)})_{1\leqslant j\leqslant d}$ respectively. Following [CG04], we define the induced representation by

$$\pi_\lambda(x)f(\xi) = \chi_\lambda\Big(\mathrm{Exp}(Q + Z + [\xi, Q] + \frac{1}{2}[P, Q])\Big)f(\xi + p).$$

Using $\lambda([P_j^{(\lambda)}, Q_j^{(\lambda)}]) = B(\lambda)(P_j^{(\lambda)}, Q_j^{(\lambda)}) = |\lambda|$, we obtain

$$\pi_\lambda(x)f(\xi) = \mathrm{e}^{i\lambda(Z) + \frac{i}{2}|\lambda|p\cdot q + i|\lambda|\xi\cdot q} f(\xi + p).$$

We can then use the scaling operator T_λ defined by

$$T_\lambda f(\xi) = |\lambda|^{d/4} f(|\lambda|^{1/2}\xi)$$

to get the equivalent representation $\pi_x^\lambda := T_\lambda^* \pi_\lambda(x) T_\lambda$ written in (4.11).

This inductive process can be generalized to the case of groups presenting more than two strata. For our purpose, it remains to prove that any irreducible representation is equivalent to one of those, which is a consequence of the Stone-Von Neumann Theorem.

Stone-Von Neumann Theorem

Let us recall the celebrated Stone-Von Neumann theorem (see [CG04, Section 2.2.9] for a proof).

Theorem 4.27. *Let ρ_1, ρ_2 be two unitary representations of $G = \mathbb{R}^d$ in the same Hilbert space $\mathcal{H}$ satisfying, for some $\alpha \neq 0$, the covariance relation*

$$\rho_1(x)\rho_2(y)\rho_1(x)^{-1} = e^{i\alpha x\cdot y}\rho_2(y), \quad \text{for all } x, y \in \mathbb{R}^d.$$

Then $\mathcal{H}$ is a direct sum $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \ldots$ of subspaces that are invariant and irreducible under the joint action of ρ_1 and ρ_2. For any k, there is an isometry $J_k : \mathcal{H}_k \to L^2(\mathbb{R}^d)$ which transforms ρ_1 and ρ_2 to the canonical actions on $L^2(\mathbb{R}^d)$:

$$[\widetilde{\rho}_1(x)f](\xi) = f(\xi + x), \quad [\widetilde{\rho}_2(y)f](\xi) = e^{i\alpha y\cdot\xi}f(\xi).$$

For each $\alpha \neq 0$, the canonical pair $\widetilde{\rho}_1, \widetilde{\rho}_2$ acts irreducibly on $L^2(\mathbb{R}^d)$, so ρ_1, ρ_2 act irreducibly on each $\mathcal{H}_k$.

Let π be an irreducible representation of G on $\mathcal{H}_\pi$. Our goal is to prove that it is equivalent either to a χ_ω or to a π_λ of the preceding section. For $Z \in \mathfrak{z}$, the operators $\pi(\mathrm{Exp}(Z))$ commute will all elements of $\{\pi_g : g \in G\}$. By Schur's Lemma (see [CG04, Lemma 2.1.1]), they are thus scalar: $\pi_{\mathrm{Exp}(Z)} = \chi(\mathrm{Exp}(Z))\mathrm{Id}_{\mathcal{H}_\pi}$ where χ is a one-dimensional representation of the center $Z(G) = \mathrm{Exp}(\mathfrak{z})$ of G. Then, two cases appear:

• If $\chi \equiv 1$, then π is indeed a representation of the Abelian quotient group $G/Z(G) = \mathrm{Exp}(\mathfrak{v})$, thus it is one-dimensional and of the form χ_ω for some $\omega \in \mathfrak{v}^*$.

• If $\chi \not\equiv 1$, there is $\lambda \in \mathfrak{z}^* \setminus \{0\}$ such that $\chi(\mathrm{Exp}(Z)) = e^{i\lambda(Z)}$. We keep the notations of (4.9), the notations $P = p_1P_1^{(\lambda)} + \ldots + p_dP_d^{(\lambda)}$, $Q = q_1Q_1^{(\lambda)} + \ldots + q_dQ_d^{(\lambda)}$ and $Z = z_1Z_1 + \ldots + z_pZ_p$ of the previous section, and we set $p = (p_1, \ldots, p_d)$, $q = (q_1, \ldots, q_d)$ and $z = (z_1, \ldots, z_p)$. The actions of the d-parameter subgroups $\rho_1(p) = \pi_{\mathrm{Exp}(P)}$ and $\rho_2(q) = \pi_{\mathrm{Exp}(Q)}$ satisfy the covariance relation

$$\begin{aligned}\rho_1(p)\rho_2(q)\rho_1^{-1}(p)\rho_2^{-1}(q) &= \pi_{\mathrm{Exp}(\frac{1}{2}(p_1q_1[P_1^{(\lambda)},Q_1^{(\lambda)}]+\ldots+p_dq_d[P_d^{(\lambda)},Q_d^{(\lambda)}]))} \\ &= e^{\frac{i}{2}|\lambda|p\cdot q}\mathrm{Id}_{\mathcal{H}_\pi}\end{aligned}$$

where we have used $[P_j^{(\lambda)}, Q_j^{(\lambda)}] = \mathcal{Z}^{(\lambda)}$ with $\lambda(\mathcal{Z}^{(\lambda)}) = |\lambda|$. The joint action of ρ_1 and ρ_2 is irreducible since the d-parameter subgroups generate G and π is irreducible. Thus, we may apply the Stone-Von Neumann theorem, which gives that there exists an isometry identifying $\mathcal{H}_\pi$ with $L^2(\mathbb{R}^d)$ such that the actions take the form

$$\begin{aligned}[\rho_1(p)f](t) &= [\pi_{\mathrm{Exp}(P)}f](\xi) = f(\xi + p), \\ [\rho_2(q)f](t) &= [\pi_{\mathrm{Exp}(Q)}f](\xi) = e^{i|\lambda|q\cdot\xi}f(\xi)\end{aligned}$$

for all $f \in L^2(\mathbb{R}^d)$ and $p, q \in \mathbb{R}^d$. Hence, in this model, the action of an arbitrary element of G is

$$[\pi_{\mathrm{Exp}(P+Q+Z)}f](\xi) = e^{i\lambda(z)+\frac{i}{2}|\lambda|p\cdot q+i|\lambda|q\cdot\xi}f(\xi + p)$$

since $\mathrm{Exp}(P+Q+Z) = \mathrm{Exp}(Z+\frac{1}{2}[P,Q])\cdot\mathrm{Exp}(Q)\cdot\mathrm{Exp}(P)$ by the Baker-Campbell-Hausdorff formula. This is just the action of π_λ modeled in $L^2(\mathbb{R}^d)$. Thus, an infinite-dimensional irreducible representation π is isomorphic to π_λ for some λ.

4-A.2 Pseudodifferential operators and semi-classical measures

In this section we focus on different aspects of the pseudodifferential calculus on quotient manifolds.

Properties of pseudodifferential operators on quotient manifolds

We prove here properties (3), (4) and (5) of Section 4.2.

• *Proof of Property (3).* We write $G = \cup_{\gamma\in\widetilde{\Gamma}} M\gamma^{-1}$ and, using the periodicity of f, we obtain

$$\int_G \kappa_x^\varepsilon(y^{-1}x)f(y)dy = \sum_{\gamma\in\widetilde{\Gamma}} \int_{y\in M\gamma^{-1}} \kappa_x^\varepsilon(y^{-1}x)f(y)dy = \sum_{\gamma\in\widetilde{\Gamma}} \int_{y\in M} \kappa_x^\varepsilon(\gamma y^{-1}x)f(y)dy.$$

As a consequence, the action of the operator $\mathrm{Op}_\varepsilon(\sigma)$ writes as a sum of convolution

$$\mathrm{Op}_\varepsilon(\sigma)f(x) = \sum_{\gamma\in\widetilde{\Gamma}} f * \kappa_x^\varepsilon(\gamma\cdot)(x).$$

• *Proof of Property (4).* By Young's convolution inequality

$$\|f * \kappa_x^\varepsilon(\gamma\cdot)\|_{L^2(M)} \leqslant \|\sup_{x\in M}|\kappa_x^\varepsilon(\gamma\cdot)|\|_{L^1(M)}\|f\|_{L^2(M)}.$$

We have

$$\|\sup_{x\in M}|\kappa_x^\varepsilon(\gamma\cdot)|\|_{L^1(M)} = \varepsilon^{-Q}\int_M \sup_{x\in M}|\kappa_x(\varepsilon\cdot\gamma y)|dy = \int_{\gamma^{-1}M}\sup_{x\in M}|\kappa_x(y)|dy.$$

Therefore

$$\|\mathrm{Op}_\varepsilon(\sigma)f\|_{L^2(M)} \leqslant \|f\|_{L^2(M)}\sum_{\gamma\in\widetilde{\Gamma}}\int_{\gamma^{-1}M}\sup_{x\in M}|\kappa_x(y)|dy = \|f\|_{L^2(M)}\int_G\sup_{x\in M}|\kappa_x(y)|dy,$$

which gives (4.19)

• *Proof of Property (5).* We argue as for the L^2 boundedness and observe that the kernel of $\mathrm{Op}_\varepsilon(\sigma) - \mathrm{Op}_\varepsilon(\sigma)\chi$ is the function

$$(x,y)\mapsto \kappa_x^\varepsilon(y^{-1}x)(1-\chi(y)).$$

Writing

$$\kappa_x^\varepsilon(y^{-1}x)(1-\chi(y)) = \kappa_x^\varepsilon(y^{-1}x)(1-\chi(x(y^{-1}x)^{-1})$$

we deduce that we can write the operator $\mathrm{Op}_\varepsilon(\sigma) - \mathrm{Op}_\varepsilon(\sigma)\chi$ as the convolution with an x-dependent function:

$$(\mathrm{Op}_\varepsilon(\sigma) - \mathrm{Op}_\varepsilon(\sigma)\chi)f(x) = \sum_{\gamma\in\widetilde{\Gamma}} f * \theta^\varepsilon(x,\gamma\cdot)$$

with $\theta^\varepsilon(x,z) = \varepsilon^{-Q}\kappa_x(\varepsilon\cdot z)(1-\chi)(xz^{-1})$. Therefore, if $K = \operatorname{supp}\sigma$ (where $\chi\equiv 1$), we have

$$\|\sup_{x\in K}\theta^\varepsilon(x,\gamma\cdot)\|_{L^1(M)} \leqslant \int_M\sup_{x\in K}|\kappa_x(\gamma z)||(1-\chi)(x(\varepsilon\cdot(\gamma z))^{-1})|dz.$$

A Taylor formula gives that there exists a constant $c>0$ such that for all $x\in K$,

$$|(1-\chi)(x(\varepsilon\cdot(\gamma z))^{-1})| \leqslant c\varepsilon^N|\gamma z|^N.$$

Therefore,

$$\|\sup_{x\in K}\theta^\varepsilon(x,\gamma\cdot)\|_{L^1(M)}\leqslant c\varepsilon^N\int_M\sup_{x\in K}|\kappa_x(\gamma z)||\gamma z|^N dz.$$

We deduce thanks to Young's convolution inequality

$$\begin{aligned}\|(\mathrm{Op}_\varepsilon(\sigma)(1-\chi)f\|_{L^2(M)}&\leqslant \varepsilon^N c\|f\|_{L^2(M)}\sum_{\gamma\in\widetilde{\Gamma}}\int_M\sup_{x\in K}|\kappa_x(\gamma z)||\gamma z|^N dz\\&=\varepsilon^N c\|f\|_{L^2(M)}\int_G\sup_{x\in K}|\kappa_x(z)||z|^N dz.\end{aligned}$$

Time-averaged semi-classical measures

We give here comments about the proof of Proposition 4.13. Note that when $\mathbb{V}=0$, Theorem 2.10(ii)(2) in [FF21] implies the statement, except for the continuity of the map $t\mapsto\Gamma_t d\gamma_t$. The key observation is that for any symbol $\sigma\in\mathcal{A}_0$,

$$\frac{1}{i\varepsilon}\left[-\frac{\varepsilon^2}{2}\Delta_M-\varepsilon^2\mathbb{V},\mathrm{Op}_\varepsilon(\sigma)\right]=\frac{1}{i\varepsilon}\left[-\frac{\varepsilon^2}{2}\Delta_M,\mathrm{Op}_\varepsilon(\sigma)\right]+O(\varepsilon)\tag{4.49}$$

in $\mathcal{L}(L^2(G))$ by the boundedness of $\mathbb{V}$. As a consequence, the results of Theorem 2.10(ii)(2) in [FF21] without potential passes to the case with a bounded potential. Note in particular that we do not need any analyticity on the potential. The two points of Proposition 4.13 derive from relation (4.49).

For (1), using Proposition 4.8 and multiplying (4.49) by ε, one gets that for any symbol $\sigma\in\mathcal{A}_0$ and $\theta\in L^1(G)$,

$$\int_{\mathbb{R}\times G\times\hat{G}}\theta(t)\mathrm{Tr}([\sigma(x,\lambda),H(\lambda)]\Gamma_t(x,\lambda))d\gamma_t(x,\lambda)dt=0,$$

which implies the commutation of $\Gamma_t(x,\lambda)$ with $H(\lambda)$ and thus the relation 4.21.

Let us now prove the transport equation and the continuity property; Let $\Pi_n^{(\lambda)}$ be the projector on the n-th eigenspace of $H(\lambda)$. We prove here the continuity of the map $t\mapsto(\Pi_n^{(\lambda)}\Gamma_t\mathbf{1}_{\mathfrak{z}^*}\Pi_n^{(\lambda)},\gamma_t\mathbf{1}_{\mathfrak{z}^*})$. Since $\Pi_n^{(\lambda)}\notin\mathcal{A}_0$, it is necessary to regularize the operator $\Pi_n^{(\lambda)}\sigma(x,\lambda)\Pi_n^{(\lambda)}$ for $\sigma\in\mathcal{A}_0$. In that purpose, we fix $\chi\in\mathcal{C}^\infty(\mathbb{R})$ such that $0\leqslant\chi\leqslant1$, $\chi(u)=1$ for on $|u|>1$ and $\chi(u)=0$ for $|u|\leqslant1/2$. We consider $\sigma\in\mathcal{A}_0$ a symbol strictly supported inside a unit cell of M and associate with it the symbol

$$\sigma^{(u,n)}(x,\lambda)=\chi(uH(\lambda))\Pi_n^{(\lambda)}\sigma(x,\lambda)\Pi_n^{(\lambda)},\quad n\in\mathbb{N},\quad u\in(0,1].$$

In view of Corollary 3.9 in [FF21], this symbol belongs to the class $S^{-\infty}$ of regularizing symbols. Besides, it is also supported inside a unit cell of M. Fix $n\in\mathbb{N}$ and consider the map

$$t\mapsto\Big(\mathrm{Op}_\varepsilon(\sigma^{(u,n)})\psi^\varepsilon(t),\psi^\varepsilon(t)\Big):=\ell_{u,\varepsilon}(t)$$

where $\psi^\varepsilon(t)$ is a family of solutions to (4.3) for some family of initial data $(\psi_0^\varepsilon)_{\varepsilon>0}$.

Lemma 4.28. *The map $t\mapsto\big(\mathrm{Op}_\varepsilon(\sigma^{(u,n)})\psi^\varepsilon(t),\psi^\varepsilon(t)\big)$ is equicontinuous with respect to the parameter $\varepsilon\in(0,1)$.*

We recall that from Theorem 2.5 (i) of [FF21] we have for all $\sigma\in\mathcal{A}_0$, χ and u as above, $\theta\in L^1(\mathbb{R})$, and $p,p'\in\mathbb{N}$ with $p\neq p'$,

$$\int_{\mathbb{R}}\theta(t)\left(\mathrm{Op}_\varepsilon(\Pi_p\chi(uH(\lambda))\sigma\Pi_{p'})\psi^\varepsilon(t),\psi^\varepsilon(t)\right)dt=O(\varepsilon)\tag{4.50}$$

Proof. For any symbol $\sigma \in \mathcal{A}_0$, we have

$$\begin{aligned} \frac{d}{dt}\left(\mathrm{Op}_\varepsilon(\sigma)\psi^\varepsilon(t), \psi^\varepsilon(t)\right) &= \frac{1}{i\varepsilon^2}\Big([\mathrm{Op}_\varepsilon(\sigma), -\frac{\varepsilon^2}{2}\Delta_M - \varepsilon^2\mathbb{V}]\psi^\varepsilon(t), \psi^\varepsilon(t)\Big) \\ &= \frac{1}{i\varepsilon^2}\left(\mathrm{Op}_\varepsilon([\sigma, H(\lambda)]\psi^\varepsilon(t), \psi^\varepsilon(t)\right) - \frac{1}{i\varepsilon}\Big(\mathrm{Op}_\varepsilon(V\cdot\pi^\lambda(V)\sigma)\psi^\varepsilon(t), \psi^\varepsilon(t)\Big) \\ &\quad - \frac{1}{2i}\left(\mathrm{Op}_\varepsilon(\Delta_M\sigma)\psi^\varepsilon(t), \psi^\varepsilon(t)\right) - \frac{1}{i}\left([\mathrm{Op}_\varepsilon(\sigma), \mathbb{V}]\psi^\varepsilon(t), \psi^\varepsilon(t)\right). \end{aligned} \tag{4.51}$$

For $\sigma^{(u,n)}$ (which commutes with $H(\lambda)$) we have

$$\begin{aligned} \frac{d}{dt}\ell_{u,\varepsilon}(t) &= \frac{1}{i\varepsilon^2}\Big([\mathrm{Op}_\varepsilon(\sigma^{(u,n)}), -\frac{\varepsilon^2}{2}\Delta_M - \varepsilon^2\mathbb{V}]\psi^\varepsilon(t), \psi^\varepsilon(t)\Big) \\ &= -\frac{1}{i\varepsilon}\Big(\mathrm{Op}_\varepsilon(V\cdot\pi^\lambda(V)\sigma^{(u,n)})\psi^\varepsilon(t), \psi^\varepsilon(t)\Big) - \frac{1}{2i}\Big(\mathrm{Op}_\varepsilon(\Delta_M\sigma^{(u,n)})\psi^\varepsilon(t), \psi^\varepsilon(t)\Big) + O(\varepsilon) \end{aligned}$$

where we used $[\mathrm{Op}_\varepsilon(\sigma^{(u,n)}), \mathbb{V}] = O(\varepsilon)$ in $\mathcal{L}(L^2(M))$ by Proposition 4.8. By Lemma 4.1 in [FF21], there exists $\sigma_1(x,\lambda)$ such that

$$\begin{aligned} V\cdot\pi^\lambda(V)\sigma^{(u,n)}(x,\lambda) &= [\sigma_1(x,\lambda), H(\lambda)] \\ (V\cdot\pi^\lambda(V)\sigma_1(x,\lambda)) &= \left((n+\frac{d}{2})i\mathcal{Z}^{(\lambda)} - \frac{1}{2}\Delta_M\right)\sigma^{(u,n)}(x,\lambda) \end{aligned} \tag{4.52}$$

The proof of these relations is discussed at the end of the proof of Proposition 4.29 where we use quite similar properties. We then write for $t, t' \in \mathbb{R}$,

$$\begin{aligned} \ell_{u,\varepsilon}(t) - \ell_{u,\varepsilon}(t') &= -\frac{1}{i\varepsilon}\int_{t'}^t \left(\mathrm{Op}_\varepsilon([\sigma_1, H(\lambda)])\psi^\varepsilon(s), \psi^\varepsilon(s)\right) ds \\ &\quad - \frac{1}{2i}\int_{t'}^t \Big(\mathrm{Op}_\varepsilon(\Delta_M\sigma^{(u,n)})\psi^\varepsilon(s), \psi^\varepsilon(s)\Big)\, ds + O(\varepsilon|t-t'|). \end{aligned}$$

Besides, using (4.51) for the symbol σ_1, we deduce

$$\begin{aligned} -\frac{1}{i\varepsilon}\left(\mathrm{Op}_\varepsilon([\sigma_1, H(\lambda)])\psi^\varepsilon(t), \psi^\varepsilon(t)\right) &= -\frac{\varepsilon}{i}\left([\mathrm{Op}_\varepsilon(\sigma_1), \mathbb{V}]\psi^\varepsilon(t), \psi^\varepsilon(t)\right) - \varepsilon\frac{d}{dt}\left(\mathrm{Op}_\varepsilon(\sigma_1)\psi^\varepsilon(t), \psi^\varepsilon(t)\right) \\ &\quad - \frac{1}{i}\Big(\mathrm{Op}_\varepsilon(V\cdot\pi^\lambda(V)\sigma_1)\psi^\varepsilon(t), \psi^\varepsilon(t)\Big) - \frac{\varepsilon}{2i}\left(\mathrm{Op}_\varepsilon(\Delta_M\sigma_1)\psi^\varepsilon(t), \psi^\varepsilon(t)\right). \end{aligned}$$

This implies

$$\begin{aligned} \ell_{u,\varepsilon}(t) - \ell_{u,\varepsilon}(t') &= -\frac{1}{i}\int_{t'}^t \Big(\mathrm{Op}_\varepsilon(V\cdot\pi^\lambda(V)\sigma_1)\psi^\varepsilon(s), \psi^\varepsilon(s)\Big)\, ds - \frac{1}{2i}\int_{t'}^t \left(\mathrm{Op}_\varepsilon(\Delta_M\sigma_1)\psi^\varepsilon(s), \psi^\varepsilon(s)\right) ds \\ &\quad + O(\varepsilon|t-t'|) \\ &= (n+\frac{d}{2})\int_{t'}^t \Big(\mathrm{Op}_\varepsilon(\mathcal{Z}^{(\lambda)}\sigma)\psi^\varepsilon(s), \psi^\varepsilon(s)\Big)\, ds + O(\varepsilon|t-t'|) \end{aligned} \tag{4.53}$$

which concludes the proof. $\square$

The continuity of the map $t \mapsto (\Pi_n^{(\lambda)}\Gamma_t\mathbf{1}_{\mathfrak{z}^*}\Pi_n^{(\lambda)}, \gamma_t\mathbf{1}_{\mathfrak{z}^*})$ follows from Lemma 4.28 and the Arzelà-Ascoli theorem. Note that, equation (4.53) of the proof of Lemma 4.28 also implies the transport equation (4.22).

Finally, let us prove Point (2) of Proposition 4.13. We use the relation

$$\frac{1}{\varepsilon}[-\varepsilon^2\Delta_M, \mathrm{Op}_\varepsilon(\sigma)] = \frac{1}{\varepsilon}\mathrm{Op}_\varepsilon([H(\lambda),\sigma]) - 2\mathrm{Op}_\varepsilon(V\cdot\pi^\lambda(V)\sigma) - \varepsilon\mathrm{Op}_\varepsilon(\Delta_M\sigma).$$

together with (4.49). We denote by ς_t the scalar measure $\Gamma_t d\gamma_t \mathbf{1}_{\mathfrak{v}^*}$ and we use that for the finite dimensional representations $\pi^{(0,\omega)}$, we have $\pi^{(0,\omega)}(V_j) = i\omega_j$. In the limit $\varepsilon\to 0$, we obtain that for any $\theta\in L^1(\mathbb{R})$ and $\sigma\in\mathcal{A}_0$ commuting with $H(\lambda)$,

$$\int_{\mathbb{R}\times M\times\mathfrak{z}^*}\theta(t)\mathrm{Tr}(V\cdot\pi(V)\sigma(x,\lambda)\Gamma_t(x,\lambda))d\gamma_t(x,\lambda)dt + \int_{\mathbb{R}\times M\times\mathfrak{v}^*}\theta(t)i\omega\cdot V\sigma(x,\omega)d\varsigma_t(x,\omega)dt = 0.$$

Since Γ_t commutes with $H(\lambda)$ and $V\cdot\pi(V)\sigma$ is off-diagonal when σ is diagonal (see (4.52)), we deduce that the first term of the left-hand side of the preceding relation is 0. Therefore,

$$\int_{\mathbb{R}\times M\times\mathfrak{v}^*}\theta(t)\omega\cdot V\sigma(x,\omega)d\varsigma_t(x,\omega)dt = 0,$$

which implies the invariance of $\varsigma_t(x,\omega)$ by the map $(x,\omega)\mapsto(\mathrm{Exp}(t\omega\cdot V)x,\omega)$.

4-A.3 Wave packet solutions to the Schrödinger equation

We assume here $\mathbb{V}=0$. We prove that the solution of (4.3) with an initial datum which is a wave packet can be approximated by a wave packet. We focus on the case where the harmonics verify $\Phi_1=\Phi_2=h_0$, see the discussion preceding Remark 4.30 for more details. We work in G, keeping in mind that by Remark 4.7, the result extends to M. Note that the results of this section give in particular a second proof of the necessary part of Theorem 4.2 in case $\mathbb{V}=0$.

Proposition 4.29. *Let $u^\varepsilon(t)$ be the solution of equation* (4.3) *with $\mathbb{V}=0$ and initial data of the form*

$$u_0^\varepsilon = WP^\varepsilon_{x_0,\lambda_0}(a,h_0,h_0),$$

where $(x_0,\lambda_0)\in M\times(\mathfrak{z}^*\setminus\{0\})$, $a\in\mathcal{S}(G)$ and h_0 is the first Hermite function. Then, there exists a map $(t,x)\mapsto a(t,x)$ in $\mathcal{C}^1(\mathbb{R},\mathcal{S}(G))$ such that for all $k\in\mathbb{N}$,

$$u^\varepsilon(t,x) = WP^\varepsilon_{x(t),\lambda_0}(a(t,\cdot),h_0,h_0) + O(\sqrt{\varepsilon})$$

in Σ^k_ε (see (4.40) *for definition), with*

$$x(t) = \mathrm{Exp}\left(\frac{d}{2}t\mathcal{Z}^{(\lambda_0)}\right)x_0.$$

In particular, this proposition means that, contrarily to what happens in Riemannian manifolds, there are wave packet solutions of the Schrödinger equation which remain localized even in very long time (of order ~ 1 independently of ε). For example, this is not the case for the torus (see [AM14, BZ12]) or semi-classical completely integrable systems (see [AFM15]).

In what follows, we use the notation $\pi^\lambda(X)$ for denoting the operator such that

$$\mathcal{F}(Xf)(\lambda) = \pi^\lambda(X)\mathcal{F}(f),\ \ \forall f\in\mathcal{H}_\lambda$$

where $X\in\mathfrak{g}$ (recall that Xf is defined in (4.2)). Using an integration by part in the definition of $\mathcal{F}(Xf)(\lambda)$ and the fact that $(\pi^\lambda_x)^* = \pi^\lambda_{-x}$, we obtain in particular

$$X(\pi^\lambda_x\Phi_1,\Phi_2) = (\pi^\lambda(X)\pi^\lambda_x\Phi_1,\Phi_2) \tag{4.54}$$

and, in view of (4.39), we have

$$\pi^\lambda(P_j^{(\lambda)}) = \sqrt{|\lambda|}\partial_{\xi_j} \text{ and } \pi^\lambda(Q_j^{(\lambda)}) = i\sqrt{|\lambda|}\xi_j. \tag{4.55}$$

We recall that extending the definition to $-\Delta_G$, we have $\pi^\lambda(-\Delta_G) = H(\lambda)$ where $H(\lambda)$ is the Harmonic oscillator

$$H(\lambda) = |\lambda| \sum_{j=1}^{d} (-\partial_{\xi_j}^2 + \xi_j^2). \tag{4.56}$$

Of course, we also have the relations

$$H(\lambda) = -\sum_{j=1}^{d} \pi^\lambda(V_j)^2 = -\sum_{j=1}^{d} \left(\pi^\lambda(P_j^{(\lambda)})^2 + \pi^\lambda(Q_j^{(\lambda)})^2\right). \tag{4.57}$$

In the sequel, in order to simplify notations, since $\lambda = \lambda_0$ is fixed, we write P_j and Q_j instead of $P_j^{(\lambda_0)}$ and $Q_j^{(\lambda_0)}$. We also use the notation Π_n instead of $\Pi_n^{(\lambda_0)}$.

Proof of Proposition 4.29. We construct a function $v^\varepsilon(t,x)$ of the form

$$v^\varepsilon(t,x) = WP^\varepsilon_{x(t),\lambda_0}(\sigma^\varepsilon(t,x)(t,\cdot), h_0, h_0) + O(\sqrt{\varepsilon}) \tag{4.58}$$

which solves for all $t \in \mathbb{R}$,

$$i\partial_t v^\varepsilon + \frac{1}{2}\Delta_g v^\varepsilon = O(\sqrt{\varepsilon}) \tag{4.59}$$

in all the spaces Σ^ε_k, $k \in \mathbb{N}$. More precisely, we look for $\sigma^\varepsilon(t,x) = \sum_{j=1}^N \varepsilon^{\frac{j}{2}}\sigma_j(t,x)$, for some $N \in \mathbb{N}$ to be fixed later and some maps $(t,x) \mapsto \sigma_j(t,x)$ that are smooth maps from $\mathbb{R} \times G$ to $L^2(\mathbb{R}^d)$, and we shall require $\sigma_0(0,x) = a(x)$ (note that, more rigorously, these operator-valued maps are the values at $\lambda = \lambda_0$ of fields of operators $\sigma_j(t,x,\lambda)$ over the spaces $\mathcal{H}_\lambda = L^2(\mathbb{R}^d)$ of representations, as the symbols of the pseudodifferential calculus). Then, an energy estimate shows that $u^\varepsilon(t) - v^\varepsilon(t) = O(\sqrt{\varepsilon})$ in $L^2(G)$ for all $t \in \mathbb{R}$.

In view of (4.36), it is equivalent to construct a family $\widetilde{v}^\varepsilon(t,x) = \varepsilon^{Q/4}v^\varepsilon(t, x(t)\delta_{\sqrt{\varepsilon}}x)$ which satisfies

$$i\varepsilon\partial_t\widetilde{v}^\varepsilon - i\frac{d}{2}\mathcal{Z}^{(\lambda_0)}\widetilde{v}^\varepsilon + \frac{1}{2}\Delta_g\widetilde{v}^\varepsilon = O(\varepsilon\sqrt{\varepsilon})$$

and

$$\widetilde{v}^\varepsilon(t,x) = \sum_{j=0}^{N} \varepsilon^{\frac{j}{2}}(\sigma_j(t,x)\pi^{\lambda_0}_{\delta_{\varepsilon^{-1/2}}(x)}h_0, h_0), \quad N \in \mathbb{N}. \tag{4.60}$$

We emphasize that if we look for operators $\sigma_j(t,x)$ which are of finite rank, then, decomposing $\sigma_j(t,x)h_0$ on the Hermite basis, the function $(\sigma_j(t,x)\pi^{\lambda_0}_{\delta_{\varepsilon^{-1/2}}(x)}h_0, h_0)$ is a sum of terms of the form

$$(a_{j,\beta}(t,x)\pi^{\lambda_0}_{\delta_{\varepsilon^{-1/2}}(x)}h_0, h_\beta),$$

which means that $v^\varepsilon(t)$ satisfying (4.58) is indeed a sum of wave packets.

Let us now construct the operators $\sigma_j(t,x)$. In order to simplify the notations, we set $S_0 = |\lambda_0|\frac{d}{2}$ and

$$\mathcal{L} = i\frac{d}{2}\mathcal{Z}^{(\lambda_0)} - \frac{1}{2}\Delta_G.$$

Note that

$$i\frac{d}{2}\mathcal{Z}^{(\lambda_0)}\pi^{\lambda_0}_x = -S_0\pi^{\lambda_0}_x$$

and that S_0 is such that $H(\lambda_0)h_0 = 2S_0h_0$. We denote by Π_0 the orthogonal projector on the eigenspace of $H(\lambda_0)$ for the eigenvalue $2S_0$. For any operator-valued $\sigma(t,x)$, we have the following result:

$$\begin{aligned}(i\varepsilon\partial_t - \mathcal{L})(\sigma(t,x)\pi^{\lambda_0}_{\delta_{\varepsilon^{-1/2}}(x)}h_0,h_0) &= \frac{S_0}{\varepsilon}(\sigma(t,x)\pi^{\lambda_0}_{\delta_{\varepsilon^{-1/2}}(x)}h_0,h_0) - \frac{1}{2\varepsilon}(\sigma(t,x)H(\lambda_0)\pi^{\lambda_0}_{\delta_{\varepsilon^{-1/2}}(x)}h_0,h_0)\\ &+ \frac{1}{\sqrt{\varepsilon}}(V\sigma(t,x)\cdot\pi^{\lambda_0}(V)\pi^{\lambda_0}_{\delta_{\varepsilon^{-1/2}}(x)}h_0,h_0) + ((i\varepsilon\partial_t - \mathcal{L})\sigma(t,x)\pi^{\lambda_0}_{\delta_{\varepsilon^{-1/2}}(x)}h_0,h_0)\end{aligned}$$

where $V\sigma\cdot\Pi^{\lambda_0}(V) = \sum_{j=1}^{2d} V_j\sigma\Pi^{\lambda_0}(V_j)$. Equivalently, we can write the latter relation under the more convenient form:

$$\begin{aligned}(i\varepsilon\partial_t - \mathcal{L})(\sigma(t,x)\pi^{\lambda_0}_{\delta_{\varepsilon^{-1/2}}(x)}h_0,h_0) &= \frac{1}{2\varepsilon}([H(\lambda_0),\sigma(t,x)]\pi^{\lambda_0}_{\delta_{\varepsilon^{-1/2}}(x)}h_0,h_0)\\ &+ \frac{1}{\sqrt{\varepsilon}}(V\sigma(t,x)\cdot\pi^{\lambda_0}(V)\pi^{\lambda_0}_{\delta_{\varepsilon^{-1/2}}(x)}h_0,h_0) + ((i\varepsilon\partial_t - \mathcal{L})\sigma(t,x)\pi^{\lambda_0}_{\delta_{\varepsilon^{-1/2}}(x)}h_0,h_0).\end{aligned} \tag{4.61}$$

Therefore, for $\sigma_0 = a \in \mathcal{C}^1(\mathbb{R},\mathcal{S}(G))$ a scalar map, we have

$$(i\varepsilon\partial_t - \mathcal{L})(\sigma_0(t,x)\pi^{\lambda_0}_{\delta_{\varepsilon^{-1/2}}(x)}h_0,h_0) = (r_0^\varepsilon(t,x)\pi^{\lambda_0}_{\delta_{\varepsilon^{-1/2}}(x)}h_0,h_0)$$

with

$$r_0^\varepsilon(t,x) = \frac{1}{\sqrt{\varepsilon}}(V\sigma_0(t,x)\cdot\pi^{\lambda_0}(V)\pi^{\lambda_0}_{\delta_{\varepsilon^{-1/2}}(x)}h_0,h_0) + ((i\varepsilon\partial_t - \mathcal{L})\sigma_0(t,x)\pi^{\lambda_0}_{\delta_{\varepsilon^{-1/2}}(x)}h_0,h_0) \tag{4.62}$$

In other words, for any $\sigma_0(t,x)$ which is scalar, the rest term is of order $\varepsilon^{-1/2}$. At the end of the proof, we will specify our choice of σ_0 in (4.67).

We now focus on constructing correction terms in order to compensate the rest term $r_0^\varepsilon(x)$. Note that since $\Pi_0h_0 = h_0$, we also have

$$r_0^\varepsilon(t,x) = \frac{1}{\sqrt{\varepsilon}}(\Pi_0V\sigma_0(t,x)\cdot\pi^{\lambda_0}(V)\pi^{\lambda_0}_{\delta_{\varepsilon^{-1/2}}(x)}h_0,h_0) + ((i\varepsilon\partial_t - \mathcal{L})\sigma_0(t,x)\pi^{\lambda_0}_{\delta_{\varepsilon^{-1/2}}(x)}h_0,h_0)$$

The second term involves the scalar operator $(i\varepsilon\partial_t - \mathcal{L})\sigma_0(t,x)$ which commutes with Π_0 while the first one depends on $\Pi_0V\sigma_0(t,x)\cdot\pi^{\lambda_0}(V)$ which does not. For constructing $\sigma_1(t,x)$, we use the computation (4.61) and the fact that for symbols $\sigma(t,x)$ that anti-commute with $H(\lambda_0)$, one can find $\theta(t,x)$ such that $\sigma(t,x) = [H(\lambda_0),\theta(t,x)]$.

• *Construction of the approximate solution up to* $\sqrt{\varepsilon}$. We have already noticed in Section 4-A.2) that if

$$\theta_0(t,x) = -\frac{1}{2i|\lambda_0|}\sum_{j=1}^{d}\Big(P_j\sigma_0(t,x)\pi^{\lambda_0}(Q_j) - Q_j\sigma_0(t,x)\pi^{\lambda_0}(P_j)\Big),$$

we have the following relations that we prove below

$$V\sigma_0(t,x)\cdot\pi^{\lambda_0}(V) = -[H(\lambda_0),\theta_0(t,x)], \tag{4.63}$$

$$\Pi_0(V\theta_0(t,x)\cdot\pi^{\lambda_0}(V))\Pi_0 = \frac{1}{2}\Pi_0\left(i\frac{d}{2}\mathcal{Z}^{\lambda_0}\sigma_0(t,x) - \frac{1}{2}\Delta_G\sigma_0(t,x)\right)\Pi_0 = \frac{1}{2}\Pi_0\mathcal{L}\sigma_0(t,x). \tag{4.64}$$

Therefore, setting

$$\sigma_1(t,x) = 2\Pi_0\theta_0(t,x),$$

and using (4.61), we obtain that

$$
\begin{aligned}
(i\varepsilon\partial_t - \mathcal{L})(\sigma_1(t,x)\pi^{\lambda_0}_{\delta_{\varepsilon^{-1/2}}(x)}h_0, h_0) &= -\frac{1}{\varepsilon}(V\sigma_0(t,x)\cdot\pi^{\lambda_0}(V)\pi^{\lambda_0}_{\delta_{\varepsilon^{-1/2}}(x)}h_0, h_0) \\
&+ \frac{1}{\sqrt{\varepsilon}}(\mathcal{L}\sigma_0(t,x)\pi^{\lambda_0}_{\delta_{\varepsilon^{-1/2}}(x)}h_0, h_0) + ((i\varepsilon\partial_t - \mathcal{L})\sigma_1(t,x)\pi^{\lambda_0}_{\delta_{\varepsilon^{-1/2}}(x)}h_0, h_0)
\end{aligned}
$$

Therefore, the function $\widetilde{v}_1^\varepsilon(t,x) = ((\sigma_0(t,x) + \sqrt{\varepsilon}\sigma_1(t,x))\pi^{\lambda_0}_{\delta_{\varepsilon^{-1/2}}(x)}h_0, h_0)$ satisfies in Σ^k_ε the equation

$$
(i\varepsilon\partial_t - \mathcal{L})\widetilde{v}_1^\varepsilon(t,x) = r_1^\varepsilon(t,x) + O(\varepsilon\sqrt{\varepsilon})
$$

with

$$
r_1^\varepsilon(t,x) = -\sqrt{\varepsilon}(\mathcal{L}\sigma_1(t,x)\pi^{\lambda_0}_{\delta_{\varepsilon^{-1/2}}(x)}h_0, h_0) + i\varepsilon(\partial_t\sigma_0(t,x)\pi^{\lambda_0}_{\delta_{\varepsilon^{-1/2}}(x)}h_0, h_0).
$$

• *Construction of the approximate solution up to ε.* We observe that by construction $\theta_0(t,x)$ and $\sigma_1(t,x)$ anticommute with $H(\lambda_0)$. Therefore, there exists $\sigma_2(t,x)$ such that

$$
\mathcal{L}\sigma_1(t,x) = \frac{1}{2}[H(\lambda_0), \sigma_2(t,x)], \tag{4.65}
$$

and the function $\widetilde{v}_2^\varepsilon(t,x) = ((\sigma_0(t,x) + \sqrt{\varepsilon}\sigma_1(t,x) + \varepsilon\sqrt{\varepsilon}\sigma_2(t,x))\pi^{\lambda_0}_{\delta_{\varepsilon^{-1/2}}(x)}h_0, h_0)$ satisfies the equation

$$
(i\varepsilon\partial_t - \mathcal{L})\widetilde{v}_2^\varepsilon(t,x) = r_2^\varepsilon(t,x) + O(\varepsilon\sqrt{\varepsilon})
$$

with

$$
r_2^\varepsilon(t,x) = \varepsilon(V\sigma_2(t,x)\cdot\pi^{\lambda_0}(V)\pi^{\lambda_0}_{\delta_{\varepsilon^{-1/2}}(x)}h_0, h_0) + i\varepsilon(\partial_t\sigma_0(t,x)\pi^{\lambda_0}_{\delta_{\varepsilon^{-1/2}}(x)}h_0, h_0).
$$

At this stage of the proof, we observe that by choosing an adequate term σ_3, the off-diagonal part of $V\sigma_2\cdot\pi^{\lambda_0}(V)$ can be treated in the same manner than the off-diagonal term $\mathcal{L}\sigma_1$. Finally we are left with

$$
\widetilde{v}_3^\varepsilon(t,x) = ((\sigma(t,x) + \sqrt{\varepsilon}\sigma_1(t,x) + \varepsilon\sqrt{\varepsilon}\sigma_2(t,x) + \varepsilon^2\sigma_3(t,x))\pi^{\lambda_0}_{\delta_{\varepsilon^{-1/2}}(x)}h_0, h_0)
$$

and the equation

$$
(i\varepsilon\partial_t - \mathcal{L})\widetilde{v}_3^\varepsilon(t,x) = r_3^\varepsilon(t,x) + O(\varepsilon^{3/2})
$$

with

$$
r_3^\varepsilon(t,x) = \varepsilon((i\partial_t\sigma_0 + \Pi_0 V\sigma_2(t,x)\cdot\pi^{\lambda_0}(V)\Pi_0)\pi^{\lambda_0}_{\delta_{\varepsilon^{-1/2}}(x)}h_0, h_0).
$$

• *Construction of the approximate solution up to $\varepsilon^{3/2}$.* For concluding the proof, we use the specific form of the term $\Pi_0 V\sigma_2(t,x)\cdot\pi^{\lambda_0}(V)\Pi_0$. We claim, and we prove below, that there exists a selfadjoint differential operator $\widetilde{\mathcal{L}}$ such that

$$
\Pi_0 V\sigma_2(t,x)\cdot\pi^{\lambda_0}(V)\Pi_0 = \widetilde{\mathcal{L}}\sigma_0(t,x)\Pi_0. \tag{4.66}
$$

Therefore, it is enough to choose the function $\sigma_0(t,x)$ as the solution of the equation

$$
i\partial_t\sigma_0(t,x) + \widetilde{\mathcal{L}}\sigma_0(t,x) = 0 \quad \sigma_0(0,x) = a(x). \tag{4.67}
$$

• *Proof of relations* (4.63)*,* (4.64) *and* (4.66)*.* Let us begin with (4.63). Using (4.55) and (4.56), we get that for $1 \leqslant j \leqslant d$ there holds

$$
[H(\lambda_0), \pi^{\lambda_0}(Q_j)] = 2i|\lambda|\pi^{\lambda_0}(P_j) \;\text{ and }\; [H(\lambda_0), \pi^{\lambda_0}(P_j)] = -2i|\lambda_0|\pi^{\lambda_0}(Q_j).
$$

Therefore

$$\begin{aligned}[H(\lambda_0),\theta_0] &= -\frac{1}{2i|\lambda|}\sum_{j=1}^{d}(P_j\sigma_0[H,\pi^{\lambda_0}(Q_j)] - Q_j\sigma_0[H,\pi^{(\lambda_0)}(P_j)]) \\ &= -\sum_{j=1}^{d}(P_j\sigma_0\pi^{\lambda_0}(P_j) + Q_j\sigma_0\pi^{(\lambda_0)}(Q_j)) \\ &= -V\sigma_0\cdot\pi^{\lambda_0}(V)\end{aligned}$$

which gives (4.63).

The relation (4.64) is a direct application of Lemma B.2 in [FF21] which states that if

$$T := \left(\sum_{j_1=1}^{2d} V_{j_1}\pi^{\lambda_0}(V_{j_1})\right)\circ\left(\sum_{j_2=1}^{d}\Big(P_{j_2}\pi^{\lambda_0}(Q_{j_2}) - Q_{j_2}\pi^{\lambda_0}(P_{j_2})\Big)\right),$$

then

$$\Pi_n T\Pi_n = |\lambda_0|\left((n+\frac{d}{2})\mathcal{Z}^{(\lambda_0)} + \frac{i}{2}\Delta_G\right)\Pi_n$$

where Π_n denotes the orthogonal projector on $\mathrm{Vect}(h_\alpha,\ \ |\alpha|=n)$ (recall that Π_n depends on λ_0 since it is defined from $H(\lambda_0)$ but we omit this fact in the notation). Note that these relations are nothing but consequences of the elementary properties of the creation-annihilation operators ∂_{ξ_j} and $i\xi_j$.

Let us now prove the claim (4.66).We use the notations of [FF21] and introduce the operators

$$R_j := \frac{1}{2}(P_j - iQ_j), \quad \text{and} \quad \bar{R}_j := \frac{1}{2}(P_j + iQ_j).$$

By (4.39), the operators $\pi^{\lambda_0}(R_j) = \frac{\sqrt{|\lambda_0|}}{2}(\partial_{\xi_j}+\xi_j)$ and $\pi^{\lambda_0}(\bar{R}_j) = \frac{\sqrt{|\lambda_0|}}{2}(\partial_{\xi_j}-\xi_j)$ are the creation-annihilation operators associated with the harmonic oscillator $H(\lambda_0)$. The well-known recursive relations of the Hermite functions give for $\alpha\in\mathbb{N}^d$ and $j=1,\dots,d$,

$$\pi^{\lambda_0}(R_j)h_\alpha = \frac{\sqrt{|\lambda_0|}}{2}\sqrt{2\alpha_j}h_{\alpha-\mathbf{1}_j} \qquad \pi^{\lambda_0}(\bar{R}_j)h_\alpha = -\frac{\sqrt{|\lambda_0|}}{2}\sqrt{2(\alpha_j+1)}h_{\alpha+\mathbf{1}_j}.$$

In the preceding formula, we use the convention $h_{\alpha-\mathbf{1}_j}=0$ as soon as $\alpha_j=0$. Actually, one has $\pi(R_j)h_0=0$. We will also use the expression of $\Pi_0\pi(\bar{R}_j)$ that derives from these formula.

Let us now compute σ_2. Starting from

$$\sum_{j=1}^{d}(P_j\pi^{\lambda_0}(Q_j) - Q_j\pi^{\lambda_0}(P_j)) = -2i\sum_{j=1}^{d}(R_j\pi^{\lambda_0}(\bar{R}_j) - \bar{R}_j\pi^{\lambda_0}(R_j)),$$

and using $\Pi_0\pi^{\lambda_0}(\bar{R}_j)=0$, we obtain

$$\sigma_1(t,x) = -\frac{2\Pi_0}{|\lambda_0|}\sum_{j=1}^{d}\bar{R}_j a(t,x)\pi^{\lambda_0}(R_j).$$

Therefore $\sigma_1 = \Pi_0\sigma_1\Pi_1$ can be written

$$\Pi_0\sigma_1\Pi_1 = -\frac{2}{|\lambda_0|}\sum_{j=1}^{d}\bar{R}_j a(t,x)\Pi_0\pi^{\lambda_0}(R_j).$$

We deduce from (4.65) that

$$\Pi_0\sigma_2\Pi_1 = -\frac{1}{|\lambda_0|}\Pi_0\mathcal{L}\sigma_1\Pi_1.$$

Therefore

$$\sigma_2(t,x) = \frac{2}{|\lambda_0|^2}\sum_{j=1}^{d}\mathcal{L}\bar{R}_j a(t,x)\Pi_0\pi^{\lambda_0}(R_j).$$

We now use that for any operator-valued $\sigma(t,x)$,

$$V\sigma\cdot\Pi^{\lambda_0}(V) = 2\sum_{k=1}^{d}(R_k\sigma\pi^{\lambda_0}(\bar{R}_k) + \bar{R}_k\sigma\pi^{\lambda_0}(R_k))$$

and we obtain

$$V\sigma_2\cdot\Pi^{\lambda_0}(V) = \frac{4}{|\lambda_0|^2}\sum_{j,k=1}^{d}(R_k\mathcal{L}\bar{R}_j a(t,x)\Pi_0\pi^{\lambda_0}(R_j)\pi^{\lambda_0}(\bar{R}_k) + \bar{R}_k\mathcal{L}\bar{R}_j a(t,x)\Pi_0\pi^{\lambda_0}(R_j)\pi^{\lambda_0}(R_k)).$$

When computing the diagonal part of the operator above or, more precisely $\Pi_0 V\sigma_2\cdot\Pi^{\lambda_0}(V)\Pi_0$, we use $\Pi_0\pi(R_j)\pi(\bar{R}_k) = \Pi_0\pi(\bar{R}_k)\pi(R_j) = 0$ when $j\neq k$ and we find

$$\Pi_0 V\sigma_2\cdot\Pi^{\lambda_0}(V)\Pi_0 = \frac{4}{|\lambda_0|^2}\sum_{j=1}^{d}R_j\mathcal{L}\bar{R}_j a(t,x)\Pi_0\pi^{\lambda_0}(R_j)\pi^{\lambda_0}(\bar{R}_j).$$

Using

$$R_j\bar{R}_j = \frac{1}{4}(P_j^2+Q_j^2) + \frac{i}{4}\mathcal{Z}^{(\lambda_0)} \quad \text{and} \quad [R_j,\bar{R}_j] = \frac{i}{2}\mathcal{Z}^{(\lambda_0)},$$

we obtain

$$R_j\mathcal{L}\bar{R}_j = (\mathcal{L} - i\mathcal{Z}^{(\lambda_0)})R_j\bar{R}_j \quad \text{and} \quad \Pi_0\pi^{\lambda_0}(R_j)\pi^{\lambda_0}(\bar{R}_j) = -\frac{|\lambda_0|}{2}\Pi_0$$

and therefore

$$\begin{aligned}
\Pi_0 V\sigma_2\cdot\Pi^{\lambda_0}(V)\Pi_0 &= -\frac{2}{|\lambda_0|}\sum_{j=1}^{d}(\mathcal{L} - i\mathcal{Z}^{(\lambda_0)})R_j\bar{R}_j a\Pi_0 \\
&= -\frac{2}{|\lambda_0|}(\mathcal{L} - i\mathcal{Z}^{(\lambda_0)})(\frac{1}{4}\Delta_G + \frac{id}{4}\mathcal{Z}^{(\lambda_0)})a\Pi_0 \\
&= -\frac{1}{2|\lambda_0|}\left(i\left(\frac{d}{2}-1\right)\mathcal{Z}^{(\lambda_0)} - \frac{1}{2}\Delta_G\right)(\Delta_G + id\mathcal{Z}^{(\lambda_0)})a\Pi_0
\end{aligned}$$

which concludes the proof of (4.66) with

$$\widetilde{\mathcal{L}} = -\frac{1}{2|\lambda_0|}\left(i\left(\frac{d}{2}-1\right)\mathcal{Z}^{(\lambda_0)} - \frac{1}{2}\Delta_G\right)(\Delta_G + id\mathcal{Z}^{(\lambda_0)})$$

that is clearly self-adjoint. □

In case the harmonics of the initial wave packet are no more equal to h_0, e.g.

$$u_0^\varepsilon = WP^\varepsilon_{x_0,\lambda_0}(a, h_\alpha, h_\alpha)$$

with $\alpha\in\mathbb{N}^d$ of length n, the operator $\Pi_n V\sigma_2\pi(V)\Pi_n$ is not scalar: it is matricial since one must add terms of the form $(b_\beta(t,x)\pi_x^{\lambda_0}h_\alpha, h_\beta)$ for all $\beta\in\mathbb{N}^d$ of length n. Equation (4.67) is then replaced by an equation with values in finite-rank operators. Setting $F(\sigma_0) = \Pi_n V \sigma_2 \pi(V) \Pi_n$, F is a linear map on the set $\mathcal{S}(G, \mathcal{L}(V_n))$ where $V_n = \mathrm{Vect}(h_\alpha,\ |\alpha| = n)$. We endow this set of matrix-valued functions with the scalar product $\langle a, b\rangle = \int_G \mathrm{Tr}_{\mathcal{L}(V_n)}(a(x)\bar{b}(x))dx$. Then, one can define two linear maps $\mathbb{A}$ and $\mathbb{S}$ such that $F = \mathbb{S} + \mathbb{A}$ with $\mathbb{S}$ self-adjoint, $\mathbb{A}$ skew symmetric and $\mathbb{A} \circ \mathbb{S} = \mathbb{S} \circ \mathbb{A}$. Observing that $\sigma_0(0) = a(x)\mathrm{Id}_{V_n} \in \mathrm{Ker}\mathbb{A}$, one then solves $i\partial_t \sigma_0 = F(\sigma_0)$ in $\mathrm{Ker}\mathbb{A}$, which induces the solution $\sigma_0(t) = \mathrm{e}^{-it\mathbb{S}}\sigma_0(0)$. As a conclusion, noticing that the argument would be the same for

$$u_0^\varepsilon = WP^\varepsilon_{x_0,\lambda_0}(a, h_\gamma, h_\alpha)$$

for $\alpha \neq \gamma$, we deduce the following remark from the linearity of the equation and the fact that the set of Hermite functions generates $L^2(\mathbb{R}^d)$.

Remark 4.30. The solution to (4.3) with $\mathbb{V} = 0$ and initial data which is a wave packet is asymptotic to a wave packet in finite time.

Chapter 5

Propagation of singularities of subelliptic wave equations

"Si cela va sans le dire, cela ira encore mieux en le disant."
Talleyrand.

This chapter is adapted from the preprint [Let21b]. It proves Theorems 4 and 5.

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We revisit the paper [Mel86] by R. Melrose, providing a full proof of the main theorem on propagation of singularities for subelliptic wave equations, and linking this result with sub-Riemannian geometry. This result asserts that singularities of subelliptic wave equations only propagate along null-bicharacteristics and abnormal extremal lifts of singular curve.

As a new consequence, for $x \neq y$ and denoting by K_G the wave kernel, we obtain that the singular support of the distribution $t \mapsto K_G(t,x,y)$ is included in the set of lengths of the normal geodesics joining x and y, at least up to the time equal to the minimal length of a singular curve joining x and y.

5.1 Introduction

In microlocal analysis, the celebrated propagation of singularities theorem describes the wave-front set $WF(u)$ of a distributional solution u to a partial (or pseudo) differential equation $Pu = f$ in terms of the principal symbol p of P: it says that if p is real and homogeneous, then $WF(u)\backslash WF(f) \subset p^{-1}(0)$, and that if additionally the characteristics are simple ($p = 0 \Rightarrow dp \neq 0$ outside the null section), then $WF(u) \setminus WF(f)$ is invariant under the bicharacteristic flow induced by the Hamiltonian vector field of p.

This result was first proved in [DH72, Theorem 6.1.1] and [Hor71a, Proposition 3.5.1]. However, it leaves open the case where the characteristics of P are not simple. In a very short and impressive paper [Mel86], Melrose sketched the proof of an analogous propagation of singularities result for the wave operator $P = D_t^2 - A$ when A is a self-adjoint non-negative real second-order differential operator which is only subelliptic. Such operators P are typical examples for which there exist double characteristic points.

Despite the potential scope of this result, we did not find in the literature any other paper quoting it. The proof provided in [Mel86] is very sketchy, and we thought it would deserve to be written in full details. This is what we do in the present note, before presenting in the last section a new application of this result. Since the publication of [Mel86] in 1986, the development of sub-Riemannian geometry (the geometry associated to subelliptic operators) has brought some tools and concepts which we use here to shed a new light on this result: for example, we explain that singular curves and their abnormal extremal lifts, which are central objects in control theory and played a key role in the discovery of so-called abnormal minimizers (see [Mon94], [Mon02]), appear naturally in [Mel86], although it is not written explicitly.

For the sake of coherence, we borrow nearly all notations to [Mel86]. A is a self-adjoint non-negative real second-order differential operator on a smooth compact manifold X without boundary:

$$\forall u \in C^\infty(X), \qquad (Au,u) = (u,Au) \geqslant 0 \tag{5.1}$$

with

$$(u,v) = \int_X u(x)\overline{v(x)}d\nu, \tag{5.2}$$

where ν is some positive C^∞ density. The associated norm is denoted by $\|\cdot\|$.

We also assume that A is subelliptic, in the following sense: there exist a (Riemannian) Laplacian Δ on X and $c, s > 0$ such that

$$\forall u \in C^\infty(X), \qquad \|(-\Delta)^{s/2}u\|^2 \leqslant c((Au,u) + \|u\|^2). \tag{5.3}$$

Finally, we assume that A has vanishing sub-principal symbol.[1]

[1] Since X is endowed with a smooth density ν, the sub-principal symbol makes sense, see Appendix A. Note also that the assumption of vanishing sub-principal symbol is not made in [Mel86], but it simplifies the presentation and it is valid in applications.

Example 5.1. An important class of examples of such operators A is given by sub-Laplacians (or Hörmander's sums of squares, see [RS76] or [LL21]), that is, operators of the form $A = \sum_{i=1}^K Y_i^* Y_i$ for some smooth vector fields Y_i on X (here, Y_i^* denotes the adjoint of Y_i for the scalar product (5.2)) satisfying Hörmander's condition: the Lie algebra generated by $Y_1, \ldots, Y_K$ is equal to the whole tangent bundle TX.

The assumption (5.1) implies that A has a self-adjoint extension with the domain

$$\mathscr{D}(A) = \{u \in \mathcal{D}'(X);\ Au \in L^2(X)\}.$$

By the spectral theorem, for any $t \in \mathbb{R}$, the self-adjoint operator

$$G(t) = A^{-1/2} \sin(tA^{1/2})$$

is a well-defined operator bounded on $L^2(X)$, in fact it maps $L^2(X)$ into $\mathscr{D}(A^{1/2})$. Together with the self-adjoint operator $G'(t) = \cos(tA^{1/2})$, this allows to solve the Cauchy problem for the wave operator (here $D_t = \frac{1}{i}\partial_t$)

$$\begin{aligned} (D_t^2 - A)u = Pu = 0 \qquad &\text{in } \mathbb{R} \times X, \\ u = u_0, \quad \partial_t u = u_1 \quad &\text{at} \quad t = 0 \end{aligned} \tag{5.4}$$

by

$$u(t,x) = G'(t)u_0 + G(t)u_1.$$

For $(u_0, u_1) \in \mathscr{D}(A^{1/2}) \times L^2(X)$, we have $u \in C^0(\mathbb{R}; \mathscr{D}(A^{1/2})) \cap C^1(\mathbb{R}; L^2(X))$.

For $f \in \mathcal{D}'(Y)$ a distribution on a manifold Y (equal to X, $\mathbb{R} \times X$ or $\mathbb{R} \times X \times X$ in the sequel), we denote by $WF(f)$ the usual Hörmander wave-front set (see [Hör71b]); in particular, $WF(f) \subset T^*Y \setminus 0$.

The first main result of [Mel86] is the following (the terminology "null-ray" is explained below):

Theorem 5.2. *Let $t \mapsto u(t)$ be a solution of* (5.4)*. For any $t > 0$, if $(x, \xi) \in WF(u(0))$ then there exists $(y, \eta) \in WF(u(-t)) \cup WF(\partial_t u(-t))$ such that (y, η) and (x, ξ) can be joined by a null-ray of length t.*

The second main result of [Mel86], which we state here only in the context of sub-Laplacians[2], concerns the Schwartz kernel K_G of G, i.e., the distribution $K_G \in \mathcal{D}'(\mathbb{R} \times X \times X)$ defined by

$$\forall u \in C^\infty(X), \qquad G(t)u(x) = \int_X K_G(t,x,y)u(y)dy. \tag{5.5}$$

Theorem 5.3. *[Mel86, Theorem 1.8] Assume that A is a sub-Laplacian (see Example 5.1). Then*

$$\begin{aligned} WF(K_G) \subset \{&(t,x,y,\tau,\xi,-\eta) \in T^*(\mathbb{R} \times X \times X) \setminus 0; \\ &\textit{there is a null-ray from } (0,\tau,y,\eta) \textit{ to } (t,\tau,x,\xi)\}. \end{aligned} \tag{5.6}$$

Comments on Theorems 5.2 and 5.3. The null-rays which appear in the statements of Theorems 5.2 and 5.3 are generalizations of the usual null-bicharacteristics (i.e., integral curves of the Hamiltonian vector field H_p of the principal symbol p of P, contained in the characteristic set $p^{-1}(0)$). Their definition will be given in Section 5.2: they are paths tangent to a family

[2] This assumption is not made in [Mel86].

of convex cones Γ_m introduced in Definition 5.4. For example, at $m \in T^*(\mathbb{R} \times X)$ which is not in the double characteristic set $p = dp = 0$, Γ_m is simply $\mathbb{R}^+ \cdot H_p(m)$ (or $\mathbb{R}^- \cdot H_p(m)$). In the double characteristic set $\Sigma_{(2)} = \{p = dp = 0\} \subset M$, their definition is more involved, but readers familiar with control theory will recognize that null-rays contained in $\Sigma_{(2)}$ are abnormal extremal lifts of singular curves (as in Pontryagin's maximum principle). That is, they are integral curves of $\ker(\omega_{\Sigma_{(2)}})$ where $\omega_{\Sigma_{(2)}} = \iota^*_{\Sigma_{(2)}}\omega$ is the pullback of the canonical symplectic form ω on $T^*(\mathbb{R} \times X)$ by the canonical injection $\iota : \Sigma_{(2)} \to M$.

As a particular case of Theorems 5.2 and 5.3, if A is elliptic, then we recover Hörmander's result [Hor71a, Proposition 3.5.1] already mentioned above (see also [Hor07a, Theorem 8.3.1 and Theorem 23.2.9] and [Ler11, Theorem 1.2.23]). In case A has only double characteristics on a symplectic submanifold it was obtained in [Mel84] (in codimension 2) and by B. and R. Lascar [Las82], [LL82] in the general case, using constructions of parametrices (and not positive commutator estimates as in [Mel86]). It is explained in Remark 5.23 how Theorem 5.3 implies these results.

Also, in [Mel86], two other results are proved, namely the finite speed of propagation for P and an estimate on the heat kernel $\exp(-tA)$, but it is not our purpose to discuss here these other results, whose proofs are written in details in [Mel86].

Organization of the paper. As said above, the goal of this note is firstly to provide a fully detailed proof of Theorems 5.2 and 5.3, and secondly to derive a new consequence on the singular support of the Schwartz kernel K_G of the wave operator.

In Section 5.2, we define the convex cones Γ_m generalizing bicharacteristics and give an explicit formula (5.14) for them, then prove their semi-continuity with respect to m, and finally introduce "time functions", which are by definition non-increasing along these cones. In this section, there is no operator, we work at a purely "classical" level.

The proof of Theorems 5.2 and 5.3 is based on a positive commutator argument: the idea, which dates back at least to [Hor71a] (see also [Ivr19, Chapter I.2]), is to derive an *energy inequality* from the computation of a quantity of the form $\mathrm{Im}(Pu, Lu)$, where L is some well-chosen (pseudodifferential) operator. In Section 5.3, we compute this quantity for $L = \mathrm{Op}(\Phi)D_t$ where Φ is a time function, we write it under the form $\frac{1}{2}(Cu, u)$ for an explicit second-order operator C which, up to remainder terms, has non-positive symbol.

In Section 5.4, we derive from this computation the sought energy inequality, which in turn implies Theorem 5.2. This proof requires to construct specific time functions and to use the powerful Fefferman-Phong inequality [FP78].

In Section 5.5, we prove Theorem 5.3: the main idea is to see K_G itself as the solution of a subelliptic wave equation.

Whether Theorem 5.3 implies a trace formula in the spirit of [DG75] for subelliptic wave operators is an open question: due to the particular role of the section $\tau = 0$, it is not clear whether the trace $K_G(t, x, x)$ is a well-defined distribution. However, in Section 5.6, for $x \neq y$, we are able to infer from Theorem 5.3 that the singular support of the distribution $t \mapsto K_G(t, x, y)$ is included in the set of lengths of the normal geodesics joining x and y, at least up to the time equal to the minimal length of a singular curve joining x and y.

In the supplementary sections 5-A.1 and 5-A.2, we prove two additional results concerning the inner semi-continuity of the cones Γ_m.

5.2 The cones Γ_m

At double characteristic points where $dp = 0$, the Hamiltonian vector field H_p vanishes, and the usual propagation of singularities result [DH72, Theorem 6.1.1] does not provide any information. In [Mel86], Melrose defines convex cones Γ_m which replace the usual propagation cone $\mathbb{R}^+ \cdot H_p$ at these points, and which will indicate the directions in which singularities of the subelliptic wave equation (5.4) may propagate.

5.2.1 First definition of the cones Γ_m

In this section, we introduce several notations, and we define the cones Γ_m.

We consider $a \in C^\infty(T^*X)$ satisfying

$$a(x,\xi) \geqslant 0, \quad a(x,r\xi) = r^2 a(x,\xi), \quad r > 0 \tag{5.7}$$

in canonical coordinates (x,ξ). Also we consider

$$p = \tau^2 - a \in C^\infty(M), \qquad \text{where} \quad M = T^*(\mathbb{R}\times X)\setminus 0.$$

Of course, a and p will be in the end the principal symbols of the operators A and P introduced in Section 5.1, but for the moment we work at a purely classical level and forget about operators.

We set

$$M_+ = \{m \in M,\ p(m) \geqslant 0, \tau \geqslant 0\}, \qquad M_- = \{m \in M,\ p(m) \geqslant 0, \tau \leqslant 0\};$$

in particular, $M_+ \cup M_- = \{p \geqslant 0\}$. Let

$$\Sigma = \{m \in M;\ p(m) = 0,\ \tau \geqslant 0\}.$$

Note that $\Sigma \subset \{\tau \geqslant 0\}$; the next few definitions also hold only at points where $\tau \geqslant 0$.

For $m \in M_+$, we consider the set

$$\mathscr{H}_m = \mathbb{R}^+ \cdot H_p(m) \subset T_m M,$$

where H_p is the Hamiltonian vector field of p verifying $\omega(H_p, Z) = -dp(Z)$ for any smooth vector field Z (recall that ω is the canonical symplectic form on the cotangent bundle M).

If m verifies $dp(m) = 0$ and $p(m) \geqslant 0$ (or equivalently $\tau = a = 0$, i.e., m is a double characteristic point), $\mathscr{H}_m = \{0\}$. We therefore extend the notion of "bicharacteristic direction" at m. This will be done first for $m \in M_+$, then also for $m \in M_-$, but never for $m \in \{p < 0\}$: the cones Γ_m are not defined for points $m \in \{p < 0\}$.

Let

$$\Sigma_{(2)} = \{m \in M, \tau = a = 0\} \subset \Sigma.$$

Note that since $a \geqslant 0$, there holds $\Sigma_{(2)} = M_+ \cap M_-$. At $m \in \Sigma_{(2)}$, we have $\tau = a = da = p = dp = 0$ (this follows from the positivity (5.7)) and the Hessian of a is well-defined: it is a quadratic form on $T_m M$. We denote by a_m *the half of this Hessian*, and by $p_m = (d\tau)^2 - a_m$ the half of the Hessian of p. For $m \in \Sigma_{(2)}$, we set

$$\Lambda_m = \{w \in T_m M;\ d\tau(w) \geqslant 0,\ p_m(w) \geqslant 0\} \tag{5.8}$$

and, still for $m \in \Sigma_{(2)}$,

$$\Gamma_m := \{v \in T_m M;\ \omega(v,w) \leqslant 0 \ \ \forall w \in \Lambda_m\}. \tag{5.9}$$

If $m \in M_+ \setminus \Sigma_{(2)}$, we set

$$\Gamma_m = \mathscr{H}_m. \tag{5.10}$$

In particular, the cones Γ_m are defined also at points m outside Σ, i.e. for which $p(m) \neq 0$. Note also that the relation (5.10) says that the cones Γ_m are only *half*-tangents.

In order to extend the definition of the cones Γ_m to M_-, we want this extension to be consistent with the previous definition at points in $M_+ \cap M_- = \Sigma_{(2)}$. We observe that M_- is the image of M_+ under the involution sending τ to $-\tau$. For $(t, \tau, \alpha) \in M_-$, we set

$$\Gamma_m = \Gamma_{m'} \qquad \text{where} \qquad m' = (t, -\tau, \alpha) \in M_+.$$

It is clear that at points of $M_+ \cap M_- = \Sigma_{(2)}$, the two definitions of Γ_m coincide. With this definition in M_-, note that for $m \in M_- \setminus \Sigma_{(2)}$, there is a sign change:

$$\Gamma_m = -\mathscr{H}_m. \tag{5.11}$$

In summary, the formulas (5.9), (5.10) and (5.11) define Γ_m at any point $m \in M_+ \cup M_-$, with different definitions for $m \in \Sigma_{(2)}$, $m \in M_+ \setminus \Sigma_{(2)}$ and $m \in M_- \setminus \Sigma_{(2)}$. The cones Γ_m are not defined for $m \notin M_+ \cup M_-$. For any $m \in M_+ \cup M_-$, the cone Γ_m is closed and convex.

Definition 5.4. *A forward-pointing ray for p is a Lipschitz curve $\gamma : I \to M_+$ defined on some interval $I \subset \mathbb{R}$ with (set-valued) derivative $\gamma'(s) \subset \Gamma_{\gamma(s)}$ for all $s \in I$. Such a ray is forward-null if $\gamma(s) \in \Sigma$ for any $s \in I$. We define backward-pointing rays similarly, with γ valued in M_-, and backward-null rays, with γ valued in $\{m \in M;\ p(m) = 0,\ \tau \leqslant 0\}$.*

Under the terminology "*ray*", we mean either a forward-pointing or a backward-pointing ray; under the terminology "*null-ray*", we mean either a forward-null or a backward-null ray.

In particular null-rays live in $\{p = 0\}$. In Definition 5.4, the fact that the curve γ is only Lipschitz explains why its derivative can be set-valued.

Remark 5.5. In the inclusion (5.6), the null-ray mentioned in the right-hand side is forward if $\tau \geqslant 0$ and backward if $\tau \leqslant 0$ (and both forward and backward if $\tau = 0$).

5.2.2 Formulas for the cones Γ_m

In this section, we derive a formula for the cones Γ_m when $m \in \Sigma_{(2)}$. It is more explicit than (5.9) and we will give in Section 5.6 an application of this formula.

It relies on the computation of the polar of a cone defined by a non-negative quadratic form:

Proposition 5.6. *Let S be a non-negative quadratic form on a real vector space Y, and let $\Theta = (ker(S))^\perp \subset Y^*$ where $\perp$ is understood in the duality sense. Let $\Lambda = \{\xi = (\xi_0, \eta) \in \mathbb{R} \times Y;\ \xi_0 \geqslant S(\eta)^{\frac{1}{2}}\}$ and $\Lambda^0 = \{\xi' \in (\mathbb{R} \times Y)^*;\ \forall \xi \in \Lambda,\ \xi'(\xi) \leqslant 0\}$. Then*

$$\Lambda^0 = \{\xi' = (\xi_0', \eta') \in (\mathbb{R} \times Y)^*;\ \eta' \in \Theta \text{ and } -\xi_0' \geqslant (S^*(\eta'))^{\frac{1}{2}}\}. \tag{5.12}$$

where $\mathbb{R}^*$ is identified with $\mathbb{R}$ and

$$S^*(\eta') = \sup_{\eta \notin ker(S)} \frac{\eta'(\eta)^2}{S(\eta)}. \tag{5.13}$$

Proof. Let $\xi' = (\xi_0', \eta') \in (\mathbb{R} \times Y)^*$ such that $\eta' \in \Theta$ and $-\xi_0' \geqslant (S^*(\eta'))^{\frac{1}{2}}$, we seek to prove that $\xi' \in \Lambda^0$. Let $\xi = (\xi_0, \eta) \in \Lambda$. In particular, $\xi_0 \geqslant (S(\eta))^{\frac{1}{2}}$. We have

$$\xi'(\xi) = \xi_0'(\xi_0) + \eta'(\eta) \leqslant -(S^*(\eta'))^{\frac{1}{2}}(S(\eta))^{\frac{1}{2}} + \eta'(\eta) \leqslant 0$$

hence $\xi' \in \Lambda^0$, which proves one inclusion.

Conversely, to prove that Λ^0 is included in the expression (5.12), we first note that if $\eta' \notin \Theta$, then $(\xi_0', \eta') \notin \Lambda^0$ for any $\xi_0' \in \mathbb{R}^*$. Indeed, if $\eta' \notin \Theta$, there exists $\eta \in Y$ such that $S(\eta) = 0$ and $\eta'(\eta) > 0$. Thus, considering $\xi = (0, \eta)$, which is in Λ by assumption, we get $\xi'(\xi) = \eta'(\eta) > 0$ for any $\xi_0' \in \mathbb{R}^*$ and $\xi' = (\xi_0', \eta')$, proving that $\xi' \notin \Lambda^0$. Now, if $\xi' = (\xi_0', \eta') \in \Lambda^0$ with $\eta' \in \Theta$, we take $\xi_n = (\xi_{0n}, \eta_n)$ with $\eta_n \notin \ker(S)$ so that $\eta'(\eta_n)^2/S(\eta_n) \to S^*(\eta')$, and $\eta'(\eta_n) \geqslant 0$ and $\xi_{0n} = S(\eta_n)^{\frac{1}{2}}$. Then $\xi_n \in \Lambda$. Therefore, $\xi'(\xi_n) \leqslant 0$, which implies that $-\xi_0' \geqslant (S^*(\eta'))^{\frac{1}{2}}$. This proves the result. □

Applying the previous proposition to $S = a_m$ yields a different definition of the cones Γ_m. First, Λ_m, which has been defined in (5.8), can be written as

$$\Lambda_m = \{w \in T_m M;\ d\tau(w) \geqslant (a_m(w))^{\frac{1}{2}}\},$$

Since the definition of Λ_m does not involve dt, we have $v(\partial_t) = 0$ for any $v \in \Lambda_m^0$. Now, using the notation a_m^* to denote (5.13) when $S = a_m$, Proposition 5.6 yields that

$$\begin{gathered}\Lambda_m^0 = \mathbb{R}^+(-d\tau + B_0),\\ B_0 = \{b_0 \in (\ker(a_m))^\perp,\ a_m^*(b_0) \leqslant 1\}.\end{gathered}$$

The duality $\perp$ is computed with respect to the space $\ker(a_m) \subset T(T^*X)$, i.e., $b_0 \in T^*(T^*X)$.

Comparing the definition of Λ_m^0 as the polar cone of Λ_m and the definition (5.9) of Γ_m, we see that Γ_m is exactly the image of Λ_m^0 through the canonical isomorphism $\omega(v, \cdot) \mapsto v$ between T_m^*M and $T_m M$. Thus,

$$\begin{gathered}\Gamma_m = \mathbb{R}^+(\partial_t + B),\\ B = \{b \in \ker(a_m)^{\perp_{\omega_X}},\ a_m^*(\mathcal{I}(b)) \leqslant 1\}.\end{gathered} \tag{5.14}$$

Here, $\perp_{\omega_X}$ designates the symplectic orthogonal with respect to the canonical symplectic form ω_X on T^*X and $\mathcal{I} : b \mapsto \omega_X(b, \cdot)$ is the canonical isomorphism between $T(T^*X)$ and $T^*(T^*X)$.

In case $A = \sum_{i=1}^K Y_i^* Y_i$ is a sum of squares, the expression $a_m^*(\mathcal{I}(b))$ which appears in (5.14) can be written in a much simpler form involving the sub-Riemannian metric associated to the vector fields Y_i, see Lemma 5.22. For more on formula (5.14), which plays a key role in the sequel, see also Section 5.6.3.

Without assuming that A is a sum of squares for the moment, we can already write (5.14) differently, and for that we introduce the "fundamental matrix" F (see [Hor07a, Section 21.5]) defined as follows:

$$\forall Y, Z \in T_m(T^*X), \qquad \omega_X(Y, FZ) = a_m(Y, Z). \tag{5.15}$$

Then, $\omega_X(FY, Z) = -\omega_X(Y, FZ)$. Note that there is a slight abuse of notations here since $T_m(T^*X)$ stands for $T_{\pi_2(m)}(T^*X)$ where $\pi_2 : M \to T^*X$ is the canonical projection on the second factor (recall that $M = T^*(\mathbb{R} \times X) \setminus 0$).

We now prove the following formula[3]:

$$\Gamma_m = \mathbb{R}^+(\partial_t + B), \qquad B = \mathrm{cxhl}\left\{ \frac{FZ}{a_m(Z)^{\frac{1}{2}}},\ Z \notin \ker(a_m) \right\}.$$

Thanks to (5.14), it is sufficient to prove that if $b \in \ker(a_m)^{\perp_{\omega_X}}$ with $a_m^*(\mathcal{I}(b)) = 1$, then $b = FZ/a_m(Z)^{\frac{1}{2}}$ for some $Z \notin \ker(a_m)$. We set $b_0 = -\mathcal{I}(b) \in \ker(a_m)^{\perp}$. By Lax-Milgram's theorem applied to the bilinear form a_m which is continuous and coercive on $T_m(T^*X)/\ker(a_m)$ and b_0 which is a linear form on this space, we get the existence of Z such that $b_0 = a_m(Z, \cdot)$. Using that $a_m^*(b_0) = 1$, we obtain $a_m(Z) = 1$, hence $b_0 = a_m(Z, \cdot)/a_m(Z)^{\frac{1}{2}}$. It follows that $b = -\mathcal{I}^{-1}(b_0) = FZ/a_m(Z)^{\frac{1}{2}}$.

Fixing a norm $|\cdot|$ on TM, the expression (5.14) implies that near any point $m \in \{p \geqslant 0\}$, there is a (locally) uniform constant $c > 0$ such that

$$v \in \Gamma_m \Rightarrow v = T\partial_t + v', \qquad |v'| \leqslant cT \tag{5.16}$$

where v' is tangent to T^*X. Thus, if $\gamma : I \to M_+$ is a forward-pointing ray (thus a Lipschitz curve) defined for $s \in I$, (5.16) implies that $dt/ds \geqslant c'|d\gamma/ds|$, hence $d\gamma/dt = (d\gamma/ds)/(dt/ds)$ is well-defined (possibly set-valued), i.e., γ can be parametrized by t.

Finally, we define the *length* of a ray $\gamma : s \in [s_0, s_1] \to M_+$ by $\ell(\gamma) := |t(s_1) - t(s_0)|$.

Remark 5.7. Thanks to the above parametrization and with a slight abuse in the terminology, we say that there is a null-ray of length $|T|$ from (y, η) to (x, ξ) if there exists a null-ray (in the sense of Definition 5.4) parametrized by t which joins $(0, \tau, y, \eta)$ to (T, τ, x, ξ), where τ verifies $\tau^2 = a(y, \eta) = a(x, \xi)$.

5.2.3 Inner semi-continuity of the cones Γ_m

Using the formula (5.14), we can prove a continuity property for the cones Γ_m, inspired by the arguments of [Mel86, Lemma 2.4].

Lemma 5.8. *Let $a \in C^\infty(T^*X)$ satisfying (5.7). The assignment $m \mapsto \Gamma_m$ is inner semi-continuous on $M_+ \cup M_- = \{p \geqslant 0\}$. In other words,*

$$\forall m_j \to m \ \ (m_j \in M_+ \cup M_-), \quad \forall v_j \in \Gamma_{m_j} \text{ such that } v_j \to v \in T_mM, \text{ there holds } v \in \Gamma_m.$$

Proof of Lemma 5.8. The assignments $\Sigma_{(2)} \ni m \mapsto \Gamma_m$ and $M_+ \cup M_- \setminus \Sigma_{(2)} \ni m \mapsto \Gamma_m$ are clearly continuous thanks to formula (5.9) (resp. (5.10) and (5.11)). Therefore, we restrict to the case where $m \in \Sigma_{(2)}$ and $m_j \in M_+ \cup M_- \setminus \Sigma_{(2)}$.

The cone Γ_{m_j} at $m_j = (t_j, \tau_j, x_j, \xi_j)$ is given by the positive multiples of the Hamiltonian vector field of p:

$$\Gamma_{m_j} = \mathbb{R}^+[2\tau_j\partial_t - H_a(m_j)] \tag{5.17}$$

where $H_a(m_j)$ is the Hamiltonian vector field of a at m_j. Dividing by $2\tau_j$, we rewrite it as

$$\Gamma_{m_j} = \mathbb{R}^+\left(\partial_t - \frac{1}{2}\frac{a(m_j)^{\frac{1}{2}}}{\tau_j}\frac{H_a(m_j)}{a(m_j)^{\frac{1}{2}}}\right) \tag{5.18}$$

We assume without loss of generality that $\tau_j > 0$, the case $\tau_j < 0$ being similar.

[3] This is formula (2.6) in [Mel86].

Since $m_j \in \{p \geqslant 0\}$, we know that $\tau_j \geqslant (a(m_j))^{\frac{1}{2}}$ (the equality would correspond to null-bicharacteristics) thus the first fraction is bounded. For the second fraction, we consider its image $da(m_j)/a(m_j)^{\frac{1}{2}}$ through the isomorphism $\mathcal{I}$ between the tangent and the cotangent bundle given by the canonical symplectic form on T^*X.

In the sequel, we work in a chart near m. If $m_j - m$ accumulates in a direction where a vanishes at order exactly $k \geqslant 2$, then a Taylor development yields

$$|H_a(m_j)| = O(\|m_j - m\|_M^k) = o(\|m_j - m\|_M^{(k+1)/2}) = o(a(m_j)^{1/2}) = o(\tau_j)$$

where $\|\cdot\|_M$ is the Euclidean norm on a chart of M near m. Hence, using (5.17), we obtain that the only limiting direction of the Γ_{m_j} is $\mathbb{R}^+ \cdot \partial_t$, which is contained in Γ_m.

Otherwise, we use the following elementary result.

Lemma 5.9. *If $\frac{m_j - m}{\|m_j - m\|_M}$ has no accumulation point in ker(a_m), then for any $v \in T_m M$, there holds $\frac{1}{2}\frac{da(m_j)(v)}{a(m_j)^{1/2}} = \frac{a_m(m_j - m, v)}{a_m(m_j - m)^{1/2}} + o(1)$.*

Proof. Recall that a_m is half the Hessian of a at m. In a chart, we have $da(m_j)(v) = 2a_m(m_j - m, v) + o(\|m_j - m\|_M)$ and $a(m_j) = a_m(m_j - m) + o(\|m_j - m\|_M^2)$, hence the result. □

In view of (5.18) and (5.14), the inner semi-continuity at m is equivalent to proving that

$$a_m^* \left(\frac{1}{2} \frac{a(m_j)^{\frac{1}{2}}}{\tau_j} \frac{da(m_j)}{a(m_j)^{\frac{1}{2}}} \right) \leqslant 1 + o(1). \tag{5.19}$$

Using the fact that $a(m_j) \leqslant \tau_j^2$ and Lemma 5.9, for any $v \in T_m M \setminus \ker(a_m)$, there holds

$$\frac{1}{a_m(v)} \left(\frac{1}{2} \frac{a(m_j)^{\frac{1}{2}}}{\tau_j} \frac{da(m_j)(v)}{a(m_j)^{\frac{1}{2}}} \right)^2 \leqslant \frac{a_m(m_j - m, v)^2}{a_m(v) a_m(m_j - m)} + o(1) \leqslant 1 + o(1)$$

by Cauchy-Schwarz, hence (5.19) holds, which concludes the proof of Lemma 5.8.

Remark 5.10. We only proved the inner semi-continuity in m, since these arguments do not seem to be sufficiently robust to prove the inner semi-continuity in a. However, we prove in Section 5-A.1 that if we make some additional assumptions, the cones Γ_m are also inner semi-continuous with respect to a (and this second proof requires no formula for the cones, just convexity arguments).

Remark 5.11. Let us explain briefly the intuition behind the semi-continuity stated in Lemma 5.8. Recall that the cones Γ_m generalize bicharacteristic directions at points where $\tau = a = da = p = dp = 0$. To define the cones Γ_m at these points, following formulas (5.8) and (5.9), we have first considered directions where p grows (since $p = dp = 0$, we consider the (half) Hessian p_m), yielding Λ_m, and then Γ_m has been defined as the (symplectic) polar cone of Λ_m. This is exactly parallel to a procedure which yields bicharacteristic directions in the non-degenerate case: the directions along which p grows, verifing $dp(v) \geqslant 0$, form a cone, and it is not difficult to check that its (symplectic) polar consists of a single direction given by the Hamiltonian vector field of p. This unified vision of the cones Γ_m (in the sense that they are obtained in a unified way, no matter whether $m \in \Sigma_{(2)}$ or not) is not used directly in the proof of Lemma 5.8, but it is at the heart of the proof of Proposition 5.30.

Remark 5.12. We prove in Section 5-A.2 that for any $m \in \Sigma_{(2)}$, the cone Γ_m is exactly given by all limits of the cones Γ_{m_j} for $m_j \notin \Sigma_{(2)}$ tending to m.

5.2.4 Time functions

In this section, we introduce time functions which are one of the key ingredients of the proof of Theorems 5.3 and 5.2.

Definition 5.13. *A C^∞ function ϕ near $\overline{m} \in \{p \geqslant 0\} \subset M$ is a time function near $\overline{m}$ if in some neighborhood N of $\overline{m}$,*

$$\phi \text{ is non-increasing along } \Gamma_m,\ m \in N \cap \{p \geqslant 0\}.$$

In particular, ϕ is non-increasing along the Hamiltonian vector field H_p in M_+ but non-decreasing along H_p in M_- (due to (5.11)).

Note that outside $\{p \geqslant 0\}$, there is no constraint on the values of ϕ. The following result asserts the existence of (local) time functions.

Proposition 5.14. *Let $\overline{m} \in \{p \geqslant 0\}$. Then there exists a (non-constant) time function near $\overline{m}$, and moreover we can choose it independent of τ and homogeneous of any fixed degree.*

Proof. We choose $\phi_t' < 0$ and ϕ independent of x, ξ, which is clearly possible even with ϕ 0-homogeneous and independent of τ.

If $\overline{m} \notin \Sigma_{(2)}$, then $m \notin \Sigma_{(2)}$ for m close to $\overline{m}$. Therefore, we want to check that ϕ is weakly decreasing along H_p when $\tau \geqslant 0$, and weakly increasing along H_p when $\tau \leqslant 0$ (because of the sign conventions (5.10) and (5.11)). This is the case: if $\tau > 0$ in a small neighborhood of m, then $H_p = 2\tau\phi_t' \leqslant 0$; and if $\tau < 0$ in a small neighborhood of m, then $H_p = 2\tau\phi_t' \geqslant 0$.

Let us now consider the case $\overline{m} \in \Sigma_{(2)}$. Firstly, for $m \notin \Sigma_{(2)}$ near $\overline{m}$, we have $d\phi(H_p) = 2\tau\phi_t'$ is $\leqslant 0$ if $m \in M_+ \setminus \Sigma_{(2)}$ and $\geqslant 0$ if $m \in M_- \setminus \Sigma_{(2)}$. Secondly, for $m \in \Sigma_{(2)}$ near $\overline{m}$, we have the inequality $d\phi(v) = dt(v)\phi_t' \leqslant 0$ for any v such that $dt(v) \geqslant 0$, which is the case for $v \in \Gamma_m$. In any case, ϕ is non-increasing along Γ_m. □

5.3 A positive commutator

The proof of Theorems 5.2 and 5.3 is based on a "positive commutator" technique, also known as "multiplier" or "energy" method in the literature. The idea is to derive an inequality from the computation of a quantity of the form $\mathrm{Im}(Pu, Lu)$ where L is some well-chosen (pseudodifferential) operator. In the present note, the operator L is related to the time functions introduced in Definition 5.13.

In the sequel, we use polyhomogeneous symbols, denoted by S^m_{phg}, and the *Weyl quantization*, denoted by $\mathrm{Op} : S^m_{\mathrm{phg}} \to \Psi^m_{\mathrm{phg}}$ (see Appendix A). For example, we consider the operator $D_t = \frac{1}{i}\partial_t = \mathrm{Op}(\tau)$ (of order 1). The operator $A \in \Psi^2_{\mathrm{phg}}$ has principal symbol $a \in C^\infty(T^*X)$ satisfying (5.7), and $P = D_t^2 - A$ has principal symbol $p = \tau^2 - a$.

Also, $\Phi(t, x, \xi)$ designates a smooth *real-valued* function on M, homogeneous of degree $\alpha \in \mathbb{R}$ in ξ, compactly supported on the base $\mathbb{R} \times X$, and independent of τ. In Section 5.4, we will take Φ to be a time function. By the properties of the Weyl quantization, $\mathrm{Op}(\Phi)$ is a compactly supported selfadjoint (with respect to ν) pseudodifferential operator of order α.

As indicated above, our goal in the next section will be to compute C defined by[4]

$$\mathrm{Im}(Pu, \mathrm{Op}(\Phi)D_t u) := \frac{1}{2}(Cu, u), \tag{5.20}$$

since this will allow us to derive the inequality (5.50) which is the main ingredient in the proof of Theorems 5.2 and 5.3.

[4] In [Mel86], C is explicitly defined as $\mathrm{Im}(\mathrm{Op}(\Phi)D_t u, Pu) := (Cu, u)$; however the formulas (6.1) and (6.2) in [Mel86] are not coherent with this definition, but they are correct if we take the definition (5.20) for C.

5.3.1 The operator C

Our goal in this section is to compute C defined by (5.20). We have

$$\mathrm{Im}(Pu, \mathrm{Op}(\Phi)D_t u) := I_1 - I_2 \tag{5.21}$$

with

$$I_1 = \mathrm{Im}(D_t^2 u, \mathrm{Op}(\Phi)D_t u) \qquad \text{and} \qquad I_2 = \mathrm{Im}(Au, \mathrm{Op}(\Phi)D_t u).$$

Noting that

$$[D_t, \mathrm{Op}(\Phi)] = \mathrm{Op}(\frac{1}{i}\Phi_t')$$

where $\Phi_t' = \partial_t \Phi$ (see [Zwo12, Theorem 4.6]), we have for I_1:

$$\begin{aligned}
I_1 &= \frac{1}{2i}\big((D_t^2 u, \mathrm{Op}(\Phi)D_t u) - (\mathrm{Op}(\Phi)D_t u, D_t^2 u)\big)\\
&= \frac{1}{2i}\big((D_t\mathrm{Op}(\Phi)D_t^2 u, u) - (D_t^2\mathrm{Op}(\Phi)D_t u, u)\big)\\
&= -\frac{1}{2i}(D_t[D_t, \mathrm{Op}(\Phi)]D_t u, u)\\
&= -\frac{1}{2i}(D_t\frac{1}{i}\mathrm{Op}(\Phi_t')D_t u, u)\\
&= \frac{1}{2}(D_t\mathrm{Op}(\Phi_t')D_t u, u)
\end{aligned} \tag{5.22}$$

Then, we write $\mathrm{Op}(\Phi)D_t = S + iT$ where

$$\begin{aligned}
S &= \frac{1}{2}(\mathrm{Op}(\Phi)D_t + D_t\mathrm{Op}(\Phi))\\
T &= \frac{1}{2i}(\mathrm{Op}(\Phi)D_t - D_t\mathrm{Op}(\Phi)) = \frac{1}{2}\mathrm{Op}(\Phi_t').
\end{aligned} \tag{5.23}$$

Using that A, S and T are selfadjoint, we compute I_2:

$$\begin{aligned}
I_2 = \mathrm{Im}(Au, (S+iT)u) &= \mathrm{Im}((S-iT)Au, u) = \frac{1}{2i}([S,A]u, u) - \mathrm{Re}((TAu, u))\\
&= \frac{1}{2i}([S,A]u, u) - \frac{1}{2}((TA + AT)u, u).
\end{aligned} \tag{5.24}$$

First,

$$[S, A] = \frac{1}{2}([\mathrm{Op}(\Phi), A]D_t + D_t[\mathrm{Op}(\Phi), A]). \tag{5.25}$$

All in all, combining (5.21), (5.22), (5.23), (5.24) and (5.25), we find that C in (5.20) is given by

$$C = D_t\mathrm{Op}(\Phi_t')D_t - \frac{i}{2}([A, \mathrm{Op}(\Phi)]D_t + D_t[A, \mathrm{Op}(\Phi)]) + \frac{1}{2}(A\mathrm{Op}(\Phi_t') + \mathrm{Op}(\Phi_t')A). \tag{5.26}$$

Note that C is of order $2+\alpha$, although we could have expected order $3+\alpha$ by looking too quickly at (5.20).

5.3.2 The principal and subprincipal symbols of C

In this section, we compute the operator C modulo a remainder term in $\Psi^{\alpha}_{\mathrm{phg}}$. All symbols and pseudodifferential operators used in the computations are polyhomogeneous (see Appendix A); we denote by $\sigma_p(C)$ the principal symbol of C. We use the Weyl quantization in the variables $y = (t, x)$, $\eta = (\tau, \xi)$, hence we have for any $b \in S^m_{\mathrm{phg}}$ and $c \in S^{m'}_{\mathrm{phg}}$:

$$\mathrm{Op}(b)\mathrm{Op}(c) - \mathrm{Op}(bc + \frac{1}{2i}\{b, c\}) \in \Psi^{m+m'-2}_{\mathrm{phg}} \tag{5.27}$$

and

$$[\mathrm{Op}(b), \mathrm{Op}(c)] - \mathrm{Op}(\frac{1}{i}\{b, c\}) \in \Psi^{m+m'-3}_{\mathrm{phg}}. \tag{5.28}$$

Note that in (5.28), the remainder is in $\Psi^{m+m'-3}_{\mathrm{phg}}$, and not only in $\Psi^{m+m'-2}_{\mathrm{phg}}$ (see [Hor07a, Theorem 18.5.4], [Zwo12, Theorem 4.12]). Finally, we recall that $\Phi(t, x, \xi)$ is homogeneous in ξ of degree α.

Now, we compute each of the terms in (5.26) modulo $\Psi^{\alpha}_{\mathrm{phg}}$. We prove the following formulas:

$$\frac{1}{2}(A\mathrm{Op}(\Phi'_t) + \mathrm{Op}(\Phi'_t)A) = \mathrm{Op}(a\Phi'_t) \mod \Psi^{\alpha}_{\mathrm{phg}} \tag{5.29}$$

$$D_t\mathrm{Op}(\Phi'_t)D_t = \mathrm{Op}(\tau^2\Phi'_t) \mod \Psi^{\alpha}_{\mathrm{phg}} \tag{5.30}$$

$$\frac{i}{2}([A, \mathrm{Op}(\Phi)]D_t + D_t[A, \mathrm{Op}(\Phi)]) = \mathrm{Op}(\tau\{a, \Phi\}) \mod \Psi^{\alpha}_{\mathrm{phg}} \tag{5.31}$$

Firstly, (5.29) follows from the fact that $A = \mathrm{Op}(a) \mod \Psi^0_{\mathrm{phg}}$ (since the subprincipal symbol of a vanishes) and from (5.27) applied once with $b = a$, $c = \Phi'_t$, and another time with $b = \Phi'_t$ and $c = a$.

Secondly, $\mathrm{Op}(\Phi'_t)D_t = \mathrm{Op}(\Phi'_t)\mathrm{Op}(\tau) = \mathrm{Op}(\Phi'_t\tau + \frac{1}{2i}\{\Phi'_t, \tau\}) + \Psi^{\alpha-1}_{\mathrm{phg}}$ thanks to (5.27). Hence, using again (5.27), we get

$$\begin{aligned} D_t\mathrm{Op}(\Phi'_t)D_t &= \mathrm{Op}(\tau)\mathrm{Op}(\Phi'_t\tau + \frac{1}{2i}\{\Phi'_t, \tau\}) \mod \Psi^{\alpha}_{\mathrm{phg}} \\ &= \mathrm{Op}(\tau^2\Phi'_t + \frac{\tau}{2i}\{\Phi'_t, \tau\} + \frac{1}{2i}\{\tau, \Phi'_t\tau\}) \mod \Psi^{\alpha}_{\mathrm{phg}} \end{aligned}$$

which proves (5.30).

Thirdly, thanks to $A = \mathrm{Op}(a) \mod \Psi^0_{\mathrm{phg}}$ and (5.28), we have

$$[A, \mathrm{Op}(\Phi)] = \mathrm{Op}\left(\frac{1}{i}\{a, \Phi\}\right) \mod \Psi^{\alpha-1}_{\mathrm{phg}}$$

(note that the remainder is in Ψ^{-1}_{phg}, not in Ψ^0_{phg}). Using (5.27), we get

$$[A, \mathrm{Op}(\Phi)]D_t + D_t[A, \mathrm{Op}(\Phi)] = \mathrm{Op}\left(\frac{2\tau}{i}\{a, \Phi\}\right) \mod \Psi^{\alpha}_{\mathrm{phg}}$$

which proves (5.31).

In particular, we get the principal symbol

$$\sigma_2(C) = \tau^2\Phi'_t - \tau H_a\Phi + \Phi'_t a.$$

Using $p = \tau^2 - a$, we can write it differently:

$$\begin{aligned}\sigma_p(C) &= \tau^2\Phi_t' - \tau\{\tau^2 - p, \Phi\} + \Phi_t' a \\ &= \tau^2\Phi_t' - \tau\{\tau^2, \Phi\} + \tau H_p\Phi + \Phi_t' a \\ &= \tau^2\Phi_t' - 2\tau^2\Phi_t' + \tau H_p\Phi + \Phi_t' a \\ &= \tau H_p\Phi - \Phi_t' p. \end{aligned} \tag{5.32}$$

Moreover, the formulas (5.29), (5.30) and (5.31) imply that the subprincipal symbol of C vanishes:

$$\sigma_{\mathrm{sub}}(C) = 0. \tag{5.33}$$

5.4 Proof of Theorem 5.2

The goal of this section is to prove Theorem 5.2. For $V \subset T^*X$ and $t \in \mathbb{R}$, we set

$$\begin{aligned}\mathscr{S}^{-t}(V) =\{(-t, y, \eta) \in \mathbb{R} \times T^*X,\ &\text{there exist } (x, \xi) \in V,\ \tau \in \mathbb{R} \text{ and a ray} \\ &\text{from } (-t, \tau, y, \eta) \text{ to } (0, \tau, x, \xi)\}.\end{aligned} \tag{5.34}$$

Also, when we replace the upper index $-t$ in the above definitions by an interval $J \subset \mathbb{R}$, this means that we allow $-t$ to vary in J. Take care that the above notation (5.34) refers to rays, and not null-rays.

With the above notations, Theorem 5.2 can be reformulated as follows: for any $t > 0$ and any $(x_0, \xi_0) \in WF(u(0))$, there exists $(y_0, \eta_0) \in WF(u(-t)) \cup WF(\partial_t u(-t))$ such that $(-t, y_0, \eta_0) \in \mathscr{S}^{-t}(\{(x_0, \xi_0)\})$ and one of the rays from (y_0, η_0) to (x_0, ξ_0) is null.

First reduction of the problem. If $a(x_0, \xi_0) \neq 0$, then Theorem 5.2 follows from the usual propagation of singularities theorem [DH72, Theorem 6.1.1] and the fact that $\Gamma_m = \mathbb{R}^{\pm} \cdot H_p(m)$ for $m \notin \Sigma_{(2)}$. Therefore, in the sequel we assume that $a(x_0, \xi_0) = 0$.

Also, note that, to prove Theorem 5.2, it is sufficient to find $T > 0$ independent of (x, ξ) (and possibly small) such that the result holds for any $t \in (0, T)$.

Idea of the proof of Theorem 5.2. To show Theorem 5.2, we will prove for $T > 0$ sufficiently small an inequality of the form

$$\|\mathrm{Op}(\Psi_0)u\|_{H^s}^2 \leqslant c(\|\mathrm{Op}(\Psi_0)u\|_{L^2}^2 + \|\mathrm{Op}(\Psi_1)u\|_{L^2}^2) + \text{ Remainder terms} \tag{5.35}$$

where Ψ_0 and Ψ_1 are functions of t, x, ξ such that

- the function Ψ_0 is supported near $t \in [-T, 0]$ and the function Ψ_1 near $t = -T$;
- on their respective supports in t, the operators $\mathrm{Op}(\Psi_0)$ and $\mathrm{Op}(\Psi_1)$ microlocalize respectively near (x_0, ξ_0) and $\mathscr{S}^{-T}(\{(x_0, \xi_0)\})$.

Then, assuming that u is smooth on the support of Ψ_1, we deduce by applying (5.35) for different functions Ψ_0 with different degrees of homogeneity in ξ that u is smooth on the support of Ψ_0.

The inequality (5.35), written more precisely as (5.50) below, will be proved by constructing a time function $\Phi(t, x, \xi)$ such that $\Phi_t' = \Psi_1^2 - \Psi_0^2$, and then by applying the Fefferman-Phong inequality to the operator C given by (5.26) (for this Φ).

Reduction to $X \subset \mathbb{R}^d$ Let us show that it is sufficient to prove Theorem 5.2 in the case $X \subset \mathbb{R}^d$. Note first that it is sufficient to prove Theorem 5.2 "locally", i.e., for sufficiently short times and in a neighborhood of a fixed point $x \in X$, since null-rays stay close from their departure points for short times (this follows from (5.10), (5.11), (5.14)). Then, working in a coordinate chart $\psi : \Omega \to \mathbb{R}^d$ where Ω is a neighborhood of x, the differential operator A is pushed forward into a differential operator $\widetilde{A}$ on $\mathbb{R}^d$ which is also real, second-order, self-adjoint, non-negative and subelliptic. Moreover, we can lift ψ to a symplectic mapping ψ_{lift} : $(x, \xi) \mapsto (\psi(x), ((d_x\psi(x))^{-1})^T\xi)$. Through the differential of ψ_{lift}, the cones Γ_m (computed with $a = \sigma_P(A)$, in X) are sent to the same cones, computed this time with $\widetilde{a} = \sigma_P(\widetilde{A})$ in $\mathbb{R}^d$. This follows from the "symplectic" definition of the cones in Section 5.2.1 and the fact that $\sigma_P(\widetilde{A})$ is the pushforward of $\sigma_P(A)$. Hence, ψ_{lift} maps also null-rays to null-rays. To sum up, if we prove the Theorem for subsets of $\mathbb{R}^d$, then pulling back the situation to X proves Theorem 5.2 in full generality.

In the sequel, we assume $X = \Omega \subset \mathbb{R}^d$.

5.4.1 Construction of the time function

As explained in the introduction of this section, we construct a time function $\Phi(t, x, \xi)$ which verifies several properties. Some time functions are also constructed in the classical proofs of Hörmander's propagation of singularities theorem [Hor71a, Proposition 3.5.1], but in the present context of subelliptic wave equations, the construction is more involved since the cones Γ_m along which time functions should be non-increasing contain much more than a single direction (compare (5.10) with (1.25)). The following lemma summarizes the properties that the time functions we need thereafter should satisfy.

Lemma 5.15. *Let $(x_0, \xi_0) \in T^*X \setminus 0$ and $V \subset V'$ be sufficiently small open neighborhoods of (x_0, ξ_0) such that $\overline{V} \subset V'$. There exist $T > 0$ and $\delta_1 \ll T$ such that for any $0 \leqslant \delta_0 \leqslant \delta_1$ and any $\alpha \in \mathbb{R}$, there exists a smooth function $\Phi(t, x, \xi)$ with the following properties:*

(1) it is compactly supported in t, x;

(2) it is homogeneous of degree α in ξ;

(3) it is independent of τ;

(4) there exists $\delta > 0$ such that at any point of M where $p \geqslant -2\delta a$, there holds $\tau H_p\Phi \leqslant 0$.

(5) its derivative in t can be written $\Phi'_t = \Psi_1^2 - \Psi_0^2$ with Ψ_0 and Ψ_1 homogeneous of degree $\alpha/2$ in ξ;

(6) $\Psi_0 = 0$ outside $\mathscr{S}^{(-T, \frac{\delta_0}{2})}(V')$ and $\Psi_1 = 0$ outside $\mathscr{S}^{(-T-\frac{\delta_0}{2}, -T+\frac{\delta_0}{2})}(V')$;

(7) $\Psi_0 > 0$ on $\mathscr{S}^{(-T+\frac{\delta_0}{2}, 0)}(V)$;

(8) Φ is a time function on $\mathscr{S}^{(-T+\frac{\delta_0}{2}, \frac{\delta_0}{2})}(V)$.

All of the above properties of Φ will be used in Sections 5.4.2 and 5.4.4 to prove Theorem 5.2. The rest of Section 5.4.1 is devoted to the proof of Lemma 5.15. The figures may be helpful to follow the explanations.

We fix $(x_0, \xi_0) \in T^*X \setminus 0$. As said in the introduction of Section 5.4, we assume that $a(x_0, \xi_0) = 0$, and we set $\overline{m} = (0, 0, x_0, \xi_0) \in \Sigma_{(2)}$ where the first two coordinates correspond to the variables t, τ. For m near $\overline{m}$, the cone $-\Gamma_m$ is the cone with base point m and containing the opposite of the directions of Γ_m.

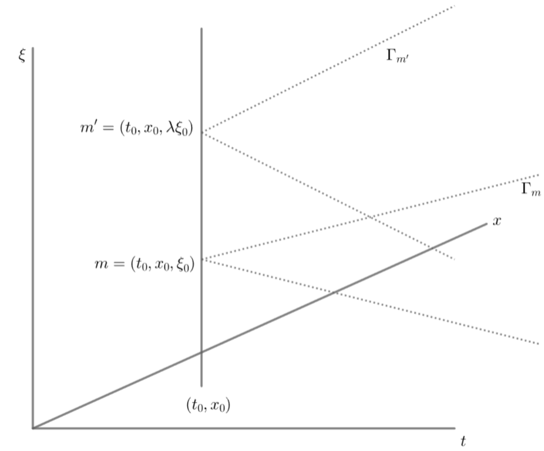
We are looking for a τ-independent time function; since any ray lives in a slice $\tau = \text{const.}$, we first construct Φ in the slice $\tau = 0$, and then we extend Φ to any τ so that it does not depend on τ. If we start from a time function in $\{\tau = 0\}$, then its extension is also a time function: indeed, the image of a ray contained in $\{\tau \neq 0, a = 0\}$ under the map $\tau \mapsto 0$ is also a ray, this follows from the fact that $\mathbb{R}^+\partial_t \subset \Gamma_m$ for any $m \in \Sigma_{(2)}$ (see (5.14)). Thus, the property of being non-increasing along Γ_m is preserved under this extension process.

After the τ variable, we turn to the ξ variable. There is a global homogeneity in ξ of the cones Γ_m and consequently of the null-rays:

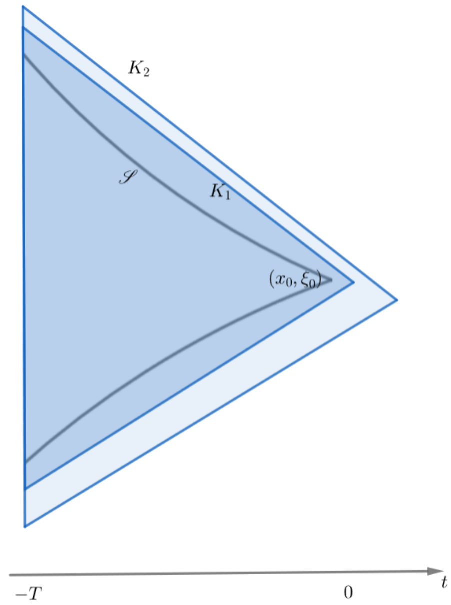
Homogeneity Property. If $[T_1, T_2] \ni t \mapsto \gamma(t) = (x(t), \xi(t)) \in \{a = 0\}$ is a null-ray parametrized by t, then for any $\lambda > 0$, $[T_1, T_2] \ni t \mapsto \gamma_\lambda(t) = (x(t), \lambda\xi(t))$ is a null-ray parametrized by t and joining the same endpoints as γ (in the same time interval $[T_1, T_2]$).

This property follows from (5.9). Thanks to this property, we will be able to find Φ satisfying Point (2) in Lemma 5.15.

Consequently, in our construction, we should have the following picture 5.1a in mind:



(a) The coordinates and the cones Γ_m. On the picture, the cone $\Gamma_{m'}$ has an aperture which is equal to λ times the aperture of Γ_m.



(b) The cones K_1 and K_2 (see (5.36)).

Figure 5.1

At this point we should say that since we are working in the slice $\{\tau = 0\}$, we will use in the sequel the following convenient abuse of notations: for $m = (t, 0, x, \xi)$, we still denote by m the projection of m on $\mathbb{R} \times T^*\Omega$ obtained by throwing away the coordinate $\tau = 0$. The fact that the whole picture is now embedded in $\mathbb{R}^{2d+1}$ (see Figure 5.1a) is very convenient: for example, after throwing away the coordinate $\tau = 0$, we see the cones Γ_m as subcones of $\mathbb{R}^{2d+1}$ (and not of its tangent space).

Also, in the sequel, we only consider points for which $t \geqslant -T$ for some (small) $T > 0$.[5] We set $\delta_1 = T/10$ and take $0 \leqslant \delta_0 \leqslant \delta_1$.

The set of all points which belong to a *backward-pointing* ray starting from (x_0, ξ_0) at time

[5] T is denoted by ε in [Mel86].

0 and stopped at time $-T$ is denoted by $\mathscr{S}$:

$$\mathscr{S} = \bigcup_{0 \leqslant t \leqslant T} \mathscr{S}^{-t}(\{(x_0, \xi_0)\}).$$

Then, $\mathscr{S}$ is closed according to the first point of the following lemma (the second point will be used later):

Lemma 5.16. *The following two properties hold:*

1. *For any closed $V \subset T^*X$ and any $T \geqslant 0$, the set $\mathscr{S}^{-T}(V)$ is closed.*
2. *The mapping $(T, x, \xi) \mapsto \mathscr{S}^{-T}(\{(x, \xi)\})$ is inner semi-continuous, meaning that when $(T_n, x_n, \xi_n) \to (T, x, \xi)$, any point obtained as a limit, as $n \to +\infty$, of points of $\mathscr{S}^{-T_n}(\{(x_n, \xi_n)\})$ belongs to $\mathscr{S}^{-T}(\{(x, \xi)\})$.*

Proof. Both properties follow from the locally uniform Lipschitz continuity (5.16) combined with the extraction of Lipschitz rays as in the Arzelà-Ascoli theorem and the fact that the cones Γ_m are closed. □

We take two closed convex cones K_1 and K_2 such that

$$\mathscr{S} \subset \mathrm{Int}(K_1) \subset \mathrm{Int}(K_2). \tag{5.36}$$

(see Figure 5.1b). It is possible to define Φ going backwards in time from time 0 to time $-T + \frac{\delta_0}{2}$, which is weakly increasing along the directions of K_2 and strictly increasing along the directions of K_1, and which is compactly supported in (t, x) with support contained in the projection of K_2 on this base.

Since $\mathscr{S} \subset \mathrm{Int}(K_1)$, Point 2. of Lemma 5.16 implies that

$$\begin{gathered} \text{if } V \text{ is a sufficiently small neighborhood of } (x_0, \xi_0), \\ \Phi \text{ is strictly increasing from time 0 to time } -T + \frac{\delta_0}{2} \\ \text{along any backward-pointing ray starting from any point } (x, \xi) \in V. \end{gathered} \tag{5.37}$$

Also, if V' is a sufficiently small neighborhood of V, then it has the property that $\mathscr{S}^{-t}(V') \subset K_2$ for any $0 \leqslant t \leqslant T$, thus Property (6) can be guaranteed.

For $t \geqslant -T + \frac{\delta_0}{2}$, we have $\Phi'_t \leqslant 0$ since $\partial_t \in \Gamma_m$, and thus we set $\Psi_0 = \sqrt{-\Phi'_t}$. Then, following the rays backwards in time, we make Ψ_0 fall to 0 between times $-T + \frac{\delta_0}{2}$ and $-T$. Similarly, following the rays backward from time $-T + \frac{\delta_0}{2}$ to time $-T - \frac{\delta_0}{2}$, we extend Φ smoothly and homogeneously (in the fibers in ξ) in a way that Φ is compactly supported in the time-interval $(-T - \frac{\delta_0}{2}, \frac{\delta_0}{2})$ and $\Phi'_t + \Psi_0^2 \geqslant 0$. Finally, we set $\Psi_1 = \sqrt{\Phi'_t + \Psi_0^2}$. It is clear that points (5), (6), (8) are satisfied. See Figure 5.2 for the profile of Φ along a ray.

In Lemma 5.15, Properties (1), (2), (3), (5), (6), (8) follow from the construction. Property (7) follows from (5.37). Finally, Property (4) follows from the fact that due to (5.36), we can replace the cones Γ_m by slightly bigger cones in a way that along the rays associated to these new cones, Φ is still non-decreasing.

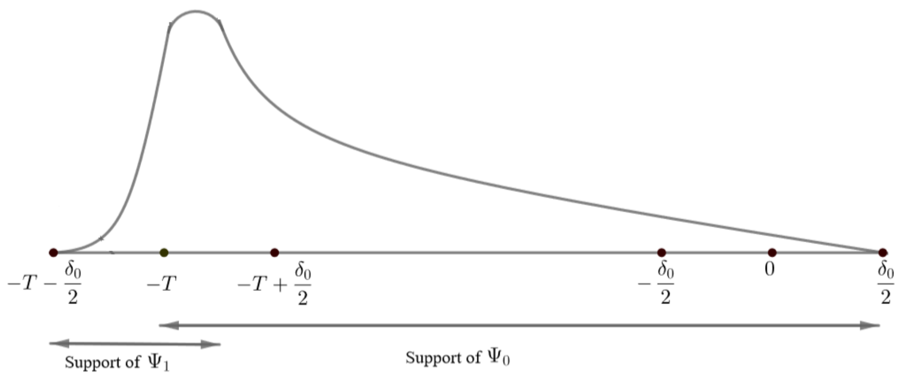


Figure 5.2: Profile of the function Φ along a ray. The abscissa indicates variable t.

5.4.2 A decomposition of C

When Φ satisfies (2), (3), (4) and (5) in Lemma 5.15, the operator C given by (5.26) can be expressed as follows:

Proposition 5.17. *If Φ satisfies (2), (3), (4) and (5) in Lemma 5.15, then writing $\Phi_t' = \Psi_1^2 - \Psi_0^2$, there holds*

$$C = R + R'P + PR' + C' - \delta(Op(\Psi_0)A\,Op(\Psi_0) + D_t\,Op(\Psi_0)^2 D_t) \tag{5.38}$$

where $\delta > 0$ is the same as in (4), $R' = -\frac{\delta}{2}Op(\Phi_t') \in \Psi^{\alpha}_{phg}$, $R = \delta\,Op(\Psi_1)(D_t^2 + A)\,Op(\Psi_1) \in \Psi^{2+\alpha}_{phg}$, and $C' \in \Psi^{2+\alpha}_{phg}$ has non-positive principal symbol and vanishing subprincipal symbol.

We start the proof of this proposition with the following corrected version of [Mel86, Lemma 5.3]:

Lemma 5.18. *Let ϕ be a time function near $\overline{m} \in \Sigma_{(2)}$ which does not depend on τ. Then, there holds*

$$\tau H_p\phi \leqslant \phi_t' p \tag{5.39}$$

in a neighborhood of $\overline{m}$.

Proof of Lemma 5.18. Recalling that $\pm\tau \geqslant 0$ on $M_\pm$, it follows from the definition of a time function that

$$q = \tau\{p, \phi\} \leqslant 0 \quad \text{on } \{p \geqslant 0\}. \tag{5.40}$$

Now, since ϕ does not depend on τ, we get that q is a quadratic polynomial in τ, vanishing at $\tau = 0$:

$$q = b\tau^2 - c\tau, \quad p = \tau^2 - a, \quad a \geqslant 0.$$

More explicitly, $b = 2\phi_t'$ and $c = \{a, \phi\}$. From (5.40), we know that $b \leqslant 0$. Moreover, (5.40) also implies that if $b = 0$, then $c = 0$, hence $\phi_t' = H_p\phi = 0$, and (5.39) is automatically satisfied. Otherwise, $b < 0$. Since $q \leqslant 0$ on $\tau \notin [-a^{1/2}, a^{1/2}]$ by (5.40), we get that the other zero of q, $\tau = c/b$, must lie in $[-a^{1/2}, a^{1/2}]$. Thus, $c^2 \leqslant b^2 a$. Then,

$$\tau\{p, \phi\} - \phi_t' p = \frac{1}{2}b(\tau - c/b)^2 + (b^2 a - c^2)/2b \leqslant 0 \tag{5.41}$$

where we used that $b < 0$. $\square$

Let us come back to the proof of Proposition 5.17. Following the proof of Lemma 5.18 and keeping its notations, we replace (5.40) by the condition that $\tau H_p\Phi \leqslant 0$ on $\{p \geqslant -2\delta a\}$ (this is Point (4) in Lemma 5.15). The proof then gives that in case $b < 0$, there holds $c/b \in [-((1-2\delta)a)^{1/2}, ((1-2\delta)a)^{1/2}]$, hence $c^2 \leqslant b^2 a(1-2\delta)$. Therefore, (5.41) yields this time

$$\tau\{p,\Phi\} - \Phi_t' p \leqslant (b^2a - c^2)/2b \leqslant ba\delta = 2\Phi_t' a\delta.$$

This inequality obviously also holds in case $b = 2\Phi_t' = 0$. Hence, setting $r' = -\frac{\delta}{2}\Phi_t'$, we have

$$\tau\{p,\Phi\} - \Phi_t' p - 2r'p \leqslant 2\Phi_t' a\delta + \Phi_t' p\delta = \Phi_t'\delta(\tau^2 + a) = \delta(\Psi_1^2 - \Psi_0^2)(\tau^2 + a). \tag{5.42}$$

We set $R = \delta \mathrm{Op}(\Psi_1)(D_t^2 + A)\mathrm{Op}(\Psi_1)$. It follows from (5.42), (5.32), (5.33) and (5.27) that the operator

$$C' = C - R - (R'P + PR') + \delta(\mathrm{Op}(\Psi_0) A \mathrm{Op}(\Psi_0) + D_t \mathrm{Op}(\Psi_0)^2 D_t) \tag{5.43}$$

has non-positive principal symbol and vanishing sub-principal symbol. This proves Proposition 5.17.

5.4.3 The Fefferman-Phong inequality

The Fefferman-Phong inequality [FP78] (see also [Ler11, Section 2.5.3]) can be stated as follows: for any pseudodifferential operator C_1' of order $2+\alpha$ whose (Weyl) symbol is non-positive, there holds for any $u \in C_c^\infty$,

$$(C_1'u, u)_{L^2} \leqslant c((\mathrm{Id} - \Delta)^{\alpha/2} u, u)_{L^2} \tag{5.44}$$

where Δ is a Riemannian Laplacian on X. The following lemma is a simple microlocalization of this inequality.

Lemma 5.19. *Let $W, W' \subset T^*(\mathbb{R}\times X)$ be conic sets such that W' is a conic neighborhood of W. Let $C' \in \Psi^{2+\alpha}_{phg}$ with essupp$(C') \subset W$ such that $\sigma_p(C') \leqslant 0$ and $\sigma_{sub}(C') \leqslant 0$. Then there exists $C_\alpha \in \Psi^{\alpha/2}_{\mathrm{phg}}$ with essupp$(C_\alpha) \subset W'$ such that*

$$\forall u \in C_c^\infty(\mathbb{R}\times X), \qquad (C'u, u)_{L^2} \leqslant c(\|C_\alpha u\|_{L^2}^2 + \|u\|_{L^2}^2). \tag{5.45}$$

Proof. Taking a microlocal cut-off χ homogeneous of order 0, essentially supported in W' and equal to 1 on a neighborhood of W, we see that

$$\begin{aligned}(C'u, u) &= (C'(\mathrm{Op}(\chi) + \mathrm{Op}(1-\chi))u, (\mathrm{Op}(\chi) + \mathrm{Op}(1-\chi))u) \\ &= (\mathrm{Op}(\chi)C'\mathrm{Op}(\chi)u, u) + (Q'u, u)\end{aligned} \tag{5.46}$$

where $Q' \in \Psi^{-\infty}$ is explicit:

$$Q' = \mathrm{Op}(1-\chi)C'\mathrm{Op}(\chi) + \mathrm{Op}(\chi)C'\mathrm{Op}(1-\chi) + \mathrm{Op}(1-\chi)C'\mathrm{Op}(1-\chi).$$

Since $Q' \in \Psi^{-\infty}$, we have in particular

$$(Q'u, u) \leqslant c\|u\|_{L^2}^2. \tag{5.47}$$

Then, we write $C' = C_1' + C_2'$ where C_1' has non-positive full Weyl symbol, and $C_2' \in \Psi^\alpha_{\mathrm{phg}}$. First, we apply (5.44) with $\mathrm{Op}(\chi)u$ instead of u: we obtain

$$(\mathrm{Op}(\chi)C_1'\mathrm{Op}(\chi)u, u) \leqslant c\|C_\alpha u\|_{L^2}^2 \tag{5.48}$$

with $C_\alpha = (\mathrm{Id} - \Delta)^{\alpha/4}\mathrm{Op}(\chi)$. Secondly, writing $C_2' = (\mathrm{Id}-\Delta)^{\alpha/4} C_2'' (\mathrm{Id}-\Delta)^{\alpha/4}$ with $C_2'' \in \Psi^0_{\mathrm{phg}}$, we see that

$$(\mathrm{Op}(\chi)C_2'\mathrm{Op}(\chi)u, u) \leqslant c\|C_\alpha u\|_{L^2}^2. \tag{5.49}$$

Combining (5.46), (5.47), (5.48) and (5.49), we get (5.45). ☐

5.4.4 End of the proof of Theorem 5.2

We come back to the proof of Theorem 5.2. We fix $(x_0, \xi_0) \in T^*X \setminus 0$ and consider u a solution of (5.4). For the moment, we assume that u is *smooth*. We consider a time function Φ as constructed in Lemma 5.15.

Using (5.38), we have

$$\begin{aligned} 0 &= 2\mathrm{Im}(Pu, \mathrm{Op}(\Phi) D_t u) \\ &= (Cu, u) \\ &= ((R + R'P + PR' + C' - \delta(\mathrm{Op}(\Psi_0) A \mathrm{Op}(\Psi_0) + D_t \mathrm{Op}(\Psi_0)^2 D_t))u, u). \end{aligned}$$

Hence, using $Pu = 0$ and applying Lemma 5.19 to C', we get:

$$\begin{aligned} (A\mathrm{Op}(\Psi_0)u, \mathrm{Op}(\Psi_0)u) + \|\mathrm{Op}(\Psi_0) D_t u\|_{L^2}^2 &\leqslant c((R_\alpha + R'P + PR' + C')u, u) \\ &\leqslant c_\alpha(\|C_\alpha u\|_{L^2}^2 + \|u\|_{L^2}^2 + (R_\alpha u, u)). \end{aligned}$$

with $c_\alpha \geqslant 1/\delta$ and $R_\alpha = R$, just to keep in mind in the forthcoming inequalities that it depends on α.

But $(A\mathrm{Op}(\Psi_0)u, \mathrm{Op}(\Psi_0)u) \geqslant \frac{1}{c}((-\Delta)^s \mathrm{Op}(\Psi_0)u, \mathrm{Op}(\Psi_0)u) - \|\mathrm{Op}(\Psi_0)u\|^2$ by subellipticity (5.3). Hence

$$\|(-\Delta)^{s/2}\mathrm{Op}(\Psi_0)u\|_{L^2}^2 + \|\mathrm{Op}(\Psi_0)D_t u\|_{L^2}^2 \leqslant c_\alpha(\|C_\alpha u\|_{L^2}^2 + \|u\|_{L^2}^2 + (R_\alpha u, u) + \|\mathrm{Op}(\Psi_0)u\|_{L^2}^2) \tag{5.50}$$

which we decompose into

$$\|(-\Delta)^{s/2}\mathrm{Op}(\Psi_0)u\|_{L^2}^2 \leqslant c_\alpha(\|C_\alpha u\|_{L^2}^2 + \|u\|_{L^2}^2 + (R_\alpha u, u) + \|\mathrm{Op}(\Psi_0)u\|_{L^2}^2) \tag{5.51}$$

and

$$\|\mathrm{Op}(\Psi_0)D_t u\|_{L^2}^2 \leqslant c_\alpha(\|C_\alpha u\|_{L^2}^2 + \|u\|_{L^2}^2 + (R_\alpha u, u) + \|\mathrm{Op}(\Psi_0)u\|_{L^2}^2). \tag{5.52}$$

Now, assume that u is a general solution of (5.4), not necessarily smooth. We have $u \in C^0(\mathbb{R}; \mathscr{D}(A^{1/2})) \cap C^1(\mathbb{R}; L^2(X))$. Recall the following definition.

Definition 5.20. *Let $s_0 \in \mathbb{R}$ and $f \in \mathcal{D}'(\Omega)$. We shall say that f is H^{s_0} at $(x,\xi) \in T^*\Omega \setminus 0$ if there exists a conic neighborhood W of (x,ξ) such that for any 0-th order pseudodifferential operator B with essupp$(B) \subset W$, we have $Bf \in H^s_{loc}(\Omega)$.*
We shall say that f is smooth at (x,ξ) of it is H^{s_0} at (x,ξ) for any $s_0 \in \mathbb{R}$.

Lemma 5.21. *Let V, V' be sufficiently small open neighborhoods of (x_0, ξ_0) such that $\overline{V} \subset V'$. Let u be a solution of (5.4). If u and $\partial_t u$ are smooth in $\mathscr{S}^{(-T-\frac{\delta_0}{2}, -T+\frac{\delta_0}{2})}(V')$, then u is smooth in*

$$U = \mathscr{S}^{(-T+\frac{\delta_0}{2}, 0)}(V).$$

When we say that u is H^{s_0} at (t, y, η), we mean that $u(t)$ is H^{s_0} at $(y, \eta) \in T^*\Omega$.

Proof of Lemma 5.21. We set $u_\varepsilon = \rho_\varepsilon * u$ where $\rho_\varepsilon = \varepsilon^{-(d+1)}\rho(\cdot/\varepsilon)$ and $\rho \in C_c^\infty(\mathbb{R}^{d+1})$ is of integral 1 (and depends on the variables t, x). Recall that d is the dimension of X (and of the coordinate patch Ω).

Applying Lemma 5.15 for any $\alpha \in \mathbb{R}$ yields a function Φ_α which is in particular homogeneous of degree α in ξ; its derivative in t can be written $\Phi'_\alpha = (\Psi_1^\alpha)^2 - (\Psi_0^\alpha)^2$ (the upper index being not an exponent). Then we apply (5.51) to u_ε and with $\alpha = 0$: we get

$$\|(-\Delta)^{s/2}\mathrm{Op}(\Psi_0^0)u_\varepsilon\|_{L^2}^2 \leqslant c_0(\|C_0 u_\varepsilon\|_{L^2}^2 + \|u_\varepsilon\|_{L^2}^2 + (R_0 u_\varepsilon, u_\varepsilon) + \|\mathrm{Op}(\Psi_0^0)u_\varepsilon\|_{L^2}^2) \tag{5.53}$$

where $R_0 = \delta \mathrm{Op}(\Psi_1^0)(D_t^2 + A)\mathrm{Op}(\Psi_1^0)$ (see Proposition 5.17) and $c > 0$ does not depend on ε. All quantities

$$\|C_0 u\|_{L^2}, \quad \|u\|_{L^2}, \quad (R_0 u, u), \quad \|\mathrm{Op}(\Psi_0^0)u\|_{L^2}^2$$

are finite. Therefore, taking the limit $\varepsilon \to 0$ in (5.53), we obtain $u \in H^{2s}$ in U. Using the family of inequalities (5.51), we can iterate this argument: first with $\alpha = 2s$, then with $\alpha = 4s, 6s$, etc, and each time we replace Ψ_0^0, R_0, C_0 by Ψ_0^α, R_α, C_α. At step k, we deduce thanks to (5.51) that $u \in H^{2ks}$. In particular, we use the fact that $\|C_\alpha u\|_{L^2}$ and $\|\mathrm{Op}(\Psi_0^\alpha)u\|_{L^2}$ are finite, which comes from the previous step of iteration since C_α is essentially supported close to the essential support of C' (whose essential support is contained in that of Φ thanks to (5.43)). Thus, $u \in \bigcap_{k\in\mathbb{N}} H^{2ks} = C^\infty$ in U.

Then, using (5.52) for any $\alpha \in \mathbb{N}$ with Ψ_0^α in place of Ψ_0, we obtain that $D_t u$ is also H^α in U. Hence, it is C^∞ in U, which concludes the proof of Lemma 5.21. □

We conclude the proof of Theorem 5.2. We assume that

$$u \text{ is smooth in } W = \mathscr{S}^{(-T-\frac{\delta_0}{2}, -T+\frac{\delta_0}{2})}(\{(x_0,\xi_0)\}). \tag{5.54}$$

Then, u is smooth in a slightly larger set W', i.e., such that $\overline{W} \subset W'$. By Lemma 5.16, there exists $V' \subset T^*X \setminus 0$ an open neighborhood of (x_0, ξ_0) such that

$$W \subset \mathscr{S}^{(-T-\frac{\delta_0}{2}, -T+\frac{\delta_0}{2})}(V') \subset W'.$$

Fix also an open set $V \subset T^*X \setminus 0$ such that

$$(x_0, \xi_0) \in V \subset \overline{V} \subset V'.$$

Lemma 5.21 implies that u is smooth in $\mathscr{S}^{(-T+\frac{\delta_0}{2}, 0)}(V)$. In particular,

$$u \text{ is smooth in } \mathscr{S}^{(-T+\frac{\delta_0}{2}, 0)}(\{(x_0,\xi_0)\}). \tag{5.55}$$

The fact that (5.54) implies (5.55) proves that singularities of (5.4) propagate only along *rays*. Using that singularities of P are contained in $\{p = 0\}$, we obtain finally Theorem 5.2.

5.5 Proof of Theorem 5.3

In the last two sections of this note, we assume that A is a sub-Laplacian. As mentioned in the introduction, it means that we assume that A has the form

$$A = \sum_{i=1}^{K} Y_i^* Y_i \tag{5.56}$$

where the global smooth vector fields Y_i are assumed to satisfy Hörmander's condition (the Lie algebra generated by $Y_1, \ldots, Y_K$ is equal to the whole tangent bundle TX). Here Y_i^* denotes the adjoint of Y_i for the scalar product (5.2).

5.5.1 The sub-Riemannian metric

In this preliminary section, we work with a general sub-Laplacian $A_\bullet$ on a smooth compact manifold $X_\bullet$ without boundary. This is because the results of this section will be used in Section 5.5 also for a sub-Laplacian defined on $X \times X$. We have

$$A_\bullet = \sum_{i=1}^{K_\bullet} Y_{\bullet i}^* Y_{\bullet i}. \tag{5.57}$$

There is a metric $g_\bullet$ on the distribution $\mathcal{D}_\bullet = \mathrm{Span}(Y_{\bullet 1}, \ldots, Y_{\bullet K})$:

$$(g_\bullet)_x(v,v) = \inf \left\{ \sum_{i=1}^{K_\bullet} u_i^2 \mid v = \sum_{i=1}^{K_\bullet} u_i Y_{\bullet i}(x) \right\}. \tag{5.58}$$

The triple $(X_\bullet, \mathcal{D}_\bullet, g_\bullet)$ is called a sub-Riemannian structure (see [Mon02]).

The principal symbol of $A_\bullet$, which is also the natural Hamiltonian, is

$$a_\bullet = \sum_{i=1}^{K_\bullet} h_{Y_{\bullet i}}^2.$$

Here, for $Y_\bullet$ a vector field on $X_\bullet$, we denoted by $h_{Y_\bullet}$ the momentum map given in canonical coordinates (x,ξ) by $h_{Y_\bullet}(x,\xi) = \xi(Y_\bullet(x))$.

Denote by $\pi_\bullet$ denotes the canonical projection $\pi_\bullet : T^*X_\bullet \to X_\bullet$ and by $\mathcal{I}_\bullet : b \mapsto \omega_\bullet(b,\cdot)$ the canonical isomorphism between $T(T^*X_\bullet)$ and $T^*(T^*X_\bullet)$. The notation $a_{\bullet m}$ stands for the Hessian of the principal symbol of $A_\bullet$ at m.

Lemma 5.22. *There holds $a_{\bullet m}^*(\mathcal{I}_\bullet(b)) = g_\bullet(d\pi_\bullet(b))$ for any $b \in (ker(a_{\bullet m}))^{\perp_{\omega_\bullet}} \subset T(T^*X_\bullet)$.*

Proof. We consider a local $g_\bullet$-orthonormal frame $Z_1, \ldots, Z_N$. In particular, the Z_j are independent, and the $H_{h_{Z_j}}$ are also independent. We have $a_{\bullet m} = \sum_{j=1}^N (dh_{Z_j})^2$. Hence, $H_{h_{Z_1}}, \ldots, H_{h_{Z_N}}$ span $(\ker(a_{\bullet m}))^{\perp_{\omega_\bullet}}$ since

$$\begin{aligned}
\ker(a_{\bullet m}) &= \bigcap_{j=1}^N \ker(dh_{Z_j}) = \{\xi \in T(T^*X_\bullet),\ dh_{Z_j}(\xi) = 0,\ \forall 1 \leqslant j \leqslant N\} \\
&= \{\xi \in T(T^*X_\bullet),\ \omega_\bullet(\xi, H_{h_{Y_N}}) = 0,\ \forall 1 \leqslant j \leqslant N\} \\
&= \mathrm{span}(H_{h_{Y_1}}, \ldots, H_{h_{Y_N}})^{\perp_{\omega_\bullet}}.
\end{aligned}$$

We fix $b \in (\ker(a_{\bullet m}))^{\perp_{\omega_\bullet}}$ and we write $b = \sum_{j=1}^N u_j H_{h_{Z_j}}$. Note that $g_\bullet(\sum_{j=1}^N u_j Z_j) = \sum_{j=1}^N u_j^2$. By definition, $\mathcal{I}_\bullet(H_{h_{Z_j}}) = -dh_{Z_j}$ and $d\pi(H_{h_{Z_j}}) = Z_j$ for any j, so there holds

$$\begin{aligned}
a_{\bullet m}^*\left(\mathcal{I}_\bullet\left(\sum_{j=1}^N u_j H_{h_{Z_j}}\right)\right) &= a_{\bullet m}^*\left(\sum_{j=1}^N u_j dh_{Z_j}\right) = \sup_{\eta \notin \ker(a_{\bullet m})} \frac{\left(\sum_{j=1}^N u_j dh_{Z_j}(\eta)\right)^2}{\sum_{j=1}^N dh_{Z_j}(\eta)^2} \\
&= \sup_{(\theta_j) \in \mathbb{R}^N} \frac{\left(\sum_{j=1}^N u_j \theta_j\right)^2}{\sum_{j=1}^N \theta_j^2} = \sum_{j=1}^N u_j^2 = g_\bullet\left(\sum_{j=1}^N u_j Z_j\right) \\
&= g_\bullet\left(d\pi_\bullet\left(\sum_{j=1}^N u_j H_{h_{Z_j}}\right)\right)
\end{aligned}$$

where, to go from line 1 to line 2, we used that the dh_{Z_j} are independent. $\square$

5.5.2 K_G as a solution of a wave equation

The rest of Section 5.5 is devoted to the proof of Theorem 5.3, i.e., we deduce the wave-front set of the Schwartz kernel K_G from the "geometric" propagation of singularities given by Theorem 5.2. The idea is to consider K_G itself as the solution of a wave equation to which we can apply Theorem 5.2.

We consider the product manifold $X \times X$, with coordinate x on its first copy, and coordinate y on its second copy. We set

$$A^{\otimes} = \frac{1}{2}(A_x \otimes \mathrm{Id}_y + \mathrm{Id}_x \otimes A_y)$$

and we consider the operator

$$P = \partial_{tt}^2 - A^{\otimes}$$

acting on functions of $\mathbb{R} \times X_x \times X_y$. Using (5.5), we can check that the Schwartz kernel K_G is a solution of

$$K_{G|t=0} = 0, \qquad \partial_t K_{G|t=0} = \delta_{x-y}, \qquad PK_G = 0.$$

The operator $A^{\otimes}$ is a self-adjoint non-negative real second-order differential operator on $X \times X$. Moreover it is subelliptic: it is immediate that the vector fields $Y_1 \otimes \mathrm{Id}_y, \ldots, Y_K \otimes \mathrm{Id}_y, \mathrm{Id}_x \otimes Y_1, \ldots, \mathrm{Id}_x \otimes Y_K$ verify Hörmander's Lie bracket condition, since it is satisfied by $Y_1, \ldots, Y_K$. Hence, Theorem 5.2 applies to P, with the null-rays being computed with $A^{\otimes}$ in $T^*(X \times X)$ (see (5.61) for the associated cones). We denote by $\sim_t$ the relation of existence of a null-ray of length $|t|$ joining two given points of $T^*(X \times X) \setminus 0$ (see Remark 5.7 for the omission of the variables t and τ in the null-rays).

Since $WF(K_G(0)) = \emptyset$ and

$$WF(\partial_t K_G(0)) = \{(z, z, \zeta, -\zeta) \in T^*(X \times X) \setminus 0\},$$

we have

$$\begin{aligned} WF(K_G(t)) \subset \{(x, y, \xi, -\eta) \in T^*(X \times X) \setminus 0, \, \exists (z, \zeta) \in T^*X \setminus 0, \\ (z, z, \zeta, -\zeta) \sim_t (x, y, \xi, -\eta)\}. \end{aligned} \tag{5.59}$$

Let us denote by g^1 the sub-Riemannian metric on X_x and by g^2 the sub-Riemannian metric on X_y. The sub-Riemannian metric on $X_x \times X_y$ is $g^{\otimes} = \frac{1}{2}(g^1 \oplus g^2)$. In other words, if $q = (q_1, q_2) \in X \times X$ and $v = (v_1, v_2) \in T_q(X \times X) \approx T_{q_1}X \times T_{q_2}X$, we have

$$g_q^{\otimes}(v) = \frac{1}{2}(g_{q_1}^1(v_1) + g_{q_2}^2(v_2)). \tag{5.60}$$

Now, the cones $\Gamma_m^{\otimes}$ associated to $A^{\otimes}$ are given by

$$\begin{aligned} \Gamma_m^{\otimes} &= \mathbb{R}^+(\partial_t + B), \\ B &= \{b \in \ker(a_m^{\otimes})^{\perp_{\omega^{\otimes}}}, \; g^{\otimes}(d\pi^{\otimes}(b)) \leqslant 1\}. \end{aligned} \tag{5.61}$$

Here, $\perp_{\omega^{\otimes}}$ designates the symplectic orthogonal with respect to the canonical symplectic form $\omega^{\otimes}$ on $T^*(X \times X)$, and $\pi^{\otimes} : T^*(X \times X) \to X \times X$ is the canonical projection.

To evaluate the right-hand side of (5.59), we denote by $\approx_t$ the relation of existence of a null-ray of length $|t|$ joining two given points of $T^*X \setminus 0$ (the cones Γ_m are subsets of $T(T^*(\mathbb{R} \times X))$ as defined in Section 5.2). Let us prove that

$$\begin{aligned} \{(x, y, \xi, -\eta) \in T^*(X \times X) \setminus 0, \; \exists (z, \zeta) \in T^*X \setminus 0, \; (z, z, \zeta, -\zeta) \sim_t (x, y, \xi, -\eta)\} \\ \subset \{(x, y, \xi, -\eta) \in T^*(X \times X) \setminus 0, \; (x, \xi) \approx_t (y, \eta)\}. \end{aligned} \tag{5.62}$$

Combining with (5.59), it will immediately follow that

$$WF(K_G(t)) \subset \{(x, y, \xi, -\eta) \in T^*(X \times X) \setminus 0, \; (x, \xi) \approx_t (y, \eta)\}. \tag{5.63}$$

5.5.3 Proof of (5.62).

We denote by $\gamma : [0,t] \to T^*(X \times X)\setminus 0$ a null-ray from $(z,z,\zeta,-\zeta)$ to $(x,y,\xi,-\eta)$, parametrized by time. Our goal is to construct a null-ray of length $|t|$ in $T^*X \setminus 0$, from (y,η) to (x,ξ). It is obtained by concatenating a null-ray from (y,η) to (z,ζ) with another one, from (z,ζ) to (x,ξ). However, there are some subtleties hidden in the parametrization of this concatenated null-ray.

We write $\gamma(s) = (\alpha_1(s), \alpha_2(s), \beta_1(s), \beta_2(s))$, and for $i = 1,2$ and $0 \leqslant s \leqslant t$, we set $\gamma_i(s) = (\alpha_i(s), \beta_i(s)) \in T^*X$. We also set $\delta_i(s) = g^i(d\pi_i(\dot{\gamma}_i(s)))$, where $\pi_i : T^*X \to X$ (here X is the i-th copy of X). The upper dot denotes here and in the sequel the derivative with respect to the time variable. Since $g^{\otimes}(d\pi^{\otimes}(\dot{\gamma}(s))) \leqslant 1$ for any $s \in [0,t]$, we deduce from (5.60) that

$$\frac{1}{2}(\delta_1(s) + \delta_2(s)) \leqslant 1.$$

We are going to construct a null-ray $\varepsilon : [0,t] \to T^*X$ of the form

$$\begin{aligned} \varepsilon(s) &= (\alpha_2(\theta(s)), -\beta_2(\theta(s))), \qquad 0 \leqslant s \leqslant s_0 \\ \varepsilon(s) &= (\alpha_1(\theta(s)), \beta_1(\theta(s))), \qquad s_0 \leqslant s \leqslant t. \end{aligned} \tag{5.64}$$

The parameter s_0 and the parametrization θ will be chosen so that the first part of ε joins (y,η) to (z,ζ) and the second part joins (z,ζ) to (x,ξ). We choose $\theta(0) = t$, hence $\varepsilon(0) = (y,\eta)$. Then, for $0 \leqslant s \leqslant s_0$, we choose $\theta(s) \leqslant t$ in order to guarantee that $g^1(d\pi_1(\dot{\varepsilon}(s))) = 1$. This defines s_0 in a unique way as the minimal time for which $\varepsilon(s_0) = (z,\zeta)$. In particular, $\theta(s_0) = 0$. A priori, we do not know that $s_0 \leqslant t$, but we will prove it below. Then, for $s_0 \leqslant s_1$, we choose $\theta(s) \geqslant 0$ in order to guarantee that $g^2(d\pi_2(\dot{\varepsilon}(s))) = 1$. This defines s_1 in a unique way as the minimal time for which $\varepsilon(s_1) = (x,\xi)$. Finally, if $s_1 \leqslant t$, we extend ε by $\varepsilon(s) \equiv (x,\xi)$ for $s_1 \leqslant s \leqslant t$.

We check that ε is a null-ray in T^*X. We come back to the definition of null-rays as tangent to the cones Γ_m. It is clear that

$$\ker(a_m^{\otimes})^{\perp_{\omega^{\otimes}}} = \ker(a_m)^{\perp_{\omega_1}} \times \ker(a_m)^{\perp_{\omega_2}}$$

where ω_i is the canonical symplectic form on T^*X_i. Therefore, $\dot{\varepsilon}(s) \in \ker(a_m)^{\perp_{\omega_i}}$ for $i = 1$ when $0 \leqslant s \leqslant s_0$ and for $i = 2$ when $s_0 \leqslant s \leqslant t$. Thanks to Lemma 5.22, the inequality in (5.14) (but for the cones in X_1 and X_2) is verified by $\dot{\varepsilon}(s)$ for any $0 \leqslant s \leqslant t$ by definition. There is a "time-reversion" (or "path reversion") in the first line of (5.64); the property of being a null-ray is preserved under time reversion together with momentum reversion. Hence ε is a null-ray in T^*X.

The fact that $s_0, s_1 \leqslant t$ follows from the following computation:

$$\begin{aligned} t \geqslant \int_0^t g^{\otimes}(d\pi^{\otimes}(\dot{\gamma}(s)))ds &= \frac{1}{2}\int_0^t g^1(d\pi_1(\dot{\gamma}_1(s)))ds + \frac{1}{2}\int_0^t g^2(d\pi_2(\dot{\gamma}_2(s)))ds \\ &= \frac{1}{2}\int_0^{s_0} g^1(d\pi_1(\dot{\varepsilon}(s)))ds + \frac{1}{2}\int_{s_0}^{s_1} g^2(d\pi_2(\dot{\varepsilon}(s)))ds \\ &= s_0 + (s_1 - s_0) = s_1. \end{aligned}$$

where the second equality follows from the fact that we ε is a reparametrization of γ_1 (resp. γ_2) for $s \in [0, s_0]$ (resp. $[s_0, s_1]$). This concludes the proof of (5.62).

5.5.4 Conclusion of the proof of Theorem 5.3

Let us finish the proof of Theorem 5.3. We fix (x_0,ξ_0), (y_0,η_0) and t_0 such that there is no null-ray from $(y_0,\eta_0) \in T^*X$ to $(x_0,\xi_0) \in T^*X$ in time t_0.

Claim. There exist a conic neighborhood V of $(x_0, y_0, \xi_0, -\eta_0)$ and a neighborhood V_0 of t_0 such that for any $N \in \mathbb{N}$ and any $t \in V_0$, $\partial_t^{2N} K_G(t)$ is smooth in V.

Proof. We choose V so that for $(x,y,\xi,-\eta)\in V$ and $t\in V_0$, there is no null-ray from (y,η) to (x,ξ) in time t. Such a V exists, since otherwise by extraction of null-rays (which are Lipschitz with a locally uniform constant, see (5.16)), there would exist a null-ray from (y_0,η_0) to (x_0,ξ_0) in time t_0. Then, we can check that for any $N\in\mathbb{N}$, $K_G^{(2N)}=\partial_t^{2N}K_G$ is a solution of

$$K_G^{(2N)}{}_{|t=0}=0,\qquad \partial_t K_G^{(2N)}{}_{|t=0}=(A^{\otimes})^N\delta_{x-y},\qquad PK_G^{(2N)}=0.$$

Repeating the above argument leading to (5.63) with $K_G^{(2N)}$ instead of K_G, we obtain

$$WF(K_G^{(2N)}(t))\subset\{(x,y,\xi,-\eta)\in T^*(X\times X)\setminus 0,\ (x,\xi)\approx_t(y,\eta)\},$$

which proves the claim. □

We deduce from the claim that if there is no null-ray from $(y_0,\eta_0)\in T^*X$ to $(x_0,\xi_0)\in T^*X$ in time t_0, then $(t_0,\tau_0,x_0,y_0,\xi_0,-\eta_0)\notin WF(K_G)$ for any $\tau_0\in\mathbb{R}$.

Finally, if there is a null-ray from (y_0,η_0) to (x_0,ξ_0) in time t_0, then $a(x_0,\xi_0)=a(y_0,\eta_0)$, and due to the fact that $WF(K_G)$ is included in the characteristic set of $\partial_{tt}^2-A^{\otimes}$, the only τ_0's for which $(t_0,\tau_0,x_0,y_0,\xi_0,-\eta_0)\in WF(K_G)$ is possible are the ones satisfying $\tau_0^2=a(x_0,\xi_0)=a(y_0,\eta_0)$. This concludes the proof of Theorem 5.3.

Remark 5.23. Theorem 5.3 allows to recover some results already known in the literature.

In the situations studied in [Las82], [LL82] and [Mel86], $\Sigma_{(2)}$ is a symplectic manifold. In this case, thanks to (5.14), we see that the only null-rays starting from points in $\Sigma_{(2)}$ are lines in t. Therefore Theorem 5.3 implies:

- the "wave-front part" of the main results of [Las82] and [LL82] (but not the effective construction of parametrices handled in these papers).
- Theorem 1.8 in [Mel84], which can be reformulated as follows: if Σ_2 (in the notations of [Mel84]) is of codimension 2, then

  singularities outside Σ_2 propagate along bicharacteristics,
  and singularities inside Σ_2 propagate along lines in t.

  This is exactly the content of Theorem 5.3 in this case. To see that Theorem 1.8 of [Mel84] can be reformulated as above, we must notice that on Σ_2, $\mathring{\chi}_t\pm$ extends as the identity for any $t\in\mathbb{R}$, which follows from the following property (denoting by $U^*_{x_0}X$ the set of covectors of norm 1 with base point x_0):

  $$\forall t>0,\ \forall x_0\in X,\ \exp_{x_0}^t: U^*_{x_0}X\to X \text{ is proper} \tag{5.65}$$

  (when restricted to minimizers), which implies that for any open neighborhood V of x_0, $(\exp_{x_0}^t)^{-1}(X\setminus V)$ is compact, at positive distance from Σ_2. The property (5.65) is always true in the absence of singular curves (defined in Section 5.6.1).

5.6 A consequence for wave equations with sub-Laplacians

We now turn to the consequences of Theorem 5.3. For that purpose, we briefly introduce notations and concepts from sub-Riemannian geometry. Our presentation is inspired by [Mon02, Chapter 5 and Appendix D]. In this last section, we continue to assume that A is a sub-Laplacian on X (see Example 5.1). The associated sub-Riemannian metric (see (5.58)) is denoted by g.

5.6.1 Sub-Riemannian geometry and horizontal curves

Fix an interval $I = [b, c]$ and a point $x_0 \in X$. We denote by $\Omega(I, x_0; \mathcal{D})$ the space of all absolutely continuous curves $\gamma : I \to X$ that start at $\gamma(b) = x_0$ and whose derivative is square integrable with respect to g, implying that the length

$$\int_I \sqrt{g_{\gamma(t)}(\dot{\gamma}(t), \dot{\gamma}(t))}dt$$

of γ is finite. Such a curve γ is called *horizontal*. The *endpoint map* is the map

$$\mathrm{End} : \Omega(I, x_0; \mathcal{D}) \to X, \quad \gamma \mapsto \gamma(c).$$

The metric (5.58) induces a distance d on X, and $d(x, y) < +\infty$ for any $x, y \in X$ thanks to Hörmander's condition (this is the Chow-Rashevskii theorem).

Two types of curves in $\Omega(I, x_0; \mathcal{D})$ will be of particular interest: the critical points of the endpoint map, and the curves which are projections of the Hamiltonian vector field H_a associated to a.

Projections of integral curves of H_a are geodesics:

Theorem 5.24. *[Mon02, Theorem 1.14] Let $\gamma(s)$ be the projection on X of an integral curve (in T^*X) of the Hamiltonian vector field H_a. Then γ is a horizontal curve and every sufficiently short arc of γ is a minimizing sub-Riemannian geodesic (i.e., a minimizing path between its endpoints in the metric space (X, d)).*

Such horizontal curves γ are called *normal* geodesics, and they are smooth.
The differentiable structure on $\Omega(I, x_0; \mathcal{D})$ described in [Mon02, Chapter 5 and Appendix D] allows to give a sense to the following notion:

Definition 5.25. *A singular curve is a critical point for the endpoint map.*

Note that in Riemannian geometry (i.e., for a elliptic), there exist no singular curves.
In the next definition, we use the notation $\mathcal{D}^\perp$ for the annihilator of $\mathcal{D}$ (thus a subset of the cotangent bundle T^*X), and $\overline{\omega}_X$ denotes the restriction to $\mathcal{D}^\perp$ of the canonical symplectic form ω_X on T^*X.

Definition 5.26. *A characteristic for $\mathcal{D}^\perp$ is an absolutely continuous curve $\lambda(t) \in \mathcal{D}^\perp$ that never intersects the zero section of $\mathcal{D}^\perp$ and that satisfies $\dot{\lambda}(t) \in ker(\overline{\omega}_X(\lambda(t)))$ at every point t for which the derivative $\dot{\lambda}(t)$ exists.*

Theorem 5.27. *[Mon02, Theorem 5.3] A curve $\gamma \in \Omega$ is singular if and only if it is the projection of a characteristic λ for $\mathcal{D}^\perp$ with square-integrable derivative. λ is then called an abnormal extremal lift of the singular curve γ.*

Normal geodesics and singular curves are particularly important in sub-Riemannian geometry because of the following fact (Pontryagin's maximum principle):

any minimizing geodesic in (X, d) is either a singular curve or a normal geodesic.

The existence of geodesics which are singular curves but not normal geodesics was proved in [Mon94].

5.6.2 The singular support of $K_G(\cdot, x, y)$

When A is a sub-Laplacian (5.56), the cones Γ_m defined in Section 5.2.1 have an additional geometric interpretation, which we now explain.

We consider a null-ray, as introduced in Definition 5.4. It is necessarily of one of the following types (depending on the value of τ, which is a constant):

- either a null-bicharacteristic with (constant) $\tau \neq 0$, since $\Gamma_m = \mathbb{R}^{\pm} \cdot H_p(m)$ in this case;
- or contained in $\Sigma_{(2)}$ and tangent to the cones Γ_m given by (5.14), with $\tau \equiv 0$ since $d\tau(v) = 0$ for any $v \in \Gamma_m$ according to (5.14).

In the second case, setting $n = \pi_2(m)$ and writing $v = c(\partial_t + b)$ as in (5.14), we have $b \in T_n\mathcal{D}^{\perp}$ since $a \equiv 0$ along the path. There holds $\ker(a_m) = T_n\mathcal{D}^{\perp}$ and, plugging into the above formula, we also get $b \in (T_n\mathcal{D}^{\perp})^{\perp_{\omega_X}}$. It follows that $b \in T_n\mathcal{D}^{\perp} \cap (T_n\mathcal{D}^{\perp})^{\perp_{\omega_X}} = \ker \overline{\omega}_X$, i.e., the trajectory of the null-ray (forgetting the time variable) is a characteristic curve.

In summary, when A is a sub-Laplacian (5.56), Theorem 5.3 asserts that singularities of the wave equation (5.4) propagate only along integral curves of H_a and characteristics for $\mathcal{D}^{\perp}$. From that, we can infer the following proposition, in the spirit of Duistermaat-Guillemin's trace formula [DG75]:

Proposition 5.28. *We fix $x, y \in X$ with $x \neq y$. We denote by $\mathscr{L}$ the set of lengths of normal geodesics from x to y and by T_s the minimal length of a singular curve joining x to y. Then $\mathscr{G} : t \mapsto K_G(t, x, y)$ is well-defined as a distribution on $(-T_s, T_s)$, and*

$$\textit{sing supp}(\mathscr{G}) \subset \mathscr{L} \cup -\mathscr{L}.$$

Note that this proposition does not say anything about times $|t| \geqslant T_s$: it could happen a priori that $t \mapsto K_G(t, x, y)$ is not a distribution after T_s.

Proof. As said above, null-rays living in $\{\tau = 0\}$ are characteristic curves (in the sense of Definition 5.26) tangent to Γ_m. Now, it follows from (5.14), Theorem 5.27 and Lemma 5.22 that the least $t > 0$ for which there exists a null-ray of length t with $\tau \equiv 0$ joining x and y is equal to the length of the shortest singular curve joining x and y.

We consider $\varphi : \mathbb{R} \to \mathbb{R} \times X \times X$, $t \mapsto (t, x, y)$ which has conormal set $N_\varphi = \{(t, x, y, 0, \xi, \eta)\}$ (in other words N_φ corresponds to $\tau = 0$). Thus, using Theorem 5.3, we see that $WF(\mathscr{G})$ does not intersect the conormal set of $\varphi_{|(-T_s,T_s)}$. Then, [Hör71b, Theorem 2.5.11'] ensures that $\mathscr{G}$, which is the pull-back of K_G by $\varphi_{|(-T_s,T_s)}$, is well-defined as a distribution over $(-T_s, T_s)$. Of course, sing supp$(\mathscr{G})$ is the projection of $WF(\mathscr{G})$ (for $|t| < T_s$).

For $|t| < T_s$, null-rays between x and y are contained in $\{\tau \neq 0\}$, thus they are tangent to the cones $\Gamma_m = \mathbb{R}^{\pm} \cdot H_p(m)$. Hence, the singularities of $\mathscr{G}$ occur at times belonging to the set $\mathscr{L}$ of lengths of normal geodesics (for $\tau > 0$, we obtain normal geodesics from y to x, and for $\tau < 0$, normal geodesics from x to y). □

Remark 5.29. If $x = y$, the same reasoning as in the proof of Proposition 5.28 says nothing more than sing supp$(K_G(\cdot, x, x)) \subset \mathbb{R}$ since for any point $(x, \xi) \in \mathcal{D}^{\perp}$ and any $t \in \mathbb{R}$, the constant path joining (x, ξ) to (x, ξ) in time t is a null-ray (with $\tau = 0$).

5.6.3 Comments on the inequality in (5.14)

In the formula (5.14) for the cones Γ_m, the inequality $a_m^*(\mathcal{I}(b)) \leqslant 1$ may seem surprising at first sight. When A is a sub-Laplacian, according to Lemma 5.22, it is equivalent to $g(d\pi(b)) \leqslant 1$. In rough terms, Theorem 5.2 does not exclude that singularities contained in $\mathcal{D}^\perp$ propagate at speeds < 1, which would be in strong contrast with the usual propagation "at speed 1" of singularities of wave equations with elliptic Laplacian. In a joint work with Yves Colin de Verdière [CL21] (see Chapter 6), we give explicit examples of initial data of a subelliptic wave equation whose singularities effectively propagate at any speed between 0 and 1 along singular curves.

5-A Further properties of the cones Γ_m

5-A.1 Inner semi-continuity of the cones Γ_m in a

In this appendix, we prove that if we make some additional assumptions, the convex cones Γ_m are inner semi-continuous with respect to a (in addition to their inner semi-continuity with respect to m proved in Lemma 5.8). For that, we introduce the following class of functions on T^*X (for $k \in \mathbb{N}$):

$$\mathscr{A}_k = \left\{ \sum h_{Y_j}^2,\ (Y_j) \in \mathscr{D}_k \right\},$$

where $\mathscr{D}_k$ is the set of families of smooth vector fields generating a regular (i.e., constant rank) distribution of rank k. Note that any $a \in \mathscr{A}_k$ automatically satisfies (5.7).

Proposition 5.30. *The mapping $(M_+ \cup M_-) \times \mathscr{A}_k \ni (m,a) \mapsto \Gamma_m^{(a)}$ is inner semi-continuous (for the C^∞ topology in $a \in \mathscr{A}_k$). In other words,*

$$\forall m_j \to m_* \quad (m_j, m_* \in M_+ \cup M_-),\ \ \forall a_j \in \mathscr{A}_k,\ a_j \xrightarrow{C^\infty} a_*, \quad \forall v_j \in \Gamma_{m_j}^{(a_j)},\ v_j \to v \in T_{m_*}M,$$
$$\text{there holds } \ v \in \Gamma_{m_*}^{(a_*)}$$

where we temporarily denoted by $\Gamma_m^{(a)}$ the cone computed with the Hamiltonian a at point m.

Proposition 5.30 follows quite directly from the computations done in the proof of Lemma 5.8. However, we give here a different proof which has the advantage of requiring no formula, and which illustrates Remark 5.11.

Definition 5.31. *Let F be a manifold and $E \subset F$ be a closed set. For $x \in E$, the tangent cone $C(x)$ is the $\mathbb{R}^+$-subcone of the tangent space T_xF consisting of all the vectors $\gamma'(0)$ where $\gamma : [0,a[\to F$ is a C^1 curve so that $\gamma(0) = x$ and $\gamma(t) \in E$ for $t \geqslant 0$ small enough. The dual tangent cone $C^o(x)$ is the subcone of T_x^*F of all covectors ξ so that $\xi(v) \leqslant 0$ for all $v \in C(x)$.*

Let us remark that if ∂E is smooth at x, then $C^o(x)$ is generated by the normal outgoing covectors at x.

Proof of Proposition 5.30. We set $Y = T^*X$. The statement clearly holds if $m_* = (t_*, \tau_*, y_*)$ does not verify $\tau_* = 0$ and $a_*(y_*) = 0$. Hence we assume in the sequel that $\tau_* = 0$ and $y_* \in a_*^{-1}(0)$. Writing $m_j = (t_j, \tau_j, y_j)$, we can also assume that for any j, $\tau_j \neq 0$ since otherwise $0 = \tau_j^2 \geqslant a_j(y_j) \geqslant 0$, meaning that all cones Γ_m are computed according to the formula (5.9), and in this case we even have continuity of the cones $\Gamma_{m_j}^{(a_j)}$ towards $\Gamma_{m_*}^{(a_*)}$. In other words, with transparent notations, we assume in the sequel that $m_j \notin \Sigma_{(2)}^{(a_j)}$ and $m_* \in \Sigma_{(2)}^{(a_*)}$.

For $a \in \mathscr{A}_k$, $p_0 \geqslant 0$, and $b = (a, p_0)$, we consider

$$E_b = \{(t, \tau, y) \in T^*\mathbb{R} \times Y;\ \tau \geqslant 0 \text{ and } \tau^2 - a(y) \geqslant p_0\}.$$

There are two steps: 1) prove that the mapping $(m, a) \mapsto C_b^o(m)$ is inner semi-continuous where $b = (a, p(m))$ and $C_b^o(m)$ is the dual tangent cone of E_b at m (which is in ∂E_b); 2) conclude the proof of Proposition 5.30.

1) Since $a_* \in \mathscr{A}_k$, the characteristic manifold $Z = a_*^{-1}(0) \subset Y$ is smooth (see [ABB19], below Definition 4.33), and it is non-degenerate. Thus, the Morse-Bott Lemma (see [BH04]) guarantees the existence of local coordinates $y = (\widetilde{y}, z) \in N$ such that $y_* = (0, 0)$ and $a_*(y) = \|\widetilde{y}\|^2$. In these coordinates (valid for $(t, \tau, y) \in N$), the set $E_{b_*} \cap N$ is convex.

The boundary ∂E_{b_j} is smooth at m_j and, for any j, the tangent cone of E_{b_j} at $m_j \in \partial E_{b_j}$ is a set $H_j \subset T_{m_j}M$ which is nearly a half tangent space[6]. Indeed, the convergence $a_j \to a_*$ (in the C^∞ topology) implies that any set H_∞ which is the limit of a convergent subsequence of (H_j) is a half-space, and $E_{b_*} \subset H_\infty$. Hence, by convexity of $E_{b_*} \cap N$, the tangent cone at m_* is contained in H_∞. By taking duals, we get the opposite inclusion: any limit of the dual tangent cones $C_{b_j}^o(m_j)$ belongs to $C_{b_*}^o(m_*)$. This proves the result: the mapping $(m, a) \mapsto C_b^o(m)$ is inner semi-continuous at (m_*, a_*).

2) Let us compute $C_b^o(m)$ depending on m and a.

If $m \notin \Sigma_{(2)}$, then its tangent cone is $C_b(m) = \{w \in T_mM,\ dp(w) \geqslant 0\}$. Hence $C_b^o(m) = \{\lambda \in T_m^*M,\ \lambda(w) \leqslant 0\ \forall w \text{ such that } dp(w) \geqslant 0\} = \{-dp\}$ where this last differential is taken at m.

If $m \in \Sigma_{(2)}$, then $C_b(m) = \{w \in T_mM;\ d\tau(w) \geqslant 0, p_m(w) \geqslant 0\} = \Lambda_m$ and $C_b^o(m) = \{\lambda \in T_m^*M;\ \lambda(w) \leqslant 0\ \forall w \in \Lambda_m\}$.

Then, identifying T_m^*M and T_mM through the isomorphism $\omega(v, \cdot) \mapsto v$, we see that in both cases $C_b^o(m)$ identifies with Γ_m (see the sign conventions for symplectic geometry in Appendix A.1). Since this identification between T_m^*M and T_mM is continuous in m, we get the result. □

5-A.2 What is there exactly in the cone Γ_m when $m \in \Sigma_{(2)}$?

Lemma 5.8 and Proposition 5.30 state that the cones Γ_m are inner semi-continuous. It is natural to wonder whether a cone Γ_m can be much bigger than the set of limits of the cones Γ_{m_j} for m_j tending to m. The answer is given by the following:

Proposition 5.32. *For any $m \in \Sigma_{(2)}$, the cone Γ_m (resp. its boundary) is exactly given by all limits of the cones Γ_{m_j} for $m_j \notin \Sigma_{(2)}$ (resp. $m_j \in \Sigma \setminus \Sigma_{(2)}$) converging to m.*

Proof. As in Section 5.2.3, we work in a chart near m. Let $v \in \Gamma_m$, which, up to multiplication by a constant, we can take equal to $\partial_t + b$ according to (5.14). According to (5.18), we have to prove that b is the limit of $\frac{1}{2}\frac{a(m_j)^{\frac{1}{2}}}{\tau_j}\frac{H_a(m_j)}{a(m_j)^{\frac{1}{2}}}$ for some well-chosen $m_j \to m$. Playing with the multiplication factor $a(m_j)^{\frac{1}{2}}/\tau_j$, it is sufficient to show that if $a_m^*(\mathcal{I}(b)) = 1$, then b is the limit of $\frac{1}{2}\frac{H_a(m_j)}{a(m_j)^{1/2}}$ for some well-chosen $m_j \to m$.

Since our computations do not depend on t, τ, we replace m_j, m by $\pi_2(m_j), \pi_2(m)$ (omitted in the notations).

[6] The formula for the tangent cone in point 2) at $m \notin \Sigma_{(2)}$ is perturbed since we take coordinates, but this perturbation is smooth since $a_j \to a_*$ in the C^∞ topology.

Following the computations of Lemma 5.9 and using the notation F for the "fundamental matrix" introduced in (5.15), we get

$$\begin{aligned}\frac{1}{2}\omega_X(H_a(m_j), w) &= -\frac{1}{2}da(m_j)(w) = -a_m(m_j - m, w) + o(m_j - m)\\ &= \omega_X(F(m_j - m), w) + o(m_j - m),\end{aligned}$$

and finally $\frac{1}{2}\frac{H_a(m_j)}{a(m_j)^{1/2}} = \frac{F(m_j-m)}{a_m(m_j-m)^{1/2}} + o(1)$. But it follows from (5.15) that

$$F : T_m(T^*X)/\ker(a_m) \to (\ker(a_m))^{\perp_{\omega_X}}$$

is an isomorphism[7]. Thus, choosing the sequence (m_j) adequately, we can take $F(m_j - m)$ colinear to $b \in (\ker(a_m))^{\perp_{\omega_X}}$, and then we compute

$$\begin{aligned}a_m^*\left(\mathcal{I}\left(F(m_j - m)\right)\right) &= \sup_{w \notin \ker(a_m)} \frac{\omega(F(m_j - m), w)^2}{a_m(w)} = \sup_{w \notin \ker(a_m)} \frac{a_m(m_j - m, w)^2}{a_m(w)}\\ &= a_m(m_j - m).\end{aligned}$$

Hence, with this choice of m_j, any limit v' of $F(m_j - m)/a_m(m_j - m)^{1/2}$ is colinear to b and the above computation implies that

$$a_m^*(\mathcal{I}(v')) = a_m^*(\mathcal{I}(b)),$$

which implies that $F(m_j - m)/a_m(m_j - m)^{1/2}$ tends to b. $\square$

[7] It follows for example from Lax-Milgram's lemma applied in the space $T_m(T^*X)/\ker(a_m)$, see Section 5.2.2.

Chapter 6

Propagation of well-prepared states along Martinet singular geodesics

"Regardez les singularités, il n'y a que cela qui compte."
Gaston Julia.

This chapter is adapted from [CL21]. Its main object is the proof of Theorem 7, restated as Theorem 6.2.

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We prove that for the Martinet wave equation with "flat" metric, which a subelliptic wave equation, singularities can propagate at any speed between 0 and 1 along any singular geodesic. This is in strong contrast with the usual propagation of singularities at speed 1 for wave equations with elliptic Laplacian.

6.1 Introduction

6.1.1 Propagation of singularities and singular curves

Restated in the language of sub-Riemannian geometry, Melrose's result [Mel86] presented in the previous chapter asserts that singularities of subelliptic wave equations propagate only along usual null-bicharacteristics (at speed 1) and along singular curves (see Definition 6.1). Along singular curves, Melrose writes in [Mel86] that the speed should be between 0 and 1, but nothing more. It is our purpose here to prove that for the Martinet wave equation, which is a subelliptic wave equation, singularities can propagate at any speed between 0 and 1 along the singular curves of the Martinet distribution. As explained in Remark 6.9, an analogous result also holds in the so-called quasi-contact case (the computations are easier in that case).

To state our main result, we consider the *Martinet sub-Laplacian*

$$\Delta = X_1^2 + X_2^2$$

on $\mathbb{R}^3$, where

$$X_1 = \partial_x, \qquad X_2 = \partial_y + x^2\partial_z.$$

Hörmander's theorem implies that Δ is hypoelliptic since X_1, X_2 and $[X_1,[X_1,X_2]]$ span $T\mathbb{R}^3$. The Martinet half-wave equation is

$$i\partial_t u - \sqrt{-\Delta}u = 0 \tag{6.1}$$

on $\mathbb{R}_t \times \mathbb{R}^3$, with initial datum $u(t=0) = u_0$. The vector fields X_1 and X_2 span the horizontal distribution

$$\mathcal{D} = \mathrm{Span}(X_1, X_2) \subset T\mathbb{R}^3.$$

Let us recall the definition of singular curves. We use the notation $\mathcal{D}^\perp$ for the annihilator of $\mathcal{D}$ (thus a subcone of the cotangent bundle $T^*\mathbb{R}^3$), and $\overline{\omega}$ denotes the restriction to $\mathcal{D}^\perp$ of the canonical symplectic form ω on $T^*\mathbb{R}^3$.

Definition 6.1. *A characteristic curve for $\mathcal{D}$ is an absolutely continuous curve $t \mapsto \lambda(t) \in \mathcal{D}^\perp$ that never intersects the zero section of $\mathcal{D}^\perp$ and that satisfies*

$$\dot{\lambda}(t) \in \ker(\overline{\omega}(\lambda(t)))$$

for almost every t. The projection of $\lambda(t)$ onto $\mathbb{R}^3$, which is an horizontal curve[1] *for $\mathcal{D}$, is called a singular curve, and the corresponding characteristic an abnormal extremal lift of that curve.*

We refer the reader to [Mon02] for more material related to sub-Riemannian geometry.

The curve $t \mapsto \gamma(t) = (0, t, 0) \in \mathbb{R}^3$ is a singular curve of the Martinet distribution $\mathcal{D}$. Denoting by (ξ, η, ζ) the dual coordinates of (x, y, z), this curve admits both an abnormal extremal lift, for which $\xi(t) = \eta(t) = 0$, and a normal extremal lift, for which $\xi(t) = 0$, $\eta(t) = 1$, $\zeta(t) = 0$ (meaning that, if $\tau = 1$ is the dual variable of t, this yields a null-bicharacteristic). Martinet-type distributions attracted a lot of attention since Montgomery showed in [Mon94] that they provide

[1] i.e., $d\pi(\dot{\lambda}(t)) \in \mathcal{D}_{\lambda(t)}$ for almost every t, where $\pi : T^*\mathbb{R}^3 \to \mathbb{R}^3$ denotes the canonical projection.

examples of singular curves which are geodesics of the associated sub-Riemannian structure, but which are not necessarily projections of bicharacteristics (in contrast with the Riemannian case, where all geodesics are obtained as projections of bicharacteristics).

In this chapter, all phenomena and computations are done (microlocally) near the abnormal extremal lift, and thus away (in the cotangent bundle $T^*\mathbb{R}^3$) from the normal extremal lift, which plays no role.

6.1.2 Main result

Let $Y \in C^\infty(\mathbb{R},\mathbb{R})$ be equal to 0 on $(-\infty,1)$ and equal to 1 on $(2,\infty)$. Take $\phi \in C_0^\infty(\mathbb{R},\mathbb{R})$ with $\phi \geqslant 0$ and $\phi \not\equiv 0$. Consider as Cauchy datum for the Martinet half-wave equation (6.1) the distribution $u_0(x,y,z)$ whose Fourier transform [2] with respect to (y,z) is

$$\mathcal{F}_{y,z}u_0(x,\eta,\zeta) = Y(\zeta)\phi(\eta/\zeta^{1/3})\psi_{\eta,\zeta}(x). \tag{6.2}$$

Here, $\psi_{\eta,\zeta}$ is the ground state of the $x-$operator

$$-d_x^2 + (\eta + x^2\zeta)^2$$

with $\psi_{\eta,\zeta}(0) > 0$ and $\|\psi_{\eta,\zeta}\|_{L^2} = 1$, and α_1 is the associated eigenvalue. Thanks to the Fourier inversion formula applied to (6.2), we note that

$$\sqrt{-\Delta}u_0(x,y,z) = \iint_{\mathbb{R}^2} Y(\zeta)\phi(\eta/\zeta^{1/3})\sqrt{\alpha_1}(\eta,\zeta)\psi_{\eta,\zeta}(x)e^{i(y\eta+z\zeta)}d\eta d\zeta.$$

We call u_0 a *well-prepared Cauchy datum.* It yields a solution of (6.1), namely

$$(U(t)u_0)(x,y,z) = \iint_{\mathbb{R}^2} Y(\zeta)\phi(\eta/\zeta^{1/3})\psi_{\eta,\zeta}(x)e^{-it\sqrt{\alpha_1}(\eta,\zeta)}e^{i(y\eta+z\zeta)}d\eta d\zeta.$$

For $\mu \in \mathbb{R}$, we set $H_\mu = -d_x^2 + (\mu + x^2)^2$ and we denote by ψ_μ its normalized ground state

$$H_\mu\psi_\mu = \lambda_1(\mu)\psi_\mu,$$

whose properties are described at the beginning of Section 6.2. We also define

$$F(\mu) = \sqrt{\lambda_1(\mu)}.$$

We assume that

$$F' \text{ is strictly monotonic on the support of } \phi, \tag{6.3}$$

which is no big restriction (choosing adequately the support of ϕ) since F is an analytic, non-affine, function[3].

We set $\eta = \zeta^{1/3}\eta_1$ and we note that $\psi_{\eta,\zeta}(x) = \zeta^{1/6}\psi_{\eta/\zeta^{1/3}}(\zeta^{1/3}x) = \zeta^{1/6}\psi_{\eta_1}(\zeta^{1/3}x)$ and $\sqrt{\alpha_1} = \zeta^{1/3}F(\eta/\zeta^{1/3})$. Hence,

$$(U(t)u_0)(x,y,z) = \iint_{\mathbb{R}^2} Y(\zeta)\zeta^{1/2}\phi(\eta_1)\psi_{\eta_1}(\zeta^{1/3}x)e^{-i\zeta^{1/3}(tF(\eta_1)-y\eta_1)}e^{iz\zeta}d\eta_1 d\zeta. \tag{6.4}$$

We denote by $WF(f) \subset T^*\mathbb{R}^3 \setminus 0$ the wave-front set of $f \in \mathcal{D}'(\mathbb{R}^3)$, whose projection onto $\mathbb{R}^3$ is the singular support Sing Supp(f) (see [Hor07a, Definition 8.1.2]). Our main result states that the speed of propagation of the singularities of u_0 is in some window determined by the support of ϕ.

[2] We take the convention $\mathcal{F}f(p) = (2\pi)^{-d}\int_{\mathbb{R}^d} f(q)e^{-iqp}dq$ for the Fourier transform in $\mathbb{R}^d$.
[3] See Point 1 of Lemma 6.10 and Proposition 6.14

Theorem 6.2. *For any $t \in \mathbb{R}$, we have*

$$WF(U(t)u_0) = \{(0,y,0;\ 0,0,\lambda) \in T^*\mathbb{R}^3,\ \lambda > 0,\ y \in tF'(I)\}\ , \tag{6.5}$$

where I is the support of ϕ. In particular,

$$\textit{Sing Supp}(U(t)u_0) = \{(0,y,0) \in \mathbb{R}^3,\quad y \in tF'(I)\}. \tag{6.6}$$

Theorem 6.2 means that

$$\begin{gathered}\textit{singularities propagate along the singular curve } \gamma \\ \textit{at speeds given by } F'(I).\end{gathered} \tag{6.7}$$

Let us comment on the notion of "speed" used throughout this paper. In the Riemannian setting, when one says that singularities propagate at speed 1, this has to be understood with respect to the Riemannian metric. In the context of the Martinet distribution $\mathcal{D}$, there is also a metric, called sub-Riemannian metric, defined by

$$g_q(v) = \inf\left\{u_1^2 + u_2^2,\quad v = u_1X_1(q) + u_2X_2(q)\right\},\quad q \in \mathbb{R}^3,\ v \in T_q\mathbb{R}^3, \tag{6.8}$$

which is a Riemannian metric on $\mathcal{D}$. This metric g induces naturally a way to measure the speed of a point moving along an horizontal curve: if $\delta : J \to \mathbb{R}^3$ is an horizontal curve describing the time-evolution of a point, i.e., $\dot{\delta}(t) \in \mathcal{D}_{\delta(t)}$ for any $t \in J$, then the speed of the point is $(g_{\delta(t)}(\dot{\delta}(t)))^{1/2}$. In the case of the curve γ, since $g_q(\partial_y) = 1$ for any q of the form $(0,y,0)$, we have $(g_q(F'(I)\partial_y))^{1/2} = F'(I)$. This is why the set $F'(I)$ is understood as a set of speeds in (6.7).

Proposition 6.3. *There holds $F'(\mathbb{R}) = [a, 1)$ for some $-1 < a < 0$.*

Together with (6.7), and choosing I adequately, this implies the following informal statement.

"Corollary" 6.4. *Any value between 0 and 1 can be realized as a speed of propagation of singularities along the singular curve γ.*

According to (1.29), the negative values in the range of F' yield singularities propagating backwards along the singular curve. This happens when $F'(I)$ contains negative values (see Proposition 1.25).

The next remarks explain possible adaptations of the statement of Theorem 6.2.

Remark 6.5. Putting in the initial Fourier data (6.2) an additional phase $e^{-iz_0\zeta}$ for some fixed $z_0 \in \mathbb{R}$, we obtain that the singularities of the corresponding solution propagate along the curve $t \mapsto (0,t,z_0)$, which is also a singular curve: for this new initial datum, we replace in (6.5) the 0 in the z coordinate by z_0.

Remark 6.6. If we consider $(u, D_tu)_{|t=0} = (u_0, 0)$ as initial data of the Martinet wave equation $\partial_t^2 u - \Delta u = 0$, the solution is given by

$$u(t) = \frac{1}{2}\left(U(t)u_0 + U(-t)u_0\right).$$

Hence, under the assumption that $F'(I)$ and $-F'(I)$ do not intersect, (6.5) must be replaced by

$$WF(u(t)) = \{(0,y,0;\ 0,0,\lambda) \in T^*\mathbb{R}^3,\ \lambda > 0,\ y \in \pm tF'(I)\}.$$

Remark 6.7. If we take $\zeta < 0$ instead of $\zeta > 0$ in the (Fourier) initial data

$$Y(|\zeta|)\phi(\eta/|\zeta|^{1/3})\psi_{\eta,\zeta}(x),$$

then we must replace $F'(I)$ by $-F'(-I)$ in the Theorem 6.2. The same if we replace X_2 by $\partial_y - x^2\partial_z$ and keep $\zeta > 0$ in the Fourier initial data. This is due to the "orientation" of the singular curve γ: for Theorem 6.2 to hold without any change, we have to take $(0,0,\zeta)(X_2) > 0$.

Remark 6.8. Instead of $\psi_{\eta,\zeta}$, we can use in the Fourier initial datum (6.2) the k-th eigenfunction of $-d_x^2 + (\eta + x^2\zeta)^2$. This yields a function F_k and the associated velocity F_k', instead of F and F'. Theorem 6.2 also holds for this initial datum with the same proof, just replacing F' by F_k' in the statement.

Remark 6.9. It is possible to establish an analogue of Theorem 6.2 for the half-wave equation associated to the quasi-contact sub-Laplacian

$$\Delta = \partial_x^2 + \partial_y^2 + (\partial_z - x\partial_s)^2$$

on $\mathbb{R}^4$. For that, we take Fourier initial data of the form

$$\mathcal{F}_{y,z,s}u_0(x,\eta,\zeta,\sigma) = \phi(\eta/\sigma^{1/2}, \zeta/\sigma^{1/2})\psi_{\eta,\zeta,\sigma}(x)$$

where $\phi \in C_0^\infty(\mathbb{R}^2, \mathbb{R})$, η, ζ, σ denote the dual variables of y, z, s, and $\psi_{\eta,\zeta,\sigma}$ is the normalized ground state of the $x-$operator $-d_x^2 + \eta^2 + (\zeta - x\sigma)^2$. Then, the singularities propagate along the curve $t \mapsto (0,t,0,0)$ which is a singular curve of the quasi-contact distribution $\mathrm{Span}(\partial_x, \partial_y, \partial_z - x\partial_s)$. The proof of this fact requires simpler computations than in the Martinet case since, instead of *quartic* oscillators, they involve usual *harmonic* oscillators. Note that the (non-flat) quasi-contact case has also been investigated in [Sav19], with other methods.

6.1.3 Comments and organization of the chapter

The singular curve $t \mapsto \gamma(t) = (0,t,0) \in \mathbb{R}^3$ of the Martinet distribution $\mathcal{D}$ has played an important role in the last decades in the development of sub-Riemannian geometry. This role is due to the fact that γ is a minimizing geodesic for the sub-Riemannian distance induced by the metric g defined in (6.8). However, we insist on the fact that in the present work,

the minimizing character of the singular curve γ plays no role.

For example, as explained in Remark 6.9, our computations can be adapted to the quasi-contact case, where singular curves are not minimizing.

It follows from Definition 6.1 that the existence of singular curves is a property of the distribution $\mathcal{D}$, and does not depend on the metric g on $\mathcal{D}$ (or on the vector fields X_1, X_2 which span $\mathcal{D}$). Besides, it was proved in [Mar70, Section II.6] that generically, a rank 2 distribution $\mathcal{D}_0$ in a $3D$ manifold M_0 is of contact type outside a surface $\mathscr{S}$, called the Martinet surface, and near any point of $\mathscr{S}$ except a finite number of them, the distribution is isomorphic to $\mathcal{D} = \ker(dz - x^2dy)$, which is exactly the distribution under study in the present work. Therefore, we expect to be able to generalize Theorem 6.2 to more generic situations.

To explain further the importance of singular curves, let us provide more context about sub-Riemannian geometry. A sub-Riemannian manifold is a triple $(M, \mathcal{D}, g)$ where M is a smooth manifold, $\mathcal{D}$ is a smooth sub-bundle of TM which is assumed to satisfy the Hörmander condition $\mathrm{Lie}(\mathcal{D}) = TM$, and g is a Riemannian metric on $\mathcal{D}$ (which naturally induces a distance d on M). Sub-Riemannian manifolds are thus a generalization of Riemannian manifolds (for which

$\mathcal{D} = TM$), and they have been studied in depth since the years 1980, see [Mon02] and [ABB19] for surveys.

As already mentioned, a particular interest has been devoted to the understanding of geodesics, i.e. absolutely continuous horizontal paths for which every sufficiently short subarc realizes the sub-Riemannian distance between its endpoints. It follows from Pontryagin's maximum principle (see also [Mon02, Section 5.3.3]) that any sub-Riemannian geodesic is

- either *normal*, meaning that it is the projection of an integral curve of the normal Hamiltonian vector field [4];
- or *singular*, meaning that it is the projection of a characteristic curve (see Definition 6.1).

A sub-Riemannian geodesic can be normal and singular at the same time, and it is indeed the case of the singular curve $t \mapsto (x, y, z) = (0, t, 0)$ in the Martinet distribution described above. But it was proved in [Mon94] that there also exist sub-Riemannian manifolds which exhibit geodesics which are singular, but not normal (they are called strictly singular).

The study of the spectral consequences of the presence of singular minimizers was initiated in [Mon95], where it was proved that in the situation where strictly singular minimizers show up as zero loci of two-dimensional magnetic fields, the ground state of a quantum particle concentrates on this curve as e/h tends to infinity, where e is the charge and h is the Planck constant. In [CHT21b], it is proved that, for 3D compact sub-Riemannian manifolds with Martinet singularities, the support of the Weyl measure is the 2D Martinet manifold: most eigenfunctions concentrate on it.

The present work gives a new illustration of the intuition that singular curves play a role "at the quantum level", this time at the level of propagation for a wave equation. However, the fact that the propagation speed is not 1, but can take any value between 0 and 1 was unexpected, since it is in strong contrast with the usual propagation of singularities at speed 1 for wave equations with elliptic Laplacians.

The chapter is organized as follows. In Section 6.2, we prove some properties of the eigenfunctions ψ_μ which play a central role in the next sections. In Section 6.3, we compute the wave-front set of the Cauchy datum u_0 thanks to stationary phase arguments; this proves Theorem 6.2 at time $t = 0$. In Section 6.4, we complete the proof of Theorem 6.2 by extending the previous computation to any $t \in \mathbb{R}$. We could have directly done the proof for any $t \in \mathbb{R}$ (thus avoiding to distinguish the case $t = 0$), but we have chosen this presentation to improve readability. In Section 6.5, to illustrate Theorem 6.2, we prove Proposition 6.3, we provide plots of F and F' and compute their asymptotics.

6.2 Some properties of the eigenfunctions ψ_μ

Let us recall that H_μ is the essentially self-adjoint operator $H_\mu = -d_x^2 + (\mu + x^2)^2$ on $L^2(\mathbb{R}, dx)$ and ψ_μ is the ground state eigenfunction with $\int_{\mathbb{R}} \psi_\mu(x)^2 dx = 1$ and $\psi_\mu(0) > 0$. We denote by $\lambda_1(\mu)$ the associated eigenvalue, $\lambda_1(\mu) = F(\mu)^2$.

Lemma 6.10. *The domain of the essentially self-adjoint operator H_μ is independent of μ. It is denoted by $D(H_0)$. Moreover, the following assertions hold:*

1. *The map $\mu \mapsto \lambda_1(\mu)$ is analytic on $\mathbb{R}$, and the map $\mu \mapsto \psi_\mu$ is analytic from $\mathbb{R}$ to $D(H_0)$;*

[4] By this, we mean the Hamiltonian vector field of g^*, the semipositive quadratic form on T_q^*M defined by $g^*(q, p) = \|p_{|\mathcal{D}_q}\|_q^2$, where the norm $\|\cdot\|_q$ is the norm on $\mathcal{D}_q^*$ dual of the norm g_q.

2. *The function ψ_μ is in the Schwartz space $\mathcal{S}(\mathbb{R})$ uniformly with respect to μ on any compact subset of $\mathbb{R}$* [5]*;*

3. *Any derivative in $D(H_0)$ of the map $\mu \mapsto \psi_\mu$ is in the Schwartz space $\mathcal{S}(\mathbb{R})$ uniformly with respect to μ on any compact subset of $\mathbb{R}$.*

Proof. The domain of H_μ is given by

$$D(H_\mu) = \{\psi \in L^2(\mathbb{R}), -\psi'' + x^4\psi \in L^2(\mathbb{R}),\ x^2\psi \in L^2(\mathbb{R})\}\ ,$$

the last property coming from the finiteness of the associated quadratic form $Q(\psi) = \int_{\mathbb{R}}((\psi')^2 + (\mu + x^2)^2\psi^2)dx$. We have hence $D(H_\mu) = D(H_0)$. The map $\mu \mapsto H_\mu$ is analytic from $\mathbb{R}$ into $\mathcal{L}(D(H_0), L^2(\mathbb{R}))$. Moreover, by [BS12, Theorem 3.1], the eigenvalues of H_μ are non-degenerate (simple). This implies (see [Kat13, Chapter VII.2] or [CR19, Proposition 5.25]) that the eigenvalues $\lambda_1(\mu)$ and eigenfunctions ψ_μ are analytic functions of μ, respectively with values in $\mathbb{R}$ and in $D(H_0)$. This proves Point 1.

Point 2 follows from Agmon estimates (precisely, [Hel88, Proposition 3.3.4] with $h = h_0 = 1$), which are uniform with respect to μ on any compact subset of $\mathbb{R}$.

This allows to start to prove Point 3 by induction. Assume that Point 3 is true for the derivatives of order $0, \dots, k-1$. Then, taking the derivatives with values in the domain $D(H_0)$ with respect to μ in the equation $(H_\mu - \lambda_1(\mu))\psi_\mu = 0$, we get

$$(H_\mu - \lambda_1(\mu))\frac{d^k}{d\mu^k}\psi_\mu = v_{k,\mu} \tag{6.9}$$

and we know, by the induction hypothesis, that $v_{k,\mu} \in \mathcal{S}(\mathbb{R})$ uniformly with respect to μ on any compact subset of $\mathbb{R}$. We now use the results of [Shu87, Section 25] (see also [Shu87, Section 23] for the notations, and [HR82] for similar results). We check that $\xi^2 + x^4$ is a symbol in the sense of Definition 25.1 of [Shu87], with $m = 4$, $m_0 = 2$ and $\rho = 1/2$. Its standard quantization (i.e., $\tau = 0$ in Equation (23.31) of [Shu87]) is H_μ. By [Shu87, Theorem 25.1], $H_\mu - \lambda_1(\mu)$ admits a parametrix B_μ; in particular, $B_\mu(H_\mu - \lambda_1(\mu)) = \mathrm{Id} + R_\mu$ where R_μ is smoothing. Hence, composing on the left by B_μ in (6.9), and noting that $B_\mu v_{k,\mu} \in \mathcal{S}(\mathbb{R})$, we obtain that $\frac{d^k}{d\mu^k}\psi_\mu \in \mathcal{S}(\mathbb{R})$ uniformly with respect to μ on any compact subset of $\mathbb{R}$, which concludes the induction and the proof of Point 3.

6.3 Wave-front of the Cauchy datum

The goal of this section is to compute the wave-front set of u_0. In other words, we prove Theorem 6.2 for $t = 0$. Recall that (see (6.4))

$$u_0(x, y, z) = \iint_{\mathbb{R}^2} Y(\zeta)\zeta^{1/2}\phi(\eta_1)\psi_{\eta_1}(\zeta^{1/3}x)e^{i(y\zeta^{1/3}\eta_1 + z\zeta)}d\eta_1 d\zeta. \tag{6.10}$$

Lemma 6.11. *The function u_0 is smooth on $\mathbb{R}^3 \setminus \{(0,0,0)\}$.*

Proof. We prove successively that u_0 is smooth outside $x = 0$, $y = 0$ and $z = 0$. Any derivative of (6.10) in x, y, z is of the form

$$\iint_{\mathbb{R}^2} Y(\zeta)\zeta^\alpha\psi_{\eta_1}^{(\gamma)}(\zeta^{1/3}x)\phi(\eta_1)\eta_1^\beta e^{i(y\zeta^{1/3}\eta_1 + z\zeta)}d\eta_1 d\zeta \tag{6.11}$$

[5] This means that for any compact $K \subset \mathbb{R}$, in the definition of $\mathcal{S}(\mathbb{R})$, the constants in the semi-norms can be taken independent of $\mu \in K$.

for some $\alpha, \beta, \gamma \geqslant 0$. By the dominated convergence theorem, locally uniform (in x, y, z) convergence of these integrals implies smoothness. Recalling that ϕ has compact support, we see that the main difficulty for proving smoothness comes from the integration in ζ in (6.11).

For $x \neq 0$ it follows from Lemma 6.10 (Point 2) that the integrand in (6.11) has a fast decay in ζ. This proves that u_0 is smooth outside $x = 0$.

If $y \neq 0$, we use the fact that the phase $y\zeta^{1/3}\eta_1 + z\zeta$ is non critical with respect to η_1 to get the decay in ζ. More precisely, (6.11) is equal to

$$\iint_{\mathbb{R}^2} Y(\zeta)\zeta^{\alpha}(y\zeta^{1/3})^{-N} D_{\eta_1}^N(\psi_{\eta_1}^{(\gamma)}(\zeta^{1/3}x)\phi(\eta_1)\eta_1^{\beta})e^{i(y\zeta^{1/3}\eta_1+z\zeta)}d\eta_1 d\zeta$$

after integration by parts in η_1 (where $D_{\eta_1} = i^{-1}\partial_{\eta_1}$). Taking N sufficiently large and using that $D_{\eta_1}^N(\psi_{\eta_1}^{(\gamma)}(\zeta^{1/3}x)\phi(\eta_1)\eta_1^{\beta})$ is bounded thanks to Lemma 6.10 (Point 3), we obtain that this integral converges when $y \neq 0$, and that this convergence is locally uniform with respect to x, y, z. This proves that u_0 is smooth outside $y = 0$.

Finally, let us study the case $z \neq 0$. We can also assume that $y \leqslant 1$ due to the previous point.

Claim. The function

$$\zeta \mapsto Y(\zeta)\zeta^{1/2}\phi(\eta_1)\psi_{\eta_1}^{(\gamma)}(\zeta^{1/3}x)e^{iy\zeta^{1/3}\eta_1} \tag{6.12}$$

is a symbol (see Definition 6.16) uniformly on every compact in (y, η_1).

Proof. The functions $\zeta \mapsto \zeta^{1/2}\phi(\eta_1)$ and $\zeta \mapsto Y(\zeta)e^{iy\zeta^{1/3}\eta_1}$ are symbols. Besides, $\zeta \mapsto \psi_{\eta_1}^{(\gamma)}(\zeta^{1/3}x)$ is also a symbol (of degree 0 with $\rho = 1$): we notice for example that the first derivative with respect to ζ writes $(1/3)\zeta^{-1}(\zeta^{1/3}x)\psi_{\eta_1}^{(\gamma+1)}(\zeta^{1/3}x)$ which is uniformly $O(1/\zeta)$ thanks to Lemma 6.10 (Point 2). Finally, since the space of symbols is an algebra for the pointwise product, we get the claim.

Integrating (6.12) in $\eta_1 \in \mathbb{R}$ and using Lemma 6.17 (in the variable ζ), we obtain that (6.10) is smooth outside $z = 0$, which concludes the proof of Lemma 6.11.

The following lemma proves Theorem 6.2 at time $t = 0$.

Lemma 6.12. *There holds $WF(u_0) = \{(0, 0, 0;\ 0, 0, \lambda) \in T^*\mathbb{R}^3,\ \lambda > 0\}$.*

Proof. The Fourier transform of u_0 is

$$U_0(\xi, \eta, \zeta) = Y(\zeta)\phi(\eta/\zeta^{1/3})\Psi_{\eta/\zeta^{1/3}}(\xi/\zeta^{1/3}) \tag{6.13}$$

where Ψ_μ is the Fourier transform of the eigenfunction ψ_μ. By Lemma 6.10 (Point 2), for any $N \in \mathbb{N}$ we get

$$|U_0(\xi, \eta, \zeta)| \leqslant C_N|\phi(\eta/\zeta^{1/3})|(1 + |\xi/\zeta^{1/3}|)^{-N}. \tag{6.14}$$

We show that U_0 is fastly decaying in any cone $C := \{|\xi| + |\eta| \geqslant c|\zeta|\}$ for c small. We split the cone into $C = C_1 \cup C_2$ with $C_1 = C \cap \{|\xi| \leqslant |\eta|\}$ and $C_2 = C \cap \{|\eta| \leqslant |\xi|\}$.
In C_1, we have $|\eta/\zeta^{1/3}| \geqslant c_1|\eta^{2/3}|$. This implies that $\phi(\eta/\zeta^{1/3})$ vanishes for η large enough. Hence, U_0 has fast decay in C_1.
In C_2, we have $|\xi/\zeta^{1/3}| \geqslant c_2|\xi|^{2/3} \geqslant c_3(1 + \xi^2 + \eta^2 + \zeta^2)^{1/3}$, hence, plugging into (6.14), we get that U_0 has fast decay in C_2.

This proves that no point of the form $(x, y, z;\ \xi, \eta, \zeta) \in T^*\mathbb{R}^3$ with $(\xi, \eta) \neq (0, 0)$ can belong to $WF(u_0)$. Moreover, due to the factor $Y(\zeta)$, necessarily $WF(u_0) \subset \{\zeta > 0\}$. Combining with Lemma 6.11, we get the inclusion $\subset$ in Lemma 6.12.

Let us finally prove that $(0, 0, 0;\ 0, 0, \lambda) \in WF(u_0)$ for $\lambda > 0$. We pick $a, b \in \mathbb{R}$ such that $\phi(a) \neq 0$ and $\Psi_a(b) \neq 0$. Then, we note that $U_0(\zeta^{1/3}a, \zeta^{1/3}b, \zeta)$ is independent of ζ and $\neq 0$, thus it is not fastly decaying as $\zeta \to +\infty$. Since $(\zeta^{1/3}a, \zeta^{1/3}b, \zeta)$ converges to the direction $(0,0,+\infty)$ as $\zeta \to +\infty$, we get that there exists at least one point of the form $(x, y, z;\ 0, 0, \lambda) \in T^*\mathbb{R}^3$ which belongs to $WF(u_0)$. By Lemma 6.11, we necessarily have $x = y = z = 0$, which concludes the proof.

6.4 Wave front of the propagated solution

In this Section, we complete the proof of Theorem 6.2. We set

$$\mathcal{G}_t = \{(0, y, 0;\ 0, 0, \lambda),\ \lambda > 0,\ y \in tF'(\mathrm{Support}(\phi))\}.$$

In Section 6.4.1, we prove the inclusion $WF(U(t)u_0) \subset \mathcal{G}_t$, and then in Section 6.4.2 the converse inclusion $\mathcal{G}_t \subset WF(U(t)u_0)$. This completes the proof of Theorem 6.2.

6.4.1 The inclusion $WF(U(t)u_0) \subset \mathcal{G}_t$

For this inclusion, we follow the same arguments as in Section 6.3: we adapt Lemma 6.11 to find out the singular support of $U(t)u_0$, and then we adapt Lemma 6.12 to determine the full wave-front set.

Lemma 6.13. *For any $t \in \mathbb{R}$, $U(t)u_0$ is smooth outside* $\{(0, y, 0) \in \mathbb{R}^3,\ y \in tF'(I)\}$.

Proof. As in Lemma 6.11, we prove successively that $U(t)u_0$ is smooth outside $x = 0$, $y \notin tF'(I)$ and $z = 0$. Any derivative of $U(t)u_0$ is of the form

$$\iint_{\mathbb{R}^2} Y(\zeta)\zeta^\alpha \psi_{\eta_1}^{(\gamma)}(\zeta^{1/3}x)\phi(\eta_1)\eta_1^\beta e^{-i\zeta^{1/3}(tF(\eta_1)-y\eta_1)} e^{iz\zeta} d\eta_1 d\zeta \tag{6.15}$$

for some $\alpha, \beta, \gamma \geqslant 0$.

For $x \neq 0$, it follows from Lemma 6.10 (Point 2) that the integrand in (6.15) has a fast decay in ζ (locally uniformly in x, y, z). This proves that $U(t)u_0$ is smooth outside $x = 0$.

If $y \notin tF'(I)$, we use the fact that the phase $\zeta^{1/3}(tF(\eta_1) - y\eta_1) - z\zeta$ is non critical with respect to η_1 to get decay in ζ. We set $R_{\eta_1}H = D_{\eta_1}(Q^{-1}H)$ where $Q = D_{\eta_1}(-i(\zeta^{1/3}(tF(\eta_1) - y\eta_1) - z\zeta)) = -\zeta^{1/3}(tF'(\eta_1) - y)$. Note that $Q \neq 0$ since $y \notin tF'(I)$. Doing N integration by parts, the above expression becomes

$$\iint_{\mathbb{R}^2} Y(\zeta)\zeta^\alpha R_{\eta_1}^N(\psi_{\eta_1}^{(\gamma)}(\zeta^{1/3}x)\phi(\eta_1)\eta_1^\beta) e^{-i\zeta^{1/3}(tF(\eta_1)-y\eta_1)} e^{iz\zeta} d\eta_1 d\zeta. \tag{6.16}$$

We set $H(x, \eta_1, \zeta) = \psi_{\eta_1}^{(\gamma)}(\zeta^{1/3}x)\phi(\eta_1)\eta_1^\beta$.

Claim. For any N, there exists C_N such that $|R_{\eta_1}^N H(x, \eta_1, \zeta)| \leqslant C_N|\zeta|^{-N/3}$ for any $\zeta \in \mathbb{R}$, any $\eta_1 \in I = \mathrm{Support}(\phi)$ and any $x \in \mathbb{R}$.

Taking N sufficiently large, the claim implies that (6.16), and thus (6.15), converge (locally uniformly), which proves the smoothness when $y \notin tF'(I)$ thanks to the dominated convergence theorem.

Proof of the claim. We prove it first for $N = 1$. We have

$$R_{\eta_1}H = \frac{D_{\eta_1}H}{Q} - H\frac{D_{\eta_1}Q}{Q^2}. \tag{6.17}$$

Since H is bounded (thanks to Point 2 of Lemma 6.10) and $|Q| \geqslant c|\zeta|^{1/3}$ with $c > 0$ and $|D_{\eta_1}Q| \leqslant C|\zeta|^{1/3}$ on the support of ϕ, we have $|H\frac{D_{\eta_1}Q}{Q^2}| \leqslant c|\zeta|^{-1/3}$. For the first term in the right-hand side of (6.17), we only need to prove that $D_{\eta_1}H$ is bounded. When D_{η_1} falls on $\phi(\eta_1)$ or η_1^β, it is immediate. When D_{η_1} falls on $\psi_{\eta_1}^{(\gamma)}(\zeta^{1/3}x)$, we use Lemma 6.10 (Point 3) and also get the result. This ends the proof of the case $N=1$. Now, we notice that our argument works not only for H, but for any function of the form $\psi_{\eta_1}^{(\gamma')}(\zeta^{1/3}x)\phi^{(\delta)}(\eta_1)\eta_1^{\beta'}$ where $\phi^{(\delta)}$ is any derivative of ϕ and $\beta',\gamma'\geqslant 0$. Hence, applying the previous argument recursively, we obtain the claim for any N.

Finally, the case $z\neq 0$ is checked in the same way as in the case $t=0$, just shifting the phase by $it\zeta^{1/3}F(\eta_1)$ in (6.12).

Let us finish the proof of the inclusion $WF(U(t)u_0)\subset\mathcal{G}_t$.

The Fourier transform of $U(t)u_0$ is

$$\mathcal{F}(U(t)u_0)(\xi,\eta,\zeta)=Y(\zeta)\phi(\eta/\zeta^{1/3})\Psi_{\eta/\zeta^{1/3}}(\xi/\zeta^{1/3})e^{-it\sqrt{\alpha_1}(\eta,\zeta)}. \tag{6.18}$$

The change of phase with respect (6.13) has no influence on the properties of decay at infinity. Hence, the proof of Lemma 6.12 allows to conclude that $WF(U(t)u_0)\subset\mathcal{G}_t$ for any $t\in\mathbb{R}$.

6.4.2 The inclusion $\mathcal{G}_t\subset WF(U(t)u_0)$

We fix $t\in\mathbb{R}$ and we prove the non smoothness at $(0,tF'(c),0)$ for any $c\in I$. We can assume that c is in the interior of I and that $\phi(c)\neq 0$. This implies thanks to (6.3) that $F''(c)\neq 0$. We want to show non-smoothness with respect to z at $x=0$, $y=tF'(c)$ and $z=0$. We set $v(z):=(U(t)u_0)(0,tF'(c),z)$. We will show that the Fourier transform of v is not fastly decaying.

Starting from (6.4), we get the explicit formula for the Fourier transform of v,

$$\mathcal{F}v(\zeta)=Y(\zeta)\zeta^{1/2}K(\zeta)$$

where

$$K(\zeta)=\int_{\mathbb{R}}\phi(\eta_1)\psi_{\eta_1}(0)e^{-i\zeta^{1/3}t(F(\eta_1)-F'(c)\eta_1)}d\eta_1.$$

The only critical point of the phase $\eta_1\mapsto -i\zeta^{1/3}t(F(\eta_1)-F'(c)\eta_1)$ located in I is c thanks to (6.3). Applying the stationary phase theorem with respect to η_1, we obtain

$$K(\zeta)=e^{-i\zeta^{1/3}t(F(c)-F'(c)c)}\sum_{j\geqslant 1}a_j(\zeta^{1/3}|t|)^{-j/2}$$

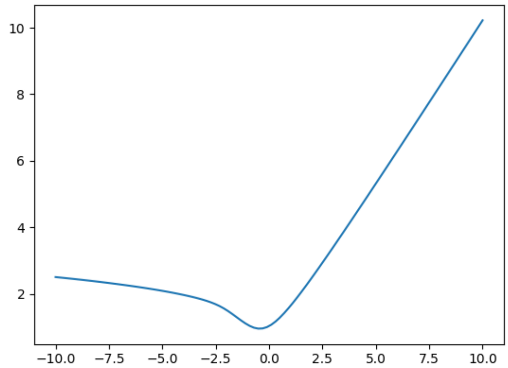
where

$$a_1=\phi(c)\psi_c(0)\left(\frac{2\pi}{|F''(c)|}\right)^{1/2}\exp(-i\frac{\pi}{4}\mathrm{sgn}(F''(c)))\neq 0.$$

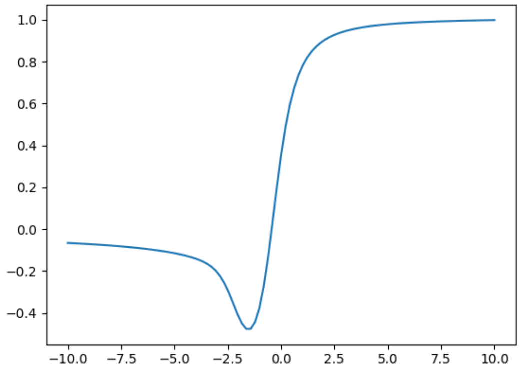
Since $\phi(c)>0$ and $\psi_c(0)>0$,we have $K(\zeta)\sim c_0(\zeta^{1/3}|t|)^{-1/2}$ where $c_0\neq 0$, and $\mathcal{F}v(\zeta)$ is not fastly decaying as $\zeta\to+\infty$. Applying Lemma 6.17 to $a=\mathcal{F}v$ which is a symbol in ζ, this implies that v is not smooth at $z=0$, thus $U(t)u_0$ is not smooth at $(0,tF'(c),0)$.

6.5 The function $F_k(\mu)=\sqrt{\lambda_k(\mu)}$

In this Section, we illustrate Theorem 6.2 with some plots and asymptotics of the functions F_k defined by $\mu\to\sqrt{\lambda_k(\mu)}$. As shown by Theorem 6.2 (and Remark 6.8), the speeds of the propagation of singularities along the singular curve are determined by the derivative $F_k'(\mu)$. Below, we plot $F = F_1$ and F' for $\mu \in (-10, 10)$[6].



(a) Plot of $F(\mu)$ for $\mu \in (-10, 10)$



(b) Plot of $F'(\mu)$ for $\mu \in (-10, 10)$

Recall that the F_k's are analytic (see Point 1 of Lemma 6.10). We state a more precise version of Proposition 6.3:

Proposition 6.14. *For any $k \in \mathbb{N} \setminus \{0\}$, there holds $F_k'(\mu) \to 1^-$ as $\mu \to +\infty$, $F_k'(\mu) \to 0^-$ as $\mu \to -\infty$, and F_k' is minimal for some value $\mu_k^\star < 0$. There exists $a_k \in (-1, 0)$ such that the range of F_k' is $[a_k, 1)$.*

Proposition 6.14 will be a consequence of the following result:

Proposition 6.15. *Denote by $\lambda_k(\mu)$ the k-th eigenvalue of $H_\mu = -d_x^2 + (\mu + x^2)^2$. Then, for $k \in \mathbb{N} \setminus \{0\}$, as $\mu \to +\infty$,*

$$\lambda_k(\mu) = \mu^2 + \sqrt{2}(2k-1)\sqrt{\mu} + \sum_{\ell=2}^{\infty} b_{\ell,k}\mu^{2-3\ell/2} \tag{6.19}$$

and

$$\frac{d}{d\mu}\sqrt{\lambda_k(\mu)} = 1 - \frac{2k-1}{2\sqrt{2}}\mu^{-3/2} + o(\mu^{-3/2}) \tag{6.20}$$

These derivatives are > 0 and converge to 1.

As $\mu \to -\infty$, for $k \in \mathbb{N} \setminus \{0\}$,

$$\lambda_{2k-1}(\mu) = 2(2k-1)\sqrt{-\mu} + \sum_{\ell=2}^{\infty} c_{\ell,k}(-\mu)^{2-3\ell/2} \tag{6.21}$$

$$\lambda_{2k}(\mu) = \lambda_{2k-1} + o\left(\mu^{-\infty}\right) \tag{6.22}$$

and

$$\frac{d}{d\mu}\sqrt{\lambda_{2k-1}(\mu)} = -\frac{\sqrt{2(2k-1)}}{4}(-\mu)^{-3/4} + o((-\mu)^{3/4}) \tag{6.23}$$

and the same for $\frac{d}{d\mu}\sqrt{\lambda_{2k}(\mu)}$. These derivative are < 0 and converge to 0.

[6]We thank Julien Guillod for his help in making the first numerical experiments.

Proof of Proposition 6.15. For $\mu > 0$, we consider the operator $T_\mu : \psi \mapsto \psi(\cdot/\mu^{1/4})$. Then $H_\mu = T_\mu^{-1} G_\mu T_\mu$ where $G_\mu = \mu^2 + \mu^{1/2}(-d_x^2 + 2x^2 + x^4/\mu^{3/2})$. The eigenvalues of $-d_x^2 + 2x^2 + hx^4$ for $h \to 0$ can be computed with the usual perturbation theory (see [RS78, Chapter XII.3]), and this yields (6.19) with $h = \mu^{-3/2}$. Moreover the formal expansion can be differentiated with respect to μ, hence we get (6.20).

For $\mu = -\mu_0 < 0$, we see that the transformation $x \mapsto \mu_0^{1/4}(x \mp \mu_0^{1/2})$ conjugates H_μ to the operator $\mu_0^{1/2}(-d_x^2 + 4x^2 \pm 4\mu_0^{-3/4}x^3 + \mu_0^{-3/2}x^4)$. Using again perturbation theory and the separation into pairs of eigenvalues in double wells (see [HS84]), we get (6.21) and (6.22), and (6.23) follows.

Proof of Proposition 6.14. The convergences at $\pm\infty$ are proved by Proposition 6.15. This behaviour at $\pm\infty$ implies the existence of $\mu_k^\star$ such that $F_k'(\mu_k^\star) = a_k$ is minimal. We denote by ψ_μ^k the normalized eigenfunction corresponding to $\lambda_k(\mu)$. Taking the first derivative (with value in the domain $D(H_0)$) with respect to μ of the eigenfunction equation $(H_\mu - \lambda_k(\mu))\psi_\mu^k = 0$, and then integrating against ψ_μ^k, we obtain $\lambda_k'(\mu) = \int_{\mathbb{R}} (\mu + x^2)\psi_\mu^k(x)^2 dx$. Thus,

$$F_k'(\mu) = \frac{1}{\sqrt{\lambda_k(\mu)}} \int_{\mathbb{R}} (\mu + x^2)\psi_\mu^k(x)^2 dx$$

which is positive for $\mu \geqslant 0$, hence $\mu_k^\star < 0$.

It remains to show that $|F_k'(\mu)| < 1$ for every μ: by the Cauchy-Schwarz inequality, we get

$$F_k'(\mu)^2 \leqslant \frac{1}{\lambda_k(\mu)} \int_{\mathbb{R}} (\mu + x^2)^2 \psi_\mu^k(x)^2 dx \int_{\mathbb{R}} \psi_\mu^k(x)^2 dx$$

and, from the quadratic form associated to H_μ,

$$\int_{\mathbb{R}} (\mu + x^2)^2 \psi_\mu^k(x)^2 dx < \lambda_k(\mu),$$

which concludes the proof.

6-A Fourier transform of symbols

Definition 6.16. *A smooth function $a : \mathbb{R}^d \to \mathbb{C}$ is called a symbol of degree $\leqslant m$ if there exists $0 < \rho \leqslant 1$ so that the partial derivatives of a satisfy*

$$\forall \alpha \in \mathbb{N}^d, \qquad |D^\alpha a(\xi)| \leqslant C_\alpha (1 + |\xi|)^{m - \rho|\alpha|}.$$

The space of symbols is an algebra for the pointwise product. If a is a real valued symbol of degree $m < 1$ and $\rho > m$, e^{ia} is a symbol of degree 0 (with a different ρ).

We will need the

Lemma 6.17. *If a is a symbol, the Fourier transform $\mathcal{F}a$ of a is smooth outside $x = 0$ and all derivatives of $\mathcal{F}a$ decay fastly at infinity. If moreover a does not belong to the Schwartz space $\mathcal{S}(\mathbb{R}^d)$, then $\mathcal{F}a$ is non smooth at $x = 0$.*

Proof. For $x \neq 0$ and for any $\alpha, \beta \in \mathbb{N}^d$, we have

$$(\mathcal{F}a)^{(\beta)}(x) = C_\beta \int_{\mathbb{R}^d} \xi^\beta a(\xi) e^{-ix\xi} d\xi = \frac{c_\beta^\alpha}{x^\alpha} \int_{\mathbb{R}^d} D_\xi^\alpha(\xi^\beta a(\xi)) e^{-ix\xi} d\xi. \tag{6.24}$$

The multi-index $\beta \in \mathbb{N}^d$ being fixed, this last integral converges for $|\alpha|$ sufficiently large since a is a symbol. By the dominated convergence theorem, this implies that $\mathcal{F}a$ is smooth outside $x = 0$. Moreover, (6.24) also implies that all derivatives of $\mathcal{F}a$ decay fastly at infinity.

Finally, if $\mathcal{F}a$ were smooth at 0, then $\mathcal{F}a$ would be in the Schwartz space as well as a.

Chapter 7

Quantum limits of sub-Laplacians via joint spectral calculus

"La chance c'est comme le Tour de France, on l'attend longtemps et ça passe vite."
Amélie Poulain.
"Combien de personnes savent que pour rester éveillé, il faut soustraire les moutons ?"
Les Marx Brothers.

This chapter is adapted from the preprint [Let20a]. It proves Theorems 8, 9 and 10.

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Warning: Section 1.5 must be read before this chapter, since we use without recalling them a lot of notations introduced in this section.

Structure of the chapter. In Section 7.1 we prove Theorem 8 using joint spectral calculus. Then, Section 7.2 is devoted to the proof of Theorem 9. In Section 7.2.2, we explain the spectral decomposition of $L^2(\mathbf{H}^m)$ according to the eigenspaces of the harmonic oscillators Ω_j. Building upon this spectral decomposition and Theorem 8, we establish in Section 7.2.3 Theorem 9. In Section 7.3, we prove Theorem 10 by constructing explicitely a sequence of eigenfunctions with prescribed Quantum Limit. In Section 7.4, we make a few remarks concerning the links of our main results with non-commutative harmonic analysis.

Then, we provide some supplementary material (this is an appendix in the preprint [Let20a]). In Section 7-A.1, we prove two elementary lemmas. In Section 7-A.2, we provide some supplementary material on Assumption (A). Finally, in Section 7-A.3, we prove a result concerning Quantum Limits of flat contact manifolds in any dimension: for such manifolds, the invariance properties of Quantum Limits are essentially the same as in the 3D case. Although this is a direct consequence of the results in [FF21], we decided to provide here a short and self-contained proof since this can be seen as a toy model for the averaging techniques used repeatedly in the proof of Theorem 9.

We also mention that in a previous version of the corresponding paper[1], we explain an alternative way to obtain the measure $Q^{\mathcal{J}}$ on $\mathbf{S}_{\mathcal{J}}$ and the family of measures $(\nu_s^{\mathcal{J}})_{s\in\mathbf{S}_{\mathcal{J}}}$ on $S\Sigma_{\mathcal{J}}$, based on pure functional analysis.

Joint spectral calculus. A key ingredient in the proof of all results of the present chapter is the joint spectral calculus (see [RS72, VII and VIII.5] and [Col79]) associated to the operators $Z_1, \ldots, Z_m$ and $-\Delta_{g,\mu}$. This joint calculus, at least for Heisenberg groups, is well-known, see for example [DS84, Section 2], or [Tha09] for the quotient case. It was used for instance in [MRS95] to prove a Marcinkiewicz multiplier theorem in H-type groups.

7.1 Proof of Theorem 8

In this Section, we prove Theorem 8. We fix a sub-Laplacian $\Delta_{g,\mu}$ satisfying Assumption (A), we fix $(\varphi_k)_{k\in\mathbb{N}^*}$ a sequence of eigenfunctions of $-\Delta_{g,\mu}$ associated with the eigenvalues $(\lambda_k)_{k\in\mathbb{N}^*}$ with $\lambda_k \to +\infty$ and $\|\varphi_k\|_{L^2} = 1$, and we consider ν, a Quantum Limit associated to the sequence $(\varphi_k)_{k\in\mathbb{N}^*}$.

Let us first give an intuition of how the proof goes. We decompose φ_k as a sum of functions which are joint eigenfunctions of $-\Delta_{g,\mu}$ and of all the $Z_j^*Z_j$ for $1 \leqslant j \leqslant m$. Each of these functions is an eigenfunction of $-\Delta_{g,\mu}$ with same eigenvalue λ_k as φ_k. Then, roughly speaking, we gather some of these functions into $\varphi_k^{\emptyset}$ or into $\varphi_k^{\mathcal{J}}$ for some $\mathcal{J} \in \mathcal{P} \setminus \{\emptyset\}$, depending on their eigenvalues with respect to the operators $Z_j^*Z_j$ (for $1 \leqslant j \leqslant m$) and $-\Delta_{g,\mu}$. More precisely, setting

$$E = -\Delta_{g,\mu} + \sum_{j=1}^{m} Z_j^*Z_j \in \Psi^2(M), \tag{7.1}$$

the functions which we select (asymptotically as $k \to +\infty$) to be in $\varphi_k^{\mathcal{J}}$ are those such that:

1. $-\Delta_{g,\mu} \ll E$;

[1] https://arxiv.org/pdf/2007.00910.pdf

2. if $i \notin \mathcal{J}$, then $Z_j^* Z_j \ll E$;

3. if $j \in \mathcal{J}$, then $Z_j^* Z_j \gtrsim E$.

Here, since we consider joint eigenfunctions of $-\Delta_{g,\mu}$, E and $Z_j^* Z_j$ for any $1 \leqslant j \leqslant m$, the above notation $A \ll B$ (resp. $A \gtrsim B$) means that the eigenvalue with respect to A is negligible compared to (resp. is greater than a constant times) the eigenvalue with respect to B.

This splitting "quantizes" the fact that $\Sigma_{\mathcal{J}}$ is the set of points (q,p) of T^*M for which $g^*(q,p) = 0$ (point 1 above) and $h_{Z_j(q,p)}$ is non-nul if and only if $j \in \mathcal{J}$ (points 2 and 3 above).

Here is the rigorous proof:

Proof of Theorem 8. For $n \in \mathbb{N}^*$, let $\chi_n \in C_c^\infty(\mathbb{R}, [0,1])$ such that $\chi_n(x) = 1$ for $|x| \leqslant \frac{1}{2n}$ and $\chi_n(x) = 0$ for $|x| \geqslant \frac{1}{n}$. We consider E given by (7.1) which, thanks to point (i) in Assumption (A), is elliptic. Its principal symbol is

$$\sigma_P(E) = g^* + \sum_{j=1}^m \sigma_P(Z_j^* Z_j).$$

Also, thanks to point (ii) in Assumption (A), we know that E commutes with Z_j, for any $1 \leqslant j \leqslant m$, and with $-\Delta_{g,\mu}$. Therefore, thanks to functional calculus (see [RS72, VII and VIII.5]), for $\mathcal{J} \in \mathcal{P} \setminus \{\emptyset\}$, it makes sense to consider the operator

$$P_n^{\mathcal{J}} = \chi_n\left(\frac{-\Delta_{g,\mu}}{E}\right) \prod_{i \notin \mathcal{J}} \chi_n\left(\frac{Z_i^* Z_i}{E}\right) \prod_{j \in \mathcal{J}} (1-\chi_n)\left(\frac{Z_j^* Z_j}{E}\right). \tag{7.2}$$

Similarly, we consider

$$P_n^{\emptyset} = (1-\chi_n)\left(\frac{-\Delta_{g,\mu}}{E}\right). \tag{7.3}$$

As we will see, for any $\mathcal{J} \in \mathcal{P}$, $P_n^{\mathcal{J}} \in \Psi^0(M)$ and, as $n \to +\infty$, its principal symbol tends either to the characteristic function $\mathbf{1}_{\Sigma_{\mathcal{J}}} : T^*M \to \mathbb{R}$ of $\Sigma_{\mathcal{J}}$, or to the characteristic function $\mathbf{1}_{U^*M}$ of U^*M if $\mathcal{J} = \emptyset$. Recall that $\Sigma_{\mathcal{J}}$ has been defined in (1.34).

For any $\mathcal{J} \in \mathcal{P}$, the following properties hold:

(1) $P_n^{\mathcal{J}} \in \Psi^0(M)$;

(2) $[P_n^{\mathcal{J}}, \Delta_{g,\mu}] = 0$;

(3) If $\mathcal{J} \neq \emptyset$, then $\sigma_P(P_n^{\mathcal{J}}) \to \mathbf{1}_{\Sigma_{\mathcal{J}}}$ pointwise as $n \to +\infty$.
If $\mathcal{J} = \emptyset$, then $\sigma_P(P_n^{\mathcal{J}}) \to \mathbf{1}_{U^*M}$ pointwise as $n \to +\infty$.

Let us prove Point (1). Since $E \in \Psi^2(M)$ is elliptic, it is invertible, and thus $-\Delta_{g,\mu} E^{-1} = -E^{-1}\Delta_{g,\mu} \in \Psi^0(M)$ is self-adjoint. Hence, by [HV00, Theorem 1(ii)], $(1-\chi_n)\left(\frac{-\Delta_{g,\mu}}{E}\right) \in \Psi^0(M)$ with principal symbol

$$(1-\chi_n)\left(\frac{g^*}{\sigma_P(E)}\right).$$

Similarly, the operators $\chi_n\left(\frac{-\Delta_{g,\mu}}{E}\right)$, $\chi_n\left(\frac{Z_i^* Z_i}{E}\right)$ and $(1-\chi_n)\left(\frac{Z_j^* Z_j}{E}\right)$ (for any $1 \leqslant i,j \leqslant m$) belong to $\Psi^0(M)$ with respective principal symbols

$$\chi_n\left(\frac{g^*}{\sigma_P(E)}\right), \qquad \chi_n\left(\frac{|h_{Z_i}|^2}{\sigma_P(E)}\right) \quad \text{and} \quad (1-\chi_n)\left(\frac{|h_{Z_j}|^2}{\sigma_P(E)}\right).$$

Hence, $P_n^{\mathcal{J}} \in \Psi^0(\mathbf{H}^m)$.

Point (2) is an immediate consequence of functional calculus, since $\Delta_{g,\mu}$ commutes with E and with Z_j for any $1 \leqslant j \leqslant m$.

Let us prove Point (3). For $\kappa > 0$, we consider the cone

$$S_\kappa := \left\{ \frac{g^*}{\sigma_P(E)} \leqslant \kappa \right\} \subset T^*M$$

and, for $1 \leqslant j \leqslant m$, we also consider the cone

$$T_\kappa^j = \left\{ \frac{|h_{Z_j}|^2}{\sigma_P(E)} \leqslant \kappa \right\} \subset T^*M.$$

For the moment, we assume $\mathcal{J} \neq \emptyset$. Then, the support of $\sigma_P(P_n^{\mathcal{J}})$ is contained in $S_{\frac{1}{n}}$, in $T_{\frac{1}{n}}^i$ for $i \notin \mathcal{J}$ and in the complementary set $(T_{\frac{1}{2n}}^j)^c$ for $j \in \mathcal{J}$. It follows that, in the limit $n \to +\infty$, $\sigma_P(P_n^{\mathcal{J}})$ vanishes everywhere outside the set of points (q,p) satisfying $g^*(q,p) = 0$,

$$\begin{aligned} h_{Z_i}(q,p) &= 0, \quad \forall i \notin \mathcal{J} \\ h_{Z_j}(q,p) &\neq 0, \quad \forall j \in \mathcal{J}. \end{aligned}$$

We note that these relations exactly define the set $\Sigma_{\mathcal{J}}$.
Conversely, let $(q,p) \in \Sigma_{\mathcal{J}}$. Our goal is to show that $\sigma_P(P_n^{\mathcal{J}})(q,p) = 1$ for sufficiently large $n \in \mathbb{N}^*$. It follows from a separate analysis of the principal symbol of each factor in the product (7.2):

- Since $(q,p) \in \Sigma$, there holds $g^*(q,p) = 0$, hence

$$\chi_n\left(\frac{g^*}{\sigma_P(E)}\right) = 1;$$

- For $i \notin \mathcal{J}$, since $h_{Z_i}(q,p) = 0$, there holds

$$\chi_n\left(\frac{|h_{Z_i}|^2}{\sigma_P(E)}\right)(q,p) = 1;$$

- For $j \in \mathcal{J}$, we know that $h_{Z_j}(q,p) \neq 0$. Hence, for n sufficiently large, at (q,p),

$$(1 - \chi_n)\left(\frac{|h_{Z_j}|^2}{\sigma_P(E)}\right)(q,p) = 1.$$

All in all, $\sigma_P(P_n^{\mathcal{J}})(q,p) = 1$ for sufficiently large n, which proves Point (3) in case $\mathcal{J} \neq \emptyset$. The proof in the case $\mathcal{J} = \emptyset$ is very similar, for the sake of brevity we do not repeat it here.

We now conclude the proof of Theorem 8. We consider, for fixed $n \in \mathbb{N}$ and $\mathcal{J} \in \mathcal{P} \setminus \{\emptyset\}$, the sequence $(P_n^{\mathcal{J}} \varphi_k)_{k \in \mathbb{N}^*}$, which, thanks to Points (1) and (2), is also a sequence of eigenfunctions of $-\Delta_{g,\mu}$ with same eigenvalues as φ_k. We denote by $\nu_n^{\mathcal{J}}$ a microlocal defect measure of $(P_n^{\mathcal{J}} \varphi_k)_{k \in \mathbb{N}^*}$ and by $\nu_n^{\emptyset}$ a microlocal defect measure of the sequence given by the eigenfunctions

$$\varphi_k - \sum_{\mathcal{J} \in \mathcal{P} \setminus \{\emptyset\}} P_n^{\mathcal{J}} \varphi_k.$$

Furthermore, we can assume thanks to the diagonal extraction process that the extraction used to obtain all these microlocal defect measures is the same for any $n \in \mathbb{N}^*$ and any $\mathcal{J} \in \mathcal{P}$.

Finally, we take $\nu^{\mathcal{J}}$ a weak-star limit of $(\nu_n^{\mathcal{J}})_{n\in\mathbb{N}}$ and $\beta\nu^{\emptyset}$ a weak-star limit of $(\nu_n^{\emptyset})_{n\in\mathbb{N}}$, with $\nu \in \mathscr{P}(S^*M)$ and $\beta \in [0,1]$. Thanks to the analysis done while proving Point (3), we know that $\nu^{\mathcal{J}}$ gives no mass to the complementary of $S\Sigma_{\mathcal{J}}$ in S^*M, and that $\nu^{\emptyset}(S\Sigma) = 0$. Again, thanks to the diagonal extraction process, up to extraction of a subsequence in $k \in \mathbb{N}^*$, we can write

$$\varphi_k = \varphi_k^{\emptyset} + \sum_{\mathcal{J}\in\mathcal{P}\setminus\{\emptyset\}} \varphi_k^{\mathcal{J}} \tag{7.4}$$

where the unique microlocal defect measure of $(\varphi_k^{\emptyset})_{k\in\mathbb{N}^*}$ is $\beta\nu^{\emptyset}$, and $\varphi_k^{\mathcal{J}} = P_{r(k)}^{\mathcal{J}}\varphi_k$ (for some function r tending (slowly) to $+\infty$ as $k \to +\infty$) has a unique microlocal defect measure as $k \to +\infty$, which is $\nu^{\mathcal{J}}$.

Let us prove that (7.4) implies (1.39). For that, we first recall an elementary lemma concerning joint microlocal defect measures (see Definition 1.30). It is proved in Section 7-A.1.

Lemma 7.1. *Let $(u_k), (v_k)$ be two sequences of functions weakly converging to 0, each with a unique microlocal defect measure, which we denote respectively by μ_{11} and μ_{22}. Then, any joint microlocal defect measures μ_{12} (resp. μ_{21}) of $(u_k)_{k\in\mathbb{N}^*}$ and $(v_k)_{k\in\mathbb{N}^*}$ (resp. of $(v_k)_{k\in\mathbb{N}^*}$ and $(u_k)_{k\in\mathbb{N}^*}$) is absolutely continuous with respect to both μ_{11} and μ_{22}.*

Using Lemma 7.1, we then notice that if $\mathcal{J}, \mathcal{J}' \in \mathcal{P}\setminus\{\emptyset\}$ are distinct, the joint microlocal defect measures of $(\varphi_k^{\mathcal{J}})_{k\in\mathbb{N}^*}$ and $(\varphi_k^{\mathcal{J}'})_{k\in\mathbb{N}^*}$ vanish since $\Sigma_{\mathcal{J}}$ and $\Sigma_{\mathcal{J}'}$ are disjoint. Similarly, the joint microlocal defect measure of $(\varphi_k^{0})_{k\in\mathbb{N}^*}$ with the sequence $(\varphi_k^{\mathcal{J}})_{k\in\mathbb{N}^*}$ vanishes for any $\mathcal{J} \in \mathcal{P}\setminus\{\emptyset\}$. Therefore, evaluating $(\mathrm{Op}(a)\varphi_k, \varphi_k)$ and using (7.4), we obtain (1.39), which finishes the proof of Theorem 8. $\square$

Remark 7.2. The above proof is inspired by the proof of a slightly different fact (see [Ger91b, Proposition 3.3]): if θ is the unique microlocal defect measure of a sequence $(\psi_k)_{k\in\mathbb{N}^*}$ of functions over a manifold M, A (resp. B) is a closed (resp. open) subset of S^*M, and A and B form a partition of S^*M, then we can write $\theta = \theta_A + \theta_B$, with θ_A (resp. θ_B) supported in A (resp. $\theta_B(A) = 0$) and $\psi_k = \psi_k^A + \psi_k^B$ such that θ_A (resp. θ_B) is a microlocal defect measure of $(\psi_k^A)_{k\in\mathbb{N}^*}$ (resp. of $(\psi_k^B)_{k\in\mathbb{N}^*}$). The proof just consists in choosing symbols $p_n \in \mathscr{S}^0(M)$ concentrating on A and taking $\psi_k^A = \mathrm{Op}(p_n)\psi_k$ as in the proof above.

In the proof of Theorem 8, we had to choose particular symbols p_n in order to ensure that $\varphi_k^{\mathcal{J}}$ and $\varphi_k^{\emptyset}$ are still eigenfunctions of $-\Delta_{g,\mu}$.

Remark 7.3. As already mentioned, the ideas underlying Theorem 8 are close to those of [Col79, Theorem 0.6], which deals with the joint spectrum of commuting pseudodifferential operators whose sum of squares is elliptic. The parallel is the following: the elliptic operator Q in [Col79] is replaced here by E, and the operators P_i in [Col79] are replaced here by the X_i and the Z_j.

With this parallel in mind and using the tools developed in the above proof, given a Riemannian Laplacian $\Delta_g = \sum X_i^2$ with all the X_i commuting and a sequence of eigenfunctions of Δ_g, one could identify which part of the eigenfunctions concentrates on each part of the cotangent bundle.

In our setting, not all X_i and Z_j commute, but $\sum_{i=1}^N X_i^* X_i$ commutes with all Z_j, which is sufficient because we do not look for any information on the QLs in U^*M. Our statement is, in some sense, more precise than [Col79, Theorem 0.6] since the splitting of eigenfunctions is made precise, but also less general because X_i and Z_j are differential, and not general pseudodifferential operators as in [Col79].

7.2 Proof of Theorem 9

This section is devoted to the proof of Theorem 9. In other words, we seek to prove that for any continuous function $a : S\Sigma \to \mathbb{R}$, there holds

$$\int_{S\Sigma} a d\nu_\infty = \sum_{\mathcal{J}\in\mathcal{P}\setminus\{\emptyset\}} \int_{\mathbf{S}_\mathcal{J}} \left(\int_{S\Sigma_\mathcal{J}} a d\nu_s^\mathcal{J} \right) dQ^\mathcal{J}(s)$$

where $\nu^\mathcal{J}, \nu_s^\mathcal{J}$ and $Q^\mathcal{J}$ are as in (1.44).

Therefore, we fix $m \geqslant 2$ and $\Delta_{g,\mu} = \Delta$ as in Section 1.5.2. The last part of Theorem 9 is an immediate consequence of the last part of Proposition 1.29, and therefore we are reduced to prove Points (1) and (2). The first step in the proof consists in reducing the analysis to the part of the QL above $\Sigma_\mathcal{J}$ for some $\mathcal{J} \in \mathcal{P} \setminus \{\emptyset\}$, and it is achieved thanks to Theorem 8 as follows.

Reduction to a fixed $\mathcal{J} \in \mathcal{P} \setminus \{\emptyset\}$. Combining Theorem 8 with Point (1) of Proposition 1.29, we see that it is enough to prove Point (2) of Theorem 9, and that it is possible to assume that $(\varphi_k)_{k\in\mathbb{N}^*}$ is a sequence of eigenfunctions with eigenvalue tending to $+\infty$, and with a unique microlocal defect measure ν, which can be assumed to be supported in $S\Sigma$. Indeed, thanks to Theorem 8, we can even assume that all the mass of ν is contained in $S\Sigma_\mathcal{J}$ for some $\mathcal{J} \in \mathcal{P}\setminus\{\emptyset\}$, i.e., $\nu = \nu^\mathcal{J}$: once we have established the decomposition

$$\nu^\mathcal{J} = \int_{\mathbf{S}_\mathcal{J}} \nu_s^\mathcal{J} dQ^\mathcal{J}(s),$$

Point (2) of Theorem 9 follows by just gluing all pieces of ν together thanks to Theorem 8.

Therefore, in order to establish Point (2) of Theorem 9, we assume that the unique microlocal defect measure of $(\varphi_k)_{k\in\mathbb{N}^*}$ has no mass outside $S\Sigma_\mathcal{J}$ for some $\mathcal{J} \in \mathcal{P} \setminus \{\emptyset\}$. By symmetry, we can even assume that $\mathcal{J} = \{1, \ldots, J\}$ with $J = \mathrm{Card}(\mathcal{J})$.

To sum up, the sequence $(\varphi_k)_{k\in\mathbb{N}^*}$ that we consider is no more a general sequence of normalized eigenfunctions with eigenvalues tending to $+\infty$, but it satisfies the following property:

Property 7.4. *$(\varphi_k)_{k\in\mathbb{N}^*}$ is a bounded sequence of eigenfunctions of $-\Delta$ labeled with increasing eigenvalues tending to $+\infty$, and with unique microlocal defect measure ν. Moreover, there exist $J \leqslant m$ and $r(k) \to +\infty$ as $k \to +\infty$ such that*

$$\varphi_k = P^\mathcal{J}_{r(k)} \varphi_k \tag{7.5}$$

for $\mathcal{J} = \{1, \ldots, J\}$ and for any $k \in \mathbb{N}^*$, where $P_n^\mathcal{J}$ is defined in (7.2). In particular, ν has no mass outside $S\Sigma_\mathcal{J}$.

Remark 7.5. Writing Σ as a disjoint union (1.34), we notice that $\Sigma_\mathcal{J}$ is indeed the set of points $(q, p) \in \Sigma$ with $p = (p_{x_1}, p_{y_1}, p_{z_1}, \ldots, p_{x_m}, p_{y_m}, p_{z_m})$ such that

$$\left(p_{z_j} \neq 0\right) \Leftrightarrow (j \in \mathcal{J}).$$

7.2.1 Illustration and sketch of proof

Since the rest of the proof is a bit involved, in this section we provide an illustration and a sketch of proof which could be helpful. The proof is written in full details in Sections 7.2.2 and 7.2.3. Logically, one may omit the discussion which follows and proceed directly to the next section.

An illustration of Point (2) of Theorem 9. A way to get an intuition of Point (2) of Theorem 9 is to fix $(n_1, \dots, n_m) \in \mathbb{N}^m$, and to consider a sequence of normalized eigenfunctions $(\psi_k)_{k\in\mathbb{N}^*}$ of $-\Delta$ given in a tensor form as in Remark 1.31, such that, for any $k \in \mathbb{N}^*$, ψ_k is also, for any $1 \leqslant j \leqslant m$, a sequence of eigenfunctions of R_j with eigenvalue tending to $+\infty$, and of Ω_j with eigenvalue $2n_j + 1$. We notice that any associated Quantum Limit ν is supported in $S\Sigma$: it follows directly from the arguments developed in the proof of Theorem 8, since for any $1 \leqslant j \leqslant m$, the eigenvalues with respect to R_j^2 are much larger than the eigenvalues with respect to $-\Delta$.

Let $\mathcal{J} = \{1, \dots, m\} \in \mathcal{P}$. Then, ν is necessarily invariant under the Hamiltonian vector field $\vec{\rho}_s^{\mathcal{J}}$, where $s = (s_1, \dots, s_m) \in \mathbf{S}_{\mathcal{J}}$ is defined by $s_j = \frac{2n_j+1}{2n_1+1+\ldots+2n_m+1}$ for $j = 1, \dots, m$. To see it, we set

$$R = \frac{\sum_{j=1}^m (2n_j + 1) R_j}{\sum_{j=1}^m 2n_j + 1}$$

and we note that for any $A \in \Psi^0(\mathbf{H}^m)$, we have

$$([A, R]\psi_k, \psi_k) = (AR\psi_k, \psi_k) - (A\psi_k, R\psi_k) = 0$$

since ψ_k is an eigenfunction of R. In the limit $k \to +\infty$, taking the principal symbol, we obtain $\int_{S\Sigma} \{a, \rho_s^{\mathcal{J}}\} d\nu = 0$ where $a = \sigma_P(A)$. Since it is true for any $a \in \mathscr{S}^0(\mathbf{H}^m)$, this implies $\vec{\rho}_s^{\mathcal{J}} \nu = 0$. Hence, for such sequences $(\psi_k)_{k\in\mathbb{N}^*}$, any QL verifies $\nu = \nu_s^{\mathcal{J}}$ (which is invariant under $\vec{\rho}_s^{\mathcal{J}}$), $Q^{\mathcal{J}}$ is a Dirac mass on s and $Q^{\mathcal{J}'} = 0$ for $\mathcal{P} \ni \mathcal{J}' \neq \mathcal{J}$.

In some sense, any QL supported on $S\Sigma$ is a linear combination of sequences as in the above example, for different $\mathcal{J} \in \mathcal{P} \setminus \{\emptyset\}$ and different $s \in \mathbf{S}_{\mathcal{J}}$.

Roles of R_j and Ω_j. The operators R_j and Ω_j play a key role in the proofs of Theorem 9 and Theorem 10. As illustrated in the previous paragraph, the operators Ω_j are linked with the parameters $s \in \mathbf{S}_{\mathcal{J}}$: in some sense, once the eigenfunctions have been orthogonally decomposed with respect to the operators R_j and Ω_j (as explained in Section 7.2.2), the ratios between the Ω_j-s determines the invariance property of the associated Quantum Limits through the parameter s and the Hamiltonian vector field $\vec{\rho}_s^{\mathcal{J}}$. On the other side, the operators R_j 'determine' the microlocal support of the associated Quantum Limits, for example the element $\mathcal{J} \in \mathcal{P} \setminus \{\emptyset\}$ (such that the QL concentrates on $S\Sigma_{\mathcal{J}}$). The next paragraph, which is devoted to a sketch of proof of Theorem 9, will make these intuitions more precise.

Sketch of proof. In order to simplify the presentation, in this sketch of proof, we assume that $\mathcal{J} = \{1, \dots, m\}$ and we omit this notation (writing for example $\mathbf{S}$ instead of $\mathbf{S}_{\mathcal{J}}$), but the ideas are similar for any $\mathcal{J} \in \mathcal{P} \setminus \{\emptyset\}$.

Le us use the decomposition (1.42) to write each φ_k as a sum of eigenfunctions of operators of the form $\sum_{j=1}^m (2n_j + 1) R_j$ for some integers $n_1, \dots, n_m$:

$$\varphi_k = \sum_{(n_1,\dots,n_m)\in\mathbb{N}^m} \varphi_{k,n_1,\dots,n_m}, \tag{7.6}$$
$$\text{with} \quad \Omega_j \varphi_{k,n_1,\dots,n_m} = (2n_j + 1)\varphi_{k,n_1,\dots,n_m}, \quad \forall\, 1 \leqslant j \leqslant m.$$

We will see in Section 7.2.2 that the decomposition (7.6) is orthogonal, and therefore each eigenfunction $\varphi_{k,n_1,\dots,n_m}$ has the same eigenvalue λ_k as φ_k. Then, we do a careful analysis of this decomposition into modes, which, in the limit $k \to +\infty$, gives the disintegration $\nu = \int_{\mathbf{S}} \nu_s dQ(s)$.

This analysis builds upon a partition of the lattice $\mathbb{N}^m$ into positive cones, each of them gathering together the modes $\varphi_{k,n_1,\ldots,n_m}$ for which the m-tuples

$$\left(\frac{2n_1+1}{2n_1+1+\ldots+2n_m+1},\ldots,\frac{2n_m+1}{2n_1+1+\ldots+2n_m+1}\right)$$

are approximately the same: each of these positive cones accounts for a small region of the simplex $\mathbf{S}$. If $\mathbb{N}^m$ is partitioned into 2^N positive cones C_ℓ^N (with $0\leqslant \ell\leqslant 2^N-1$), this gathering defines eigenfunctions

$$\varphi_{k,\ell}^N=\sum_{(n_1,\ldots,n_m)\in C_\ell^N}\varphi_{k,n_1,\ldots,n_m}$$

of $-\Delta$ such that

$$\varphi_k=\sum_{\ell=0}^{2^N-1}\varphi_{k,\ell}^N \tag{7.7}$$

for any $N\in\mathbb{N}^*$.

Taking a microlocal defect measure ν_ℓ^N in each sequence $(\varphi_{k,\ell}^N)_{k\in\mathbb{N}^*}$ and making $N\to+\infty$ (i.e., taking the limit where the positive cones degenerate to half-lines parametrized by $s\in\mathbf{S}$), we obtain from (7.7) the disintegration $\nu=\int_{\mathbf{S}}\nu_s dQ(s)$.

Given a certain $s=(s_1,\ldots,s_m)\in\mathbf{S}$, $dQ(s)$ accounts for the relative importance, in the limit $N\to+\infty$, of the eigenfunction $\varphi_{k,\ell(N)}^N$ in the sum (7.7), where $\ell(N)$ is chosen so that the positive cone $C_{\ell(N)}^N$ converges to the half-line with parameter s as $N\to+\infty$.

The invariance property $\vec{\rho}_s\nu_s=0$ can be seen from the fact that, for any large N and any $0\leqslant\ell\leqslant 2^N-1$, each eigenfunction $\varphi_{k,n_1,\ldots,n_m}$ with $(n_1,\ldots,n_m)\in C_\ell^N$ is indeed an eigenfunction of the operator

$$\sum_{i=1}^m\left(\frac{2n_i+1}{2n_1+1+\ldots+2n_m+1}\right)R_i$$

which, by definition of $\varphi_{k,\ell}^N$, is approximately equal to $R_s=s_1R_1+\ldots+s_mR_m$ if $s=(s_1,\ldots,s_m)\in\mathbf{S}$ denotes the parameter of the limiting half-line of the positive cones C_ℓ^N as $N\to+\infty$. Hence, $\varphi_{k,\ell}^N$ is an approximate eigenfunction of R_s, from which it follows by a classical argument that ν_s is invariant under the Hamiltonian vector field $\vec{\rho}_s$ of $\rho_s=(\sigma_P(R_s))_{|\Sigma}$.

7.2.2 Spectral decomposition of $-\Delta$

In this section, we start the proof of Theorem 9 with a detailed study of the action of $-\Delta$ on $L^2(\mathbf{H}^m)$, writing it under the form of an orthogonal decomposition of eigenspaces.

Let us recall that, for $1\leqslant j\leqslant m$, we set $R_j=\sqrt{\partial_{z_j}^*\partial_{z_j}}$ and we made a Fourier expansion with respect to the z_j-variable. On the eigenspaces corresponding to non-zero modes of this Fourier decomposition, we defined the operator $\Omega_j=-R_j^{-1}\Delta_j=-\Delta_jR_j^{-1}$ where $\Delta_j=X_j^2+Y_j^2$. For example, $-\Delta$ acts as

$$-\Delta=\sum_{j=1}^m R_j\Omega_j$$

on any eigenspace of $-\Delta$ on which $R_j\neq 0$ for any $1\leqslant j\leqslant m$. Moreover, R_j and Ω_j are pseudodifferential operators of order 1 in any cone of $T^*\mathbf{H}^m$ whose intersection with some conic neighborhood of the set $\{p_{z_j}=0\}$ is reduced to 0 (for example in small conic neighborhoods of $\Sigma_{\mathcal{J}}$ for $\mathcal{J}$ containing j).

The operator Ω_j, seen as an operator on the j-th copy of $\mathbf{H}$, is an harmonic oscillator, having in particular eigenvalues $2n+1$, $n \in \mathbb{N}$ (see [CHT18, Section 3.1]). Moreover, the operators Ω_i (considered this time as operators on $\mathbf{H}^m$) commute with each other and with the operators R_j.

Recall that $\mathcal{P}$ stands for the set of all subsets of $\{1, \ldots, m\}$. We fix $\mathcal{J} \in \mathcal{P}$. In the sequel, we think of $\mathcal{J}$ as the set of j for which $R_j \neq 0$. For $j \in \mathcal{J}$ and $n \in \mathbb{N}$, we denote by $E_n^j \subset L^2(\mathbf{H})$ the eigenspace of Ω_j corresponding to the eigenvalue $2n+1$. For $(n_j) \in \mathbb{N}^{\mathcal{J}}$, we set

$$\mathcal{H}_{(n_j)}^{\mathcal{J}} = F^1 \otimes \ldots \otimes F^m \subset L^2(\mathbf{H}^m)$$

where $F^j = E_{n_j}^j$ for $j \in \mathcal{J}$ and $F^j = L^2(\mathbf{H})$ otherwise.

We have the orthogonal decomposition

$$L^2(\mathbf{H}^m) = \bigoplus_{\mathcal{J} \in \mathcal{P}} \bigoplus_{(n_j) \in \mathbb{N}^{\mathcal{J}}} \mathcal{H}_{(n_j)}^{\mathcal{J}}. \tag{7.8}$$

We can also write the associated decomposition of $-\Delta$:

$$\begin{aligned} -\Delta &= \bigoplus_{\mathcal{J} \in \mathcal{P}} \bigoplus_{(n_j) \in \mathbb{N}^{\mathcal{J}}} H_{(n_j)}^{\mathcal{J}} \\ \text{with} \quad H_{(n_j)}^{\mathcal{J}} &= \sum_{j \in \mathcal{J}} (2n_j + 1)\, R_j - \sum_{i \notin \mathcal{J}} (\partial_{x_i}^2 + \partial_{y_i}^2). \end{aligned}$$

From this, we deduce

$$\begin{aligned} \mathrm{sp}(-\Delta) &= \bigcup_{\mathcal{J} \in \mathcal{P}} \bigcup_{(n_j) \in \mathbb{N}^{\mathcal{J}}} \mathrm{sp}(H_{(n_j)}^{\mathcal{J}}) \\ &= \Bigg\{ \sum_{j \in \mathcal{J}} (2n_j + 1)\, |\alpha_j| + 2\pi \sum_{i \notin \mathcal{J}} (k_i^2 + \ell_i^2), \\ &\qquad\qquad \text{with } k_i, \ell_i \in \mathbb{Z},\ \mathcal{J} \in \mathcal{P},\ n_j \in \mathbb{N},\ \alpha_j \in (\mathbb{Z} \setminus \{0\}) \Bigg\} \end{aligned}$$

where sp denotes the spectrum.

Remark 7.6. The particularly rich structure of the Quantum Limits of the sub-Laplacian $-\Delta$ described in Theorem 9 is due to the high degeneracy of this spectrum. To make an analogy with the Riemannian case, the QLs of the usual flat Riemannian torus $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$ have a rich structure (see [Jak97]), whereas the QLs of irrational Riemannian tori are much simpler to describe.

7.2.3 Step 2: End of the proof of Point (2) of Theorem 9

In the sequel, the notation $(\cdot, \cdot)$ stands for the $L^2(\mathbf{H}^m)$ scalar product, and the associated norm is denoted by $\| \cdot \|_{L^2}$.

Positive cones. We set $V = \left(-\frac{1}{2}, \ldots, -\frac{1}{2}\right) \in \mathbb{R}^J$ and we consider the quadrant

$$V + \mathbb{R}_+^J = \left\{ (x_1, \ldots, x_J) \in \mathbb{R}^J \mid x_j \geqslant -\frac{1}{2} \text{ for any } 1 \leqslant j \leqslant J \right\}.$$

We now define a series of partitions of $V + \mathbb{R}_+^J$ into positive cones with vertex at V, each of these partitions (indexed by N) being composed of 2^N thin positive cones, with the property that each partition is a refinement of the preceding one.

More precisely, these positive cones $C_\ell^N \subset V + \mathbb{R}_+^J$, for $N \in \mathbb{N}^*$ and $0 \leqslant \ell \leqslant 2^N - 1$, satisfy the following properties, some of which are illustrated on Figure 7.1 below:

(1) For any $N \in \mathbb{N}^*$ and any $0 \leqslant \ell \leqslant 2^N - 1$, C_ℓ^N is a positive cone with vertex at V, i.e.,

$$V + \lambda(W - V) \in C_\ell^N, \quad \forall \lambda > 0,\ \forall W \in C_\ell^N;$$

(2) For any $N \in \mathbb{N}^*$, $(C_\ell^N)_{0 \leqslant \ell \leqslant 2^N - 1}$ is a partition of $V + \mathbb{R}_+^J$, i.e.,

$$\bigcup_{\ell=0}^{2^N-1} C_\ell^N = V + \mathbb{R}_+^J \quad \text{and} \quad C_\ell^N \cap C_{\ell'}^N = \emptyset,\ \forall \ell \neq \ell';$$

(3) Each partition is a refinement of the preceding one: for any $N \geqslant 2$ and any $0 \leqslant \ell \leqslant 2^N - 1$, there exists a unique $0 \leqslant \ell' \leqslant 2^{N-1} - 1$ such that $C_\ell^N \subset C_{\ell'}^{N-1}$.

Denote by $\mathscr{L}$ the set of half-lines issued from V and contained in $V + \mathbb{R}_+^J$. Note that $\mathscr{L}$ is parametrized by $s \in \mathbf{S}_{\mathcal{J}}$. We also assume the following property:

(4) For any $L \in \mathscr{L}$ parametrized by $s \in \mathbf{S}_{\mathcal{J}}$, there exists a subsequence $(C_{\ell(s,N)}^N)_{N \in \mathbb{N}^*}$ which converges to $\mathscr{L}$, in the following sense. There exists $d : \mathbb{N} \to \mathbb{R}^+$ with $d \to 0$ as $N \to +\infty$, such that, for any $s' \in \mathbf{S}_{\mathcal{J}}$ parametrizing a half-line $L' \in \mathscr{L}$ contained in $\mathbf{S}_{\ell(s,N)}^N$, we have

$$\|s' - s\|_1 \leqslant d(N). \tag{7.9}$$

This last property is equivalent to saying that the size of the positive cones tends uniformly to 0 as $N \to +\infty$.

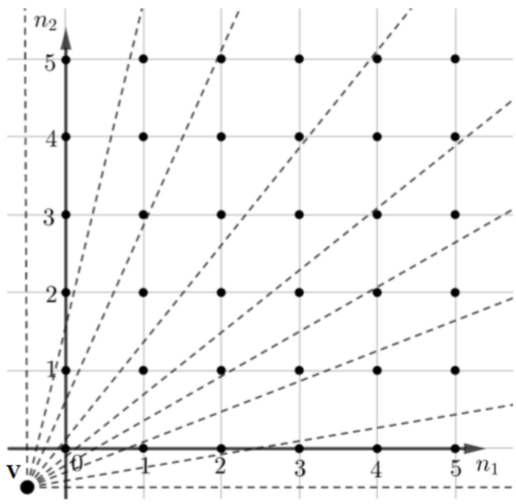

Figure 7.1: The positive cones C_ℓ^N, for $J = 2$, $N = 3$.

Remark 7.7. The positive cones C_ℓ^N can be seen as positive sub-cones of the Heisenberg fan (whose definition is recalled in Section 7.4).

Spectral decomposition. Decomposing φ_k on the spaces $\mathcal{H}^{\mathcal{J}}_{(n_j)}$ defined in Section 7.2.2, we write

$$\varphi_k = \sum_{\ell=0}^{2^N-1} \varphi^N_{k,\ell} \tag{7.10}$$

where

$$\varphi^N_{k,\ell} = \sum_{(n_1,\dots,n_J)\in C^N_\ell} \varphi_{k,n_1,\dots,n_J}$$

and, for any $(n_j) \in \mathbb{N}^{\mathcal{J}}$, $k \in \mathbb{N}^*$ and $j \in \mathcal{J}$,

$$\Omega_j \varphi_{k,n_1,\dots,n_J} = (2n_j+1)\varphi_{k,n_1,\dots,n_J}.$$

For any $N \in \mathbb{N}^*$ and any $0 \leqslant \ell \leqslant 2^N-1$, we take ν^N_ℓ to be a microlocal defect measure of the sequence $(\varphi^N_{k,\ell})_{k\in\mathbb{N}^*}$. By diagonal extraction in $k \in \mathbb{N}^*$ (which we omit in the notations), we can assume that any of these microlocal defect measures is obtained with respect to the same subsequence.

Lemma 7.8. *The following properties hold:*

(1) All the mass of ν^N_ℓ is contained in $S\Sigma_{\mathcal{J}}$ for any $N \in \mathbb{N}^*$ and any $0 \leqslant \ell \leqslant 2^N-1$;

(2) For $N \in \mathbb{N}^*$ and $\ell \neq \ell'$ with $0 \leqslant \ell, \ell' \leqslant 2^N-1$, the joint microlocal defect measure (see Definition 1.30) of $(\varphi^N_{k,\ell})_{k\in\mathbb{N}^*}$ and $(\varphi^N_{k,\ell'})_{k\in\mathbb{N}^*}$ vanishes. In particular, for any $N \in \mathbb{N}^*$,

$$\nu = \sum_{\ell=0}^{2^N-1} \nu^N_\ell. \tag{7.11}$$

Proof. The proof mainly relies on averaging techniques (see also Section 7-A.3 for a result obtained by these techniques in the much simpler context of flat contact sub-Laplacians).

We first prove Point (1). Using (7.5), (7.10) and the fact that $P^{\mathcal{J}}_n \in \Psi^0(\mathbf{H}^m)$ commutes with the operators Ω_j and R_j, we get that

$$\varphi^N_{k,\ell} = P^{\mathcal{J}}_{r(k)} \varphi^N_{k,\ell}.$$

Point (1) now follows from the fact that $\sigma_P(P^{\mathcal{J}}_{r(k)}) \to \mathbf{1}_{\Sigma_{\mathcal{J}}}$ as $k \to +\infty$ (see the proof of Theorem 8).

We now turn to the proof of Point (2). Let N, ℓ, ℓ' be as in the statement. By Point (1) and Lemma 7.1, we know that the joint microlocal defect measure of $(\varphi^N_{k,\ell})_{k\in\mathbb{N}^*}$ and $(\varphi^N_{k,\ell'})_{k\in\mathbb{N}^*}$ has no mass outside $S\Sigma_{\mathcal{J}}$.

Let $b \in \mathscr{S}^0(\mathbf{H}^m)$ which is microlocally supported in a conic set in which R_j, Ω_j act as first-order pseudodifferential operators for any $j \in \mathcal{J}$. A typical example of microlocal support for b is given by any conic subset of $T^*\mathbf{H}^m$ whose intersection with some conic neighborhood of the set $\{p_{z_j} = 0\}$ is reduced to 0, for any $j \in \mathcal{J}$. We set $U(t) = U(t_1,\dots,t_J) = e^{i(t_1\Omega_1+\dots+t_J\Omega_J)}$ for $t = (t_1,\dots,t_J) \in (\mathbb{R}/2\pi\mathbb{Z})^J$.

The average of $\mathrm{Op}(b)$ is then defined by

$$A = \int_{(\mathbb{R}/2\pi\mathbb{Z})^J} U(-t)\mathrm{Op}(b)U(t)dt$$

(see [Wei77]). For $1 \leqslant j \leqslant J$, since

$$\frac{d}{dt_j} U(-t)\mathrm{Op}(b)U(t) = U(-t)[\mathrm{Op}(b), \Omega_j]U(t),$$

integrating in the t_j variable, using that all Ω_i commute together, and that $\exp(2i\pi\Omega_j) = \mathrm{Id}$ (since the eigenvalues of Ω_j belong to $\mathbb{N}$), we get that $[A, \Omega_j] = 0$ for any $1 \leqslant j \leqslant J$.

By a bracket computation, A has principal symbol

$$a := \sigma_P(A) = \int_{(\mathbb{R}/2\pi\mathbb{Z})^J} b \circ \theta_1(t_1) \circ \ldots \circ \theta_J(t_J)\ dt.$$

Here, $\theta_j(\cdot)$ denotes, for $1 \leqslant j \leqslant J$, the 2π-periodic flow of the Hamiltonian vector field of $\sigma_P(\Omega_j)$ (see [CHT18, Lemma 6.1] for similar arguments).

Remark 7.9. If D is a 0^{th}-order pseudodifferential operator on $\mathbf{H}^m$ which satisfies $[D, \Omega_j] = 0$ for any $j \in \mathcal{J}$, then D leaves $\mathcal{H}^{\mathcal{J}}_{(n_j)}$ invariant for any $(n_j) = (n_1, \ldots, n_J) \in \mathbb{N}$. It follows that for any $f \in \mathcal{H}^{\mathcal{J}}_{(n_j)}$ and any $g \in \mathcal{H}^{\mathcal{J}}_{(n'_j)}$ such that $(n_1, \ldots, n_J) \neq (n'_1, \ldots, n'_J)$, we have $(Df, g) = 0$.

We know that $\sigma_P(A) = b$ on $S\Sigma_{\mathcal{J}}$. Therefore,

$$(\mathrm{Op}(b)\varphi^N_{k,\ell}, \varphi^N_{k,\ell'}) - (A\varphi^N_{k,\ell}, \varphi^N_{k,\ell'}) \underset{k\to+\infty}{\longrightarrow} 0.$$

Since A commutes with Ω_j for any $1 \leqslant j \leqslant J$, by Remark 7.9, we know that $(A\varphi^N_{k,\ell}, \varphi^N_{k,\ell'}) = 0$. Hence, $(\mathrm{Op}(b)\varphi^N_{k,\ell}, \varphi^N_{k,\ell'})$ tends to 0 as $k \to +\infty$. Using this result for all possible b with microlocal support satisfying the property recalled at the beginning of the proof, we obtain that the joint microlocal defect measure of $(\varphi^N_{k,\ell})_{k\in\mathbb{N}^*}$ and of $(\varphi^N_{k,\ell'})_{k\in\mathbb{N}^*}$ vanishes. Evaluating $(\mathrm{Op}(b)\varphi_k, \varphi_k)$ in the limit $k \to +\infty$ and using (7.10), we conclude the proof of Point (2). $\square$

Approximate invariance. We fix $N \in \mathbb{N}^*$ and $0 \leqslant \ell \leqslant 2^N - 1$ and we consider $s \in \mathbf{S}_{\mathcal{J}}$ such that the half-line issued from V and defined by the J equations $\frac{2x_j+1}{2x_1+1+\ldots+2x_J+1} = s_j$ (and $x_j \geqslant -1/2$) lies in C^N_ℓ.

Let A be a 0-th order pseudodifferential operator microlocally supported in a conic set where R_j, Ω_j act as first-order pseudodifferential operators for any $j \in \mathcal{J}$. Assume moreover that A commutes with $\Omega_1, \ldots, \Omega_J$ and with $\partial_{x_j}, \partial_{y_j}$ and ∂_{z_j} for any $J+1 \leqslant j \leqslant m$. Recall that R_s was defined in (1.43). Using that $[A, R_s]$ commutes with $\Omega_1, \ldots, \Omega_J$ in order to kill crossed terms (see Remark 7.9), we have

$$\begin{aligned} ([A, R_s]\varphi^N_{k,\ell}, \varphi^N_{k,\ell}) &= ([A, R_s] \sum_{(n_1,\ldots,n_J)\in C^N_\ell} \varphi_{k,n_1,\ldots,n_J}, \sum_{(n_1,\ldots,n_J)\in C^N_\ell} \varphi_{k,n_1,\ldots,n_J}) \\ &= \sum_{(n_1,\ldots,n_J)\in C^N_\ell} ([A, R_s]\varphi_{k,n_1,\ldots,n_J}, \varphi_{k,n_1,\ldots,n_J}) \end{aligned} \tag{7.12}$$

Let us fix $(n_1, \ldots, n_J) \in C^N_\ell$ and prove that

$$\begin{aligned} &([A, R_s]\varphi_{k,n_1,\ldots,n_J}, \varphi_{k,n_1,\ldots,n_J}) \\ &\quad = \sum_{j=1}^J \left(s_j - \frac{2n_j + 1}{\sum_{i=1}^J 2n_i + 1} \right) ([A, R_j]\varphi_{k,n_1,\ldots,n_J}, \varphi_{k,n_1,\ldots,n_J}) \end{aligned} \tag{7.13}$$

We set

$$R = \frac{\sum_{j=1}^{J}(2n_j+1)R_j - \sum_{i=J+1}^{m}\Delta_i}{\sum_{j=1}^{J}2n_j+1}.$$

and, for the sake of simplicity of notations, $\varphi = \varphi_{k,n_1,\ldots,n_J}$. Using that R is selfadjoint (since R_j is selfadjoint for any j) and that φ is an eigenfunction of R, we get

$$([A,R]\varphi,\varphi) = (AR\varphi,\varphi) - (A\varphi,R\varphi) = 0$$

and therefore, since A commutes with $\Delta_{J+1},\ldots,\Delta_m$, we get

$$([A,R_s]\varphi,\varphi) = ([A,R_s-R]\varphi,\varphi) = \sum_{j=1}^{J}\left(s_j - \frac{2n_j+1}{\sum_{i=1}^{J}2n_i+1}\right)([A,R_j]\varphi,\varphi)$$

which is exactly (7.13).

Thanks to our choice of microlocal support for A, we know that $[A,R_j] \in \Psi^0(\mathbf{H}^m)$ for $1 \leqslant j \leqslant J$. Combining (7.12) and (7.13), we obtain

$$\begin{aligned}\left|([A,R_s]\varphi_{k,\ell}^N,\varphi_{k,\ell}^N)\right| &\leqslant C\sum_{(n_1,\ldots,n_J)\in C_\ell^N}\sum_{j=1}^{J}\left|s_j - \frac{2n_j+1}{\sum_{i=1}^{J}2n_i+1}\right|\|\varphi_{k,n_1,\ldots,n_J}\|_{L^2}^2 \\ &\leqslant Cd(N)\|\varphi_{k,\ell}^N\|_{L^2}^2\end{aligned} \tag{7.14}$$

where in the last line, we used (7.9) and the fact that the decomposition (7.8) is orthogonal.

In order to pass to the limit $k \to +\infty$ in these last inequalities, we note that

$$\sigma_P([A,R_s])_{|\Sigma_{\mathcal{J}}} = \{a_{|\Sigma_{\mathcal{J}}},\rho_s\}_{\omega_{|\Sigma_{\mathcal{J}}}} \tag{7.15}$$

(see [CHT18, Lemma 6.2] for a similar identity). Here, the Poisson bracket $\{\cdot,\cdot\}_{\omega_{|\Sigma_{\mathcal{J}}}}$ is the Poisson bracket on the manifold $(\Sigma_{\mathcal{J}},\omega_{|\Sigma_{\mathcal{J}}})$ which is symplectic as it is defined as a product of symplectic manifolds (recall that for $m=1$, the 4-dimensional manifold Σ is symplectic, see for example [CHT18]).

Since all the mass of ν_ℓ^N is contained in $S\Sigma_{\mathcal{J}}$ by Lemma 7.8, we finally deduce from (7.14) the upper bound

$$\left|\int_{S\Sigma_{\mathcal{J}}}\{a_{|\Sigma_{\mathcal{J}}},\rho_s\}_{\omega_{|\Sigma_{\mathcal{J}}}}\, d\nu_\ell^N\right| \leqslant Cd(N)\nu_\ell^N(S\Sigma_{\mathcal{J}}). \tag{7.16}$$

The upper bound (7.16) has been established only for $a_{|\Sigma_{\mathcal{J}}}$ the restriction to $\Sigma_{\mathcal{J}}$ of the symbol of an operator A of order 0 *which commutes with* $\Omega_1,\ldots,\Omega_J$ *and* $\partial_{x_j},\partial_{y_j}$ *and* ∂_{z_j} *for any* $J+1 \leqslant j \leqslant m$, and we would like to remove this commutation assumption. Let $b \in \mathscr{S}^0(\mathbf{H})$ of the form

$$b(q,p) = b_{\mathcal{J}}(q_1,\ldots,q_J,p_1,\ldots,p_J)$$

where (q,p) denote the coordinates in $T^*\mathbf{H}^m$, (q_j,p_j) the coordinates in the cotangent bundle of the j-th copy of $\mathbf{H}$, and $b_{\mathcal{J}} \in \mathscr{S}^0(\mathbf{H}^{\mathcal{J}})$ is an arbitrary 0-th order symbol supported in a subset of $T^*\mathbf{H}^{\mathcal{J}}$ where R_j,Ω_j act as first-order pseudodifferential operators for any $j \in \mathcal{J}$. We consider the operator

$$A = \int_{(\mathbb{R}/2\pi\mathbb{Z})^J} U(-t)\mathrm{Op}(b)U(t)dt \ \in \Psi^0(\mathbf{H}^m)$$

where $U(t) = U(t_1,\ldots,t_J) = e^{i(t_1\Omega_1+\ldots+t_J\Omega_J)}$ for $t=(t_1,\ldots,t_J) \in (\mathbb{R}/2\pi\mathbb{Z})^J$. By an argument that we have already in the proof of Point (2) of Lemma (7.8), A commutes with Ω_j for any

$1 \leqslant j \leqslant J$, and it also commutes with $\partial_{x_j}, \partial_{y_j}$ and ∂_{z_j} for any $J+1 \leqslant j \leqslant m$. Moreover, the principal symbol of A on $S\Sigma_{\mathcal{J}}$ coincides with $b_{\mathcal{J}}$ by the Egorov theorem. Using (7.16) for A, this proves that (7.16) is valid for any symbol a of order 0 on $\mathbf{H}^m$ supported far from the sets $\{p_{z_j} = 0\}$ for $j \in \mathcal{J}$, without any assumption of commutation on A.

Disintegration of measures. From the equality (7.11) taken in the limit $N \to +\infty$, we will deduce that $\nu^{\mathcal{J}} = \int_{\mathbf{S}_{\mathcal{J}}} \nu_s^{\mathcal{J}} dQ^{\mathcal{J}}(s)$. Note that a simple Fubini argument does not suffice since $Q^{\mathcal{J}}$ is not the Lebesgue measure in general (it may contain Dirac masses, see Section 7.2.1). Instead, we have to adapt the proof of the classical disintegration of measure theorem (see [Roh62]).

First of all, we define a measure $Q^{\mathcal{J}}$ over $\mathbf{S}_{\mathcal{J}}$ as follows. It was explained at the beginning of Section 7.2.3 that the set $\mathscr{L}$ of half-lines issued from V and contained in $V+\mathbb{R}_+^J$ is parametrized by $s \in \mathbf{S}_{\mathcal{J}}$. For $N \in \mathbb{N}^*$ and $0 \leqslant \ell \leqslant 2^N - 1$, we consider the subset of $\mathbf{S}_{\mathcal{J}}$ given by

$$\mathbf{S}_\ell^N = \left\{ s \in \mathbf{S}_{\mathcal{J}},\ s \text{ parametrizes a half-line of } \mathscr{L} \text{ contained in } C_\ell^N \right\}. \tag{7.17}$$

Then we define

$$Q^{\mathcal{J}}(\mathbf{S}_\ell^N) = \nu_\ell^N(S\Sigma). \tag{7.18}$$

This definition is consistent thanks to the partition of $V+\mathbb{R}_+^J$ into nested positive cones: $Q^{\mathcal{J}}$ is well-defined on any $\mathbf{S}_\ell^N$ and it is also additive. By the properties of the positive cones C_ℓ^N, for any $s \in \mathbf{S}_{\mathcal{J}}$, there exists a sequence $(\ell(s,N))_{N\in\mathbb{N}^*}$ such that $\mathbf{S}_{\ell(s,N)}^N \subset \mathbf{S}_{\mathcal{J}}$ converges to s, in the sense that any sequence $(s^N)_{N\in\mathbb{N}^*}$ such that $s^N \in \mathbf{S}_{\ell(s,N)}^N$ for any $N \in \mathbb{N}^*$ converges to s as $N \to +\infty$. Therefore, by extension, (7.18) defines a (unique) non-negative Radon measure $Q^{\mathcal{J}}$ on $\mathbf{S}_{\mathcal{J}}$.

Given $N \geqslant 1$, $0 \leqslant \ell \leqslant 2^N - 1$ and a continuous function $f : S\Sigma_{\mathcal{J}} \to \mathbb{R}$, we set

$$f_\ell^N = \frac{1}{\nu_\ell^N(S\Sigma_{\mathcal{J}})} \int_{S\Sigma_{\mathcal{J}}} f d\nu_\ell^N \tag{7.19}$$

if $\nu_\ell^N(S\Sigma_{\mathcal{J}}) \neq 0$, and $f_\ell^N = 0$ otherwise.

Proposition 7.10. *Given any continuous function $f : S\Sigma \to \mathbb{R}$, for $Q^{\mathcal{J}}$-almost all $s \in \mathbf{S}_{\mathcal{J}}$, there exists a real number $e(f)(s)$ such that*

$$f_{\ell(s,N)}^N \underset{N\to+\infty}{\longrightarrow} e(f)(s),$$

where, for any $N \in \mathbb{N}^*$, $\ell(s,N)$ is the unique integer $0 \leqslant \ell(s,N) \leqslant 2^N - 1$ such that $s \in \mathbf{S}_{\ell(s,N)}^N$. In the sequel, we call $\ell(s,N)$ the approximation at order N of s.

Proof. By linearity of formula (7.19), it is sufficient to prove the statement for $f \geqslant 0$. Therefore, in the sequel, we fix $f \geqslant 0$. For $N \geqslant 1$, we define the function $f^N : \mathbf{S}_{\mathcal{J}} \to \mathbb{R}$ by $f^N(s) = f_{\ell(s,N)}^N$, where $\ell(s,N)$ is the approximation at order N of s. Note that f^N is constant on $\mathbf{S}_\ell^N$ for $0 \leqslant \ell \leqslant 2^N - 1$.

For $0 \leqslant \alpha < \beta \leqslant 1$, we define $S(\alpha,\beta)$ as the set of $s \in \mathbf{S}_{\mathcal{J}}$ such that

$$\liminf_{N\to+\infty} f^N(s) < \alpha < \beta < \limsup_{N\to+\infty} f^N(s).$$

To prove Proposition 7.10, it is sufficient to prove that $S(\alpha,\beta)$ has $Q^{\mathcal{J}}$-measure 0 for any $0 \leqslant \alpha < \beta \leqslant 1$. Fix such α, β. For $s \in S(\alpha,\beta)$, take a sequence $1 \leqslant N_1^\alpha(s) < N_1^\beta(s) < N_2^\alpha(s) <$

$N_2^\beta(s) < ... < N_k^\alpha(s) < N_k^\beta(s) < ...$ of integers such that $f^{N_k^\alpha(s)}(s) < \alpha$ and $f^{N_k^\beta(s)}(s) > \beta$ for any $k \geqslant 1$. We finally define the following sets:

$$A_k = \bigcup_{s \in S(\alpha,\beta)} \mathbf{S}_{\ell(s,N_k^\alpha(s))}^{N_k^\alpha(s)}$$

$$B_k = \bigcup_{s \in S(\alpha,\beta)} \mathbf{S}_{\ell(s,N_k^\beta(s))}^{N_k^\beta(s)}$$

We have $S(\alpha,\beta) \subset A_{k+1} \subset B_k \subset A_k$ for every $k \geqslant 1$. In particular,

$$S(\alpha,\beta) \subset \widetilde{S}(\alpha,\beta) := \bigcap_{k \in \mathbb{N}^*} A_k = \bigcap_{k \in \mathbb{N}^*} B_k. \tag{7.20}$$

Given any two of the sets $\mathbf{S}_{\ell(s,N_k^\alpha(s))}^{N_k^\alpha(s)}$ that form A_k, either they are disjoint or one is contained in the other. Consequently, A_k may be written as a disjoint union of such sets, denoted by $A_k^{k'}$. Therefore,

$$\int_{A_k} f dQ^{\mathcal{J}} = \sum_{k'} \int_{A_k^{k'}} f dQ^{\mathcal{J}} < \sum_{k'} \alpha Q^{\mathcal{J}}(A_k^{k'}) = \alpha Q^{\mathcal{J}}(A_k)$$

and analogously, with similar notations,

$$\int_{B_k} f dQ^{\mathcal{J}} = \sum_{k'} \int_{B_k^{k'}} f dQ^{\mathcal{J}} > \sum_{k'} \beta Q^{\mathcal{J}}(B_k^{k'}) = \beta Q^{\mathcal{J}}(B_k).$$

Since $B_k \subset A_k$, we get $\alpha Q^{\mathcal{J}}(A_k) > \beta Q^{\mathcal{J}}(B_k)$. Taking the limit $k \to +\infty$, it yields $\alpha Q^{\mathcal{J}}(\widetilde{S}(\alpha,\beta)) > \beta Q^{\mathcal{J}}(\widetilde{S}(\alpha,\beta))$, which is possible only if $Q^{\mathcal{J}}(\widetilde{S}) = 0$. Therefore, using (7.20), we get $Q^{\mathcal{J}}(S) = 0$, which concludes the proof of the proposition. □

From (7.11) and (7.19), we infer that for any $N \geqslant 1$,

$$\int_{S\Sigma_{\mathcal{J}}} f d\nu^{\mathcal{J}} = \sum_{\ell=0}^{2^N-1} \int_{S\Sigma_{\mathcal{J}}} f d\nu_\ell^N = \sum_{\ell=0}^{2^N-1} f_\ell^N \nu_\ell^N(S\Sigma_{\mathcal{J}}),$$

and the dominated convergence theorem together with the definition of $Q^{\mathcal{J}}$ and Proposition 7.10 yield

$$\int_{S\Sigma_{\mathcal{J}}} f d\nu^{\mathcal{J}} = \int_{\mathbf{S}_{\mathcal{J}}} e(f)(s) dQ^{\mathcal{J}}(s). \tag{7.21}$$

We see that for a fixed $s \in \mathbf{S}_{\mathcal{J}}$,

$$C^0(S\Sigma_{\mathcal{J}}, \mathbb{R}) \ni f \mapsto e(f)(s) \in \mathbb{R}$$

is a non-negative linear functional on $C^0(S\Sigma_{\mathcal{J}}, \mathbb{R})$. By the Riesz-Markov theorem, there exists a unique Radon probability measure $\nu_s^{\mathcal{J}}$ on $S\Sigma_{\mathcal{J}}$ such that

$$e(f)(s) = \int_{S\Sigma_{\mathcal{J}}} f d\nu_s^{\mathcal{J}}. \tag{7.22}$$

Putting (7.21) and (7.22) together, we get

$$\int_{S\Sigma_{\mathcal{J}}} f d\nu^{\mathcal{J}} = \int_{\mathbf{S}_{\mathcal{J}}} \left(\int_{S\Sigma_{\mathcal{J}}} f d\nu_s^{\mathcal{J}} \right) dQ^{\mathcal{J}}(s)$$

which is the desired disintegration of measures formula.

Conclusion of the proof. There remains to show that $\nu_s^{\mathcal{J}}$ is invariant by $\vec{\rho}_s^{\mathcal{J}}$. Let $a \in \mathscr{S}^0(\mathbf{H}^m)$ be supported in cone of $T^*\mathbf{H}^m$ whose intersection with some conic neighborhood of the set $\{p_{z_j} = 0\}$ is reduced to 0, for any $j \in \mathcal{J}$. For $Q^{\mathcal{J}}$-almost every $s \in \mathbf{S}_{\mathcal{J}}$, we have

$$\begin{aligned}\int_{S\Sigma_{\mathcal{J}}} \{a, \rho_s^{\mathcal{J}}\} d\nu_s^{\mathcal{J}} &= e(\{a, \rho_s^{\mathcal{J}}\})(s) \qquad \text{(by (7.22))} \\ &= \lim_{N \to +\infty} \frac{1}{\nu_{\ell(s,N)}^N(S\Sigma_{\mathcal{J}})} \int_{S\Sigma_{\mathcal{J}}} \{a, \rho_s^{\mathcal{J}}\} d\nu_{\ell(s,N)}^N \\ &\leqslant \lim_{N \to +\infty} Cd(N) \qquad \text{(by (7.16))} \\ &= 0\end{aligned} \tag{7.23}$$

with the convention that if the denominator in (7.23) is null, then the whole expression is null. For an arbitrary $a \in \mathscr{S}^0(\mathbf{H}^m)$, taking a sequence $a_n \in \mathscr{S}^0(\mathbf{H}^m)$ whose support has the above property and such that $a_n \to a$ in $S\Sigma_{\mathcal{J}}$ (in the space of symbols) as $n \to +\infty$, we see that the above quantity also vanishes since $\nu_s^{\mathcal{J}}$ has finite mass and $\{a_n, \rho_s^{\mathcal{J}}\} \to \{a, \rho_s^{\mathcal{J}}\}$ in $S\Sigma_{\mathcal{J}}$ as $n \to +\infty$. This implies that $\nu_s^{\mathcal{J}}$ is invariant by the flow $e^{t\vec{\rho}_s^{\mathcal{J}}}$, which concludes the proof of Theorem 9.

Remark 7.11. Contrarily to those of flat tori (see [Jak97]), the Quantum Limits of $\mathbf{H}^m$ (or, more precisely, their pushforward under the canonical projection onto $\mathbf{H}^m$) are not necessarily absolutely continuous. It was already remarked in the case $m = 1$ in [CHT18, Proposition 3.2(2)].

7.3 Proof of Theorem 10

In this section, we prove Theorem 10. The four steps are the following:

1. In Lemma 7.13 and Lemma 7.14, we prove the result for a fixed $\mathcal{J} \in \mathcal{P} \setminus \{\emptyset\}$, $Q^{\mathcal{J}}$ the Dirac mass at some $s \in \mathbf{S}_{\mathcal{J}}$, and $\nu_s^{\mathcal{J}} \in \mathscr{P}(S^*\mathbf{H}^m)$

   (i) has no mass outside $S\Sigma_{\mathcal{J}}$,

   (ii) is invariant under the flow of $\vec{\rho}_s^{\mathcal{J}}$,

   (iii) and *is in a simple tensor form* that we make precise below.

   In other words, if $\nu_\infty = \nu_s^{\mathcal{J}}$ with $\nu_s^{\mathcal{J}}$ satisfying (i), (ii) and (iii), then it is a QL.

2. In Lemma 7.16, we extend the result of Step 1 to the case where (iii) is not necessarily satisfied, i.e., $\nu_\infty = \nu_s^{\mathcal{J}}$ satisfies only (i) and (ii).

3. In Lemma 7.18, we extend the result of Steps 1 and 2 to the case where $\nu_\infty \in \mathscr{P}_{S\Sigma}$ has no mass outside $S\Sigma_{\mathcal{J}}$ for some $\mathcal{J} \in \mathcal{P} \setminus \{\emptyset\}$, i.e., $\nu_\infty = \nu^{\mathcal{J}}$.

4. Finally, using the previous result for all $\mathcal{J} \in \mathcal{P} \setminus \{\emptyset\}$, we prove Theorem 10 in full generality (i.e., for arbitrary $\nu_\infty \in \mathscr{P}_{S\Sigma}$).

The specific algebraic structure of $\mathrm{sp}(-\Delta)$ plays a key role at each of these four steps. Note that similar roadmaps have been followed in different but related contexts, see [JZ96] and [Stu19].

The map $\Sigma \to \mathbf{H}^m \times \mathbb{R}^m$, $(q, p) \mapsto (q, p_{z_1}, \ldots, p_{z_m})$ is an isomorphism, and thus, in the sequel, we consider the coordinates $(q, p_{z_1}, \ldots, p_{z_m})$ on Σ and the coordinates $(q, p_{z_1} : \cdots : p_{z_m})$ on $S\Sigma$, where the notation $p_{z_1} : \cdots : p_{z_m}$ stands for homogeneous coordinates.

Let us summarize the proof, which uses in a key way the precise description of the spectrum of $-\Delta$ (see Section 7.2.2) and the knowledge of the flows of the Hamiltonian vector fields $\vec{\rho}_s^{\mathcal{J}}$.

Remark 7.12. Projecting the flow of $\vec{\rho}_s^{\mathcal{J}}$ on M, we obtain straight lines described by changes proportional to s_j in the z_j coordinates, for $j \in \mathcal{J}$. Once all coordinates x_i, y_i (for $1 \leqslant i \leqslant m$) and z_i (for $i \notin \mathcal{J}$) have been fixed - since they are preserved by the flow -, these straight lines are similar to the lines given by the geodesic flow on the flat $|\mathcal{J}|$-dimensional Riemannian torus in the variables z_j (for $j \in \mathcal{J}$).

We fix $\mathcal{J} \in \mathcal{P} \setminus \{\emptyset\}$. Since any two of the operators R_j and $\Omega_{j'}$ for $j, j' \in \mathcal{J}$ commute, the orthogonal decomposition (7.8) can be refined: more precisely, given $(n_j) \in \mathbb{N}^{\mathcal{J}}$ and $(\alpha_j) \in (\mathbb{Z} \setminus \{0\})^{\mathcal{J}}$, we consider the joint eigenspace $\mathcal{H}^{\mathcal{J}}_{(n_j),(\alpha_j)} \subset L^2(\mathbf{H}^m)$ on which the operator $\frac{1}{i}\partial_{z_j}$ acts as α_j and Ω_j acts as $2n_j + 1$.

ν_∞ is obtained as a QL of a sequence of normalized eigenfunctions $(\varphi_k)_{k\in\mathbb{N}^*}$ which is described through its components in these eigenspaces. Moreover, each of the four steps is achieved by taking linear combinations of eigenfunctions (with same eigenvalues) used in the previous step. Therefore, the number of eigenspaces $\mathcal{H}^{\mathcal{J}}_{(n_j),(\alpha_j)}$ used for building $(\varphi_k)_{k\in\mathbb{N}^*}$ increases at each step.

In order to achieve Step 1, we focus on the eigenspaces $\mathcal{H}^{\mathcal{J}}_{(n_j),(\alpha_j)}$ corrreponding to

$$\frac{2n_j+1}{\sum_{i\in\mathcal{J}}(2n_i+1)} \approx s_j \quad \text{and} \quad \frac{\alpha_j}{\alpha_{j'}} \approx \frac{p_{z_j}}{p_{z_{j'}}}$$

for any $j, j' \in \mathcal{J}$.

For Step 2, we add the results of the previous step for different $p \in S\Sigma_{\mathcal{J}}$, and we take care that each term in the sum corresponds to the same value of $-\Delta$. Hence, $(n_j) \in \mathbb{N}^{\mathcal{J}}$ is the same as in Step 1, but we use various $(\alpha_j) \in (\mathbb{Z} \setminus \{0\})^{\mathcal{J}}$ to reach all p.

For Step 3, we add the results of Step 2 for different $s \in \mathbf{S}_{\mathcal{J}}$. Therefore, we use the eigenspaces $\mathcal{H}^{\mathcal{J}}_{(n_j),(\alpha_j)}$ also for different $(n_j) \in \mathbb{N}^{\mathcal{J}}$. Finally, in Step 4, we sum the sequences obtained at Step 3 for $\mathcal{J}$ ranging over $\mathcal{P} \setminus \{0\}$.

In order to describe the measures in a "tensor form" which we consider for Step 1, we need to introduce a few notations.

Notations. For the first three steps, we fix $\mathcal{J} \in \mathcal{P} \setminus \{\emptyset\}$. Any $s \in \mathbf{S}_{\mathcal{J}}$ can be identified to some homogeneous coordinate $p_{z_1} : \cdots : p_{z_m}$ (with $p_{z_i} = 0$ for $i \notin \mathcal{J}$), in a way which does not depend on $q \in \mathbf{H}^m$. Thus, for any $q \in \mathbf{H}^m$, $t \in \mathbb{R}$ and $s \in \mathbf{S}_{\mathcal{J}}$, it makes sense to consider the point $q + ts \in \mathbf{H}^m$, which has the same coordinates x_j and y_j as q for any $1 \leqslant j \leqslant m$ (only the coordinates z_j for $j \in \mathcal{J}$ change).

Let us consider the set

$$M_q^s = \overline{\{q + ts,\ t \in \mathbb{R}\}} \subset \mathbf{H}^m$$

where the bar denotes the closure in $\mathbf{H}^m$. The set M_q^s is a submanifold of $\mathbf{H}^m$ of dimension $d_q^s \leqslant m$, and we denote by $\mathscr{H}_q^s$ the Hausdorff measure of dimension d_q^s on M_q^s.

For any $(q, p) \in S\Sigma$ and any $q' \in \mathbf{H}^m$, it makes sense to consider the point $(q', p) \in S\Sigma$, which is the point in the fiber of $S\Sigma$ over q that has the same homogeneous coordinates $p_{z_1} : \cdots : p_{z_m}$ as p.

Lemma 7.13. *Let $(q, p) \in S\Sigma_{\mathcal{J}}$ and $s \in \mathbf{S}_{\mathcal{J}}$ be such that there exists a J-tuple $(n_j) \in \mathbb{N}^{\mathcal{J}}$ with*

$$s_j = \frac{2n_j + 1}{\sum\limits_{i\in\mathcal{J}}(2n_i+1)} \tag{7.24}$$

for any $j \in \mathcal{J}$. Then, the measure $\mathscr{H}_q^s \otimes \delta_p$ is a Quantum Limit. [The associated sequence of normalized eigenfunctions is specified in the proof, see also Remark 7.15.]

Proof. Since the s_j are pairwise rationally related, the mapping $t \mapsto q+ts$ is periodic and $d_q^s = 1$. Without loss of generality, we assume that $\mathcal{J} = \{1, \ldots, J\}$ for some $1 \leqslant J \leqslant m$.

We construct a sequence of eigenfunctions $(\varphi_k)_{k\in\mathbb{N}^*}$ of $-\Delta$ which admits $\mu_{q,p}^s$ as unique Quantum Limit. In our construction, for any $k \in \mathbb{N}^*$, φ_k belongs to the eigenspace $\mathcal{H}^{\mathcal{J}}_{(n_j),(\alpha_j)}$ for some $(n_j) \in \mathbb{N}^{\mathcal{J}}$ and some $(\alpha_j) \in (\mathbb{Z}\setminus\{0\})^{\mathcal{J}}$, and it does not depend on the variables in the i-th copy of $\mathbf{H}$ for $i \notin J$. Our goal is to choose adequately the J-tuples (n_j) and (α_j). Note that a similar argument for $m = 1$ is done in the proof of Point 2 of Proposition 3.2 in [CHT18].

We fix a sequence of J-tuples $(\alpha_{1,k}, \ldots, \alpha_{J,k}) \in (\mathbb{Z}\setminus\{0\})^J$, for $k \in \mathbb{N}^*$, such that:

- For any $1 \leqslant j \leqslant J$, $\alpha_{j,k} \to +\infty$ as $k \to +\infty$, so that for any $1 \leqslant j, j' \leqslant J$, there holds
$$\frac{n_{j'}}{\alpha_{j,k}} \underset{k\to+\infty}{\longrightarrow} 0; \tag{7.25}$$
- For any $1 \leqslant j, j' \leqslant J$,
$$\frac{\alpha_{j,k}}{\alpha_{j',k}} \underset{k\to+\infty}{\longrightarrow} \frac{p_{z_j}}{p_{z_{j'}}}, \tag{7.26}$$
where $p_{z_1} : \cdots : p_{z_m}$ are the homogeneous coordinates of p in $S\Sigma$.

Now, for any $k \in \mathbb{N}^*$, denoting by $\mathbf{1}$ the constant function equal to 1 (on some copy of $\mathbf{H}$), we define
$$\varphi_k = \Phi_k^1 \otimes \ldots \otimes \Phi_k^J \otimes \underbrace{\mathbf{1} \otimes \ldots \otimes \mathbf{1}}_{m-J \text{ times}}, \tag{7.27}$$
where, for $1 \leqslant j \leqslant J$,
$$\Phi_k^j(x_j, y_j, z_j) = \phi_{j,k}(x_j, y_j) e^{i\alpha_{j,k} z_j}$$
is an eigenfunction of $-\Delta_j$ (on the j-th copy of $\mathbf{H}$) with eigenvalue $(2n_j+1)|\alpha_{j,k}|$. The precise form of $\phi_{j,k}$ will be given below.

Using (7.25) and the proof of Theorem 8, notably the pseudodifferential operators $P_n^{\mathcal{J}}$ introduced in (7.2), we obtain that the mass of any Quantum Limit of $(\varphi_k)_{k\in\mathbb{N}^*}$ is contained in $S\Sigma_{\mathcal{J}}$. Moreover, from the decomposition into cones done in Section 7.2.3 and the equality (7.24), we infer that any Quantum Limit of $(\varphi_k)_{k\in\mathbb{N}^*}$ is invariant under $\vec{\rho}_s^{\,\mathcal{J}}$.

In the next paragraphs, we explain how to choose $\phi_{j,k}$ with eigenvalue $2n_j + 1$ in order to ensure that $(\varphi_k)_{k\in\mathbb{N}^*}$ has a unique QL, which is $\mu_{0,p}^s$. For the sake of simplicity of notations, we set $\alpha = \alpha_{j,k}$. The eigenspace of $-\Delta_j$ corresponding to the eigenvalue $(2n_j+1)|\alpha|$ is of the form $(A_\alpha^*)^{n_j}(\ker(A_\alpha))e^{i\alpha z}$, where $A_\alpha = \partial_{x_j} + i\partial_{y_j} + i\alpha x_j$ locally, and, accordingly, $A_\alpha^* = -\partial_{x_j} + i\partial_{y_j} + i\alpha x_j$ locally (see for example [Col84, Section 2]). This follows from a Fourier expansion in the z_j variable, which gives
$$-\Delta_j = \bigoplus_{\gamma\in\mathbb{Z}} B_\gamma, \qquad \text{where } \ B_\gamma = A_\gamma^* A_\gamma + \gamma \text{ for } \gamma \in \mathbb{Z}.$$

We note that the function $f_{j,k}(x_j, y_j) = c_k \exp(-\alpha\frac{x_j^2}{2} + \frac{\alpha}{4}(x_j + iy_j)^2)$ (normalized to 1 thanks to c_k) is a quasimode of A_α, as $\alpha \to +\infty$, for the eigenvalue 0. Moreover, a well-known computation on coherent states (see Example 1 of Chapter 5 in [Zwo12]) guarantees that for any $a \in \mathscr{S}^0(\mathbb{R}^{2m})$,
$$(\mathrm{Op}(a)(A_\alpha^*)^{n_j} f_{j,k}, (A_\alpha^*)^{n_j} f_{j,k}) \underset{k\to+\infty}{\longrightarrow} a(0,0).$$

In other words, $(A_\alpha^*)^{n_j} f_{j,k}$, seen as a sequence of functions of $\mathbb{R}^{2m}$, has a unique Quantum Limit, which is $\delta_{0,0}$.

Now, using that the spectrum of B_α has gaps that are uniformly bounded below, this property is preserved when we consider eigenfunctions of $-\Delta_j$: when α varies, the projection Φ_k^j of $(A_\alpha^*)^{n_j} f_{j,k} e^{i\alpha z}$ onto the eigenspace of $-\Delta_j$ corresponding to the eigenvalue $(2n_j+1)|\alpha|$ has a unique QL, which is $\mathscr{H}_0^s \otimes \delta_p$. The Dirac mass at p comes from (7.26) and from Lemma 7.23 applied, for any $1 \leqslant i,j \leqslant J$, to the operator $\frac{R_i}{R_j} - \frac{p_i}{p_j}$. Note that the point $q=0$ plays no specific role, and therefore any measure $\mathscr{H}_q^s \otimes \delta_p$ can be obtained as a QL, when $d_q^s = 1$ and under (7.24). □

Lemma 7.14. *Let $(q,p) \in S\Sigma_{\mathcal{J}}$ and $s \in \mathbf{S}_{\mathcal{J}}$ be arbitrary. Then, the measure $\mathscr{H}_q^s \otimes \delta_p$ is a Quantum Limit. [See Remark 7.15 for the description of the associated sequence of normalized eigenfunctions.]*

Proof. We still assume that $\mathcal{J} = \{1, \ldots, J\}$. Using Lemma 7.13, we can assume that $q \in \mathbf{H}^m$ and $s \in \mathbf{S}_j$ verify either $d_q^s \geqslant 2$, or $d_q^s = 1$ but (7.24) is not satisfied. In both cases, the following fact holds:

Fact 1. The measure $\mathscr{H}_q^s$ is in the weak-star closure of the set of measures $\mathscr{H}_{q'}^{s'}$ for which $d_{q'}^{s'} = 1$ and (7.24) is satisfied.

Let us denote by $\mathbb{T}^{\mathcal{J}} = (\mathbb{R}/2\pi\mathbb{Z})^{\mathcal{J}}$ the Riemannian torus of dimension $\#\mathcal{J}$ equipped with the flat metric. Due to Remark 7.12, proving Fact 1 is equivalent to proving the following fact, called Fact 2 in the sequel: if γ is a geodesic of $\mathbb{T}^{\mathcal{J}}$ and $\mathscr{H}_\gamma$ is the Hausdorff measure on γ, then $\mathscr{H}_\gamma$ is in the weak-star closure of the set of measures $\mathscr{H}_{\gamma'}$ with γ' a periodic geodesic of $\mathbb{T}^{\mathcal{J}}$ of slope $(s_1, \ldots, s_J)$ verifying (7.24) for some J-tuple $(n_1, \ldots, n_J)$. Let us prove Fact 2.

In case $d_q^s \geqslant 2$, possibly restricting to the flat torus given by the closure of γ, we can assume that γ is a dense geodesic in $\mathbb{T}^{\mathcal{J}}$. To prove Fact 2 in this elementary case, we take a sequence of geodesics $(\gamma_n')_{n\in\mathbb{N}^*}$ contained in $\mathbb{T}^{\mathcal{J}}$, with rational slopes given by J-tuples $(s_1^n, \ldots, s_J^n)$ of the form (7.24), and which become dense in $\mathbb{T}^{\mathcal{J}}$ as $n \to +\infty$.

For the case $d_q^s = 1$ where (7.24) is not satisfied, similarly, we take a sequence of geodesics with rational slopes which converges to γ. This proves Fact 2 and hence Fact 1 follows.

Since the set of QLs is closed, Fact 1 implies Lemma 7.14. □

Remark 7.15. Note that, following the proofs of Lemma 7.13 and Lemma 7.14, any measure $\mathscr{H}_q^s \otimes \delta_p$ is a Quantum Limit associated to a sequence of normalized eigenfunctions $(\varphi_k)_{k\in\mathbb{N}^*}$ such that, for any $k \in \mathbb{N}^*$, φ_k belongs to some eigenspace $\mathcal{H}^{\mathcal{J}}_{(n_{j,k}),(\alpha_{j,k})}$. In particular, φ_k is an eigenfunction of Ω_j for any $j \in \mathcal{J}$.

Note also that to guarantee this last property, it is not sufficient to invoke, at the end of the proof of Lemma 7.14, the closedness of the set of QLs: it is necessary to follow the proof of this fact, which consists in a simple extraction argument.

Lemma 7.16. *Let $s \in \mathbf{S}_{\mathcal{J}}$ and $\nu_s^{\mathcal{J}} \in \mathscr{P}(S^*\mathbf{H}^m)$ having no mass outside $S\Sigma_{\mathcal{J}}$ and being invariant under $\vec{\rho}_s^{\,\mathcal{J}}$. Then $\nu_s^{\mathcal{J}}$ is a Quantum Limit. [See Remark 7.17 for the description of the associated sequence of normalized eigenfunctions.]*

Proof. Let us consider the set $\mathscr{P}_s^{\mathcal{J}} \subset \mathscr{P}(S^*\mathbf{H}^m)$ of probability measures

$$\nu_s^{\mathcal{J}} = \sum_{(q_i,p_i)\in\mathscr{E}} \beta_i \mathscr{H}_{q_i}^s \otimes \delta_{p_i} \tag{7.28}$$

where i ranges over some finite set $\mathcal{F}$, $\mathscr{E}$ is a set of pairs $(q_i,p_i) \in S\Sigma$, and $\beta_i \in \mathbb{R}$.

We consider $\nu_s^{\mathcal{J}} \in \mathscr{P}_s^{\mathcal{J}}$ defined by (7.28). Note that if $i \neq i'$, either $\mathscr{H}_{q_i}^s \otimes \delta_{p_i} = \mathscr{H}_{q_i'}^s \otimes \delta_{p_i'}$, or the supports of $\mathscr{H}_{q_i}^s \otimes \delta_{p_i}$ and $\mathscr{H}_{q_i'}^s \otimes \delta_{p_i'}$ are disjoint. Therefore, possibly gathering terms in the above sum, we assume that the supports of $\mathscr{H}_{q_i}^s \otimes \delta_{p_i}$ and $\mathscr{H}_{q_i'}^s \otimes \delta_{p_i'}$ are disjoint as soon as $i \neq i'$.

For $i \in \mathcal{F}$, using Lemma 7.13 and Lemma 7.14, we consider a sequence of eigenfunctions $(\varphi_k^i)_{k\in\mathbb{N}^*}$ with eigenvalues $(\lambda_k^i)_{k\in\mathbb{N}^*}$ and whose unique QL is $\mathscr{H}_{q_i}^s \otimes \delta_{p_i}$. According to the proof of these lemmas (see also Remark 7.15), we can also assume that $\varphi_k^i \in \mathcal{H}^{\mathcal{J}}_{(n_{j,k}),(\alpha^i_{j,k})}$ for some J-tuples such that

$$\lambda_k^i := \sum_{j\in\mathcal{J}} (2n_{j,k}+1)|\alpha^i_{j,k}|$$

does not depend on $i \in \mathcal{F}$. In other words,

- for any $1 \leqslant j \leqslant J$, φ_k^i is also an eigenvalue of Ω_j with eigenvalue $n_{j,k}$ which does not depend on $i \in \mathcal{F}$;
- for any $i, i' \in \mathcal{F}$, $\lambda_k^i = \lambda_k^{i'}$ and we denote this common value by λ_k. This means that for any $i \in \mathcal{F}$, φ_k^i belongs to the eigenspace of $-\Delta$ corrresponding to the eigenvalue λ_k.

Since $\mathscr{H}_{q_i}^s \otimes \delta_{p_i}$ and $\mathscr{H}_{q_i'}^s \otimes \delta_{p_i'}$ have disjoint supports, the joint microlocal defect measure of $(\varphi_k^i)_{k\in\mathbb{N}^*}$ and $(\varphi_k^{i'})_{k\in\mathbb{N}^*}$ vanishes for $i \neq i'$ by Lemma 7.1. It follows that

$$\varphi_k := \sum_{i\in\mathcal{F}} \beta_i \varphi_k^i$$

is an eigenfunction of $-\Delta$ with eigenvalue λ_k, and that in the limit $k \to +\infty$, it admits $\nu_s^{\mathcal{J}}$ as unique Quantum Limit.

Finally, we note that any $\nu_s^{\mathcal{J}} \in \mathscr{P}(S^*\mathbf{H}^m)$ having all its mass contained in $S\Sigma_{\mathcal{J}}$ and being invariant under $\vec{\rho}_s^{\mathcal{J}}$ is in the closure of $\mathscr{P}_s^{\mathcal{J}}$. Since the set of QLs is closed, Lemma 7.16 is proved. □

Remark 7.17. The above proof shows that $\nu_\infty = \nu_s^{\mathcal{J}}$ is a QL for a sequence $(\varphi_k)_{k\in\mathbb{N}^*}$ such that φ_k belongs to

$$\bigoplus_{(\alpha_j)\in(\mathbb{Z}^*)^{\mathcal{J}}} \mathcal{H}^{\mathcal{J}}_{(n_{j',k'}),(\alpha_j)}$$

for some J-tuple $(n_{j',k'}) \in \mathbb{N}^{\mathcal{J}}$ which depends only on $k \in \mathbb{N}^*$.

Lemma 7.18. *Let $\mathcal{J} \in \mathcal{P} \setminus \{\emptyset\}$, and*

$$\nu^{\mathcal{J}} = \int_{\mathbf{S}_{\mathcal{J}}} \nu_s^{\mathcal{J}} dQ^{\mathcal{J}}(s)$$

for some $Q^{\mathcal{J}} \in \mathscr{P}(\mathbf{S}_{\mathcal{J}})$ and $\nu_s^{\mathcal{J}} \in \mathscr{P}(S^*\mathbf{H}^m)$ having no mass outside $S\Sigma_{\mathcal{J}}$ and such that, for $Q^{\mathcal{J}}$-almost any $s \in \mathbf{S}_{\mathcal{J}}$, $\vec{\rho}_s^{\mathcal{J}} \nu_s^{\mathcal{J}} = 0$. Then $\nu^{\mathcal{J}}$ is a Quantum Limit. [See Remark 7.19 for the description of the associated sequence of normalized eigenfunctions.]

Proof. As in the previous proofs, we assume without loss of generality that $\mathcal{J} = \{1, \ldots, J\}$ for some $1 \leqslant J \leqslant m$. Let $(s^\ell)_{\ell\in\mathcal{L}}$ be a finite family of distinct elements of $\mathbf{S}_{\mathcal{J}}$ indexed by $\mathcal{L}$, and let $\gamma_\ell \in \mathbb{R}$ for $\ell \in \mathcal{L}$. For any $\ell \in \mathcal{L}$, let also ν_{s^ℓ}, with mass only in $S\Sigma_{\mathcal{J}}$, be invariant under the flow of $\vec{\rho}_{s^\ell}^{\mathcal{J}}$. Let us prove that

$$\nu^{\mathcal{J}} = \sum_{\ell\in\mathcal{L}} \gamma^\ell \nu_{s^\ell} \tag{7.29}$$

is a Quantum Limit. This corresponds to the case where the measure $Q^{\mathcal{J}}$ on $\mathbf{S}_{\mathcal{J}}$ is given by

$$Q^{\mathcal{J}} = \sum_{\ell \in \mathcal{L}} \gamma^\ell \delta_{s^\ell}.$$

For any $\ell \in \mathcal{L}$, we take $(\varphi_k^\ell)_{k\in\mathbb{N}^*}$ to be a sequence of eigenfunctions of $-\Delta$ whose unique QL is ν_{s^ℓ}. As emphasized in the proof of Lemma 7.16, it is possible to assume that φ_k^ℓ is an eigenfunction of Ω_j for any $1 \leqslant j \leqslant J$, with eigenvalue $2n_{j,k}^\ell + 1$ such that

$$\frac{2n_{j,k}^\ell + 1}{\sum_{i=1}^{J}(2n_{i,k}^\ell) + 1} \underset{k\to+\infty}{\longrightarrow} s_j^\ell \tag{7.30}$$

where $s^\ell = (s_1^\ell, \ldots, s_J^\ell)$.

Let us prove that the joint microlocal defect measure $\nu_{\ell,\ell'}$ of $(\varphi_k^\ell)_{k\in\mathbb{N}^*}$ and $(\varphi_k^{\ell'})_{k\in\mathbb{N}^*}$ vanishes for $\ell \neq \ell'$: we note that for $\mathrm{Op}(a)$ commuting with $\Omega_1, \ldots, \Omega_m$, with $a \in \mathscr{S}^0(\mathbf{H}^m)$,

$$\begin{aligned}(2n_{j,k}^\ell + 1)(\mathrm{Op}(a)\varphi_k^\ell, \varphi_k^{\ell'}) &= (\mathrm{Op}(a)\Omega_j\varphi_k^\ell, \varphi_k^{\ell'})\\ &= (\mathrm{Op}(a)\varphi_k^\ell, \Omega_j\varphi_k^{\ell'})\\ &= (2n_{j,k}^{\ell'} + 1)(\mathrm{Op}(a)\varphi_k^\ell, \varphi_k^{\ell'})\end{aligned}$$

From (7.30) and the fact that $s^\ell \neq s^{\ell'}$, we deduce that, for any sufficiently large $k \in \mathbb{N}^*$, there exists $1 \leqslant j \leqslant J$ such that $n_{j,k}^\ell \neq n_{j,k}^{\ell'}$. Hence, the above computation shows that $(\mathrm{Op}(a)\varphi_k^\ell, \varphi_k^{\ell'}) = 0$ for sufficiently large $k \in \mathbb{N}^*$. Therefore,

$$\int_{S^*\mathbf{H}^m} a d\nu_{\ell,\ell'} = 0.$$

Since ν_{s^ℓ} and $\nu_{s^{\ell'}}$ give no mass to the complementary set of $S\Sigma_{\mathcal{J}}$ in $S^*\mathbf{H}^m$, we know that it is also the case for $\nu_{\ell,\ell'}$ by Lemma 7.1. Therefore, if $b \in \mathscr{S}^0(\mathbf{H}^m)$ is arbitrary, averaging $\mathrm{Op}(b)$ with respect to the operators $\Omega_1, \ldots, \Omega_J$ as in Lemma 7.8, we obtain an operator $A \in \Psi^0(\mathbf{H}^m)$ such that $\sigma_P(A)$ coincides with b on $\Sigma_{\mathcal{J}}$, and A commutes with $\Omega_1, \ldots, \Omega_J$. Therefore,

$$\int_{S^*\mathbf{H}^m} b d\nu_{\ell,\ell'} = \int_{S\Sigma_{\mathcal{J}}} b d\nu_{\ell,\ell'} = \int_{S\Sigma_{\mathcal{J}}} \sigma_P(A) d\nu_{\ell,\ell'} = 0,$$

and since this is true for any $b \in \mathscr{S}^0(\mathbf{H}^m)$, we conclude that $\nu_{\ell,\ell'} = 0$.

This implies that the sequence given by

$$\varphi_k^{\mathcal{J}} = \sum_{\ell\in\mathcal{L}} \gamma^\ell \varphi_k^\ell$$

admits $\nu^{\mathcal{J}}$ as unique QL, where $\nu^{\mathcal{J}}$ is defined by (7.29). Note that to ensure that $\varphi_k^{\mathcal{J}}$ is still an eigenfunction of $-\Delta$, it is necessary, as in the proof of Lemma 7.16, to adjust the sequences $(n_{j,k}^\ell)$ and $(\alpha_{j,k}^\ell)$ in order to guarantee that all φ_k^ℓ (for $\ell \in \mathcal{L}$) are eigenfunctions of $-\Delta$ with same eigenvalue.

We notice that the closure of the set of Radon measures on $S\Sigma_{\mathcal{J}}$ which may be written as a finite linear combination (7.29) is exactly the subset of $\mathscr{P}_{S\Sigma}$ for which $Q^{\mathcal{J}'} = 0$ for any $\mathcal{J}' \neq \mathcal{J}$. Using that the set of QLs is closed, Lemma 7.18 is proved. $\square$

Remark 7.19. The above proof shows that $\nu_\infty = \nu^{\mathcal{J}}$ is a QL for a sequence of normalized eigenfunctions $(\varphi_k)_{k\in\mathbb{N}^*}$ such that φ_k belongs to

$$\bigoplus_{(n_j)\in\mathbb{N}^{\mathcal{J}}} \bigoplus_{(\alpha_j)\in(\mathbb{Z}^*)^{\mathcal{J}}} \mathcal{H}^{\mathcal{J}}_{(n_j),(\alpha_j)}.$$

Let us now finish the proof of Theorem 10. Let $\nu_\infty \in \mathscr{P}_{S\Sigma}$,

$$\nu_\infty = \sum_{\mathcal{J}\in\mathcal{P}\setminus\{\emptyset\}} \nu^{\mathcal{J}}.$$

Note that the measures $\nu^{\mathcal{J}}$ are non-negative, but are not necessarily probability measures.

Let $(\varphi_k^{\mathcal{J}})_{k\in\mathbb{N}^*}$ be a sequence of eigenfunctions of $-\Delta$ whose unique microlocal defect measure is $\nu^{\mathcal{J}}$. The proof of Lemma 7.18 guarantees that, for any $k \in \mathbb{N}^*$, one may choose all $\varphi_k^{\mathcal{J}}$, for $\mathcal{J}$ running over $\mathcal{P} \setminus \{\emptyset\}$, to have the same eigenvalue with respect to $-\Delta$. Therefore,

$$\varphi_k = \sum_{\mathcal{J}\in\mathcal{P}\setminus\{\emptyset\}} \varphi_k^{\mathcal{J}}$$

is also an eigenfunction of $-\Delta$. Moreover, for any distinct $\mathcal{J}, \mathcal{J}' \in \mathcal{P} \setminus \{\emptyset\}$, the joint microlocal defect measure of $(\varphi_k^{\mathcal{J}})_{k\in\mathbb{N}^*}$ and $(\varphi_k^{\mathcal{J}'})_{k\in\mathbb{N}^*}$ vanishes (see Lemma 7.1). Computing $(\mathrm{Op}(a)\varphi_k, \varphi_k)$ for any $a \in \mathscr{S}^0(\mathbf{H}^m)$ in the limit $k \to +\infty$, we obtain that the unique Quantum Limit of $(\varphi_k)_{k\in\mathbb{N}^*}$ is ν_∞. Note that, as already explained in Remarks 7.15, 7.17 and 7.19, the sequence $(\varphi_k)_{k\in\mathbb{N}^*}$ is fully explicit in our construction.

Finally, we note that the invariance properties of ν_∞ can be established separately on each $S\Sigma_{\mathcal{J}}$ since $([A, R_s]\varphi_k^{\mathcal{J}}, \varphi_k^{\mathcal{J}'}) \to 0$ as $k \to +\infty$ for $\mathcal{J} \neq \mathcal{J}'$ (the bracket $[A, R_s]$ is the natural operator to consider for establishing invariance properties, see Section 7.2.3). This concludes the proof of Theorem 10.

Remark 7.20. The exact converse of Theorem 9 would guarantee that all measures $\nu \in \mathscr{P}(S^*\mathbf{H}^m)$ of the form $\nu = \beta\nu^{\emptyset} + (1-\beta)\nu_\infty$ with the same assumptions on β, $\nu^{\emptyset}$ and ν_∞ as in Theorem 9 are Quantum Limits. Our statement is weaker since it does not say anything about the measures ν for which $\beta \neq 0$ (which are rare, as stated in Theorem 9), but we do not think that a stronger converse statement for Theorem 9 holds.

Remark 7.21. Theorems 9 and 10 remain true for slightly more general sub-Laplacians than those considered here. Indeed, for any $d \in \mathbb{N}$, one can consider the (2d+1)-dimensional Heisenberg group $\widetilde{\mathbf{H}}_d$ and its quotient $\mathbf{H}_d = \Gamma_d \backslash \widetilde{\mathbf{H}}_d$ by the discrete cocompact subgroup $\Gamma_d = (\sqrt{2\pi}\mathbb{Z})^{2d} \times 2\pi\mathbb{Z}$. Then, one can define as in Section 1.5.2 a natural sub-Laplacian $\Delta_{\mathbf{H}_d}$ on $\mathbf{H}_d$ (see Section 7-A.3). Given a finite sequence of positive integers $d_1, \ldots, d_m$, one can consider the associated sub-Laplacian on $\mathbf{H}_{d_1} \times \ldots \times \mathbf{H}_{d_m}$ defined as in (1.40). Then, Theorems 9 and 10 are still true in this setting (mutatis mutandis). However, for the sake of clarity of presentation, we found it preferable to write full details only in the case $d_1 = \ldots = d_m = 1$, since it already contains the key ideas.

Remark 7.22. The problem of identifying other families of sub-Laplacians for which a full characterization of QLs is possible is open; it requires to identify a family of 1-homogeneous Hamiltonians on Σ replacing the family $(\rho_s^{\mathcal{J}})$. E.g., for the quasi-contact sub-Laplacian $\partial_x^2 + (\partial_y - x\partial_z)^2 + \partial_w^2$, defined on $\mathbf{H} \times (\mathbb{R}/2\pi\mathbb{Z})$, it does not seem possible to identify such a family because of the additional ∂_w^2 term which is separated from the $R\Omega$-factorization of the rest of the sub-Laplacian.

7.4 Links with non-commutative harmonic analysis.

The point of view taken in this chapter is definitely Euclidean, meaning that we do not use pseudodifferential calculus adapted to the stratified Lie algebra which possibly shows up while studying sub-Laplacians. However, our results share connexions with important problems in non-commutative Fourier analysis.

It is possible to use the stratified Lie algebra structure to study the spectral theory of (nilpotent) sub-Laplacians, as done for example in [FF21]. This work builds upon non-commutative harmonic analysis (see [Tay86]) to develop a pseudodifferential calculus and semiclassical tools "naturally attached to the sub-Laplacian". It is very likely that one could have given a proof of Theorems 9 and 10 based on similar tools as in [FF21]. The point of view we adopt in the present chapter is different: it only requires "classical" pseudodifferential calculus (briefly recalled in Appendix A.2) since there is still enough commutativity and ellipticity from the choice of operators under study. Beside making the results more accessible to some readers, it allows us to isolate in each eigenfunction the piece which is responsible, in the high-frequency limit, for a given part of the QL. Moreover, our method only builds upon abstract commutation arguments, at least for Theorem 8, and in particular it avoids the computation of irreducible representations which are always specific to certain families of groups (e.g., H-type groups in Chapter 4 and [FF21]).

Part of our results can be reinterpreted through the light of noncommutative harmonic analysis. For example, the part of the QL in U^*M, namely $\beta\nu^\emptyset$ (see (1.39)), is described in [FF21] as the part of the semiclassical measure supported above the finite dimensional representations $\pi_x^{0,\omega}$ (see [FF21, Section 2.2.1]), and the fact that $\beta\nu^\emptyset = 0$ for "almost all" QLs (see Proposition 1.29) can be recovered from the fact that the Plancherel measure denoted by $|\lambda|^d d\lambda$ in [FF21] gives no mass to finite-dimensional representations.

Also, in the setting covered by Theorems 9 and 10, i.e., products of quotients of the Heisenberg group, the joint spectrum of $(\Delta_1, \dots, \Delta_m, i^{-1}\partial_{z_1}, \dots, i^{-1}\partial_{z_m})$, which can be drawn in $\mathbb{R}^{2m}$, is called "Heisenberg fan". This terminology was introduced in [Str91] for the 3D Heisenberg sub-Laplacian; in our case, this fan consists in a discrete set of points which can be gathered into lines (see [Str91, Figure 1]). In case $m = 1$, the subset of points (or joint eigenvalues) corresponding to $\varphi_k^\emptyset$ and $\nu^\emptyset$ in the statement of Theorem 9 can be seen as points close to the vertical line $\{0\} \times \mathbb{R} \subset \mathbb{R}^2$. Similar descriptions can be given in case $m \geqslant 2$. Also, let us mention that we could derive from the proof of Theorem 8 a generalization of the definition of the Heisenberg fan to any sub-Laplacian satisfying Assumption (A), as the joint spectrum of $(-\Delta_{g,\mu}, |Z_1|, \dots, |Z_m|)$.

Let us also mention that sub-Laplacians on products of Heisenberg groups (and, more generally, on "decomposable groups") were analysed in [BFG16] with a non-commutative harmonic analysis point of view in order to establish Strichartz estimates (see notably [BFG16, Section 1.4 and Corollary 1.6]).

7-A Supplementary material

7-A.1 Proof of two lemmas

Let us prove Lemma 7.1 of Section 7.1.

Proof of Lemma 7.1. If $a \in \mathscr{S}^0(M)$ is such that $a \geqslant 0$ and a is supported in a set where $\mu_{11} = 0$, then, setting $a_\varepsilon = a + \varepsilon$ for any $\varepsilon > 0$, we get

$$(\mathrm{Op}(a_\varepsilon)u_k, v_k) = (\mathrm{Op}(a_\varepsilon^{1/2})u_k, \mathrm{Op}(a_\varepsilon^{1/2})v_k) + o(1) \leqslant \|\mathrm{Op}(a_\varepsilon^{1/2})u_k\|_{L^2}\|\mathrm{Op}(a_\varepsilon^{1/2})v_k\|_{L^2} + o(1)$$

where $a_\varepsilon^{1/2} \in \mathscr{S}^0(M)$. We know that

$$\|\mathrm{Op}(a_\varepsilon^{1/2})u_k\|_{L^2}^2 = (\mathrm{Op}(a_\varepsilon)u_k, u_k) + o(1) = (\mathrm{Op}(a)u_k, u_k) + \varepsilon\|u_k\|_2^2 + o(1) = \varepsilon\|u_k\|^2 + o(1)$$

and that $\|\mathrm{Op}(a_\varepsilon^{1/2})v_k\|_{L^2}^2 \leqslant (C+\varepsilon)\|v_k\|^2$ where C does not depend on ε. Therefore $(\mathrm{Op}(a_\varepsilon)u_k, v_k) \lesssim \varepsilon$. Hence $(\mathrm{Op}(a)u_k, v_k) \to 0$. The same result holds for $a \leqslant 0$ supported in a set where $\mu_{11} = 0$. Therefore, decomposing any symbol as $a = a^+ + a^- + r$, where $a^+, a^-, r \in \mathscr{S}^0(M)$, $a^+ \geqslant 0$, $a^- \leqslant 0$, and $|r| \leqslant \delta$ for some small $\delta > 0$, we get that μ_{12} is absolutely continuous with respect to μ_{11}. The rest of the lemma follows by symmetry. □

Lemma 7.23. *Let us assume that $\ell \in \mathbb{N}$ and $P \in \Psi^\ell(M)$ is elliptic in any cone contained in the complementary of a closed conic set $F \subset T^*M$. Assume that $(u_k)_{k\in\mathbb{N}^*}$ is a bounded sequence in $L^2(M)$ weakly converging to 0 and such that $Pu_k \to 0$ strongly in $L^2(M)$. Then any microlocal defect measure of $(u_k)_{k\in\mathbb{N}^*}$ is supported in F.*

Proof. Let μ be a microlocal defect measure of $(u_k)_{k\in\mathbb{N}^*}$, i.e.,

$$(\mathrm{Op}(a)u_{\sigma(k)}, u_{\sigma(k)}) \underset{k\to+\infty}{\longrightarrow} \int_{S^*M} a d\mu$$

for any $a \in \mathscr{S}^0(M)$, where σ is an extraction. Let $a \in \mathscr{S}^0(M)$ be supported outside F. Let $Q \in \Psi^{-\ell}(M)$ be such that $PQ - I \in \Psi^{-1}(M)$ on the support of a. Then $Q\mathrm{Op}(a)P \in \Psi^0(M)$ has principal symbol a, and therefore

$$(Q\mathrm{Op}(a)Pu_{\sigma(k)}, u_{\sigma(k)}) \underset{k\to+\infty}{\longrightarrow} \int_{S^*M} a d\mu.$$

Using that $Pu_{\sigma(k)} \to 0$, we get $(Q\mathrm{Op}(a)Pu_{\sigma(k)}, u_{\sigma(k)}) \to 0$ as $k \to +\infty$, and therefore $\int_{S^*M} a d\mu = 0$. Hence, μ is supported in F. □

7-A.2 Supplementary material on Assumption (A)

The Martinet sub-Laplacian

In this Section, we provide an example of a sub-Laplacian on a compact manifold which satisfies Assumption (A) but which is not step 2, meaning that brackets of length $\geqslant 3$ of the X_i are required to generate the whole tangent bundle, see (1.1).

For that, we consider $M = (\mathbb{R}/2\pi\mathbb{Z})^3$ with coordinates x, y, z, endowed with the Lebesgue measure $d\mu = dxdydz$. Let A be a smooth 1-form $A = A_x dx + A_y dy$, where A_x and A_y depend only on x and y. The 2-form $B = dA = (\partial_x A_y - \partial_y A_x) dx \wedge dy$ is the "magnetic field" and $b = \partial_x A_y - \partial_y A_x$ is its "strength". We consider the sub-Riemannian structure associated to the vector fields $X_1 = \partial_x + A_x\partial_z$ and $X_2 = \partial_y + A_y\partial_z$. Then, $[X_1, X_2] = b\partial_z$. Now, we choose A so that b vanishes along a closed curve in $(\mathbb{R}/2\pi\mathbb{Z})^2_{x,y}$, and $(\partial_x b, \partial_y b) \neq 0$ along this curve. This construction is classical, see Example 1.6 and [Mon95]. When adding the z-variable, this yields a surface $\mathscr{S} \subset M$, called Martinet surface, on which $[X_1, X_2] = 0$ but some bracket of length 3 of X_1, X_2 generates the missing direction of the tangent bundle thanks to $(\partial_x b, \partial_y b) \neq 0$. In other words, the sub-Riemannian structure has step 3 on $\mathscr{S}$. Nevertheless, Assumption (A) is satisfied with $Z_1 = \partial_z$.

7-A.3 Quantum Limits of flat contact manifolds

The study of Quantum Limits of higher dimensional contact manifolds is also an interesting problem. In this section, we prove that for the sub-Laplacian defined on the quotient of the Heisenberg group $\mathbf{H}_d$ of dimension $2d+1$ by one of its discrete cocompact subgroups, the invariance properties of Quantum Limits are much simpler than those described in Theorem 9, even though "frequencies" show up: the part of the QL which lies in $S\Sigma$ is invariant under the lift of the Reeb flow, as in the 3D case.

For $d \geqslant 1$, we consider the group law on $\mathbb{R}^{2d+1}$ given by

$$(x, y, z) \star (x', y', z') = (x + x', y + y', z + z' - x \cdot y')$$

where $x, x', y, y' \in \mathbb{R}^d$ and $z, z' \in \mathbb{R}$. The Heisenberg group $\widetilde{\mathbf{H}}_d$ is the group $\widetilde{\mathbf{H}}_d = (\mathbb{R}^{2d+1}, \star)$. We consider the subgroup $\Gamma_d = (\sqrt{2\pi}\mathbb{Z})^{2d} \times 2\pi\mathbb{Z}$ of $\widetilde{\mathbf{H}}_d$, and the left quotient $\mathbf{H}_d = \Gamma_d \backslash \widetilde{\mathbf{H}}_d$. We also define the $2d$ left invariant vector fields on $\mathbf{H}_d$ given by

$$X_j = \partial_{x_j}, \qquad Y_j = \partial_{y_j} - x_j \partial_z$$

for $1 \leqslant j \leqslant d$. We fix $\beta_1, \ldots, \beta_d > 0$ satisfying $\prod_{j=1}^d \beta_j = 1$, we set $\beta = (\beta_1, \ldots, \beta_d)$ and we consider the sub-Laplacian

$$\Delta_\beta = \sum_{j=1}^d \beta_j (X_j^2 + Y_j^2) \tag{7.31}$$

which is an operator acting on functions on $\mathbf{H}_d$. The positive real numbers β_j are sometimes called frequencies, see [Agr96].

We set $\rho = h_Z|_\Sigma$, which is the Hamiltonian lift of the Reeb vector field $Z = \partial_z$ to Σ (see [CHT18, Section 2.3] for properties of the Reeb vector field).

Proposition 7.24. *Let $(\varphi_k)_{k\in\mathbb{N}^*}$ be a sequence of $L^2(\mathbf{H}_d)$ consisting of normalized eigenfunctions of $-\Delta_\beta$. Then, any Quantum Limit ν_∞ associated to $(\varphi_k)_{k\in\mathbb{N}^*}$ and supported in $S\Sigma$ is invariant under $e^{t\vec{\rho}}$, the lift of the Reeb flow.*

Remark 7.25. This result follows from [FF21, Theorem 2.10(ii)(2)], but we provide here a simple self-contained proof which illustrates the averaging techniques used in Section 7.2.3.

Remark 7.26. We do not expect such a result to be true when the frequencies β_j are not constant on the manifold.

Proof of Proposition 7.24. Denoting by (q, p) the canonical coordinates in $T^*\mathbf{H}_d$, i.e., $q = (x_1, \ldots, x_d, y_1, \ldots, y_d, z)$ and $p = (p_{x_1}, \ldots, p_{x_d}, p_{y_1}, \ldots, p_{y_d}, p_z)$, we know that

$$\Sigma = \{(q, p) \in T^*\mathbf{H}_d,\ p_{x_j} = p_{y_j} - x_j p_{z_j} = 0\}$$

is isomorphic to $\mathbf{H}_d \times \mathbb{R}$.

Up to extraction of a subsequence, we may assume that $(\varphi_k)_{k\in\mathbb{N}^*}$ has a unique QL ν_∞, which is supported in $S\Sigma$. We set $R = \sqrt{\partial_z^* \partial_z}$ and, on its eigenspaces corresponding to non-zero eigenvalues, we define $\Omega_j = -R^{-1}(X_j^2 + Y_j^2) = -(X_j^2 + Y_j^2)R^{-1}$ for $1 \leqslant j \leqslant d$. On these eigenspaces, the sub-Laplacian acts as

$$-\Delta_\beta = R\Omega = \Omega R \qquad \text{with} \quad \Omega = \sum_{j=1}^d \beta_j \Omega_j$$

and $[R,\Omega]=0$.

Let V be a (small) conic microlocal neighborhood of Σ, and let us consider R,Ω as acting on functions microlocally supported in V (meaning that their wave-front set is contained in V). If $B\in\Psi^0(\mathbf{H}_d)$ is microlocally supported in V and commutes with Ω, then

$$\begin{aligned}([B,R]\varphi_k,\varphi_k) &= \frac{1}{\lambda_k}(BR\varphi_k,-\Delta_\beta\varphi_k)-\frac{1}{\lambda_k}(RB(-\Delta_\beta)\varphi_k,\varphi_k)\\ &= \frac{1}{\lambda_k}(BR\varphi_k,R\Omega\varphi_k)-\frac{1}{\lambda_k}(RBR\Omega\varphi_k,\varphi_k)\\ &= \frac{1}{\lambda_k}([\Omega,RBR]\varphi_k,\varphi_k)\\ &= 0.\end{aligned}$$

Let $U(t)=U(t_1,\dots,t_d)=e^{i(t_1\Omega_1+\ldots+t_d\Omega_d)}$ for $t=(t_1,\dots,t_d)\in(\mathbb{R}/2\pi\mathbb{Z})^d$. For $A\in\Psi^0(\mathbf{H}_d)$ microlocally supported in V, we consider

$$\widetilde{A}=\int_{(\mathbb{R}/2\pi\mathbb{Z})^d}U(-t)AU(t)dt$$

As in the proof of Lemma 7.8, we know that $[\widetilde{A},\Omega]=0$ and that $\sigma_P(A)$ and $\sigma_P(\widetilde{A})$ coincide on Σ. Therefore, using the previous computation with $B=\widetilde{A}$, we obtain

$$\int_\Sigma\{\sigma_P(A),\rho\}_{\omega_{|\Sigma}}d\nu_\infty=\int_\Sigma\{\sigma_P(\widetilde{A}),\rho\}_{\omega_{|\Sigma}}d\nu_\infty=\lim_{k\to+\infty}([\widetilde{A},R]\varphi_k,\varphi_k)=0.$$

Since it is true for any A microlocally supported in V, this implies that ν_∞ is invariant under the flow $e^{t\vec{\rho}}$. □

Chapter 8

Perspectives and open questions

"J'ai souhaité comprendre le coeur des hommes;
j'ai souhaité comprendre pourquoi les étoiles brillent."
Bertrand Russell.

This concluding chapter gathers some perspectives and open questions related to the present manuscript.

8.1 Singular curves

At the microlocal level, the two main specificities of sub-Riemannian geometry, compared to Riemannian geometry, are the existence of the *characteristic cone* Σ (see (1.5)) and, in some but not all sub-Riemannian distributions, the existence of *singular curves*, and hence of abnormal extremal lifts.

The characteristic cone. The characteristic cone is by now well understood. At the "*classical*" level (at the level of geometry, and not of operators), it is responsible, for example, for a "spiraling" of normal geodesics around curves transverse to the distribution, see [CHT21a], [Let20b].

Several works have also been devoted to the influence of the characteristic cone on the "*quantum*" level, i.e., that of operators: see for example [MS78a], [CHT21b] for asymptotics of eigenvalues, [CHT18], [CHT21b], [Let20a] for the repartition of eigenfunctions (in particular, quantum ergodicity), and [BS19], [LS20], [FL21] for the propagation of energy (observability).

Singular curves. Singular curves, in particular when they are minimizing, remain more mysterious. Since their discovery in the 90's by Montgomery [Mon94], many efforts have been devoted to understand the "*classical*" aspects of singular minimizers: for example, are they always smooth? (still open, see [HL16] for a recent breakthrough); do they exist generically? (see [CJT06]); how do they influence the regularity of the sub-Riemannian distance? (see [AG01]), ...

But the effects *at the level of operators* ("*quantum*" level) of the presence of singular (minimizing) curves are still poorly understood. This is due to the fact that many usual tools do not detect the presence of singular curves. For example, in [CHT21b], precise Weyl laws (i.e., the asymptotic distribution of eigenvalues and eigenfunctions) are established even in the presence of singular curves, but singular curves have no real influence on these asymptotics: roughly speaking, what only matters in these asymptotics is the growth vector. Indeed, the problem of establishing Weyl laws is related to that of establishing small-time asymptotics of subelliptic heat kernels (see [CHT20]), but the heat kernel "is too rough to see singular curves".

In Riemannian geometry, the correspondance between classical and quantum aspects relies on the Hamiltonian framework (think of Egorov's theorem for example), but in sub-Riemannian geometry, singular curves are not directly related to the *Hamiltonian framework*: their existence is dictated by the distribution, not the metric (or equivalently the cometric). The only works available which show the effects at the quantum level of singular curves, namely [Mon95], [Sav19] and [CL21], are thus devoted to only particular sub-Laplacians, for which, in some sense, explicit computations or "normal forms" can be derived (for example, quasi-contact or magnetic sub-Laplacians).

The work [Mel86], revisited in [Let21b], could pave the way to a general understanding of the interplay between classical and quantum level in the presence of singular curves, thanks to the cones Γ_m (see Chapter 5), which generalize the usual Hamiltonian framework.

8.2 Magnetic fields

The first examples of minimizing singular curves, exhibited in [Mon94], were inspired by the study of magnetic fields. Indeed, there is a dictionary between some $\mathbb{S}^1$-invariant sub-Riemannian geometries on a manifold M with a codimension one distribution transverse to the action of $\mathbb{S}^1$, and magnetic fields on the quotient $X = M/\mathbb{S}^1$. A first illustration has been given in Example 1.6.

Let us describe this correspondance, following [Col16] (unpublished). We assume that M is equipped with a free action of $\mathbb{S}^1$ (with coordinate θ) given by $m \mapsto \theta \cdot m$. Let $\mathcal{D}$ be a distribution on M which is transverse to the action of $\mathbb{S}^1$ and invariant under the differential of the action, and g be a metric on $\mathcal{D}$ which is also invariant under this differential. Then, M is a (principal) bundle over $X = M/\mathbb{S}^1$ with projection $p : M \to X$, and $\mathcal{D}$ is an Ehresmann connection on this bundle (see [Mon02, Section 11.1]). The distribution $\mathcal{D}$ is the kernel of a 1-form Θ, which is normalized by $\Theta(\partial_\theta) = 1$. On every open subset U of X where the bundle is trivialized, i.e., $M = U \times \mathbb{S}^1$ with the action $(x, \phi) \mapsto (x, \theta + \phi)$, the form Θ is given by $\Theta = d\phi + p^*A$ where A is a 1-form on X. The 2-form $dA = B$ on X is called the magnetic field, and it does not depend on the choice of the trivialization. For more more on this, see [Mon94], [Mon95], [Mon02].

This approach relates results in sub-Riemannian geometry with results for magnetic fields, both at the "classical" level (sub-Riemannian geodesics versus trajectories in magnetic fields) and at the "quantum" level (spectral asymptotics of sub-Laplacians versus magnetic Schrödinger operators). This is interesting because the classical motion of a charged particle in a varying magnetic field is a well-studied problem, especially in view of the important applications to physics (charged particles in the earth magnetic dipole, plasma physics, ...).

If X is oriented and of dimension 2 (thus M has dimension 3), then $B = b\, \mathrm{dvol}_X$ for some function b on X. The sub-Riemannian metric g is contact at any point where $b \neq 0$ and it has a Martinet singularity along the curves where $b = 0$ and $db \neq 0$. Also, the quasi-contact case (in which M has dimension 4) is related to magnetic fields in dimension 3. In this case, there exist some nontrivial singular curves that correspond to lines of the magnetic field.

This powerful dictionary could serve as a motivation and as an inspiration to study physical phenomena with a hidden sub-Riemannian geometry, such as magnetic mirrors (see Part 2 of [Mon02] for other physical examples).

8.3 Propagation of singularities

Our third focus in this concluding chapter is not related to singular curves; it illustrates on an example the problem of adapting classical notions of Riemannian geometry (or elliptic PDEs) to the "stratified setting" given by the sub-Riemannian flag. This example is the problem of propagation of singularities, already addressed in Chapters 5 and 6. Recall that, using for the definition of singularities the usual notion of wave-front set due to Hörmander,

- Chapter 5 explains how singularities of general subelliptic wave equations propagate;
- Chapter 6 constructs an explicit example where singularities propagate along abnormal extremals. It can probably be extended to more general geometries, starting with non-flat quasi-contact metrics and non-flat Martinet metrics.

We speculate that a different notion of singularity, adapted to sub-Riemannian geometry (i.e., taking into account the number of brackets needed to generate each direction), would yield more refined results for what concerns the propagation of singularities contained in the characteristic set $(g^*)^{-1}(0)$. To say it differently, Theorem 4 implies that in the absence of abnormal extremals, singularities pointing in a characteristic direction do not move as time evolves. But this might be due to the fact that the right notion of singularity in this region of phase-space is not the usual one with Hörmander's wave-front set. Semiclassical tools adapted to the graded structure of the sub-Riemannian tangent space as in Chapter 4 might indicate what should be this refined notion of singularity.

However, the transposition to the non-group setting of the tools of Chapter 4 is not straightforward. By that, we mean that if the sub-Riemannian manifold under study does not derive from a group (e.g., H-type groups), representations do not make sense, and non-commutative harmonic analysis cannot directly help. The hope is that the tangent space of sub-Riemannian manifolds has a group structure (at least at regular points, see [Bel96, Section 5.5]); in general, this group structure changes from point to point[1], in which case there is no reason to hope that non-commutative harmonic analysis could help. But it is sometimes possible to relate the tangent spaces at different points (in 3D contact manifolds, for example): in these manifolds, all hopes are permitted.

8.4 Spectral invariants and trace formulas

Another beautiful question is the following: can one hear something in a sub-Riemannian manifold? This is the counterpart of Mark Kac's well-known question "Can one hear the shape of a drum?"

Mathematically, spectral invariants (also called "audible quantities") are geometric quantities which are determined by the knowledge of the spectrum. In Riemannian geometry, for instance, if the spectrum of the Laplace-Beltrami operator is given, then one can compute just from this data the volume of the manifold, and, in some cases, the lengths of its closed geodesics.

The same question can be raised in sub-Riemannian geometry, i.e., for sub-Laplacians. For instance, in the equiregular case[2], is the rank of the distribution a spectral invariant? And its non-honolomic order? Or even its full growth vector? These simple questions have received no

[1] To say it differently, in contrast to the Riemannian case where tangent spaces at neighbor points are isometric, this is far from being true in sub-Riemannian geometry. This induces moduli in the normal forms of Carnot groups, as soon as the dimension is larger than or equal to 5. These normal forms are known in small dimension, see [ABB12, Theorem 29].

[2] That is, when the growth vector does not depend on the point.

answer for the moment, it is only known that the Hausdorff dimension is a spectral invariant. Of course, not only the vector fields $X_1, \dots, X_m$, but also the volume μ on M may play a role in the answers.

An exemple of known spectral invariant for sub-Laplacians is the following. When $M = \mathbb{S}^3$ and P is the Popp probability measure (see [Mon02], [CHT18]), then $1/P(M)$ is the asymptotic Hopf invariant of the Reeb vector field Z (with respect to P) introduced in [Arn86]. It follows from the Weyl formula proved in [CHT18] that the asymptotic Hopf invariant is a spectral invariant.

A common way to find audible quantities in Riemannian geometry is to establish trace formulas. It consists in computing quantities of the form

$$\sum_{n\in\mathbb{N}} f(\lambda_n)$$

where f is a (possibly complex-valued) function and λ_n describes the spectrum (with multiplicities) of $-\Delta$, i.e., $-\Delta\varphi_n = \lambda_n\varphi_n$ for smooth functions φ_n. Classical choices for f are the following: $f(x) = e^{-tx}$ (heat equation), $f(x) = |x|^{-s}$ (zeta functions), $f(x) = \cos(t\sqrt{x})$ (wave equation), $f(x) = e^{-itx/h}$ (semi-classical Schrödinger equation).

The literature on trace formulas in Riemannian manifolds is vast. But in the sub-Riemannian case, only few trace formulas have been established, and most of them are formulated with the heat kernel. It would be of interest to prove trace formulas for other kernels. A possible conjecture is the following:

Conjecture. *In the absence of singular curves, the Duistermaat-Guillemin trace formula [DG75, Corollary 1.2 and Theorem 4.5] holds for the wave trace distribution*

$$W(t) = \sum_{n\in\mathbb{N}} e^{it\sqrt{\lambda_n}}$$

outside $t = 0$: the singular support of $W(t)$ is included in the set of lengths of periodic geodesics and, assuming some non-degeneracy assumption, the principal term is given explicitly in terms of the Poincaré map and the Morse index of the periodic geodesics.

The paper [Mel84] proves the above conjecture in a particular case.

A simple question, asked by Yves Colin de Verdière, is the following: in the 3D contact case, are the periods of the closed Reeb orbits spectral invariants? Although several computations support this conjecture, no proof has been found for the moment.

Another question is: are the periods of the closed singular curves of a 4D Engel distribution spectral invariants? To answer this question, the first step is probably to compute the (semi-classical) Schrödinger kernel in quotients of the Engel group. But this is not an easy task, since elliptic functions come into play.

8.5 Observability and controllability

There are also interesting open questions that are still unanswered in the field of observability/controllability of subelliptic PDEs.

Heat equation. As explained in Section 1.3.2, the observability properties of subelliptic heat equations are known only in particular geometries. More general results would require a deeper understanding of the geometric meaning of the solutions constructed in [BCG14] or [Koe17]. Let us formulate two conjectures:

1. *For any sub-Laplacian of step 2, if $M \setminus \omega$ has non-empty interior, the observability property for the associated heat equation fails for sufficiently small times $T > 0$;*

2. *For any sub-Laplacian of step $\geqslant 3$, if $M \setminus \omega$ has non-empty interior, the observability property for the associated heat equation fails for any time $T > 0$.*

These conjectures are inspired by the results mentioned in Section 1.3.2 and by the paper [LL20] (see notably Section 1.4).

Schrödinger equation. Even in the Riemannian case, the observability properties of the Schrödinger equation remain mysterious: although (GCC) is known to be a sufficient condition for observability, it is not a necessary condition (see Section 1.3.1). In the sub-Riemannian case, the problem is even "more open", since no general sufficient condition is known for the moment, except trivial ones: only very particular geometries have been explored (see Theorems 1.21, 2 and 3), and they rely on tools which are not robust enough to cover general (in particular non-flat) sub-Riemannian geometries.

In consonance with Section 8.1, we can also ask the following question: how does the energy of solutions of subelliptic Schrödinger equations propagate along singular curves, when the latter exist?

8.6 Eigenfunctions and quasimodes

Despite recent progresses (recalled in Chapter 7), the properties of eigenfunctions and quasimodes of sub-Laplacians remain mostly unknown. Here are a few simple cases which could be interesting:

- **Higher-dimensional contact case.** The Quantum Limits of 3D contact sub-Laplacians have been studied in detail in [CHT18], but the higher dimensional contact case (see Example 1.5) remains open, except in a flat case handled in the Appendix of [Let20a].

- **Sasaki case.** For Sasaki sub-Laplacians, which are a particular family of contact sub-Laplacians of arbitrary dimension, we can however formulate a conjecture. To state it, we recall the definition of this family.

  Let (X, h) be a compact Riemannian manifold, and let $M = S^*X$ be the unit cotangent bundle of X, which is naturally endowed with the contact form α defined as the restriction of the Liouville 1-form $\Lambda = pdq$ to M. Let Z be the associated Reeb vector field. Identifying the tangent and cotangent bundles of X thanks to the Riemannian metric h, the set M is viewed as the unit tangent bundle TX of X. Using a metric g, for example the canonical metric (or "Sasaki metric", see [Sas58]), such that the restriction of the symplectic form to $\mathcal{D} = \ker(\alpha)$ is the volume form of g, Z is identified with the vector field on the unit tangent bundle of X generating the geodesic flow on S^*X. Therefore, with this identification, the Reeb flow is the geodesic flow on M.

  Sasaki sub-Laplacians, i.e., defined with such a contact metric g, are known to have all fundamental frequencies[3] equal to 1. This leads us to the following question:

  Is any Quantum Limit of a Sasaki sub-Laplacian invariant under the Reeb flow?

[3] see [Agr96, Section 2.1] for a definition.

8.7 Nodal sets

Let us finish with a totally open and beautiful question, which seems difficult. A nodal set is a set $\varphi_\lambda^{-1}(0)$ where φ_λ is an eigenfunction of a (sub-)Laplacian: $-\Delta\varphi_\lambda = \lambda\varphi_\lambda$. For Riemannian Laplacians, when the manifold and the metric are analytic, it is known since the work of Donnelly and Fefferman [DF88] that there exist $c, C > 0$ such that

$$c\sqrt{\lambda} \leqslant \mathscr{H}^{n-1}(\varphi_\lambda^{-1}(0)) \leqslant C\sqrt{\lambda}$$

where $\mathscr{H}^{n-1}$ is the $(n-1)$-dimensional Hausdorff measure (the dimension of the manifold being n). Yau's conjecture asserts that these bounds remain true if the manifold and the metric are only assumed to be C^∞. Many recent progress have been made recently in this field, see [LM18] for a review.

One can wonder what happens to these bounds in the sub-Riemannian case. Indeed, Hausdorff measures are already known to play an important role in the metric geometry of sub-Riemannian structures: the topological dimension and the Hausdorff dimension of a sub-Riemannian manifold do not coincide in general (see [Mit85]). Here is a possible conjecture:

Conjecture. *Let $\Delta = -\sum_{i=1}^m X_i^* X_i$ be a sub-Riemannian Laplacian on a compact manifold M endowed with a smooth volume μ. We assume that $X_1, \ldots, X_m$ span an equiregular distribution (see Section 8.4). Then there exists $c > 0$ such that*

$$c\sqrt{\lambda} \leqslant \mathscr{H}_{\mathrm{sph}}^{\mathcal{Q}-1}(\varphi_\lambda^{-1}(0)).$$

Here, $\mathscr{H}_{\mathrm{sph}}^{\mathcal{Q}-1}$ denotes the $\mathcal{Q}-1$-dimensional spherical Hausdorff measure, where $\mathcal{Q}$ is the homogeneous (or Hausdorff) dimension of the manifold M.

What could be the upper bound is not clear: maybe $c'\sqrt{\lambda}$, but it could also be $c'\lambda^{k/2}$, where k is the step of the distribution. In any case, we expect that the proof of such bounds would require the development of new tools, in particular in the geometric measure theory of sub-Riemannian structures.

Appendix A

Technical tools and conventions

A.1 Symplectic geometry

Given a smooth d-dimensional manifold M, the canonical symplectic form on the cotangent bundle T^*M is

$$\omega = d\xi \wedge dx$$

in local symplectic coordinates (x, ξ). The Hamiltonian vector field H_f of a function $f \in C^\infty(M)$ is defined by the relation

$$\omega(H_f, \cdot) = -df(\cdot).$$

Alternatively, we use the notation

$$\vec{f} = H_f.$$

In the coordinates (x, ξ), it reads

$$H_f = \sum_{j=1}^{d} (\partial_{\xi_j} f)\partial_{x_j} - (\partial_{x_j} f)\partial_{\xi_j}.$$

In these coordinates, the Poisson bracket is

$$\{f, g\} = \omega(H_f, H_g) = \sum_{j=1}^{d} (\partial_{\xi_j} f)(\partial_{x_j} g) - (\partial_{x_j} f)(\partial_{\xi_j} g),$$

which is also equal to $H_f g$ and $-H_g f$.

The Hamiltonian lift of a vector field X on M is the function defined by $h_X(x, \xi) = \xi(X(x))$. Given two vector fields X and Y on M, we have $\{h_X, h_Y\} = h_{[X,Y]}$.

A.2 Pseudodifferential calculus

This section is a short reminder on basic properties of pseudodifferential operators. Most proofs can be found in [Hor07a].

A.2.1 Pseudodifferential operators in $\mathbb{R}^d$

Definition A.1. *Let $m \in \mathbb{R}$. The class of symbols of order m, denoted by $S^m(\mathbb{R}^d)$, is the set of complex-valued functions $a \in C^\infty(T^*\mathbb{R}^d)$ such that, for any $\alpha, \beta \in \mathbb{N}^d$, there exists $C_{\alpha\beta} > 0$ such that*

$$\forall (x, \xi) \in T^*\mathbb{R}^d, \qquad |\partial_x^\alpha \partial_\xi^\beta a(x, \xi)| \leqslant C_{\alpha\beta}(1 + |\xi|)^{m-|\beta|}.$$

We set

$$S^{-\infty} = \bigcap_{m\in\mathbb{R}} S^m.$$

Also, $C_b^\infty(\mathbb{R}^d)$ denotes the set of smooth functions on $\mathbb{R}^d$ which are bounded and all of whose derivatives are bounded, and $\mathcal{S}(\mathbb{R}^d)$ denotes the Schwartz space.

Definition A.2 (Elliptic symbols)**.** *Let $m \in \mathbb{R}$. A symbol $a \in S^m(\mathbb{R}^d)$ is elliptic if there exist $C, R > 0$ such that*

$$\forall (x,\xi) \in T^*\mathbb{R}^d, \quad |\xi| \geqslant R \Rightarrow |a(x,\xi)| \geqslant C|\xi|^m,$$

Theorem A.3. *Let $m \in \mathbb{R}$. If $a \in S^m(\mathbb{R}^d)$ and $u \in \mathcal{S}(\mathbb{R}^d)$, the formula*

$$Op_{\mathbb{R}^d}(a)u(x) = (2\pi)^{-d} \iint_{\mathbb{R}^d\times\mathbb{R}^d} e^{i(x-x')\cdot\xi} a\left(\frac{x+x'}{2}, \xi\right) u(x')dx'd\xi$$

defines a function $Op_{\mathbb{R}^d}(a)u$ of $\mathcal{S}(\mathbb{R}^d)$. Moreover, $Op_{\mathbb{R}^d}(a)$ is continuous from $\mathcal{S}(\mathbb{R}^d)$ to $\mathcal{S}(\mathbb{R}^d)$.

The map $a \mapsto \mathrm{Op}_{\mathbb{R}^d}(a)$ is called the *Weyl quantization*. We say that $\mathrm{Op}_{\mathbb{R}^d}(a)$ is *a pseudodifferential operator with Weyl symbol a*. We denote by $\Psi^m(\mathbb{R}^d)$ the set of pseudodifferential operators of order m and we set

$$\Psi^{-\infty} = \bigcap_{m\in\mathbb{R}} \Psi^m(\mathbb{R}^d).$$

A.2.2 Pseudodifferential operators on manifolds

The definitions of the previous section can be extended to manifolds. We consider M a smooth *compact* d-dimensional manifold without boundary, and μ a smooth volume on M. Let $\pi : T^*M \to M$ be the canonical projection.

Definition A.4. *[AG07, Chapter 7] A linear operator $A : C^\infty(M) \to C^\infty(M)$ is called pseudodifferential of order m if, for any local chart $\kappa : U \to V \subset \mathbb{R}^d$, the operator $\widetilde{A} : u \mapsto [A(u\circ\kappa)]\circ\kappa^{-1}$ from $C^\infty(V)$ into $C^\infty(V)$ is pseudodifferential of order m in V, i.e., $\forall \varphi, \psi \in C^\infty(V)$, $\varphi\widetilde{A}\psi \in \Psi^m(\mathbb{R}^d)$. We then write $A \in \Psi^m(M)$.*

Proposition A.5. *If a is* real-valued*, then $Op(a)^* = Op(a)$.*

To a pseudodifferential operator $A \in \Psi^m(M)$, we can associate its *principal symbol* $\sigma_P(A)$ and its *sub-principal symbol* $\sigma_{\mathrm{sub}}(A)$. The subprincipal symbol is usually defined for operators acting on half-densities (this was a discovery of Leray [GKL64], see also [Zwo12, Section 9.1]); here we make the identification $f \leftrightarrow f d\mu^{1/2}$ between functions and half-densities, taking into account that the manifold M is equipped with a half-density. The principal and subprincipal symbols are characterized by the action of pseudodifferential operators on oscillating functions: if $A \in \Psi^m(M)$ and $u(x) = b(x)e^{ikS(x)}$ with b, S smooth and real-valued, then

$$\int_M A(u)\overline{u}d\mu = k^m \int_M \left(\sigma_P(A)(x, S'(x)) + \frac{1}{k}\sigma_{\mathrm{sub}}(A)(x, S'(x))\right) |u(x)|^2 d\mu(x) + O(k^{m-2}).$$

The map

$$(\sigma_P, \sigma_{\mathrm{sub}}) : \Psi^m(M)/\Psi^{m-2}(M) \to S^m_{\mathrm{hom}}(T^*M) \oplus S^{m-1}_{\mathrm{hom}}(T^*M) \tag{A.1}$$

is bijective, where $S^k_{\mathrm{hom}}(T^*M)$ is the space of smooth homogeneous functions of order k defined on the cone $T^*M \setminus 0$ (see Appendix A.2.3). We have the following properties:

- If m is an integer and $A = \sum_{|\alpha|\leqslant m} a_\alpha(x) D^\alpha$ with the convention $D = \frac{1}{i}\partial_x$, then
$$\sigma_P(A) = \sum_{|\alpha|=m} a_\alpha(x)\xi^\alpha.$$
- If $A \in \Psi^\ell(M)$ and $B \in \Psi^m(M)$, then $AB \in \Psi^{\ell+m}(M)$ and $\sigma_P(AB) = \sigma_P(A)\sigma_P(B)$.
- If $A \in \Psi^\ell(M)$ and $B \in \Psi^m(M)$, then $[A,B] \in \Psi^{\ell+m-1}(M)$ and
$$\sigma_P([A,B]) = \frac{1}{i}\{\sigma_P(A), \sigma_P(B)\}.$$
- If X is a vector field on M and X^* is its formal adjoint in $L^2(M,\mu)$, then $X^*X \in \Psi^2(M)$, with $\sigma_P(X^*X) = h_X^2$ and $\sigma_{\mathrm{sub}}(X^*X) = 0$.
- If $a \in S^m(M)$, then, for any $s \in \mathbb{R}$, $\mathrm{Op}(a)$ maps continuously the space $H^s(M)$ to the space $H^{s-m}(M)$.

The characteristic set of $A \in \Psi^m(M)$ is defined by
$$\mathrm{Char}(A) = \{(x,\xi) \in T^*M \setminus \{0\},\ \sigma_P(A)(x,\xi) = 0\}.$$
Finally, the essential support of $A \in \Psi^m(M)$, denoted by $\mathrm{essupp}(A)$, is the complement in T^*M of the points (x,ξ) which have a conic-neighborhood W so that A is of order $-\infty$ in W, i.e.,
$$\forall (N,\alpha,\beta) \in \mathbb{N} \times \mathbb{N}^d \times \mathbb{N}^d, \qquad \sup_{(x,\xi)\in W} |(\partial_\xi^\alpha \partial_x^\beta a)(x,\xi)||\xi|^N < +\infty$$
(this definition depends indeed only on A).

A.2.3 Polyhomogeneous pseudodifferential operators

Sometimes, it is preferable to work with classes of polyhomogeneous symbols and operators.

The class of polyhomogeneous symbols S^m_{phg} is slightly smaller than the class S^m, but it has two main advantages (see [Hor07a], the paragraph before Section 18.6, and [GL20, Appendix A]):

- the principal and subprincipal symbols of a polyhomogeneous pseudodifferential operator are functions (and not equivalence classes as in (A.1)) and they can be read easily on the Weyl symbol;
- they are particularly suited for the definition of microlocal defect measures.

We write $S^m_{\mathrm{hom}}(T^*M)$ for the set of positively homogeneous degree m functions on T^*M: that is, $a \in S^m_{\mathrm{hom}}(T^*M)$ if $a \in C^\infty(T^*M)$ and there exists $R > 0$ such that for any $(x,\xi) \in T^*M$ with $|\xi| \geqslant R$, and any $\lambda \geqslant 1$, we have $a(x,\lambda\xi) = \lambda^m a(x,\xi)$.

We also denote by $S^m_{\mathrm{phg}}(T^*M)$ the set of polyhomogeneous symbols of degree m. Hence, $a \in S^m_{\mathrm{phg}}(T^*M)$ if $a \in C^\infty(T^*M)$, and for any $j \in \mathbb{N}$ there exists $a_j \in S^{m-j}_{\mathrm{hom}}(T^*M)$ such that for any $N \in \mathbb{N}$, $a - \sum_{j=0}^N a_j \in S^{m-N-1}_{\mathrm{phg}}(T^*M)$. We denote by $\Psi^m_{\mathrm{phg}}(M)$ the space of polyhomogeneous pseudodifferential operators of order m on M (see [GL20, Appendix A] for the detailed properties).

Since we work with the Weyl quantization, the principal and subprincipal symbols of $A = \mathrm{Op}(a)$ with $a \sim \sum_{j\leqslant m} a_j$ are simply $\sigma_P(A) = a_m$ and $\sigma_{\mathrm{sub}}(A) = a_{m-1}$.

A.2.4 Wave-front set

Definition A.6. *Let $u \in \mathcal{D}'(M)$. A point $(x_0, \xi_0) \in T^*M \setminus \{0\}$* is not *in the wave-front set $WF(u)$ if there exists a conic neighborhood U of (x_0, ξ_0) such that for any smooth function $\chi \in C_c^\infty(\pi(U))$, in any set of local coordinates, one has*

$$\forall N \in \mathbb{N}, \quad \sup_{\xi \in U} |\widehat{\chi u}(\xi)||\xi|^N < +\infty.$$

This definition is independent of the choice of local coordinates. We say that $u \in \mathcal{D}'(M)$ is *smooth at* (x_0, ξ_0) if $(x_0, \xi_0) \notin WF(u)$. An equivalent definition is

$$WF(u) = \bigcap \{\mathrm{Char}(P),\ Pu \in C^\infty(M)\}$$

where P runs over all pseudodifferential operators of all orders. Therefore:

Theorem A.7. *Singularities are contained in the characteristic manifold:*

$$Pu = 0 \Rightarrow WF(u) \subset Char(P).$$

Proposition A.8. *For any $u \in \mathcal{D}'(M)$, there holds*

$$\pi(WF(u)) = Sing\ supp(u)$$

where $Sing\ supp(u)$ denotes the singular support of u.

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Titre : Équations sous-elliptiques : contrôle, singularités et théorie spectrale.

Mots Clefs : Équations aux dérivées partielles, géométrie sous-Riemannienne, analyse microlocale, théorie spectrale, théorie du contrôle.

Résumé : Dans cette thèse à la frontière entre analyse et géométrie, nous étudions des équations aux dérivées partielles (EDPs) sous-elliptiques en utilisant des outils récents de géométrie sous-Riemannienne et d'analyse microlocale.

Nous étudions tout d'abord la contrôlabilité et l'observabilité d'EDPs sous-elliptiques, en montrant que plus une direction demande de crochets de Lie pour être engendrée, plus la propagation de l'énergie (et donc l'observabilité) se fait lentement dans cette direction. Nos résultats s'appliquent de façon générale aux équations d'ondes sous-elliptiques linéaires, mais aussi à des équations de type Schrödinger et à des équations d'ondes amorties.

Ensuite, nous étudions la propagation des singularités dans les équations d'ondes sous-elliptiques : nous montrons que les singularités ne se propagent que le long des bicaractéristiques nulles et le long des relèvements anormaux extrémaux de courbes singulières. Ce résultat fait donc le lien avec des notions classiques de géométrie sous-Riemannienne. Nous l'illustrons dans le cas Martinet, en construisant des données initiales dont les singularités se propagent le long des courbes singulières à n'importe quelle vitesse entre 0 et 1.

Enfin, nous étudions les fonctions propres de certaines familles de Laplaciens sous-elliptiques, dans la limite des hautes fréquences : nous montrons que leurs limites, appelées limites quantiques, peuvent être décomposées en une infinité de morceaux, correspondant à une infinité de dynamiques classiques sur la variété sous-jacente.

Title: Subelliptic equations: control, singularities and spectral theory.

Keys words: Partial differential equations, sub-Riemannian geometry, microlocal analysis, spectral theory, control theory.

Abstract: In this thesis at the boundary between analysis and geometry, we study some subelliptic partial differential equations (PDEs) with modern tools coming from sub-Riemannian geometry and microlocal analysis.

We first study the controllability and observability of some subelliptic PDEs: we show that in directions requiring more brackets to be generated, the propagation of energy (and hence the observability) takes more time. Our results apply with full generality to linear subelliptic wave equations, but also to some Schrödinger-type and damped wave equations.

Then, we study the propagation of singularities in subelliptic wave equations: we show that singularities propagate only along null-bicharacteristics and abnormal extremal lifts of singular curves. This result makes a bridge with classical notions in sub-Riemannian geometry. We illustrate it in the Martinet case: we construct initial data whose singularities propagate along any singular curve at any speed between 0 and 1.

Finally, we study the eigenfunctions of some families of subelliptic Laplacians, in the high-frequency limit: we show that their limits, called quantum limits, can be decomposed in an infinite number of pieces, corresponding to an infinite number of dynamics on the underlying manifold.