Quelques aspects de l'équation d'Euler incompressible en deux dimensions.

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Introduction

L'équation d'Euler est, avec l'équation de Navier-Stokes, l'une des deux équations fondamentales de la mécanique des fluides, qui est la branche de la physique dédiée à l'étude du comportement des liquides, des gaz et des plasmas, traditionnellement regroupés sous le terme de "fluides". Cette équation a été posée pour la première fois en 1757 par Leonhard Euler pour décrire le mouvement des fluides dits "parfaits", c'est-à-dire non-visqueux. Lorsque le fluide est de plus supposé incompressible, ce qui est en première approximation le cas de l'eau par exemple, elle s'écrit en termes mathématiques

$$\partial_t u + u \cdot \nabla u = -\nabla p \tag{1}$$

où $u : \mathbb{R}^3 \to \mathbb{R}^3$ est la vitesse du fluide et $p : \mathbb{R}^3 \to \mathbb{R}$ modélise la pression du fluide. Cette équation est couplée à la condition d'incompressibilité

$$\nabla \cdot u = 0. \tag{2}$$

Le système (1)-(2) constitue l'équation d'Euler incompressible (en trois dimensions), et il peut être dérivé à partir des principes fondamentaux de la physique. Il ne décrit en principe qu'une approximation des phénomènes physiques observés, puisqu'aucun fluide n'est vraiment parfaitement non-visqueux (à l'exception du cas très particulier des superfluides), ni parfaitement incompressible. Concrètement, il est souvent utilisé en cosmologie pour modéliser les différentes formes de matière qui emplissent l'univers, ou encore en aérodynamique pour les phénomènes de turbulence.

Plus de 250 ans après sa découverte, ce système reste très mal compris, tout comme l'équation de Navier-Stokes incompressible qui constitue son analogue visqueux (c'est-à-dire avec un terme supplémentaire $-\nu\Delta u$ dans le membre de gauche de l'équation (1)). D'un point de vue physique par exemple, certains phénomènes observés dans la vie quotidienne et régis en première approximation par ces équations, comme la formation des vagues ou la turbulence au voisinage des ailes d'avion restent très mystérieux et sont des sujets de recherche actifs. Sur le plan mathématique, c'est encore pire puisque le sens à donner à ces équations n'est même pas clair. Nous allons illustrer maintenant ce dernier point par deux exemples.

D'un côté, des théorèmes de non-unicité de certains types de solutions de l'équation d'Euler (dites solutions "faibles") ont fleuri au cours des vingt dernières années ([Sch93], [Shn97], [DLSJ09]). Ces résultats très paradoxaux décrivent, sur le plan théorique uniquement, des fluides au repos à un certain instant, qui se mettent à s'agiter pendant quelques secondes, avant de retourner au repos. Ces théorèmes n'ont aucune valeur prédictive (selon toute vraisemblance, de tels fluides n'existent dans la nature !), mais montrent seulement la richesse mathématique de l'équation d'Euler ainsi que les limites de la modélisation mathématique.

D'un autre côté, l'existence de solutions "fortes" globales (c'est-à-dire définies pour tout temps) n'a jamais été prouvée. L'énoncé précis de ce problème est pourtant extrêmement simple. Il s'agit de savoir si, pour tout champ de vitesse initial $u_0 \in C^\infty(\mathbb{R}^3) \cap L^2(\mathbb{R}^3)$ avec la condition de décroissance à l'infini $|\partial_x^\alpha u_0(x)| \leq C_{\alpha K}(1+|x|)^{-K}$ pour tous α, K, il existe une solution $u(t,x) \in C^\infty(\mathbb{R}^+ \times \mathbb{R}^3)$ du système (1)-(2) telle que $u(0,\cdot) = u_0$. Notons au passage que l'appartenance de u_0 à l'espace $L^2(\mathbb{R}^3)$ signifie simplement que le fluide doit avoir une énergie finie.

Comme on le voit sur ces exemples, l'équation d'Euler en trois dimensions est, dans toute sa généralité, très difficile à comprendre. Cependant, certaines hypothèses simplificatrices souvent issues d'observations physiques permettent d'en tirer des versions simplifiées, dont on espère pouvoir mieux comprendre la structure. La plus célèbre de ces simplifications est certainement celle qui consiste à considérer que les fluides considérés ont en fait un mouvement planaire. On parle alors de fluides bidimensionnels, et le système (1)-(2), toujours valable mais posé dans $\mathbb{R}^2$ et pas dans $\mathbb{R}^3$, constitue l'équation d'Euler incompressible en deux dimensions. Cette approximation est courante en météorologie où l'on peut parfois considérer que le mouvement de l'air varie peu sur de petites échelles verticales. Même si cela n'est pas tout de suite visible, l'équation d'Euler en deux dimensions ("Euler 2d") est beaucoup plus abordable que son analogue tridimensionnel ("Euler 3d").

C'est précisément sur l'équation d'Euler 2d incompressible que porte ce mémoire. Il existe des espaces de Banach relativement simples dans lesquels cette équation est bien posée, comme $L^\infty \cap L^1(\mathbb{R}^2)$ (théorème de Yudovich [Yud63b]), c'est-à-dire que pour des données initiales dans ces espaces, on a existence et unicité de la solution du système (1)-(2), et continuité de la solution par rapport à la donnée initiale. Cela constitue une différence majeure avec le cas tridimensionnel, et le problème n'est donc pas de montrer l'existence ou l'unicité de solutions, mais plutôt d'étudier leurs propriétés. Cela peut signifier par exemple étudier leur comportement en temps long, comprendre les trajectoires de particules qui suivent le flot des solutions, ou encore montrer des propriétés statistiques sur les solutions.

Une différence importante entre l'équation d'Euler 2d incompressible et son analogue tridimensionnel est sa structure particulièrement simple d'équation de "transport". En fait, en prenant le rotationnel de l'équation (1), il est facile de voir que résoudre Euler 2d incompressible revient à résoudre

$$\partial_t \omega + u \cdot \nabla \omega = 0 \tag{3}$$

où $\omega = \mathrm{rot}\ u$ est un scalaire, qui est appelé vorticité. Cette équation traduit mathématiquement le fait que la vorticité est conservée le long du flot de u. En trois dimensions, si l'on prend aussi le rotationnel de (1), on obtient l'équation (3) avec un terme supplémentaire $\omega \cdot \nabla u$ dans le membre de droite, et c'est de ce terme, parfois appelé d'"étirement de la vorticité" ("vorticity stretching"), que proviennent les difficultés de la 3d.

Le plan de ce mémoire est le suivant. Les parties sont presque entièrement indépendantes et associent des résultats connus à des résultats originaux.

Dans le chapitre 1, nous expliquons des résultats récents [KŠ14], [Zla15] qui quantifient la croissance de $||\nabla\omega||_{L^\infty(\Omega)}$ au cours de l'évolution de l'équation d'Euler 2d. Malgré son apparence purement académique, ce problème est en réalité fondamental dans la compréhension des problèmes d'existence et d'explosion dans les équations d'Euler en 2d et 3d. En 2d, il est classique que la norme $||\nabla\omega||_{L^\infty(\Omega)}$ croît au plus double exponentiellement en temps, mais la question de savoir si cette borne est optimale n'a été résolue que récemment [KŠ14]. Ici, nous améliorons un résultat de Zlatos [Zla15] et nous prouvons que pour $\alpha > 0$, une norme $C^{1,\alpha}$ de la vorticité croît exponentiellement dans le tore pour des données initiales lisses bien choisies.

Dans le chapitre 2, nous étudions une nouvelle classe de solutions autosimilaires dans le domaine $\Omega = \mathbb{R}^2 \backslash \{0\}$. Précisément, nous trouvons toutes les solutions d'Euler 2d de la forme $u(t,x) = t^\alpha U(t^\beta x)$ où $U : \Omega \to \mathbb{R}^2$ est un profil qui ne dépend que de la distance à l'origine (c'est-à-dire de la coordonnée radiale, et pas de la coordonnée angulaire).

Dans le chapitre 3, nous nous intéressons à l'équation d'Euler 2d dans des domaines peu réguliers. Un théorème classique établit l'existence et l'unicité de solutions d'Euler 2d dans $L^\infty(\Omega)$ dès que le domaine Ω a un bord de classe $C^{1,1}$. Récemment, Gérard-Varet et Lacave ont prouvé dans [GVL13] et [GVL15] un résultat d'existence pour Euler 2d dans une classe d'ouverts Ω très généraux, qui peuvent être très irréguliers. Cependant, ils n'ont pas prouvé de résultat d'unicité. Ici, nous décrivons une piste de recherche infructueuse mais toutefois intéressante pour prouver l'unicité dans les domaines à bord lipschitzien, en combinant des idées de transport optimal et de γ-convergence.

Dans le chapitre 4, nous nous intéressons au problème de la préservation de la symétrie par l'équation d'Euler 2d. Dans des domaines symétriques par rapport à un axe, et pour une vorticité initiale ω_0 impaire par rapport à cet axe, on peut montrer que l'équation d'Euler préserve cette symétrie au cours du temps. Ici, on montre que cette configuration est extrêmement instable en étudiant l'exemple du domaine $\mathbb{R}^2 \backslash \{x_1 \leq 0, x_2 = 0\}$.

Enfin, dans le chapitre 5, nous étudions les solutions stationnaires d'Euler 2d, c'est-à-dire les solutions pour lesquelles u (et donc ω) ne dépend pas du temps t. Ces solutions sont intéressantes à double titre. Tout d'abord, elles peuvent s'obtenir par des problèmes de minimisation ([Arn13], [Shn93]) et ont une structure géométrique très riche [CŠ12]. Deuxièmement, il est souvent conjecturé qu'elles constituent des sortes d'attracteurs pour les solutions instationnaires, et qu'elles sont donc fondamentales pour comprendre le comportement en temps long d'Euler 2d. Dans ce chapitre, nous allons nous intéresser à leur stabilité par perturbation de domaine, en utilisant les techniques d'analyse complexe de [GVL13].

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Chapter 1

Growth of vorticity gradient

1.1 Introduction and motivations

The vorticity formulation of the 2d incompressible Euler equation in a domain Ω reads

$$\partial_t \omega + u \cdot \nabla \omega = 0 \tag{1.1}$$

where $u : \Omega \to \mathbb{R}^2$ denotes the velocity of the fluid and $\omega := \operatorname{curl} u : \Omega \to \mathbb{R}$ its vorticity. Compared to the standard formulation (1)-(2), it has the advantage of making the pressure disappear. The unknown is ω and the velocity u appearing in (1.1) is recovered from ω through Biot-Savart law

$$u(x) = \int_\Omega \nabla^\perp G_\Omega(x, y)\omega(y)dy \tag{1.2}$$

where G_Ω is the Green function of the domain Ω solving $-\Delta_y G_\Omega(x, \cdot) = \delta_x$ with Dirichlet conditions on the boundary $\partial\Omega$ and $\nabla^\perp$ is the operator $\nabla^\perp = (\partial_{x_2}, -\partial_{x_1})$. The formulations (1)-(2) and (1.1) can be shown to be equivalent for example in the case where Ω is of regularity $C^{1,1}$ and $\omega \in L^\infty \cap L^1(\Omega)$.

In this chapter, we will quantify the growth in time of $||\nabla\omega||_{L^\infty(\Omega)}$ for particular choices of initial data and domains. At first sight, it may seem unclear why this question is relevant in the study of the 2d Euler equation. We will now give one of its main motivations.

As explained in the introduction, it is still unknown whether every smooth initial datum $u_0 \in C^\infty(\mathbb{R}^3)$ gives rise to a globally defined solution u of 3d Euler. However, Beale, Kato and Majda established in [BKM84] the following far-reaching theorem.

Theorem 1 ([BKM84])**.** *Recall that if an initial velocity field u_0 is in $H^s, s \geq 3$, with $||u_0||_{H^3} \leq N_0$ for some $N_0 > 0$, then there exists $T_0 > 0$ depending only on N_0 such that equations (1)-(2) have a solution in the class*

$$u \in C([0, T]; H^s) \cap C^1([0, T]; H^{s-1}) \tag{1.3}$$

at least for $T = T_0(N_0)$. Let u be such a solution and suppose there is a time T_* such that the solution cannot be continued in the class (1.3) to $T = T_*$. Assume that T_* is the first such time.

Then

$$\int_0^{T_*} ||\omega(t, \cdot)||_{L^\infty} dt = \infty, \tag{1.4}$$

and in particular

$$\limsup_{t \uparrow T_*} ||\omega(t, \cdot)||_{L^\infty} = \infty. \tag{1.5}$$

In 2d, this criterion could not be true, since the L^∞ norm of ω is preserved during the evolution of 2d Euler. But one can notice that, taking the gradient of equation (1.1), we obtain

$$\partial_t \nabla\omega + (u\cdot\nabla)(\nabla\omega) = -\nabla\omega\cdot\nabla u, \tag{1.6}$$

which is very similar to the vorticity formulation of the 3d Euler equation

$$\partial_t \omega + (u\cdot\nabla)\omega = \omega\cdot\nabla u. \tag{1.7}$$

Heuristically, equations (1.6) and (1.7) show that $\nabla\omega$ should play in 2d approximately the same role as ω in 3d. In 3d, theorem 1 roughly says that the growth of ω in L^∞ norm controls the rate of blow-up of the Euler equation. One could therefore imagine that understanding the mechanism of explosion in 3d should be approximately as difficult as understanding the growth of $\nabla\omega$ in 2d.

The following theorem, established by Yudovich [Yud63a], makes the picture clearer in 2d.

Theorem 2 ([Yud63a], [KŠ14]). *Let Ω be a bounded domain with $C^{1,1}$ boundary. Let $\omega_0(x)$ be a C^∞ initial datum for the 2D Euler equation. Then the solution $\omega(t,x)$ satisfies*

$$1+\log\left(1+\frac{||\nabla\omega(t,x)||_{L^\infty}}{||\omega_0||_{L^\infty}}\right) \leq \left(1+\log\left(1+\frac{||\nabla\omega_0||_{L^\infty}}{||\omega_0||_{L^\infty}}\right)\right)\exp(C||\omega_0||_{L^\infty}t) \tag{1.8}$$

for some constant C which may depend only on the domain Ω.

One could ask whether this bound double exponential bound on $||\nabla\omega||_{L^\infty}$ is optimal. The answer to this question was found only very recently, in the seminal paper by Kiselev and Sverak [KŠ14]. They constructed an example of C^∞ solution of (1)-(2) in the unit disk which exhibits the aforementioned double exponential growth.

Zlatoš adapted their construction in [Zla15] to the case of the torus $\mathbb{T}^2$, constructing a solution of (1)-(2) with (single) exponential growth of the vorticity gradient. However, the regularity of the vorticity in his example is only $C^{1,\alpha}$ for $0<\alpha\leq 1$. In the same paper [Zla15], Zlatoš constructs a C^∞ solution of (1)-(2) with a (single) exponential growth of the hessian of the vorticity, that is of $||D^2\omega||_{L^\infty(\mathbb{T}^2)}$. Our goal is to prove that his example also displays an exponential growth of $||D^{1+\alpha}\omega||_{L^\infty(\mathbb{T}^2)}$ for all $0<\alpha\leq 1$, where we obviously have to give a precise definition of what is meant by $||D^{1+\alpha}\omega||_{L^\infty(\mathbb{T}^2)}$. In a certain sense, the limiting case $\alpha=0$ quantifies the growth of the vorticity gradient, and it is therefore the most interesting one, although we do not reach it with this construction.

Let us define the norm that we will use to quantify the growth of $D^{1+\alpha}\omega$.

Definition 1.9. *Let $\Omega\subset\mathbb{R}^n$ be an open set, $f:\Omega\to\mathbb{R}$ be a scalar function and $0<\alpha\leq 1$. We define the homogeneous Hölder semi-norm*

$$|f|_{\dot{C}^{1,\alpha}} = \sup_{x\neq y}\frac{||\nabla f(x)-\nabla f(y)||_\infty}{||x-y||_2^\alpha},$$

where, for a vector $v=(v_1,...,v_n)$, we have set $||v||_\infty = \sup_i|v_i|$ and $||v||_2$ is the usual euclidean norm.

Following Zlatoš [Zla15], we work in $(2\mathbb{T})^2$ instead of the torus $\mathbb{T}^2$ in order to simplify notations. $(2\mathbb{T})^2$ is the square $[-1,1)$ with identified opposite sides.

Theorem 3. *For any $A < \infty$ and any $0 < \alpha \leq 1$, there exist $T_0 \geq 0$ and $\omega_0 \in C^\infty((2\mathbb{T})^2)$ with $||\omega_0||_{L^\infty} = 1$ such that the solution of (1)-(2) satisfies for all $T \geq T_0$,*

$$\sup_{t \leq T} ||\omega(t,\cdot)||_{\dot{C}^{1,\alpha}} \geq e^{AT}.$$

The next section will be devoted to the proof of this theorem.

1.2 Proof of theorem 3

For the sake of completeness, we will recall the whole construction done by Zlatoš in [Zla15]. However, before starting, we prove the following elementary lemma, which will be needed in the proof.

Lemma 1.10. *Let $I \subset \mathbb{R}$ be an open interval containing 0 and $a > 0$ be a positive real number such that $[0,a] \subset I$. If $b > 0$ and $f \in C^\infty(I)$ verify $f(0) = 0$, $f'(0) = 0$ and $f(a) = b$, then for all $0 < \alpha \leq 1$,*

$$|f|_{\dot{C}^{1,\alpha}(I)} \geq ba^{-1-\alpha}$$

Proof. If $b = 0$, there is nothing to prove. We therefore only consider the case $b > 0$. We will proceed by contradiction. Suppose $|f'(x) - f'(y)| < ba^{-1-\alpha}|x-y|^\alpha$ for all $x, y \in I$. Since $f'(0) = 0$, we then have $f'(x) < ba^{-1-\alpha}x^\alpha$ for all $x \in [0,a]$, so that

$$f(a) \leq \int_0^a ba^{-1-\alpha}s^\alpha ds = ba^{-1-\alpha}\frac{a^{1+\alpha}}{\alpha+1} \leq \frac{b}{\alpha+1} < b.$$

This contradicts the assumption $f(a) = b$ and concludes the proof. ☐

Keeping in mind this preliminary lemma, we can start the proof of theorem 3. For $x \in [0,1]^2$ we introduce the notation $Q(x) := [x_1, 1] \times [x_2, 1]$.

The starting point is the following lemma, proved in [Zla15], and which was strongly inspired by an analoguous lemma in [KŠ14].

Lemma 1.11 ([Zla15])**.** *Let $\omega(t,\cdot) \in L^\infty((2\mathbb{T})^2)$ be odd in both x_1 and x_2. If $x_1, x_2 \in [0, 1/2]$, then*

$$u_j(t,x) = (-1)^j \left(\frac{4}{\pi} \int_{Q(2x)} \frac{y_1 y_2}{|y|^4} \omega(t,y) dy + B_j(t,x) \right) x_j \qquad (j = 1,2) \tag{1.12}$$

where, for some universal C,

$$|B_1(t,x)| \leq C||\omega(t,\cdot)||_{L^\infty} \left(1 + \min\left(\log\left(1 + \frac{x_2}{x_1}\right), x_2 \frac{||\nabla\omega(t,\cdot)||_{L^\infty([0,2x_2]^2)}}{||\omega(t,\cdot)||_{L^\infty}} \right) \right),$$

$$|B_2(t,x)| \leq C||\omega(t,\cdot)||_{L^\infty} \left(1 + \min\left(\log\left(1 + \frac{x_1}{x_2}\right), x_1 \frac{||\nabla\omega(t,\cdot)||_{L^\infty([0,2x_1]^2)}}{||\omega(t,\cdot)||_{L^\infty}} \right) \right).$$

It roughly means that in the configuration where the vorticity is odd with respect to both x_1 and x_2, the dynamics of a point driven by the velocity field u is nearly hyperbolic. We will not recall the proof of this lemma, since it is written down in full details in [Zla15].

We now come back to the proof of theorem 3, and firstly fix $A < \infty$ and $0 < \alpha \leq 1$. We will also need a parameter δ, whose value will be chosen at the end of the proof, and which has to be thought as being very small. Pick $\omega_0 : (2\mathbb{T})^2 \to [-1,1]$ which is smooth, odd in both x_1 and x_2, non-negative on $[0,1]^2$ and equal to 1 on a subset of $[0,1]^2$ of measure $1-\delta$, and with $\omega_0(x_1,x_2) = \sin^3(\pi x_1)\sin(\pi x_2)$ when $\min\{|x_1|,|x_2|\} \le \frac{\delta}{4}$.

Let us first prove that

$$\partial_{x_1}\omega(t,0,x_2) = 0 \tag{1.13}$$

for all $t \ge 0$ and $x_2 \in 2\mathbb{T} = [-1,1)$. We set $v(t,x) = \partial_{x_1}\omega(t,x)$. Differentiating the relation $\partial_t\omega + u\cdot\nabla\omega = 0$ with respect to x_1, we get that v satisfies

$$\partial_t v + \partial_{x_1}u_1 v + u_1\partial_{x_1}v + \partial_{x_1}u_2\partial_{x_2}\omega + u_2\partial_{x_2}v = 0. \tag{1.14}$$

We notice that by the symmetry assumptions, $u_1 = 0$ and $\omega = 0$ along the line $x_1 = 0$. Therefore, restricting equation (1.14) to the line $x_1 = 0$, we get

$$\partial_t w + \partial_{x_1}u_1 w + u_2\partial_s w = 0,$$

where we have set $w(t,s) = v(t,0,s)$ for $s \in 2\mathbb{T}$. Since the vector field u is smooth and $w(0,\cdot) \equiv 0$, the only solution of this equation is $w \equiv 0$. Coming back to ω, we see that we have obtained $\partial_{x_1}\omega(t,0,x_2) = 0$ for all $t \ge 0$ and $x_2 \in 2\mathbb{T}$, which is what we wanted to show. Note that by symmetry, we even have $\nabla\omega(t,0) = 0$ for all $t \ge 0$.

We now take $T \ge T_0 = \frac{1}{A}\left|\log\frac{\delta}{4}\right|$, so that $e^{-AT} \le \frac{\delta}{4}$ and consider $X(t)$ solving $X'(t) = u(t,X(t))$ with $X(0) = (e^{-AT},\eta)$, where $\eta \le e^{-AT}$ will be fixed later. We also introduce $T' := \min\{T,T^*\}$, with T^* the exit time of X from the square $[0,e^{-AT}]^2$.

In the case where

$$\sup_{t\le T} ||\nabla\omega(t,\cdot)||_{L^\infty([0,2\exp(-AT)]^2)} > e^{AT},$$

since $\nabla\omega(t,0) = 0$, we obtain $\sup_{t\le T}||\omega(t,\cdot)||_{\dot{C}^{1,\alpha}} \ge (2\sqrt{2})^{-\alpha}e^{(1+\alpha)AT}$, which is greater than e^{AT} for $T \ge T_0$, possibly after having reduced δ. This concludes the proof of theorem 3 in this case.

Let us therefore suppose in the sequel

$$\sup_{t\le T} ||\nabla\omega(t,\cdot)||_{L^\infty([0,2\exp(-AT)]^2)} \le e^{AT}.$$

Since $X(t) \in [0,e^{-AT}]^2$ for $t \le T'$, it follows that

$$x_2||\nabla\omega(t,\cdot)||_{L^\infty([0,2x_2]^2)} \le 1 \tag{1.15}$$

when $t \le T'$ and $x = X(t)$. The same estimate also holds with x_2 replaced by x_1. Hence, equality (1.12) holds with $|B_j(t,x)| \le 2C$ for $t \le T'$.

An important remark that what first made in [KŠ14] is that for the kind of ω_0 that we have chosen (odd in both x_1 and x_2, and with a proportion at least $1-\delta$ of the square $[0,1]^2$ covered by points of vorticity 1), the integral in (1.12) multiplied by $4/\pi$ is not less than $\frac{1}{D}|\log\delta|$ for $x \in [0,\delta]^2$ and some universal constant $D > 0$. Hence, for $t \le T'$,

$$u_1(t,X(t)) \le -\left(\frac{1}{D}|\log\delta| - 2C\right)X_1(t),$$

$$u_2(t, X(t)) \geq \left(\frac{1}{D}|\log\delta| - 2C\right) X_2(t). \tag{1.16}$$

From now, we suppose $\delta < e^{-2CD}$, so that $u_1(t, X(t)) \leq 0$ for $t \leq T'$, and hence $X_1(T') < e^{-AT}$. The equation (1.16) gives that for $t \leq T'$,

$$X_2(t) \geq \eta \exp\left(\left(\frac{1}{D}|\log\delta| - 2C\right) t\right) \tag{1.17}$$

We now fix $\eta = \exp\left(-AT + 2CT - \frac{T}{D}|\log\delta|\right)$, which is less than $\exp(-AT)$ for δ small enough. For this value of η, we directly infer from (1.17) that $T' \leq T$ and hence $X_2(T') = e^{-AT}$ by definition of T'.

By lemma 1.11 and (1.15), we note that

$$\left|\frac{d}{dt}[\log X_1(t) + \log X_2(t)]\right| \leq 4C$$

for $t \leq T'$. Therefore

$$\log X_1(T') \leq \log X_1(0) - \log X_2(T') + \log X_2(0) + 4CT' \leq \left(-A + 6C - \frac{1}{D}|\log\delta|\right) T$$

Since the vorticity is transported along the flow of u, we also notice that

$$\omega(T', X(T')) = \omega_0(X(0)) = \sin^3(\pi e^{-AT})\sin(\pi\eta) \geq \eta e^{-3AT} = \exp\left(-4AT + 2CT - \frac{T}{D}|\log\delta|\right).$$

We can now apply lemma 1.10 to $f(s) = \omega(T', s, e^{-AT})$. We know by (1.13) that $f'(0) = 0$ and, by definition of ω_0, we also have $f(0) = 0$. Using for a the point $X(T')$, we obtain

$$||\omega(t,\cdot)||_{\dot{C}^{1,\alpha}} \geq ||f||_{\dot{C}^{1,\alpha}} \geq \exp\left(-4AT + 2CT - \frac{T}{D}|\log\delta| + (1+\alpha)(AT - 6CT + \frac{T}{D}|\log\delta|)\right)$$

Finally, we choose δ very small, so that this last expression is greater than e^{AT}, which means

$$\delta < \exp\left(\frac{D}{\alpha}((4+6\alpha)C + (4-\alpha)A)\right).$$

For such a choice,

$$\sup_{t\leq T} ||\omega(t,\cdot)||_{\dot{C}^{1,\alpha}} \geq e^{AT}.$$

for $T \geq T_0$. This concludes the proof of theorem 3.

Chapter 2

Self-similar solutions in $\mathbb{R}^2\backslash\{0\}$

The study of self-similar solutions of the equations of fluid mechanics traces back at least to Leray [Ler34]. In this paper, Leray raises the question of the existence of solutions of the Navier-Stokes equation

$$\partial_t u + u\cdot\nabla u - \nu\Delta u = -\nabla p, \qquad \nabla\cdot u = 0 \tag{2.1}$$

of the form

$$u(t,x) = \frac{1}{\sqrt{2a(T-t)}}U\left(\frac{x}{\sqrt{2a(T-t)}}\right) \tag{2.2}$$

where $T\in\mathbb{R}$ and $a>0$. One of the interesting features of such possible solutions is that they develop a singularity at time T. However, it was proved in [NRŠ96] that under the additional assumption (which is in particular verified if u is of finite energy) that $U\in L^3(\mathbb{R}^3)$, there is no such self-similar solution.

The quest for self-similar solutions is much more natural in the case of the Navier-Stokes equation than for the Euler equation. One can see it through scaling arguments, and we will explain it now, partly following Terence Tao in [Tao07]. The main observation, which is a mathematical translation of the phenomenon of turbulence, is that the behaviour of the 3d Navier-Stokes equation at fine scales is much more nonlinear than at coarse scales. In fact, the 3d Navier-Stokes equation obeys a scaling invariance, which means that if (u,p) is a solution of the Navier-Stokes equation (2.1), then $u^{(\lambda)}(t,x)=\frac{1}{\lambda}u(\frac{t}{\lambda^2},\frac{x}{\lambda})$ is also a solution of (2.1) with associated pressure $p^{(\lambda)}(t,x)=\frac{1}{\lambda^2}p(\frac{t}{\lambda^2},\frac{x}{\lambda})$. Moreover, this scaling is the only one for which such a property holds. The parameter $\lambda>1$ should be thought as being large, and, still following Terence Tao [Tao07], the transformation from u to $u^{(\lambda)}$ can be imagined as a magnifying glass, "taking fine-scale behaviour of u and matching it with an identical (but rescaled, and slowed down) coarse-scale behaviour of $u^{(\lambda)}$". For the Euler equation, there is a whole one-parameter family of scalings $u^{(\lambda,\alpha)}(t,x)=\frac{1}{\lambda}u(\frac{t}{\lambda^{\alpha+1}},\frac{x}{\lambda^{\alpha}})$ which preserve the fact of being a solution. Therefore, in the case of the Euler equation, there is no natural relation to couple the space and time variables as in (2.2) in order to look for self-similar solutions.

In the sequel, we will be working in $\Omega=\mathbb{R}^2\backslash\{0\}$. We fix $\alpha>0$ a positive real number and we look for all solutions of 2d Euler in Ω of the form

$$u(t,x) = t^{\alpha}U(t^{\beta}x) \tag{2.3}$$

with $\alpha,\beta\in\mathbb{R}$ and U a vector field only depending on the distance to the origin, which means that for $r>0$ and $\theta\in[0,2\pi[$,

$$U(r,\theta) = f(r)\overrightarrow{e_r} + g(r)\overrightarrow{e_\theta}, \tag{2.4}$$

where $(\overrightarrow{e_r}, \overrightarrow{e_\theta})$ is the standard basis of the polar coordinates.

We will in fact show the following theorem.

Theorem 4. *The field u is a (globally defined) self-similar solution of 2d Euler in $\mathbb{R}^2\backslash\{0\}$ verifying (2.3)-(2.4) if and only if we are in one of the following three cases.*

1. *There exists $K \in \mathbb{R}$ such that*
$$f \equiv 0 \quad \text{and} \quad g(r) = Kr^{-\frac{\alpha}{\beta}}. \tag{2.5}$$

2. *$\beta = 0$, $\alpha = -1$, and there exists $c \neq 0$ and $K \in \mathbb{R}$ such that*
$$f(r) = \frac{c}{r} \quad \text{and} \quad g(r) = \frac{K}{r}\exp\left(\frac{r^2}{c}\right). \tag{2.6}$$

3. *$\beta \neq 0$, $\alpha + \beta = -1$ and there exists $c \neq 0$ such that $\frac{c}{\beta} > 0$ and $K \in \mathbb{R}$ and*
$$f(r) = \frac{c}{r} \quad \text{and} \quad g(r) = \frac{K}{r}\left(\frac{\beta r^2 + c}{\beta + c}\right)^{1+\frac{1}{2\beta}}. \tag{2.7}$$

Remark 2.8. *We must already say that the tangency condition $u \cdot n = 0$ which is usually required for the solutions of the Euler equation in domains with boundary is only satisfied for (2.5).*

2.1 The ODE satisfied by g

Since the field u is incompressible, we must have $\nabla \cdot U = 0$, and therefore
$$\frac{1}{r}\frac{\partial(rf)}{\partial r} + \frac{\partial g}{\partial \theta} = 0, \tag{2.9}$$
from which it immediately follows that $f(r) = c/r$ for some $c \in \mathbb{R}$. Hence, we obtain
$$U(r,\theta) = \frac{c}{r}\overrightarrow{e_r} + g(r)\overrightarrow{e_\theta}, \tag{2.10}$$
and we can use it in equation (2.3) to get
$$u(t,x) = \frac{ct^{\alpha-\beta}}{r}\overrightarrow{e_r} + t^\alpha g(t^\beta r)\overrightarrow{e_\theta}. \tag{2.11}$$

We now compute $\partial_t u$ and $u \cdot \nabla u$:
$$\partial_t u = (\alpha - \beta)t^{\alpha-\beta-1}\frac{c}{r}\overrightarrow{e_r} + \alpha t^{\alpha-1}g(t^\beta r)\overrightarrow{e_\theta} + \beta t^{\alpha+\beta-1} r g'(t^\beta r)\overrightarrow{e_\theta}. \tag{2.12}$$

$$\begin{aligned} u \cdot \nabla u &= \left(\frac{ct^{\alpha-\beta}}{r}\frac{\partial}{\partial r} + \frac{t^\alpha g(t^\beta r)}{r}\frac{\partial}{\partial \theta}\right)\left(\frac{ct^{\alpha-\beta}}{r}\overrightarrow{e_r} + t^\alpha g(t^\beta r)\overrightarrow{e_\theta}\right) \\ &= -t^{2(\alpha-\beta)}\frac{c^2}{r^3}\overrightarrow{e_r} + t^{2\alpha}\frac{c}{r}g'(t^\beta r)\overrightarrow{e_\theta} + t^{2\alpha-\beta}\frac{c}{r^2}g(t^\beta r)\overrightarrow{e_\theta} - t^{2\alpha}\frac{g(t^\beta r)^2}{r}\overrightarrow{e_r}, \end{aligned}$$
where we used the rules of derivation
$$\partial\overrightarrow{e_r}/\partial r = 0, \quad \partial\overrightarrow{e_r}/\partial\theta = \overrightarrow{e_\theta}, \quad \partial\overrightarrow{e_\theta}/\partial r = 0 \;\text{ et }\; \partial\overrightarrow{e_\theta}/\partial\theta = -\overrightarrow{e_r}.$$

If we put these expressions in the Euler equation

$$\partial_t u + u \cdot \nabla u = \nabla p = \frac{\partial p}{\partial r}\overrightarrow{e_r} + \frac{1}{r}\frac{\partial p}{\partial \theta}\overrightarrow{e_\theta}, \tag{2.13}$$

we obtain the derivatives for the pressure

$$\frac{\partial p}{\partial r} = (\alpha - \beta)t^{\alpha-\beta-1}\frac{c}{r} - t^{2(\alpha-\beta)}\frac{c^2}{r^3} - t^{2\alpha}\frac{g(t^\beta r)^2}{r} \tag{2.14}$$

$$\frac{\partial p}{\partial \theta} = \alpha r t^{\alpha-1} g(t^\beta r) + \beta t^{\alpha+\beta-1} r^2 g'(t^\beta r) + c t^{2\alpha} g'(t^\beta r) + t^{2\alpha-\beta}\frac{c}{r} g(t^\beta r). \tag{2.15}$$

Of course, it is necessary that $\int_0^{2\pi} \frac{\partial p}{\partial \theta} d\theta = 0$, which can be rewritten, since thanks to equation (2.15) the quantity $\frac{\partial p}{\partial \theta}$ does not depend on θ,

$$\alpha r t^{\alpha-1} g(t^\beta r) + \beta t^{\alpha+\beta-1} r^2 g'(t^\beta r) + c t^{2\alpha} g'(t^\beta r) + t^{2\alpha-\beta}\frac{c}{r} g(t^\beta r) = 0. \tag{2.16}$$

We can simplify this equation by performing the change of variables $t^\beta r \to r$, and we get the following ODE on g :

$$\alpha r t^{\alpha-\beta-1} g(r) + \beta t^{\alpha-\beta-1} r^2 g'(r) + c t^{2\alpha} g'(r) + t^{2\alpha}\frac{c}{r} g(r) = 0. \tag{2.17}$$

If $c \neq 0$, since this equation has to be satisfied for all $t > 0$, it is necessary that $\alpha - \beta - 1 = 2\alpha$, that is $\alpha + \beta + 1 = 0$. In the sequel, we suppose that it is the case. We finally obtain the following ODE on g :

$$(\beta r^2 + c)g'(r) + \left(\alpha r + \frac{c}{r}\right) g(r) = 0. \tag{2.18}$$

If $c = 0$, equation (2.18) also derives from (2.17), but without the condition that $\alpha + \beta + 1 = 0$. We will deal with this case separately.

2.2 Solution of the ODE on g

The form of the solutions of equation (2.18) on $\mathbb{R}^+$ depends on the parameters α, β and c. We have to distinguish several cases in order to find all possible solutions.

The first case is the case $c = 0$. We notice that if g is not trivial, then $\beta \neq 0$. This is assumed to hold in the sequel. Let us also suppose for the moment that $\alpha \neq 0$. Then the solutions to (2.18) are given by

$$g(r) = g(1) r^{-\frac{\alpha}{\beta}}. \tag{2.19}$$

In the case $\alpha = 0$, it not difficult to see that the solutions to (2.18) also have this form. This finishes the first case $c = 0$. In all other cases, we therefore implicitly assume that $c \neq 0$.

The second case is the case $\beta = 0$, $c \neq 0$. In this case, the solutions of (2.18) are easily found to be

$$g(r) = \frac{g(1)}{r} \exp\left(\frac{r^2 - 1}{c}\right). \tag{2.20}$$

The third case is the case $\beta \neq 0$, $c \neq 0$ and $\frac{c}{\beta} > 0$. As before, the equation (2.18) is integrable and its solutions are the $g(r)$ verifying

$$g(r) = \frac{g(1)}{r} \left(\frac{\beta r^2 + c}{\beta + c}\right)^{1+\frac{1}{2\beta}}. \tag{2.21}$$

The last case is the case $\beta \neq 0$, $c \neq 0$ and $\frac{c}{\beta} < 0$. Here, we formally find that

$$g(r) = \frac{g(1)}{r} \exp\left(\frac{\beta - \alpha}{\beta} \int_1^r \frac{s}{s^2 + \frac{c}{\beta}} ds\right),$$

but the inner integral cannot make sense for every $r > 0$. Hence, in this case, there is no solution of (2.18) defined on $(0, +\infty)$.

We have to verify that the expressions (2.19), (2.20) and (2.21) derive from a pressure, which means that there exists p verifying the equations (2.14) and (2.15) for the expression of g given by (2.19), (2.20) or (2.21). Since we already know that the right-hand side of (2.15), namely

$$\alpha r t^{\alpha-1} g(t^\beta r) + \beta t^{\alpha+\beta-1} r^2 g'(t^\beta r) + c t^{2\alpha} g'(t^\beta r) + t^{2\alpha-\beta} \frac{c}{r} g(t^\beta r),$$

is equal to zero, it is sufficient to find $p(t, r, \theta)$ not depending on θ and satisfying

$$\frac{\partial p}{\partial r} = (\alpha - \beta) t^{\alpha-\beta-1} \frac{c}{r} - t^{2(\alpha-\beta)} \frac{c^2}{r^3} - t^{2\alpha} \frac{g(t^\beta r)^2}{r}. \tag{2.22}$$

With the expressions of g that were found above in (2.19), (2.20) and (2.21), at a fixed time t, equation (2.22) reduces to an ODE in r which can be integrated for $r \in (0, +\infty)$.

Hence, the proof of theorem 4 is complete.

Remark 2.23. *One can note that for any $\alpha \in \mathbb{R}$, the self-similar solution*

$$u(t, x) = \frac{t^\alpha}{r} \overrightarrow{e_r} \tag{2.24}$$

has the additional property that it is irrotational. It is due to the fact that we removed the origin.

Remark 2.25. *As already mentioned, the tangency to the boundary condition $u \cdot n = 0$ is NOT satisfied, even in a weak sense, except for the solutions given by (2.5).*

Remark 2.26. *None of the solutions listed in theorem 4 lies in any L^p space, and in particular none of them has finite energy. However, they all have "weak singularities" in 0, in the sense they belong to some $L^{p,\infty}_{loc}$ spaces. For example, the solution (2.6) is in $L^{2,\infty}(K)$ for every compact set K.*

Chapter 3

Uniqueness in rough domains

The Cauchy problem for the 2d Euler equation in bounded domains is often considered as being well understood since Wolibner established in [Wol33] its well-posedness for smooth initial data and Yudovich proved in [Yud63a] the existence and uniqueness of weak solutions for bounded vorticities. However, these results only apply to the case of regular enough domains, and the critical regularity for the boundary of the domain happens to be $C^{1,1}$. The reason for it is that the proofs heavily rely upon elliptic regularity estimates that do not hold for more singular domains.

Remark 3.1. *In the context of smooth bounded domains Ω, a solution u of the 2d Euler equation in Ω with initial datum u_0 is a vector field $u(t,x)$ which satisfies*

$$\partial_t u + u \cdot \nabla u = -\nabla p, \quad \textit{and} \quad \nabla \cdot u = 0, \tag{3.2}$$

supplemented with the initial and tangency conditions

$$u(0,\cdot) = u_0 \quad \textit{and} \quad u \cdot n = 0, \tag{3.3}$$

where n is a unitary vector normal to Ω.

Recently, Gérard-Varet and Lacave proved in [GVL13] and [GVL15] a result of existence of weak solutions for bounded vorticities in a very large set of singular domains. Let us describe their result. The open sets Ω considered are obtained by removing from a simply connected domain $\tilde{\Omega}$ a finite number of obstacles $C^1, ..., C^k$. More precisely they can be written

$$\Omega := \tilde{\Omega} \backslash \left(\cup_{i=1}^{k} C^i \right), \quad k \in \mathbb{N} \tag{3.4}$$

with the following assumptions

(H1) $\tilde{\Omega}$ is a bounded simply connected domain.

(H2) $C^1, ..., C^k$ are disjoint connected compact subsets of $\tilde{\Omega}$, none of which is reduced to a point.

Within this setting, it is possible to establish the existence of global weak solutions of the Euler equation with L^p vorticity. In our case, we will only focus on the case $p = \infty$, since this is the only case for which uniqueness is known in smooth domains [Yud63b]. We consider initial data satisfying

$$u^0 \in L^2(\Omega), \quad \text{curl } u^0 \in L^\infty(\Omega), \quad \text{div } u^0 = 0, \quad u^0 \cdot n|_{\partial\Omega} = 0. \tag{3.5}$$

Because of the irregularity of Ω, the conditions $u^0 \cdot n|_{\partial\Omega} = 0$ and div u^0 have to be understood in a weak sense : for any $\phi \in C^1_c(\mathbb{R}^2)$,

$$\int_\Omega u^0 \cdot \nabla\phi = -\int_\Omega \text{div } u^0 \phi = 0. \tag{3.6}$$

Similarly to (3.6), the weak form of the divergence free condition and of the tangency condition on the Euler solution u will read :

$$\forall \phi \in \mathcal{D}\left([0,+\infty); C^1_c(\mathbb{R}^2)\right), \quad \int_{\mathbb{R}^+}\int_\Omega u \cdot \nabla\phi = 0. \tag{3.7}$$

Finally, the weak form of the momentum equation on u is :

$$\forall \phi \in \mathcal{D}\left([0,+\infty) \times \Omega\right) \text{ with div } \phi = 0, \quad \int_0^\infty \int_\Omega (u \cdot \partial_t \phi + (u \otimes u) : \nabla\phi) = -\int_\Omega u^0 \cdot \phi(0,\cdot). \tag{3.8}$$

The main theorem of [GVL13] is

Theorem 5. *Assume that Ω is of type (3.4), with (H1)-(H2). Let u^0 be as in (3.5). Then there exists*

$$u \in L^\infty(\mathbb{R}^+; L^2(\Omega)), \quad \textit{with curl } u \in L^\infty(\mathbb{R}^+ \times \Omega)$$

which is a global weak solution of (3.2)-(3.3) in the sense of (3.6)-(3.7)-(3.8).

In a few words, this existence theorem follows from a property of domain continuity for the Euler equations. Namely, the strong solutions u_n of the Euler equations (with smooth initial vorticity ω_0^n approximating ω_0) in smooth approximate domains Ω_n converge to a solution u in Ω. By approximate domains, we mean converging to Ω in the Hausdorff topology. These approximate domains read

$$\Omega_n := \tilde{\Omega}_n \backslash \left(\cup_{i=1}^k \overline{O_n^i}\right) \tag{3.9}$$

for some smooth Jordan domains $\tilde{\Omega}_n$ and O_n^i. A keypoint is the so-called γ-convergence of Ω_n to Ω.

We will now give the precise definition of regularization that we use in the sequel.

Definition 3.10. *Assume that Ω is of type (3.4), with (H1)-(H2) and u_0 in an initial datum in Ω satisfying (3.5). We call regularization of Ω and u_0 any pair of sequences $(\Omega_n)_{n\in\mathbb{N}}$, $(\omega_0^n)_{n\in\mathbb{N}}$ verifying the following properties.*

1. *For all $n \in \mathbb{N}$ and $i = 1, ..., k$, $\tilde{\Omega}_n$ and $\overline{O_n^i}$ are smooth Jordan domains such that the number of connected components of $D\backslash\Omega_n$ is bounded in n.*

2. *The convergences $d_H(\Omega_n, \Omega) \to 0$ and $d_H(O_n^i, O^i) \to 0$ hold, where d_H is the Hausdorff distance between open sets (cf appendix B of [GVL13]).*

3. *For all $n \in \mathbb{N}$, $\omega_0^n \in L^\infty(\Omega_n)$. Moreover, for all $p \in [1, +\infty]$, we have $\omega_0^n \rightharpoonup \omega_0 :=$ curl weakly in $L^p(D)$ and $||\omega_0^n||_{L^p(D)} \leq ||\omega_0||_{L^p(D)}$ where D is a large compact set containing Ω and all Ω_n's.*

Our goal was initially to prove that in bounded domains Ω of type (3.4), with (H1)-(H2), and for bounded vorticity ω, the solution u to the 2d Euler equation constructed above does not depend on the choice of the open sets Ω_n of regularization of Ω or on the vorticities ω_n which approximate ω. This would have been a big step towards proving uniqueness for all such domains Ω. But we did not manage to prove it. However, we feel that it is worth explaining the few things we understood on this problem, and therefore we devote this chapter to it. Since there is no positive result at the end, this chapter is not as formal as the others, and sometimes we will rely on purely heuristic arguments. However, we made efforts to try to keep it as comprehensible and rigorous as possible.

Let us first make precise the problem that we consider.

Problem 3.11. *Assume that Ω is of type (3.4), with (H1)-(H2). Let u^0 as in (3.5). Determine whether there exists $u \in L^\infty(\mathbb{R}^+; L^2(\Omega))$ with curl $u \in L^\infty(\mathbb{R}^+ \times \Omega)$ that satisfies the following property : for every regularization (in the sense of Definition 3.10) Ω_n, ω_0^n of Ω, ω_0, the unique global solution u_n of the 2d Euler equation in Ω_n with initial vorticity ω_0^n strongly converges to u without any extraction of subsequence. The strong convergence used in this context is the strong $L^2(0,T;D)$ convergence for all $T > 0$ and for a large open set D fixed from the beginning, which contains Ω and all the Ω_n's for n large enough.*

The use of the strong $L^2(0,T;D)$ convergence is natural in this context since the proof of Theorem 5 consists in considering a regularization $(\Omega_n), (\omega_0^n)$ of Ω, ω_0 and in using compactness to find an extraction σ for which $u_{\sigma(n)}$ strongly converges in $L^2(0,T;D)$ for all $T > 0$ to some u, which finally happens to be a weak solution of the 2d Euler equation in Ω with initial datum ω_0. In some sense, our problem is to show that u does not depend on the regularization and that σ can be taken to be the identity.

3.1 Heuristic considerations

Heuristically, if the answer to Problem 3.11 is yes, any proof should use a theorem of the following flavour.

(Meta)-theorem 3.12. *Fix $T > 0$. If U and V are two smooth open sets at very small Hausdorff distance of Ω (possibly depending on T) and $\omega_0 \in C^\infty(U)$ and $\omega_0' \in C^\infty(V)$ are two initial data that are close in some chosen norm, then the solutions of the Euler equation in U (resp. V) with initial data ω_0 (resp. ω_0') denoted by $\omega(t,\cdot)$ and $\omega'(t,\cdot)$ remain close in the same norm on the whole time interval $[0,T]$.*

This meta-theorem means that ω and ω' cannot diverge from each other too quickly if they were very close at the beginning. Since ω is driven by the vector field u, we see that to prove such a result, it would be necessary to show that if two distributions of vorticity $\omega \in C^\infty(U)$ and $\omega' \in C^\infty(V)$ are close enough in some norm, then the vector fields u and v that they generate respectively in U and V are also somehow close in some other norm. It happens that this result holds true if one restricts to domains Ω (approximated by U and V) which satisfy a property of ε-cone, as proved by Savaré and Schimperna in [SS02].

To state their result properly, we need to recall the definition of the ε-cone property.

Definition 3.13 ([HP06], Definition 2.4.1)**.** *Let y be a point of $\mathbb{R}^2$, ξ be a unitary vector and $\varepsilon > 0$. We call cone of vertex y, of direction ξ and of dimension ε the cone denoted by $C(y,\xi,\varepsilon)$ defined by*

$$C(y,\xi,\varepsilon) = \{z \in \mathbb{R}^2, (z-y,\xi) \geq \cos(\varepsilon)|z-y| \quad \text{and} \quad 0 < |z-y| < \varepsilon\}.$$

Note that the vertex y does not belong to $C(y,\xi,\varepsilon)$. We say that an open set Ω has the ε-cone property if

$$\forall x \in \partial\Omega,\ \exists \xi_x \text{ unitary vector such that } \quad \forall y \in \overline{\Omega} \cap B(x,\varepsilon), \quad C(y,\xi_x,\varepsilon) \subset \Omega.$$

Remark 3.14. *It is possible to show ([HP06], Theorem 2.4.7) that an open set Ω with a bounded frontier has the ε-cone property if and only if its boundary is lipschitzian. Therefore, the ε-cone property is far weaker than the classical condition of being $C^{1,1}$ needed to apply Yudovich's theory.*

We can now state (a slightly simplified version of) the theorem of Savaré and Schimperna [SS02].

Theorem 6 ([SS02], Theorem 1)**.** *Let Ω be an open set satisfying the ε-cone property, Ω_n a sequence of open sets all contained in a large open set D, $\omega \in L^2(D)$. Then there exists a constant C depending only on Ω such that*

$$||\psi^{\omega}_{\Omega_n} - \psi^{\omega}_{\Omega}||_{H^1(D)} \leq C||\omega||^{1/2}_{H^{-1}(D)}||\omega||^{1/2}_{L^2(D)} d_H(\Omega_n, \Omega)^{1/2}, \tag{3.15}$$

where we denoted by ψ^{ω}_{Ω} the solution of $\Delta\psi = \omega$ in Ω with Dirichlet boundary condition (and similarly for Ω_n).

Remark 3.16. *The H^{-1} norm which is used in $D \subset \mathbb{R}^N$ in this section is the norm given by*

$$||\omega||_{H^{-1}(D)} := \inf\{\sum_{0\leq i\leq N} ||f_i||^2_{L^2}\}$$

where the inf is taken over all the decompositions

$$\omega = f_0 + \sum_{1\leq i\leq N} \frac{\partial f_i}{\partial x_i}, \quad f_0, ..., f_N \in L^2(D).$$

It might seem unclear why Theorem 6 is related to our problem. In fact, since the velocity field u in the Euler equation satisfies $\mathrm{div}(u) = 0$, there exists $\psi : \mathbb{R} \times \Omega \to \mathbb{R}$ such that $u = \nabla^{\perp}\psi$ where $\nabla^{\perp} = (\partial_{x_2}, -\partial_{x_1})$. The function ψ is called the stream function. The definition of the vorticity $\omega = \mathrm{curl}\ u$ implies that ω and ψ are related by $\Delta\psi = \omega$. We now see how theorem 6 comes into play. Keeping the same notation as in the theorem, the left hand side of inequality (3.15) is in fact directly related to the L^2 norm of the difference of the velocities computed in Ω_n and in Ω. In other words, the inequality (3.15) gives us a control on the difference of two velocities computed in different domains from the same distribution of vorticity.

We borrow from [HP06] a second result that allows to extend theorem 6 to the case where the distribution of vorticities are also distinct.

Theorem 7 ([HP06], proposition 3.2.1)**.** *There exists a constant C depending only on the big open set D such that for all open set $W \subset D$, we have*

$$||\psi^f_W||_{H^1_0(D)} \leq C||f||_{H^{-1}(D)}. \tag{3.17}$$

We now use the triangular inequality to get

$$||\psi^{\omega_n}_{\Omega_n} - \psi^{\omega}_{\Omega}||_{H^1(D)} \leq ||\psi^{\omega_n}_{\Omega_n} - \psi^{\omega}_{\Omega_n}||_{H^1(D)} + ||\psi^{\omega}_{\Omega_n} - \psi^{\omega}_{\Omega}||_{H^1(D)},$$

which we combine with inequalities (3.15) and (3.17) to obtain

$$||\psi^{\omega_n}_{\Omega_n} - \psi^{\omega}_{\Omega}||_{H^1(D)} \leq C||\omega_n - \omega||_{H^{-1}(D)} + C||\omega||^{1/2}_{H^{-1}(D)}||\omega||^{1/2}_{L^2(D)} d_H(\Omega_n, \Omega)^{1/2}. \tag{3.18}$$

For a fixed Ω and a fixed ω, we set $\varepsilon_n = C||\omega||^{1/2}_{H^{-1}(D)}||\omega||^{1/2}_{L^2(D)} d_H(\Omega_n, \Omega)^{1/2}$, and we know that $\varepsilon_n \to 0$ as $n \to \infty$. We write the velocities $u_n = \nabla^{\perp}\psi^{\omega_n}_{\Omega_n}$ and $u = \nabla^{\perp}\psi^{\omega}_{\Omega}$, so that the inequality (3.18) becomes

$$||u_n - u||_{L^2(D)} \leq C||\omega_n - \omega||_{H^{-1}(D)} + \varepsilon_n. \tag{3.19}$$

This inequality is precisely what we were looking for : it controls the difference of the velocity fields by the difference of the vorticities. Moreover, the norms seem to be particularly well adapted : for the velocities, the energy norm is the one we want to control to get the strong $L^2(0,T;D)$ convergence which is involved in the statement of problem 3.11, and for the vorticities, the $H^{-1}(D)$ seems to be the natural counterpart to the L^2 norm for the velocities because of the relation $\omega = \text{curl } u$.

Recall that our initial idea was to establish an inequality like (3.19) and then to prove that this inequality implies that the vorticities ω_n and ω cannot diverge from each other too quickly by a kind of Gronwall lemma. However, we will now explain why inequality (3.19) alone cannot imply that the vorticities stay close, and show that a second inequality, which does not hold in our case, is necessary to establish this result.

To get intuition, let us imagine that at a given time t, the vorticities ω_n and ω in inequality (3.19) are very concentrated bumps centered on different points x and y which are very close. We suppose moreover that the supports of these bumps do not intersect. We focus on the short time evolution of these two bumps under the velocity fields u_n and u. The point is that inequality (3.19) alone (which means that we do not use anything else than this inequality and the fact that the vorticities are transported by the velocities) does not tell us that ω_n and ω stay close to each other, even during a very short time. To see it, imagine that $u_n = u$. Therefore inequality (3.19) is automatically satisfied. However, if $u_n = u$ varies very quickly in time, it can happen that its values around the bump of ω are very different from its values around the bump of ω_n since we supposed that the supports of ω and ω_n do not intersect, although very close. Hence, even if they are very close and driven by the same velocity field, the vorticities may quickly diverge.

To put it into a rigorous form, we will now describe a very interesting paper of Loeper [Loe06], which uses all these ideas to give a short proof of the uniqueness part of Yudovich's theorem in the case - and this is precisely where our argument fails - of a bounded $C^{1,1}$ domain Ω.

3.2 Rigorous arguments through optimal transport

The precise statement of the theorem which is reproved in [Loe06] is the following.

Theorem 8 ([Yud63b] ; [Loe06], theorem 4.6)**.** *Let ω_0 belong to $L^1 \cap L^\infty(\mathbb{R}^2)$. There exists a unique solution in $\mathbb{R}^2$ of the system*

$$\partial_t \omega + \nabla^\perp \Psi \cdot \nabla \omega = 0, \quad -\Delta \Psi = \omega, \quad \omega|_{t=0} = \omega_0. \tag{3.20}$$

such that $\omega(t) \in L^\infty(\mathbb{R}^+ \times \mathbb{R}^2)$.

Following [Loe06], we introduce the quadratic Wasserstein distance, which will be the right notion to quantify the distance between two distributions of vorticity.

Definition 3.21. *Let ω_1, ω_2 be two positive measures on $\mathbb{R}^d$ of same total mass, which we denote by $TM(\omega_1) = TM(\omega_2) = M$. We define*

$$W_2(\omega_1, \omega_2) = \inf_\gamma \left(\int_{\mathbb{R}^d \times \mathbb{R}^d} |x-y|^2 d\gamma(x,y) \right)^{1/2}, \tag{3.22}$$

where γ runs on all positive measures on $\mathbb{R}^d \times \mathbb{R}^d$ with marginals ω_1 and ω_2.

An interesting property of the Wasserstein distance is that it is somehow comparable to some H^{-1} norm. The H^{-1} norm which we use in this context is slightly different from the one of the previous section. It is given by

$$||f||_{H^{-1}(\mathbb{R}^d)} = \sup\left(\int_{\mathbb{R}^d} fg,\ g \in C_c^\infty(\mathbb{R}^d),\ \int |\nabla g|^2(x)dx \leq 1\right). \tag{3.23}$$

Proposition 3.24 ([Loe06], proposition 2.8)**.** *Let ρ_1, ρ_2 be in $\mathcal{P}(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$. Then*

$$||\rho_1 - \rho_2||_{H^{-1}(\mathbb{R}^d)} \leq \max\{||\rho_1||_{L^\infty}, ||\rho_2||_{L^\infty}\}^{1/2}\ W_2(\rho_1, \rho_2).$$

As a direct consequence, one can show the following proposition.

Proposition 3.25 ([Loe06], theorem 4.4)**.** *Let ω_1, ω_2 be two positive measures on $\mathbb{R}^d$ of same finite total mass M and with densities in L^∞ with respect to the Lebesgue measure. Let $\Psi_i, i = 1, 2$, solve $-\Delta\Psi_i = \omega_i$ with $\Psi_i(x) \to 0$ when $|x| \to \infty$. Then*

$$||\nabla\Psi_1 - \nabla\Psi_2||_{L^2(\mathbb{R}^d)} \leq [\max\{||\omega_1||_{L^\infty}, ||\omega_2||_{L^\infty}\}]^{1/2}\ W_2(\omega_1, \omega_2). \tag{3.26}$$

Let now ω_0 be a given initial vorticity as in the statement of theorem 8. Still following [Loe06], we introduce the quantity

$$Q(t) = \frac{1}{2}\int_{R^2} |\omega_0(x)||X_2(t,x) - X_1(t,x)|^2 dx, \tag{3.27}$$

where X_i, $i = 1, 2$ denote the characteristics associated to the flows $\nabla^\perp\Psi_i$, which means

$$\partial_t X_i(t,x) = \nabla^\perp\Psi_i(t, X_i(t,x)), \quad X(0,x) = x.$$

At this point, we shall emphasize that the results of Di Perna and Lions [DL89] ensure that these characteristics are uniquely defined for a given $u = \nabla^\perp\Psi_i \in W^{1,1}$. This unique solution can be represented as $\omega_i(t) = X_i(t)_\# \omega_0$.

Our goal is to prove that if $Q(0) = 0$, then $Q(t) = 0$ for all $t \geq 0$. Therefore we compute the derivative of Q,

$$\frac{dQ}{dt} = \int_{\mathbb{R}^2} |\omega_0(x)|(X_1 - X_2)\cdot(\nabla^\perp\Psi_1(X_1) - \nabla^\perp\Psi_2(X_2))(t,x)dx.$$

We now use the triangular inequality and the Cauchy-Schwarz inequality to get

$$\begin{aligned}\frac{dQ}{dt} &\leq \int_{\mathbb{R}^2} |\omega_0(x)||X_1 - X_2||\nabla\Psi_1(t,X_1) - \nabla\Psi_2(t,X_1)|(t,x)dx \\ &\qquad + \int_{\mathbb{R}^2} |\omega_0(x)||X_1 - X_2||\nabla\Psi_2(t,X_1) - \nabla\Psi_2(t,X_2)|(t,x)dx \\ &\leq Q(t)^{1/2}\left(T_1(t)^{1/2} + T_2(t)^{1/2}\right),\end{aligned}$$

where

$$T_2(t) = \int_{\mathbb{R}^2} |\omega_1(t,x)||\nabla\Psi_1(t,x) - \nabla\Psi_2(t,x)|^2 dx,$$

$$T_1(t) = \int_{\mathbb{R}^2} |\omega_0(x)||\nabla\Psi_2(t, X_1(t,x)) - \nabla\Psi_2(t, X_2(t,x))|^2 dx.$$

Proposition 3.25 gives an upper bound for T_2, which turns out to be $T_2(t) \leq 4||\omega_0||^2_{L^\infty} Q(t)$, with the factor 4 coming from the separation between the positive and the negative part of ω. For $T_1(t)$, we need an upper bound on the spatial variations of $\nabla\Psi_2$. This is given by the following lemma.

Lemma 3.28 ([Loe06], lemma 3.1). *For all $T>0$, there exists C_T depending only on $||\omega||_{L^\infty}+||\omega||_{L^1}$ and T such that*

$$\forall t\in[0,T),\forall(x,y)\in\mathbb{R}^3\times\mathbb{R}^3, |x-y|\le\frac{1}{2},\ |\nabla\Psi(t,x)-\nabla\Psi(t,y)|\le C|x-y|\log\frac{1}{|x-y|}.$$

From this lemma, we deduce

$$\begin{aligned}T_1(t) &\le C_T^2\int_{\mathbb{R}^2}|\omega_0(x)|\left(|X_1-X_2|^2\log^2\frac{1}{|X_1-X_2|}\right)dx\\ &= \frac{C_T^2}{4}\int_{\mathbb{R}^2}|\omega_0(x)|\left(|X_1-X_2|^2\log^2|X_1-X_2|^2\right)dx\end{aligned}$$

Then we choose T small enough such that $||X_1-X_2||_{L^\infty([0,T]\times\mathbb{R}^2)}\le\frac{1}{e}$. Since the map $x\mapsto x\log^2 x$ is concave for $0\le x\le 1/e$ we get by Jensen's inequality

$$\begin{aligned}T_1(t) &\le \frac{C_T^2}{4}\left(\int_{\mathbb{R}^2}|\omega_0(x)||X_1(t,x)-X_2(t,x)|^2\right)\log^2\left(\int_{\mathbb{R}^2}|\omega_0(x)||X_1(t,x)-X_2(t,x)|^2\right)\\ &= \frac{C_T^2}{2}Q(t)\log^2(2Q(t)).\end{aligned}$$

Finally, adding all previous results, we get

$$\frac{d}{dt}Q(t)\le CQ(t)\left(1+\log\frac{1}{Q(t)}\right).$$

Since $Q(0)=0$ and $\int_0^\delta\frac{1}{s(1-\log s)}ds=+\infty$ for all sufficiently small $\delta>0$, we can conclude by Osgood's criterion that $Q\equiv 0$. This finishes the proof of theorem 8.

3.3 Conclusion

We now draw general conclusions on the previous section.

The main conclusion is that an inequality like (3.19) or proposition 3.25 is not sufficient to obtain uniqueness of Yudovich solutions in rough domains. It is absolutely necessary to also control the small-scale space variations of u, with an inequality similar to the one of lemma 3.28, which is available only for $C^{1,1}$ domains.

The second conclusion is that the proof of uniqueness of the solutions constructed by Gérard-Varet and Lacave [GVL13] is probably as difficult as to directly prove uniqueness of Yudovich solutions in rough domains : it requires the understanding of the small-scale variations of u near the boundary of the domain.

The third conclusion is that the Wasserstein distance, thanks to its relation to H^{-1} norms, is particularly well-adapted to the measure of distance between velocities, or more generally to the measure of distances between objects whose shape is quite similar, but whose supports move with respect to each other.

Chapter 4

Instability of symmetric configurations

An interesting feature of the Euler equation (as well as the Navier-Stokes equation) is that it preserves symmetry. To make this into a precise mathematical statement, we could write the following proposition, stated only in 2d for the sake of simplicity.

Proposition 4.1. *Let $\Omega \subset \mathbb{R}^2$ be a smooth domain, supposed to be symmetric with respect to the line $x_1 = 0$. Let $u_0 : \Omega \to \mathbb{R}^2$ be a smooth velocity field which is also symmetric with respect to the line $x_1 = 0$ and which verifies $u_0 \in L^2(\Omega)$ and $\omega_0 := \mathit{curl}\ u_0 \in L^\infty \cap L^1(\Omega)$. Then the unique Yudovich solution $u(t,x)$ of the 2d Euler equation with initial datum u_0 is also symmetric with respect to the line $x_1 = 0$ for all $t > 0$.*

Proof. We set $\Omega' = \{x \in \Omega, x_1 > 0\}$ (which is not smooth) and u_0' the restriction of u_0 to Ω'. The symmetry of u_0 with respect to line $x_1 = 0$ implies that the tangency condition (3.6) is satisfied and therefore by theorem 5 there exists a solution $u'(t,x)$ of 2d Euler in Ω' with initial condition u_0'. If we symmetrize it with respect to the line $x_1 = 0$, we get a solution $u(t,x)$ of 2d Euler in Ω with initial condition u_0. Because of Yudovich's theorem in Ω which is smooth, we know that it is the unique solution of 2d Euler in Ω with initial datum u_0. $\square$

Remark 4.2. *This proof emphasizes the fact that if we did not know that uniqueness holds in the domain Ω, there possibly could happen some symmetry breaking : we could imagine that there would exist an other solution of 2d Euler in Ω with initial datum u_0 that loses the symmetry with respect to the line $x_1 = 0$ in finite time. This situation somehow recalls the symmetry breaking numerically appearing in the computations of Guillod and Šverák [GŠ17] in the context of the 3d Navier-Stokes solutions. To push this idea further, we tried to consider the situation where Ω is not smooth but possesses some cusp on its symmetry axis (and therefore non-uniqueness could possibly happen since Yudovich's theory does not apply in this case anymore), but we did not manage to find explicitly any symmetry breaking solution. Chapters 2, 3 and 4 were inspired by this problem.*

4.1 Loss of continuity for the vorticity

An interesting result related to the preservation of symmetry was found by Kiselev and Zlatoš [KZ15]. They exhibited a particular symmetric domain with two cusps on its symmetry axis, an initial continuous distribution of vorticity ω_0 and a solution ω of 2d Euler (in vorticity formulation), with this initial condition, that loses its continuity in finite time. We will briefly explain their proof to motivate our result.

Let

$$\tilde{D}_r^{\pm} = [(-r, r) \times (\pm 1 - r, \pm + r)] \cup B_r(-r, \pm 1) \cup B_r(r, \pm 1)$$

be the "stadium" domains in $\mathbb{R}^2$ with width $r > 0$ and centered at $(0, \pm 1)$. By smoothing a little bit their connections between their circular arcs and their horizontal intervals, we can define $D_r^{\pm}$ that are also rescaled copies of each other, centered at $(0, \pm 1)$ and are infinitely smooth. Finally, let

$$D' := D_1^+ \cup D_1^- \cup [(-1, 1) \times \{0\}]$$

and $D := D_1^+$ be the upper part of D'.

If $\omega_0 \in L^\infty(D')$ is odd in x_2, proceeding exactly as in the proof of proposition 4.1, we see that there is a unique odd-in-x_2 solution ω to 2d Euler on $D' \times \mathbb{R}$. Moreover, if $\omega_0 \in C(D')$, then it follows from Yudovich theory that $\omega(t, \cdot) \in C(D)$ for all $t > 0$. The main theorem of [KZ15] provides a necessary and sufficient condition for ω to remain continuous on the whole domain D'.

Theorem 9. *[KZ15] Let $\omega_0 \in C(\overline{D}')$ be non-negative on D and odd in x_2. Then the unique odd-in-x_2 solution ω to 2d Euler on $D' \times \mathbb{R}$ is continuous for all $t > 0$ if and only if $\omega_0 = 0$ on ∂D.*

Proof. If $\omega_0 = 0$ on ∂D, then $\omega(t, \cdot) = 0$ for all $t > 0$ on ∂D. Combined to the fact that ω is continuous in D_1^+ and D_1^-, it implies that ω is continuous on $D' \times \mathbb{R}$.

Suppose now $\omega_0(x_0) > 0$ for some $x_0 \in \partial D$. We will make the whole reasoning in D. By continuity of ω_0, there exists in D a non-nul area of points with $\omega_0 \geq \eta$ for some $\eta > 0$. Invoking the transport of the vorticity and the conservation of D by the flow, we can choose δ such that $D_{1-2\delta}^+$ contains for all $t > 0$ an area at least δ of points y verifying $\omega(y, t) \geq \delta$.

We now show that $|K_D(x, y)|$ can be uniformly bounded by below for $x \in \partial D$ and $y \in D_{1-2\delta}^+$. We set

$$\kappa := \inf\{G_D(x, y) \mid x \in \partial D_{1-\delta}^+, \quad y \in \overline{D}_{1-2\delta}^+\}$$

and we notice that by compactness, $\kappa > 0$. To establish a principle of comparison, we take v a solution of $\Delta v = 0$ on $D \backslash \overline{D}_{1-\delta}^+$ with $v = 0$ on ∂D and $v = \kappa$ on $\partial D_{1-\delta}^+$. By the maximum principle, we know that $v > 0$ in the interior of this domain and thanks to Hopf's lemma we get

$$\varepsilon := \inf\{|\nabla v(x)| \mid x \in \partial D\} > 0.$$

Hence $G_D(x, y) \geq v(x)$ for $x \in D \backslash \overline{D}_{1-\delta}^+$ and $y \in \overline{D}_{1-2\delta}^+$ by the maximum principle since this inequality holds on $\partial(D \backslash \overline{D}_{1-\delta}^+)$ and both functions are harmonic in x. We know that $K_D(x, y)$ is tangent to ∂D, and this also holds for $\nabla^\perp v$ since $v = 0$ on ∂D. Using $G_D(x, y) \geq v(x)$, we can conclude that $|K_D(x, y)| \geq |\nabla^\perp v(x)| \geq \varepsilon$ and the desired result follows. In particular, $K_D(x, y)$ "always goes in the same sense" on the boundary of the domain D.

We now use this last result coupled to the Biot-Savart law and to the fact that ω is positive on D : for $x \in \partial D$,

$$|u(t, x)| = 2 \left| \int_D K_D(x, y) \omega(t, y) dy \right| \geq 2 \left| \int_{\overline{D}_{1-2\delta}^+} K_D(x, y) \omega(t, y) dy \right| \geq 2\varepsilon\delta^2.$$

Therefore $u(t, x)$ "also always goes in the same sense" and is bounded by below in norm.

Finally, let T be such that $X(T, x_0)$ is at the origin, where X is the flow of u. The previous results show that $T < 0$. We take a sequence of points y_n converging to x_0 in the interior of D and z_n their images by the symmetry with respect to x_2. We have $X(T, y_n) \to 0$ and $X(T, z_n) \to 0$ but, since the vorticity is carried by the flow, $\omega(X(T, y_n)) \to \omega_0(x_0)$ and $\omega(X(T, z_n)) \to -\omega_0(x_0)$ by oddness of ω_0. This shows that the vorticity $\omega(T, \cdot)$ is discontinuous at the origin. $\square$

A natural question is then to try to extend theorem 9 to the non-symmetric case. More precisely, the problem could be the following.

Problem 4.3. *Does there exist a non-symmetric domain Ω and ω a solution of the 2d Euler equation in Ω with continuous initial datum ω_0 which loses its continuity in finite time ?*

We shall already say that we will not give any definitive answer to this problem, but we will show a computation which suggests that the strategy of [KZ15] cannot be adapted to this general case. An adaptation of the strategy of [KZ15] would have been to work in a "nearly symmetric" domain Ω and a "nearly-symmetric" distribution of vorticity that would push some small domains of distinct vorticities on both sides of the cusp to its vertex, therefore generating the loss of continuity, exactly as described in the proof of theorem 9. Such a scenario would have required a kind of "stability" of the configuration studied in [KZ15], but our computations show that it is actually very unstable.

4.2 An explicit computation in $\mathbb{R}^2 \backslash \{x_1 \leq 0, x_2 = 0\}$

We consider the following situation, which can be considered as a toy model in order to understand problem 4.3. Let $S := \mathbb{R}^2 \backslash \{(x_1, x_2) \in \mathbb{R}^2, x_1 \leq 0, x_2 = 0\}$ and $H := \{(x_1, x_2) \in \mathbb{R}^2, x_1 > 0\}$ be a half-plane. The biholomorphism which sends S to H is the square root, denoted by $z \to z^{1/2}$, with the standard definition of the complex logarithm. In order to compute the velocity field generated on the boundary of S by a given distribution of vorticity, we first aim at computing the value of $\nabla^\perp G(x, y)$ where G is the Green function of S and $x \in \partial S$.

The Green function of H is

$$G_H(x, y) = \frac{1}{2\pi}(\log|x - y| - \log|\tilde{x} - y|),$$

with $\tilde{x} = (-x_1, x_2)$. Still denoting by G the Green function of S, we deduce that

$$G(x, y) = \frac{1}{2\pi}(\log|x^{1/2} - y^{1/2}| - \log|\tilde{x}^{1/2} - y^{1/2}|)$$

Our goal is to compute $\nabla^\perp G(x, y)$ for x in a neighborhood of $D = \{(x_1, x_2) \in \mathbb{R}^2, x_1 \leq 0, x_2 = 0\}$, and especially around 0. We have to distinguish the case when x is above the half-line D, and the case when it is below D. In the sequel, we fix $y \in S$ and we write $y^{1/2} = (a, b) \in H$.

If $k > 0$ is fixed, $\varepsilon > 0$ is some (small) varying parameter and $x = -k + i\varepsilon$, then $x^{1/2} = i\sqrt{k} + \frac{\varepsilon}{2\sqrt{k}} + \varepsilon^2\psi_k(\varepsilon) + i\varepsilon^2\phi_k(\varepsilon)$ for some functions $\phi_k, \psi_k : \mathbb{R}^+ \to \mathbb{C}$ which are bounded and have bounded derivative in the limit $\varepsilon \to 0$. Therefore

$$\begin{aligned}
4\pi G(x,y) &= \log\left|i\sqrt{k} + \frac{\varepsilon}{2\sqrt{k}} + \varepsilon^2\psi_k(\varepsilon) + i\varepsilon^2\phi_k(\varepsilon) - y^{1/2}\right|^2 \\
&\qquad - \log\left|i\sqrt{k} + i\varepsilon^2\phi_k(\varepsilon) - \frac{\varepsilon}{2\sqrt{k}} - \varepsilon^2\psi_k(\varepsilon) - y^{1/2}\right|^2 \\
&= \log\left(\left(-a + \frac{\varepsilon}{2\sqrt{k}} + \varepsilon^2\psi_k(\varepsilon)\right)^2 + (\sqrt{k} - b + \varepsilon^2\phi_k(\varepsilon))^2\right) \\
&\qquad - \log\left(\left(a + \frac{\varepsilon}{2\sqrt{k}} + \varepsilon^2\psi_k(\varepsilon)\right)^2 + (\sqrt{k} - b + \varepsilon^2\phi_k(\varepsilon))^2\right)
\end{aligned}$$

We deduce the expression of the partial derivative which is of interest for us (the one which generates the tangential velocity) :

$$4\pi\partial_2 G(x,y) = \frac{\frac{-a}{\sqrt{k}} + \mathcal{O}(\varepsilon)}{a^2 + (\sqrt{k} - b)^2 + \mathcal{O}(\varepsilon)} - \frac{\frac{a}{\sqrt{k}} + \mathcal{O}(\varepsilon)}{a^2 + (\sqrt{k} - b)^2 + \mathcal{O}(\varepsilon)}.$$

Evaluating in $\varepsilon = 0$, we obtain

$$\partial_2 G(x,y) = \frac{-a}{2\pi\sqrt{k}(a^2 + (\sqrt{k} - b)^2)}. \tag{4.4}$$

If we do the same computation for $x = -k - i\varepsilon$, which means that we are below D, we find :

$$\partial_2 G(x,y) = \frac{a}{2\pi\sqrt{k}(a^2 + (\sqrt{k} - b)^2)},$$

which is the opposite of the previous result. One could have expected the change of sign : since G is positive in the interior of the domain and equal to 0 on the boundary, the gradient of G on the boundary points towards the interior, and therefore the sign of $\partial_2 G$ has to change when one crosses D. We also notice that the a appearing in the expression of $\partial_2 G$ is always positive since $y^{1/2} = (a,b)$ is an element of H. Finally, let us mention the fact that in the expression of $\partial_2 G$, there is a $\sqrt{k}$ in the denominator, which means that the velocity should behave as $k^{-1/2}$ near the origin. It was already highlighted for example in [LMW14]. This denominator will play a key role in the sequel.

We immediately deduce from the previous results that $\nabla^\perp G$ is tangent to D but that it is not in the same direction just above and below D. Then, how is it possible that in the proof of theorem 9, it is found that around the cusp of the double-stadium, the velocity has the same direction on both sides of the cusp ? To answer this question, we have to compute precisely the velocity field generated by a symmetric pair of dirac masses of vorticity, i.e. a vorticity of the form $\omega = \delta_{(x,y)} - \delta_{(x,-y)}$.

Again, we have to separate the case where x is just above and just below D. Slightly above D, with the expression of G, we get :

$$u(x) = \frac{-a}{2\pi\sqrt{k}} \times \frac{1}{a^2 + (\sqrt{k} - b)^2} + \frac{a}{2\pi\sqrt{k}} \times \frac{1}{a^2 + (\sqrt{k} + b)^2},$$

which means (we detail all lines since key cancellations happen) :

$$\begin{aligned}
u(x) &= \frac{a}{2\pi\sqrt{k}}\left(\frac{1}{a^2+(\sqrt{k}+b)^2}-\frac{1}{a^2+(\sqrt{k}-b)^2}\right)\\
&= \frac{a}{2\pi\sqrt{k}}\times\frac{(a^2+(\sqrt{k}-b)^2)-(a^2+(\sqrt{k}+b)^2)}{(a^2+(\sqrt{k}+b)^2)(a^2+(\sqrt{k}-b)^2)}\\
&= \frac{a}{2\pi\sqrt{k}}\times\frac{-4b\sqrt{k}}{(a^2+(\sqrt{k}+b)^2)(a^2+(\sqrt{k}-b)^2)}\\
&= \frac{-2ab}{\pi(a^2+(\sqrt{k}+b)^2)(a^2+(\sqrt{k}-b)^2}
\end{aligned}$$

Similarly, below D, we find

$$u(x)=\frac{a}{2\pi\sqrt{k}}\left(\frac{1}{a^2+(\sqrt{k}+b)^2}-\frac{1}{a^2+(\sqrt{k}-b)^2}\right)=\frac{-2ab}{\pi(a^2+(\sqrt{k}+b)^2)(a^2+(\sqrt{k}-b)^2}.$$

Therefore, although $\partial_2 G$ is not in the same sense just above and just below D, the velocity field u generated by an odd distribution of vorticity remains the same while crossing D. This is due to several cancellations which appear in the computations above.

Remark 4.5. *In the case of a general symmetric domain Ω, we can adapt without any difficulty these computations and obtain analoguous results since there exists a biholomorphism $\mathcal{T}$ which sends H to Ω and preserves the symmetry. The image of the pair of Dirac masses of vorticity by T is also symmetric, and the same cancellations happen.*

We now perturb the previous configuration : in the domain S, we will put a vorticity $+1$ in y and a vorticity -1 in z, but this time y and z will not be symmetric with respect to the axis x_1. We could imagine that if y and z are "nearly symmetric", the velocity field will not be very different of what we found in the symmetric case. But we will see that it is not the case, and that the whole picture dramatically changes. In the sequel, we consider two points $y=(a,b)\in S$ and $z=(c,d)\in S$. In the case of a symmetric pair, we would have $a=c$ and $b=-d$.

Firstly, we compute u just above D. Using (4.4), we get :

$$\begin{aligned}
u(x) &= \frac{-a}{2\pi\sqrt{k}(a^2+(\sqrt{k}-b)^2)}+\frac{c}{2\pi\sqrt{k}(c^2+(\sqrt{k}-d)^2)}\\
&= \frac{1}{\sqrt{k}}\times\frac{c(a^2+(\sqrt{k}-b)^2)-a(c^2+(\sqrt{k}-d)^2)}{(a^2+(\sqrt{k}-b)^2)(c^2+(\sqrt{k}-d)^2)}\\
&= \frac{1}{\sqrt{k}}\times\frac{a^2c+b^2c-ac^2-ad^2+2\sqrt{k}(ad-bc)+k(c-a)}{(a^2+(\sqrt{k}-b)^2)(c^2+(\sqrt{k}-d)^2)}
\end{aligned}\tag{4.6}$$

Similarly, below D, we have

$$\begin{aligned}
u(x) &= \frac{a}{\sqrt{k}(a^2+(\sqrt{k}+b)^2)}-\frac{c}{\sqrt{k}(c^2+(\sqrt{k}+d)^2)}\\
&= \frac{1}{\sqrt{k}}\times\frac{a(c^2+(\sqrt{k}+d)^2)-c(a^2+(\sqrt{k}+b)^2)}{(a^2+(\sqrt{k}+b)^2)(c^2+(\sqrt{k}+d)^2)}\\
&= \frac{1}{\sqrt{k}}\times\frac{ac^2+ad^2-a^2c-b^2c+2\sqrt{k}(ad-bc)+k(a-c)}{(a^2+(\sqrt{k}+b)^2)(c^2+(\sqrt{k}+d)^2)}
\end{aligned}\tag{4.7}$$

We see that in the symmetric case $a = c$, $b = -d$, the first term of the numerators of (4.6) and (4.7) both equal 0, and there only remain the terms with $\sqrt{k}$ or k. Moreover the term in $\sqrt{k}$ is the same above and below D. But as soon as the symmetry is broken, $u(x)$ blows up in $1/\sqrt{k}$, and the term in $1/\sqrt{k}$ has not the same sign in (4.6) and (4.7). The following table sums up the results.

Symmetric distribution and symmetric domain	No symmetry
u keeps the same direction around the cusp.	Typically, u may change its direction around the cusp
No blow-up of $\lvert u \rvert$ around the cusp.	Strong blowp of $\lvert u \rvert$ around the cusp.

To conclude with this toy model, it is now clear that around a cusp, some minor modifications in the distribution of vorticity may lead to huge effects. In particular, any non-symmetric perturbation of some symmetric configuration can generate big singularities in the norm and the direction of the velocity field. Therefore, the sketch of proof given at the end of the previous section cannot conclusive, because the symmetric configuration becomes very unstable if one allows symmetry breaking. However, it seems to us that the computations of this chapter have some interest by themselves because they clarify the very specific features of the symmetric distributions of vorticity.

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