CENTRAL LIMIT THEOREMS FOR THE $\mathbb{Z}^2$-PERIODIC LORENTZ GAS

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Abstract. This paper is devoted to the study of the stochastic properties of dynamical systems preserving an infinite measure. More precisely we prove central limit theorems for Birkhoff sums of observables of $\mathbb{Z}^2$-extensions of dynamical systems (satisfying some nice spectral properties). The motivation of our paper is the $\mathbb{Z}^2$-periodic Lorentz process for which we establish a functional central limit theorem for Hölder continuous observables (in discrete time as well as in continuous time).

1. Introduction

A measure preserving dynamical system is given by a transformation T or a flow $(Y_t)_{t\geq 0}$ preserving a measure. When the measure is a probability, the study of the stochastic properties of such a dynamical system consists in studying the probabilistic properties of families of stationary random variables of the form $(\phi\circ T^k)_{k\geq 0}$ or $(\phi\circ Y_t)_{t\geq 0}$ for reasonnable observables, with a particular interest in the study of the Birkhoff sums, which are given by

$$S_n\phi=\sum_{k=0}^{n-1}\phi\circ T^k \quad \text{or} \quad S_t\phi=\int_0^t \phi\circ Y_s\,\mathrm{d}s.$$

When the measure is a probability, the study of these quantities have been intensively studied in the last half a century, with an always increasing interest. A first question is the law of large number (LLN), that is the almost sure convergence of $(S_t\phi/t)_{t>0}$ to the integral $I(\phi)$ of ϕ as $t\to+\infty$, which happens to be true for any integrable function f as soon as the system is ergodic (due to the Birkhoff-Khinchin theorem). A second natural question is the establishment of central limit theorems (CLT), i.e. of the convergence in distribution of $\left(S_t\phi/\sqrt{t}\right)_t$ (as $t\to+\infty$) to a Gaussian random variable for centered square integrable observables ϕ, or, even more, the functional CLT (FCLT) that is the convergence of the family of processes $\left((S_{st}\phi/\sqrt{t})_s\right)_t$ to a Brownian motion $(B_s)_s$, as $t\to+\infty$. In practice, CLT and FCLT hold true for smooth observables when the system is chaotic enough (satisfying nice mixing properties, see [23, 13, 14], etc.).

When the measure is infinite, it is natural to adress analogous questions, but the results are of different nature (we refer to [1] for a general reference on dynamical systems preserving an infinite measure). The first analogue of the LLN is given by the Hopf theorem, which states the almost everywhere convergence of $(S_t\phi/S_t\psi)_n$ to the ratio $I(\phi)/I(\psi)$ of the integrals of ϕ and ψ, for all couples of integrable functions (ϕ,ψ) with $\psi\not\equiv 0$, as soon as the system is conservative ergodic. A second analogue of the LLN is the convergence in distribution in the strong sense of $(S_t\phi/\mathfrak{a}_t)_t$ to a random variable (convergence in distribution in the strong sense means convergence in distribution with respect to any probability measure absolutely continuous with respect to the invariant measure). Note that, due to the Hopf theorem, it is enough to prove this result for a specific function $\phi\not\equiv 0$ to extend it to any integrable function. This second kind of analogue of LLN requires additional assumptions on the dynamical system. Analogues of CLT for dynamical systems preserving an infinite invariant measure are non-degenerate limit theorems for $(S_t\phi)_t$ for null integral observables f. A classical analogue to the CLT in this context consists in establishing the convergence in distribution of $\left(S_t\phi/\mathfrak{a}_t^{\frac{1}{2}}\right)_t$ to some random variable, with $\mathfrak{a}_t$ as in the above second analogue of the LLN.

The case of dynamical systems that can be represented by a $\mathbb{Z}^d$-extension over a probability preserving dynamical system is of particular interest. As mentionned in [27, 28, 29, 22], in this specific context, the question of the behaviour of Birkhoff sums is related to the study of occupation times of d-dimensional random walks or Markov chains (see [11, 16, 17, 19]). Indeed, in the case of a transformation T, when the observable

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ϕ depends only on the $\mathbb{Z}^d$-label in the $\mathbb{Z}^d$-extension, the ergodic sum $S_n\phi$ is the exact analogue of additive functionals of d-dimensional random walks or Markov chains. Outside the cases of random walks or Markov chains, first results have been obtained by the second-named author [27, 28, 29] for Pomeau-Manneville maps, for $\mathbb{Z}^d$-extensions of Gibbs-Markov maps, for geodesic flows on a $\mathbb{Z}^d$-cover of a compact Riemannian variety with negative curvature. In [22], we established CLT in a general context of $\mathbb{Z}^d$-extensions of a dynamical system with nice spectral properties, including the $\mathbb{Z}^2$-periodic billiard model, but for observables depending only on the $\mathbb{Z}^d$-label.

The aim of the present paper is to study ergodic sums of Hölder observables of the $\mathbb{Z}^2$-periodic Sinai billiard and of the $\mathbb{Z}^2$-periodic Lorentz process, both with finite horizon. A first step in this direction is the property of conservativity and ergodicity which comes from [7, 26] (thanks to [24, 5, 4]) and which, combined with the Hopf theorem, ensures the above mentioned first analogue to the LLN. A second step is the proof by Dolgopyat, Szász and Varjú in [10] of the above mentioned second analogue to the LLN, that in this context is

$$\forall \phi\in L^1,\quad \frac{S_t\phi}{\ln t}\Longrightarrow I(\phi)\,\mathcal{E}\,,\quad \text{as } t\to+\infty\,, \tag{1.1}$$

where $\mathcal{E}$ is an exponential random variable and where $\Longrightarrow$ means the convergence in distribution in the strong sense. A third step in this direction is the CLT with a normalization in $\sqrt{\ln t}$ obtained in [22] for the billiard map and for observables depending only on the $\mathbb{Z}^d$-level. In the present paper, our main result is a CLT and even a FCLT for Hölder observables ϕ (with null expectation) of the $\mathbb{Z}^2$-periodic Sinai billiard and of the $\mathbb{Z}^2$-periodic Lorentz process of the following form

$$\frac{S_t\phi}{\sqrt{\ln(t)}}\Longrightarrow \widetilde{\sigma}_\phi\sqrt{\mathcal{E}}\,\mathcal{N}\,,\quad \text{as } t\to+\infty\,,$$

with $\mathcal{E}$ as in (1.1) and with $\mathcal{N}$ a standard gaussian random variable independent of $\mathcal{E}$, where $\widetilde{\sigma}_\phi$ is given by a Green-Kubo formula. The above convergence result holds true providing ϕ satisfies some decay property at infinity. So it holds true at least for compactly supported Hölder functions with null integral. More precisely, under the same assumptions and for any integrable function ψ, we prove the following joint FCLT

$$\left(\frac{S_{ts}\psi}{\ln(t)},\frac{S_{ts}\phi}{\sqrt{\ln(t)}}\right)_{s>0}\Longrightarrow\left(I(\psi)\,\mathcal{E},\,\widetilde{\sigma}_\phi\sqrt{\mathcal{E}}\,\mathcal{N}\right)_{s>0}\,,$$

with the same notations as above. Roughly speaking, this means that, in distribution,

$$S_t\phi\approx \ln t\, I(\phi)\mathcal{E}+\sqrt{\ln t\,\mathcal{E}}\,\tilde{\sigma}_{\phi-I(\phi)\phi_0}\mathcal{N}+o(\sqrt{\ln t})\,,\quad \text{as}\quad t\to+\infty\,,$$

for ϕ,ϕ_0 two Hölder observables decaying quickly enough at infinity, with $I(\phi_0)=1$. Note that, contrarily to the case of the classical FCLT, the limit we obtain is a process constant in time. To prove our results, we use two methods producing different formulas for the "asymptotic variance" $\widetilde{\sigma}^2_\phi$ appearing in the CLT.
First, using the method of [22], we establish a general FCLT for $\mathbb{Z}^2$-extensions over a dynamical system satisfying general nice spectral properties (namely such that the step function satisfies a spectral local limit theorem). The fact that we restrict our study to $\mathbb{Z}^2$-extension with square integrable step functions (satisfying a classical limit theorem) simplifies greatly the proof, makes its ideas appear much clearer than in [22] and allows the generalization to Hölder functions, without adding technical complications.
Second, applying the method of [27, 28, 29], we obtain another way, based on induction, to prove the CLT for Hölder observables of the $\mathbb{Z}^2$-periodic billiard, under a slightly weaker assumption.

The article is organized as follows. In Section 2, we present our context and results. We start by introducing in Section 2.1 our general context of $\mathbb{Z}^2$-extensions of dynamical systems (in discrete time as well as in continuous time). In Section 2.2, we introduce the $\mathbb{Z}^2$-periodic Lorentz gas (in discrete time as well as in continuous time). The rest of Section 2 is devoted to the exposure of our main results, with a discussion on our technical assumptions. In Sections 3 and 4, we prove our FCLT by the first method for dynamical systems (first in the case of transformations in Section 3 and then in the case of flows in Section 4). In Section 5, we prove the CLT via the second method (using induction).

2. Context and main results

2.1. **General context.** Given a probability preserving dynamical system (A, μ, T) and a function $F : A \to \mathbb{Z}^2$, we consider the infinite measure preserving dynamical system $(\widetilde{A}, \widetilde{\mu}, \widetilde{T})$ given by the $\mathbb{Z}^2$-extension of (A, μ, T) with step function F, i.e. $\widetilde{A} := A \times \mathbb{Z}^2$, $\widetilde{\mu} = \mu \otimes \mathfrak{m}$, where $\mathfrak{m}$ is the counting measure on $\mathbb{Z}^2$ and $\widetilde{T}(x, a) = (T(x), a + F(x))$. Then, for all (x, a) in $\widetilde{A}$ and $n \geq 0$,

$$\widetilde{T}^n(x, a) = (T^n(x), a + S_n F(x)) ,$$

where $S_n F$ is the ergodic sum:

$$S_n F := \sum_{k=0}^{n-1} F \circ T^k .$$

We are interested in the asymptotic behaviour of the ergodic sums

$$\widetilde{S}_n f := \sum_{k=0}^{n-1} f \circ \widetilde{T}^k ,$$

for observables $f : \widetilde{A} \to \mathbb{R}$ in the particular case when $\int_{\widetilde{A}} f \, \mathrm{d}\widetilde{\mu} = 0$.

In the context of this article, the system (A, μ, T) shall be chaotic in a strong sense. More precisely, we shall assume that $(S_n F)_n$ satisfies a standard central limit theorem and, even more, a *spectral local limit theorem* (see Assumption (2.2) below), which is a strengthening of the more classical local limit theorem:

$$\mu(S_n F = a) \sim \frac{\Phi(a/\sqrt{n})}{n}$$

for all $a \in \mathbb{Z}^2$, where Φ is the density of the Gaussian that is the limit distribution of $(S_n F/\sqrt{n})_n$. By Lemma 2.7, Assumption (2.2) holds when the transfer operator P of T, dual to the Koopman operator, acts nicely on a Banach space $\mathcal{B}$ of integrable functions or distributions. This assumption is, in particular, satisfied by the collision map for Sinai billiards.

We shall also consider continuous-time versions of this problem, in two ways. The first way consists in defining the ergodic sums $\widetilde{S}_t f$ for real $t > 0$ by linearization:

$$\begin{aligned} \widetilde{S}_t f &:= (\lfloor t \rfloor + 1 - t) S_{\lfloor t \rfloor} f + (t - \lfloor t \rfloor) S_{\lfloor t \rfloor + 1} f \\ &= S_{\lfloor t \rfloor} f + (t - \lfloor t \rfloor) f \circ \widetilde{T}^{\lfloor t \rfloor}, \end{aligned}$$

which can be used to state functional limit theorems.

The second way consists in working directly with a continuous-time system. Given a measurable function $\tau : A \to (0, +\infty)$, the suspension flow $(\widetilde{\mathcal{M}}, \widetilde{\nu}, (\widetilde{Y}_t)_t)$ of $(\widetilde{A}, \widetilde{\mu}, \widetilde{T})$ with roof function $(x, a) \mapsto \tau(x)$ is the system:

$$\begin{cases} \widetilde{\mathcal{M}} &:= \{(x, a, s) \in A \times \mathbb{Z}^2 \times (0, +\infty) \, : \, s \in (0, \tau(x))\} , \\ \widetilde{\nu} &:= (\widetilde{\mu} \otimes \mathrm{Leb})_{|\widetilde{\mathcal{M}}} , \\ \widetilde{Y}_t(x, a, s) &:= \left(\widetilde{T}^{n_{t+s}(x)}(x, a), s + t - S_{n_{t+s}(x)} \tau(x) \right) , \end{cases}$$

where Leb is the Lebesgue measure on $(0, +\infty)$ and $n_u(x) := \max\{n \geq 0 \, : \, S_n \tau(x) \leq u\}$ for every $u \geq 0$. In this case, we define:

$$\widetilde{S}_t f := \int_0^t f \circ \widetilde{Y}_s \, \mathrm{d}s .$$

2.2. **$\mathbb{Z}^2$-periodic Lorentz gas.** We consider the displacement of a particle moving at unit speed in $\mathbb{R}^2$ with elastic reflection on a $\mathbb{Z}^2$-periodic configuration of dispersing obstacles, in finite horizon.

More precisely the billiard domain is given by $\mathbb{R}^2 \setminus \bigcup_{a \in \mathbb{Z}^2} \bigcup_{i=1}^I (O_i + a)$, with obstacles $\{O_i + a \, ; \, i = 1, \ldots, I, \; a \in \mathbb{Z}^2\}$ for some $I \geq 2$. We assume that $(O_i)_{1 \leq i \leq I}$ is a finite family of precompact open convex sets in $\mathbb{R}^2$, whose boundaries are $\mathcal{C}^3$ with non vanishing curvature. We assume that the closure of the sets $O_i + a$ are pairwise disjoint. We assume moreover that Q contains no line (finite horizon assumption).

We are interested in the behaviour of a point particle moving in Q at unit speed, going straight inside Q and obeying the Descartes reflection law at reflection times off $\partial Q = \bigcup_{a \in \mathbb{Z}^2} \bigcup_{i=1}^I (\partial O_i + a)$.

A configuration is a couple position-speed $(q,\vec{v}) \in Q\times\mathbb{S}^1$. To avoid ambiguity, we allow only post-collisional vectors at reflection times, so that the configuration space is

$$\widetilde{\mathcal{M}} := \{(q,\vec{v}) \in Q \times \mathbb{S}^1 : q \in \partial Q \ \Rightarrow \ \langle \vec{n}_q, \vec{v}\rangle \geq 0\},$$

where $\vec{n}_q$ denotes the unit vector normal to ∂Q at q oriented into Q.

The **Lorentz process** is the flow $(\widetilde{Y}_t)_t$ on $\widetilde{\mathcal{M}}$ such that $\widetilde{Y}_t(q,\vec{v}) = (q_t,\vec{v}_t)$ is the configuration at time t of a point particle that has configuration $(q,\vec{v})$ at time 0. This flow preserves the restriction $\tilde{\nu}$ on $\widetilde{\mathcal{M}}$ of the Lebesgue measure Leb on $\mathbb{R}^2 \times \mathbb{S}^1$, normalized so that:

$$\mathrm{Leb}([0,1[^2\times\mathbb{S}^1) = \frac{\pi}{\sum_{i=1}^{I} |\partial O_i|}.$$

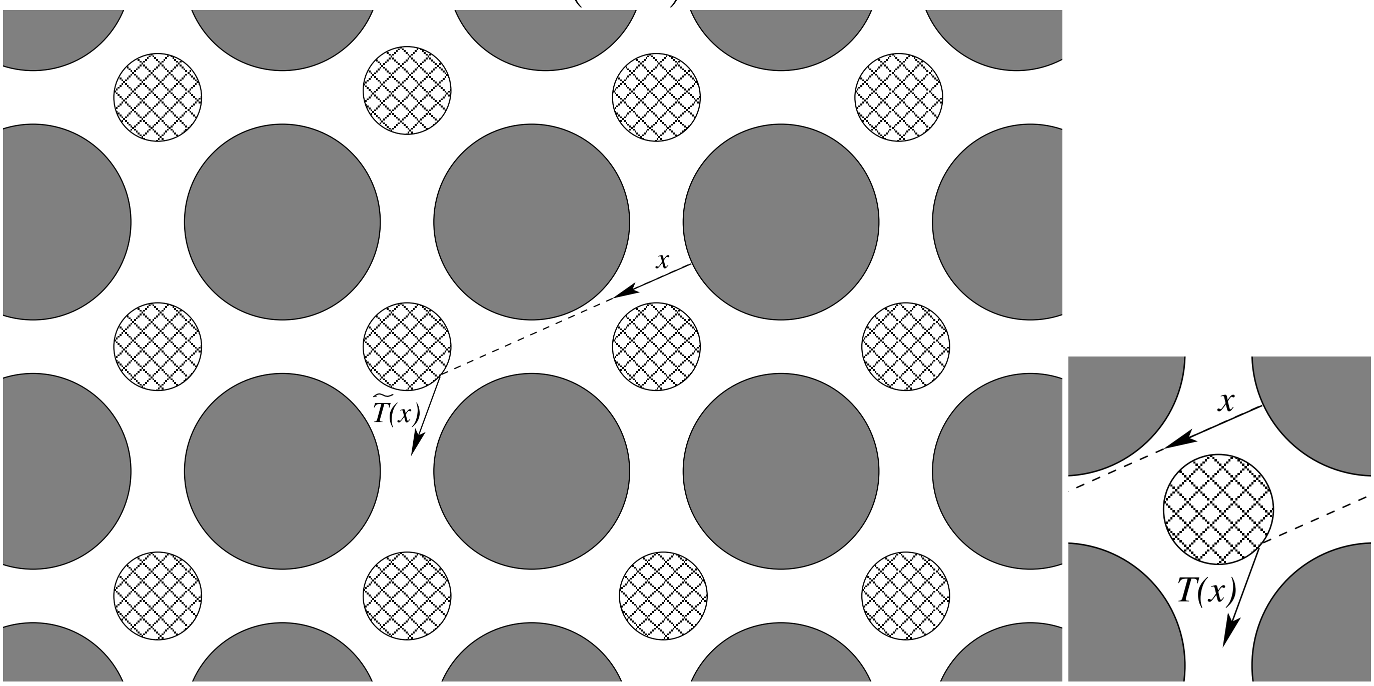
This normalization will allow us to identify canonically this flow with a $\mathbb{Z}^2$-periodic suspension flow over a $\mathbb{Z}^2$-extension of a chaotic probability preserving dynamical system, as described in Subsection (2.1).

The dynamics at reflection times is the $\mathbb{Z}^2$**-periodic billiard system** $\left(\widetilde{A},\widetilde{\mu},\widetilde{T}\right)$, that is the first return map of the flow $\widetilde{Y}$ to the Poincaré section $\partial Q \times \mathbb{S}^1$. Let $\widetilde{A} := \{(q,\vec{v}) \in \widetilde{\mathcal{M}} : q \in \partial Q\}$ be the set of configurations of post-collisional vectors off ∂Q. The map $\widetilde{T} : \widetilde{A} \to \widetilde{A}$ is the billiard transformation mapping a post-collisional configuration to the next post-collisional configuration. The measure $\widetilde{\mu}$ is given by:

$$\mathrm{d}\widetilde{\mu}(q,\vec{v}) = \frac{\cos\varphi}{2\sum_{i=1}^{I} |\partial O_i|}\, \mathrm{d}r \ \mathrm{d}\varphi\,, \tag{2.1}$$

where r is the curvilinear absciss of q on ∂Q, and φ is the angular measure in $[-\pi/2,\pi/2]$ of the angle $\widehat{(\vec{n}_q,\vec{v})}$.

This $\mathbb{Z}^2$-periodic billiard system $\left(\widetilde{A},\widetilde{\mu},\widetilde{T}\right)$ is a $\mathbb{Z}^2$-extension of the corresponding **Sinai billiard system** (A,μ,T). This Sinai billiard is the quotient of $\left(\widetilde{A},\widetilde{\mu},\widetilde{T}\right)$ modulo the action of $\mathbb{Z}^2$ on the position.



More explicitely, the configuration set A is the image of $\widetilde{A}$ by $\mathfrak{p} : \mathbb{R}^2\times\mathbb{S}^1 \to \mathbb{T}^2\times\mathbb{S}^1$ given by $\mathfrak{p}(q,\vec{v}) = (\bar{q},\vec{v})$ where $\bar{q} = q+\mathbb{Z}^2 \in \mathbb{T}^2 := \mathbb{R}^2/\mathbb{Z}^2$. By $\mathbb{Z}^2$-periodicity of Q, there exists a map $T : A \to A$ such that $T\circ\mathfrak{p} = \mathfrak{p}\circ\widetilde{T}$, which is the billiard map in the domain $\overline{Q} \subset \mathbb{T}^2$ image of Q by the canonical projection $\mathbb{R}^2 \to \mathbb{T}^2$. The measure μ has the same expression as in Equation (2.1).

The function $F : A \to \mathbb{Z}^2$ giving the size of the jumps is defined by $F(q,\vec{v}) = b - a$ whenever $\widetilde{T}(q,\vec{v},a) \in \bigsqcup_{i=1}^{I}(\partial O_i + b) \times \mathbb{S}^1$; the $\mathbb{Z}^2$-periodicity of the billiard table ensures that this function is well-defined.

Let $\tau : A \to \mathbb{R}_+^*$ be the free path length of a particle on $\overline{Q}$:

$$\tau(q,\vec{v}) = \min\{s > 0 : q + s\vec{v} \in \partial\overline{Q}\}.$$

By $\mathbb{Z}^2$-periodicity of Q, the function $\widetilde{\tau}:\,(q,\vec{v},a)\mapsto\tau(q,\vec{v})$ defined on $\widetilde{A}$ is the free path length of a particle on Q. The Lorentz gas $\left(\widetilde{\mathcal{M}},\tilde{\nu},(Y_t)_t\right)$ is then canonically identified with the suspension flow over $\left(\widetilde{A},\widetilde{\mu},\widetilde{T}\right)$ with roof function $\widetilde{\tau}$.

2.3. **Results for transformations.** We state our main limit theorem under abstract conditions; our other results – applications to billiards or to continuous-time systems – will follow from that.

Theorem 2.1. *Assume that $(\widetilde{A},\widetilde{\mu},\widetilde{T})$ is conservative and ergodic. Let $\eta>0$ and p, $p^*\in[1,\infty]$ such that $p^{-1}+(p^*)^{-1}=1$. Let $(\mathcal{B},\|\cdot\|_{\mathcal{B}})$ be a Banach space (of functions or distributions) containing $\mathbf{1}:=\mathbf{1}_A$ and such that*

- *either $p=1$ (so $p^*=\infty$) and $\mathbb{E}_\mu[\cdot]$ is continuously extended from $\mathcal{B}\cap L^1(A,\mu)$ to $\mathcal{B}$;*
- *or $p>1$ and $\mathcal{B}\hookrightarrow\mathbb{L}^p(A,\mu)$ (where $\hookrightarrow$ is a continous injection).*

Assume moreover the following spectral local limit condition:

$$\sup_{a\in\mathbb{Z}^d,\ h\in\mathcal{B}:\|h\|_{\mathcal{B}}\le 1}\left\|P^\ell\left(\mathbf{1}_{\{S_\ell F=a\}}\,h\right)-\frac{\Phi\left(\frac{a}{\sqrt{\ell}}\right)}{\ell}\mathbb{E}_\mu[h]\right\|_{\mathcal{B}}=O(\ell^{-1-\eta})\,, \tag{2.2}$$

where Φ is a two-dimensional non-degenerate Gaussian density function.

Let f, $g:\widetilde{A}\to\mathbb{R}$ be such that:

- *$\sum_{a\in\mathbb{Z}^2}(1+|a|^\varkappa)\,\|f(\cdot,a)\times\cdot\|_{\mathcal{L}(\mathcal{B},\mathcal{B})}<+\infty$ for some $\varkappa>0$;*
- *$\sum_{a\in\mathbb{Z}^2}\|f(\cdot,a)\|_{\mathbb{L}^{p^*}(A,\mu)}<+\infty$;*
- *$\int_{\widetilde{A}}f\,\mathrm{d}\widetilde{\mu}=0$;*
- *$g\in\mathbb{L}^1(\widetilde{A},\widetilde{\mu})$.*

Then the following sum over k is absolutely convergent:

$$\tilde{\sigma}^2(f):=\int_{\widetilde{A}}f^2\,\mathrm{d}\widetilde{\mu}+2\sum_{k\ge1}\int_{\widetilde{A}}f\cdot f\circ\widetilde{T}^k\,\mathrm{d}\widetilde{\mu}. \tag{2.3}$$

Moreover, for every $0<T_1<T_2$, as $n\to+\infty$,

$$\left(\frac{\widetilde{S}_{nt}g}{\ln(n)},\frac{\widetilde{S}_{nt}f}{\sqrt{\ln(n)}}\right)_{t\in[T_1,T_2]}\longrightarrow\left(\int_{\widetilde{A}}g\,\mathrm{d}\widetilde{\mu}\,\Phi(0)\mathcal{E},\,\widetilde{\sigma}(f)\sqrt{\Phi(0)\mathcal{E}}\,\mathcal{N}\right)_{t\in[T_1,T_2]} \tag{2.4}$$

in distribution in $C([T_1,T_2])$, with respect to any probability measure absolutely continuous[1] *with respect to $\widetilde{\mu}$, and where $\mathcal{E}$ and $\mathcal{N}$ are two independent random variables, with respectively standard exponential and standard Gaussian distributions.*

Note that, since $\mathbf{1}\in\mathcal{B}$, the condition $\sum_{a\in\mathbb{Z}^2}(1+|a|^\varkappa)\,\|f(\cdot,a)\times\|_{\mathcal{L}(\mathcal{B},\mathcal{B})}<+\infty$ implies that $\sum_{a\in\mathbb{Z}^2}(1+|a|^\varkappa)\,\|f(\cdot,a)\|_{\mathcal{B}}<+\infty$. Note also that the random variable $\sqrt{\mathcal{E}}\mathcal{N}$ in Equation (2.4) follows a standard (centered with variance 1) Laplace distribution.

The condition $\sum_{a\in\mathbb{Z}^2}\|f(\cdot,a)\|_{\mathbb{L}^{p^*}(A,\mu)}<+\infty$ in Theorem 2.1 is only used in the proof of the functional convergence, and is not necessary for the convergence in distribution of $\left(\frac{\widetilde{S}_n g}{\ln(n)},\frac{\widetilde{S}_n f}{\sqrt{\ln(n)}}\right)_{n\ge0}$.

We shall prove Theorem 2.1 using the method of moments. The same strategy was used in [22]. However, our setting provides some welcome simplifications, allowing us to apply our method to more general observables f than the ones considered in [22]. These simplifications come namely from the summability in ℓ in Equation (2.2), as well as the summability of other error terms.

As proved in Lemma 2.7, the hypothesis (2.2) is satisfied under quite general spectral assumptions, which are stated in Hypothesis 2.6. In particular, Hypothesis 2.6 holds for the collision map associated with Sinai billiards [8], from which we deduce:

[1] The property of convergence in distribution with respect to any absolutely continuous probability measure is sometimes called *strong convergence in distribution* [1].

Corollary 2.2. *Let $(\widetilde{A}, \widetilde{\mu}, \tilde{F})$ be the $\mathbb{Z}^2$-periodic billiard system presented in Subsection 2.2. Let f, $g : \widetilde{A} \to \mathbb{R}$ be such that:*

- *f is η-Hölder on the continuity domain of $\widetilde{T}$ for some $\eta > 0$;*
- *$\sum_{a\in\mathbb{Z}^2} |a|^{\varkappa} \|f(\cdot, a)\|_\eta < +\infty$ for some $\varkappa > 0$;*
- *$\int_{\widetilde{A}} f \, \mathrm{d}\widetilde{\mu} = 0$;*
- *$g \in \mathbb{L}^1(\widetilde{A}, \widetilde{\mu})$,*

where $\|\cdot\|_\eta$ is the maximal η-Hölder norm with respect to the euclidean metric on M on the continuity domains of T.

Then

$$\left(\frac{\widetilde{S}_n g}{\ln(n)}, \frac{\widetilde{S}_n f}{\sqrt{\ln(n)}}\right) \longrightarrow \left(\frac{\int_{\widetilde{A}} g \, \mathrm{d}\widetilde{\mu}}{2\pi\sqrt{\det(\Sigma^2)}} \mathcal{E}, \ \frac{\widetilde{\sigma}(f)}{\sqrt{2\pi}(\det(\Sigma^2))^{\frac{1}{4}}} \sqrt{\mathcal{E}}\, \mathcal{N}\right),$$

in distribution with respect to any probability measure absolutely continuous with respect to $\widetilde{\mu}$, and where $\mathcal{E}$ and $\mathcal{N}$ are two independent random variables, with respectively standard exponential and standard Gaussian distributions, with $\tilde{\sigma}^2(f)$ given by Equation (2.3) and with Σ the invertible symmetric positive matrix such that $\Sigma^2 = \sum_{k\in\mathbb{Z}} \mathbb{E}\left[F \otimes (F \circ T^k)\right]$.

Proof. This corollary is a direct consequence of Theorem 2.1 and Lemma 2.7 thanks to [8, Theorem 3.17] (ensuring Hypothesis 2.6 with $p = 1$) and to [9, Lemma 5.3] or [8, Lemma 4.5] (ensuring that $\|f(\cdot, a)\times\|_{\mathcal{L}(\mathcal{B},\mathcal{B})} \leq C \|f(\cdot, a)\|_\eta$). □

While the results above are proved using the method of moments, we shall also show how to prove similar propositions using induced systems. The strategy follows closely that of [29, Proposition 6.12], which was in the setting of geodesic flows in negative curvature, with some improvements from [22]. The main difference in the present article is that, in the context of Sinai billiards, we use Young tower in order to introduce a symbolic coding of the trajectories.

Proposition 2.3. *Let $(\widetilde{A}, \widetilde{\mu}, \widetilde{T})$ be the $\mathbb{Z}^2$-periodic billiard system presented in Subsection 2.2. Let $f : \widetilde{A} \to \mathbb{R}$ be such that:*

- *f is η-Hölder on the continuity domains of $\widetilde{T}$, with a uniformly bounded η-Hölder norm, for some $\eta > 0$;*
- *$\sum_{a\in\mathbb{Z}^2} (1 + \ln_+ |a|)^{\frac{1}{2}+\varkappa} \|f(\cdot, a)\|_\infty < +\infty$ for some $\varkappa > 0$;*
- *$\int_{\widetilde{A}} f \, \mathrm{d}\widetilde{\mu} = 0$.*

Then

$$\frac{\widetilde{S}_n f}{\sqrt{\ln(n)}} \longrightarrow \frac{\widehat{\sigma}(f)}{\sqrt{2\pi}(\det(\Sigma^2))^{\frac{1}{4}}} L,$$

in distribution with respect to any probability measure absolutely continuous with respect to $\widetilde{\mu}$, where L follows a centered Laplace distribution of variance 1, with $\widehat{\sigma}(f)$ given by Equation (5.2) and with Σ the invertible symmetric positive matrix such that $\Sigma^2 = \sum_{k\in\mathbb{Z}} \mathbb{E}\left[F \otimes (F \circ T^k)\right]$.

In addition, $\widehat{\sigma}(f) = 0$ if and only if f is a coboundary.

Comparing the conclusions of Corollary 2.2 and Proposition 2.3, one has $\widetilde{\sigma}(f) = \widehat{\sigma}(f)$ whenever f satisfies the assumptions of Corollary 2.2. This equality has a deep dynamical consequences [22].

The assumptions of Proposition 2.3 are slightly weaker than those of Corollary 2.2. The conclusions of the former are also weaker, dealing with the limit distribution of $S_n f$ and not the limit joint distribution of $(S_n g, S_n f)$. The stronger result should hold under the assumptions of Proposition 2.3, but one would start from [27, Theorem 1.7], which is beyond the scope of this article. On the other hand, it should also be possible to weaken the assumptions of Corollary 2.2 (dynamically Hölder observables satisfying a weaker decay condition expressed only in terms of supremum norm), with a less direct and more technical proof (using approximations).

2.4. **Results for flows.** Theorem 2.1 admits a version for semiflows:

Theorem 2.4. *Assume that $(\widetilde{A}, \widetilde{\mu}, \widetilde{T})$ is conservative and ergodic. Let $\eta > 0$ and p, $p^* \in [1, \infty]$ such that $p^{-1} + (p^*)^{-1} = 1$. Let $(\mathcal{B}, \|\cdot\|_{\mathcal{B}})$ be a Banach space satisfying the assumptions of Theorem 2.1.*

Let ϕ, $\psi : \widetilde{\mathcal{M}} \to \mathbb{R}$ be such that:

- $\sum_{a\in\mathbb{Z}^2} (1 + |a|^{\varkappa}) \|G(\phi)(\cdot, a) \times \cdot\|_{\mathcal{L}(\mathcal{B},\mathcal{B})} < +\infty$ *for some* $\varkappa > 0$*;*
- $\sum_{a\in\mathbb{Z}^2} \|G(|\phi|)(\cdot, a)\|_{\mathbb{L}^{p^*}(A,\mu)} < +\infty$*;*
- $\int_{\widetilde{M}} \phi \, \mathrm{d}\widetilde{\mu} = 0$*;*
- $\psi \in \mathbb{L}^1(\widetilde{\mathcal{M}}, \widetilde{\nu})$.

Then, for every $0 < s_1 < s_2$, as $t \to \infty$,

$$\left(\frac{\widetilde{S}_{ts}\psi}{\ln(t)}, \frac{\widetilde{S}_{ts}\phi}{\sqrt{\ln(t)}} \right)_{s\in[s_1,s_2]} \longrightarrow \left(\int_{\widetilde{\mathcal{M}}} \psi \, \mathrm{d}\widetilde{\nu} \, \Phi(0)\mathcal{E}, \, \widetilde{\sigma}(G(\phi)) \sqrt{\Phi(0)\mathcal{E}} \, \mathcal{N} \right)_{s\in[s_1,s_2]}, \tag{2.5}$$

in distribution in $\mathcal{C}([s_1, s_2], \mathbb{R})$ with respect to any probability measure absolutely continuous with respect to $\widetilde{\nu}$. In Equation (2.5), $\mathcal{E}$ and $\mathcal{N}$ are two independent random variables with respectively standard exponential and standard Gaussian distributions.

As Theorem 2.1 was applied to the collision map for Sinai billiards, so does Theorem 2.4 to the flow on Sinai billiards (i.e. to the two-dimensional Lorentz gas model). In order for the Lorentz gas to fit our setting, we see it as a suspension flow of height $\widetilde{\tau}$ over its collision map $(\widetilde{A}, \widetilde{\mu}, \widetilde{T})$, which was the object of Corollary 2.2.

Corollary 2.5. *Let $(\widetilde{\mathcal{M}}, \widetilde{\nu}, (\widetilde{Y}_t)_t)$ be the $\mathbb{Z}^2$-periodic Lorentz gaz described above. Let η, $\varkappa > 0$, and denote by $\|\cdot\|_\eta$ the η-Hölder norm on $\widetilde{\mathcal{M}}$.*

Let ϕ, $\psi : \widetilde{\mathcal{M}} \to \mathbb{R}$ be such that:

- $\sum_{a=(a_1,a_2)\in\mathbb{Z}^2} |a|^{\varkappa} \left\|\phi_{|[a_1,a_1+1]\times[a_2,a_2+1]}\right\|_\eta < +\infty$*;*
- $\int_{\widetilde{\mathcal{M}}} \phi \, \mathrm{d}\widetilde{\nu} = 0$*;*
- $\psi \in \mathbb{L}^1(\widetilde{\mathcal{M}}, \widetilde{\nu})$*;*

Then, for every $0 < s_1 < s_2$, as $t \to +\infty$,

$$\left(\frac{\widetilde{S}_{ts}\psi}{\ln(t)}, \frac{\widetilde{S}_{ts}\phi}{\sqrt{\ln(t)}} \right)_{s\in[S_1,S_2]} \longrightarrow \left(\frac{\int_{\widetilde{\mathcal{M}}} \psi \, \mathrm{d}\widetilde{\nu}}{2\pi\sqrt{\det(\Sigma^2)}} \mathcal{E}, \, \frac{\widetilde{\sigma}(G(\phi))}{\sqrt{2\pi}(\det(\Sigma^2))^{\frac{1}{4}}} \sqrt{\mathcal{E}} \, \mathcal{N} \right)_{s\in[s_1,s_2]}, \tag{2.6}$$

with

$$\widetilde{\sigma}(G(\phi))^2 := \sum_{k\in\mathbb{Z}} \int_{\widetilde{A}} G(\phi) \cdot G(\phi) \circ \widetilde{T}^k \, \mathrm{d}\widetilde{\mu} \, .$$

2.5. **Spectral hypotheses.** All the results above hold whenever Assumption (2.2) is satisfied. To finish this introduction, we now relate this assumption to more classical spectral conditions on the transfert operator associated with (A, μ, T).

The transfer operator P is defined, for $f \in \mathbb{L}^1(A, \mu)$, by:

$$\int_A P(f) \cdot g \, \mathrm{d}\mu = \int_A f \cdot g \circ T \, \mathrm{d}\mu \quad \forall g \in \mathbb{L}^\infty(A, \mu) \, .$$

Recall that $F : A \to \mathbb{Z}^2$. Let $\mathbb{T}^2 := \mathbb{R}^2/(2\pi\mathbb{Z})^2$. We define a family of twisted transfer operators $(P_u)_{u\in\mathbb{T}^2}$ by:

$$P_u(h) := P(e^{i\langle u,F\rangle} h)$$

for all $h \in \mathbb{L}^1(A, \mu)$. Note that:

$$P_u^k(h) = P^k \left(e^{i\langle u,S_k F\rangle} h \right) \, . \tag{2.7}$$

The idea to study the spectral properties of P_u to establish limit theorems goes back to the seminal works by Nagaev [20, 21] and Guivarc'h [13] and has been deeply generalized by Keller and Liverani in [18]. We refer to the book by Hennion and Hervé [14] for an overview of the important results that can be proved by this method.

The more usual spectral conditions are:

Hypothesis 2.6 (Spectral hypotheses)**.** *There exists a complex Banach space $(\mathcal{B}, \|\cdot\|_{\mathcal{B}})$ of functions or of distributions defined on A, on which P acts continuously, and such that:*

- $\mathbf{1} \in \mathcal{B}$ *and* $\mathbb{E}_\mu[\cdot]$ *extends continuously from* $\mathcal{B} \cap L^1(A, \mu)$ *to* $\mathcal{B}$*;*
- *for every* $a \in \mathbb{Z}^2$*, the multiplication by* $f(\cdot, a)$ *belongs to* $\mathcal{L}(\mathcal{B}, \mathcal{B})$*;*
- *There exist an open ball* $U \subset \mathbb{T}^2$ *containing* 0*, two constants* $C > 0$ *and* $r \in (0,1)$*, continuous functions* $\lambda_\cdot : U \to \mathbb{C}$ *and* $\Pi_\cdot, R_\cdot : U \to \mathcal{L}(\mathcal{B}, \mathcal{B})$ *such that*

$$P_u^n = \lambda_u^n \Pi_u + R_u^n \tag{2.8}$$

with:

$$\|\Pi_u - \mathbb{E}_\mu[\cdot]\|_{\mathcal{L}(\mathcal{B},\mathcal{B})} \leq C|u| \quad \forall u \in U, \tag{2.9}$$

$$\sup_{u \in U} \left\| R_u^k \right\|_{\mathcal{L}(\mathcal{B},\mathcal{B})} + \sup_{u \in \mathbb{T}^2 \setminus U} \left\| P_u^k \right\|_{\mathcal{L}(\mathcal{B},\mathcal{B})} \leq C r^k, \tag{2.10}$$

- *there exists an invertible positive symmetric matrix* Σ *and* $\varepsilon > 0$ *such that, as* $u \to 0$*,*

$$\lambda_u = e^{-\frac{\langle \Sigma^2 u, u\rangle}{2}} + O\left(|u|^{2+\varepsilon}\right) . \tag{2.11}$$

Lemma 2.7. *Assume that the Hypotheses 2.6 hold. Let* $\Phi(x) = \frac{e^{-\frac{\langle \Sigma^{-2} x, x\rangle}{2}}}{2\pi\sqrt{\det(\Sigma^2)}}$ *and* $\eta \in (0, \varepsilon/2]$*. Then Equation* (2.2) *holds:*

$$\sup_{\substack{a \in \mathbb{Z}^2,\ h \in \mathcal{B} \\ \|h\|_{\mathcal{B}} \leq 1}} \left\| P^\ell \left(\mathbf{1}_{\{S_\ell F = a\}} h \right) - \frac{\Phi\left(\frac{a}{\sqrt{\ell}}\right)}{\ell} \mathbb{E}_\mu[h] \right\|_{\mathcal{B}} = O(\ell^{-1-\eta}) .$$

Proof. Let $Q_{\ell,a}$ be the operator acting on any $h \in \mathbb{L}^1(A, \mu)$ by:

$$Q_{\ell,a}(h)(x) := P^\ell \left(\mathbf{1}_{\{S_\ell F = a\}} h \right)(x) .$$

Due to Equation (2.7),

$$Q_{\ell,a}(h) = \frac{1}{(2\pi)^2} \int_{\mathbb{T}^2} e^{-i\langle u, a\rangle} P_u^\ell(h) \, \mathrm{d}u , \tag{2.12}$$

and in particular $Q_{\ell,a}$ acts on $\mathcal{B}$. From Hypothesis 2.6, and up to taking a smaller neighborhood U, there exist constants C_0, $c_0 > 0$ such that $\|P_u\|_{\mathcal{L}(\mathcal{B},\mathcal{B})} \leq C_0$ and

$$\max \left\{ |\lambda_u|, \left| e^{-\frac{\langle \Sigma^2 u, u\rangle}{2}} \right| \right\} \leq e^{-c_0 |u|^2}$$

for all $u \in U$. Due to Equations (2.12) and (2.10),

$$\sup_{a \in \mathbb{Z}^2} \left\| Q_{\ell,a} - \frac{1}{(2\pi)^2} \int_U e^{-i\langle u, a\rangle} \lambda_u^\ell \Pi_u \, \mathrm{d}u \right\|_{\mathcal{L}(\mathcal{B},\mathcal{B})} = O(r^\ell). \tag{2.13}$$

In addition, there exists $C_1 > 0$ such that, for every $u \in U$,

$$\begin{aligned} \left\| \lambda_u^\ell \Pi_u - e^{-\ell \frac{\langle \Sigma^2 u, u\rangle}{2}} \Pi_0 \right\|_{\mathcal{L}(\mathcal{B},\mathcal{B})} &\leq |\lambda_u|^\ell \|\Pi_u - \Pi_0\|_{\mathcal{L}(\mathcal{B},\mathcal{B})} + \left| \lambda_u^\ell - e^{-\ell \frac{\langle \Sigma^2 u, u\rangle}{2}} \right| \|\Pi_0\|_{\mathcal{L}(\mathcal{B},\mathcal{B})} \\ &\leq C_1 (|u| + \ell |u|^{2+\varepsilon}) e^{-\ell \frac{c_0 |u|^2}{2}} , \end{aligned}$$

due to the asymptotic expansion of $u \mapsto \lambda_u$ and to Equation (2.9). Hence, using the change of variable $u = v/\sqrt{\ell}$,

$$\begin{aligned}\sup_{a\in\mathbb{Z}^2}\left\|\frac{1}{(2\pi)^2}\int_U e^{-i\langle u,a\rangle}\lambda_u^\ell\Pi_u\,\mathrm{d}u - \frac{1}{(2\pi)^2}\int_U e^{-i\langle u,a\rangle}e^{-\ell\frac{\langle\Sigma^2 u,u\rangle}{2}}\Pi_0\,\mathrm{d}u\right\|_{\mathcal{L}(\mathcal{B},\mathcal{B})}\\ \leq C_1\int_U(|u|+\ell|u|^{2+\varepsilon})e^{-\ell\frac{c_0|u|^2}{2}}\,\mathrm{d}u\\ \leq \frac{C_1}{\ell}\int_{\mathbb{R}^2}\left(\frac{|v|}{\sqrt{\ell}}+\ell\frac{|v|^{2+\varepsilon}}{\ell^{1+\frac{\varepsilon}{2}}}\right)e^{-\frac{c_0|v|^2}{2}}\,\mathrm{d}v = O\left(\frac{1}{\ell^{1+\frac{\varepsilon}{2}}}\right).\end{aligned} \tag{2.14}$$

Finally, using the same change of variable,

$$\begin{aligned}\sup_{a\in\mathbb{Z}^2}\left|\frac{1}{(2\pi)^2}\int_U e^{-i\langle u,a\rangle}e^{-\ell\frac{\langle\Sigma^2u,u\rangle}{2}}\,\mathrm{d}u-\frac{1}{\ell}\Phi\left(\frac{a}{\sqrt{\ell}}\right)\right|\\ =\sup_{a\in\mathbb{Z}^2}\left|\frac{1}{(2\pi)^2\ell}\int_{\sqrt{\ell}U}e^{-i\frac{\langle v,a\rangle}{\sqrt{\ell}}}e^{-\frac{\langle\Sigma^2v,v\rangle}{2}}\,\mathrm{d}v-\frac{1}{(2\pi)^2\ell}\int_{\mathbb{R}^2}e^{-i\frac{\langle v,a\rangle}{\sqrt{\ell}}}e^{-\frac{\langle\Sigma^2u,u\rangle}{2}}\,\mathrm{d}v\right|\\ \leq\left|\frac{1}{(2\pi)^2\ell}\int_{\mathbb{R}^2\setminus\sqrt{\ell}U}e^{-\frac{\langle\Sigma^2v,v\rangle}{2}}\,\mathrm{d}v\right| = O\left(\ell^{-2}\right).\end{aligned} \tag{2.15}$$

The lemma follows from Equations (2.13), (2.14) and (2.15). □

3. Proof of Theorem 2.1

This section is devoted to the proof of Theorem 2.1. We proceed in two steps. First, we prove the convergence in distribution for $t = 1$ and then we shall extend the convergence in distribution to a functional convergence. The method we use here is close to the one used in [22]. In [22], we considered a wide family of dynamical systems ($\mathbb{Z}^d$-extensions with $d \in \{1, 2\}$ and $(S_nF)_n$ satisfying a standard or nonstandard central limit theorem involving a stable distribution), but we considered also a specific family of observables f (which were assumed to be constant on each cell, i.e. satisfying $f(x, a) = f(y, a)$ for every $x, y \in A$). In the present paper, we focus on more specific dynamical systems (with $d = 2$ and $(S_nF)_n$ satisfying a standard central limit theorem), which includes the Lorentz process. This more strigent context allows significative simplifications (due to summable error terms) which make much clearer the understanding of our argument and allow us to generalize the method used in [22] to more general observables.

3.1. **Convergence in distribution for $t = 1$.** This section is devoted to the proof of Theorem 2.1 for $t = 1$. In other words, under the hypotheses of Theorem 2.1, we shall show that:

$$\left(\frac{\widetilde{S}_ng}{\ln(n)},\frac{\widetilde{S}_nf}{\sqrt{\ln(n)}}\right)\longrightarrow\left(\int_{\widetilde{A}}g\,\mathrm{d}\widetilde{\mu}\,\Phi(0)\mathcal{E},\,\widetilde{\sigma}(f)\sqrt{\Phi(0)\mathcal{E}}\,\mathcal{N}\right),\quad\text{as } n\to+\infty\,, \tag{3.1}$$

where the convergence is in distribution as $n \to +\infty$, with respect to any absolutely continuous probability measure.

Proof of Theorem 2.1 for $t = 1$. Since $\widetilde{T}$ is ergodic, due to Hopf's ergodic theorem [15, §14, Individueller Ergodensatz für Abbildungen], we assume without any loss of generality that $g(x, a) = \mathbf{1}_0(a)$, which shall significantly simplify the computations in the proof of Lemma 3.2.

Set $\mathfrak{a}_n = \ln(n)$, so that $\mathfrak{a}_n \sim \sum_{k=1}^n k^{-1}$ as $n \to +\infty$. Due to [31, Theorem 1], it is enough to prove the convergence in distribution with respect to $\widetilde{T}_*(\mu\otimes\delta_0)$, i.e. the convergence in distribution of $(\frac{\mathcal{Z}_ng}{\mathfrak{a}_n}, \frac{\mathcal{Z}_nf}{\sqrt{\mathfrak{a}_n}})_n$ with respect to μ, where:

$$\mathcal{Z}_nh(x) := (\widetilde{S}_nh)\circ\widetilde{T}(x,0) = \sum_{k=1}^n h\left(T^kx, S_kF(x)\right).$$

The convergence in distribution of $(\frac{\mathcal{Z}_ng}{\mathfrak{a}_n}, \frac{\mathcal{Z}_nf}{\sqrt{\mathfrak{a}_n}})_n$ is equivalent to the convergence in distribution of $\alpha\frac{\mathcal{Z}_ng}{\mathfrak{a}_n}+\beta\frac{\mathcal{Z}_nf}{\sqrt{\mathfrak{a}_n}}$ for every $\alpha, \beta \in \mathbb{R}$. Let us fix α, $\beta \in \mathbb{R}$ for the remainder of the proof.

We use the method of moments. Setting $h_n(x,a) := \frac{\alpha}{\mathfrak{a}_n} g(x,a) + \frac{\beta}{\sqrt{\mathfrak{a}_n}} f(x,a)$, due to Carleman's criterion [12, Chap. XV.4], it is enough to prove that, for all $m \geq 0$,

$$\lim_{n\to+\infty} \mathbb{E}_\mu\left[(\mathcal{Z}_n h_n)^m\right] = \mathbb{E}\left[(\alpha\Phi(0)\mathcal{E} + \beta\sqrt{\Phi(0)\mathcal{E}}\,\widetilde{\sigma}(f)\mathcal{N})^m\right] . \tag{3.2}$$

Let us fix an integer $m \geq 0$ for the remainder of the proof. Then, for all n:

$$\begin{aligned}\mathbb{E}_\mu\left[(\mathcal{Z}_n h_n)^m\right] &= \mathbb{E}_\mu\left[\left(\sum_{k=1}^n (h_n(T^k(\cdot), S_k F(\cdot))\right)^m\right] \\ &= \sum_{k_1,\dots,k_m=1}^n \sum_{d_1,\dots,d_m\in\mathbb{Z}^2} \mathbb{E}_\mu\left[\prod_{s=1}^m h_n(T^{k_s}(\cdot), d_s)\mathbf{1}_{\{S_{k_s}F(\cdot)=d_s\}}\right] .\end{aligned}$$

Gathering the terms for which the k_s (with their multiplicities) are the same, we obtain

$$\mathbb{E}_\mu\left[(\mathcal{Z}_n h_n)^m\right] = \sum_{q=1}^m \sum_{\substack{N_j\geq 1\\ N_1+\dots+N_q=m}} c_{\mathbf{N}} A_{n;q,\mathbf{N}} , \tag{3.3}$$

where $\mathbf{N} = (N_j)_{1\leq j\leq q}$, where $c_{\mathbf{N}}$ is the cardinal of the set of maps $\phi: \{1,\dots,m\} \to \{1,\dots,q\}$ such that $|\phi^{-1}(\{j\})| = N_j$ for all $j \in \{1,\dots,q\}$, and where

$$\begin{aligned}A_{n;q;\mathbf{N}} &:= \sum_{1\leq n_1<\dots<n_q\leq n} \sum_{\mathbf{a}\in(\mathbb{Z}^2)^q} \mathbb{E}_\mu\left[\prod_{j=1}^q \left(h_n(T^{n_j}(\cdot), a_j)^{N_j}\mathbf{1}_{\{S_{n_j}F(\cdot)=a_j\}}\right)\right] \\ &= \sum_{1\leq n_1<\dots<n_q\leq n} \sum_{\mathbf{a}\in(\mathbb{Z}^2)^q} \mathbb{E}_\mu\left[\prod_{j=1}^q \left(h_n(T^{n_j}(\cdot), a_j)^{N_j}\mathbf{1}_{\{S_{n_j}F(\cdot)-S_{n_{j-1}}F(\cdot)=a_j-a_{j-1}\}}\right)\right] \\ &= \sum_{\mathbf{a}\in(\mathbb{Z}^2)^q}\left[\sum_{\boldsymbol{\ell}\in E_{q,n}} \mathbb{E}_\mu\left[\prod_{j=1}^q \left(h_n(T^{\ell_j}(\cdot), a_j)^{N_j}\mathbf{1}_{\{S_{\ell_j}F(\cdot)=a_j-a_{j-1}\}}\right)\circ T^{\ell_1+\dots+\ell_{j-1}}(\cdot)\right]\right] ,\end{aligned}$$

with the notations $\mathbf{a} = (a_1,\dots,a_q)$, $n_0 := 0$, $a_0 := 0$ and

$$E_{q,n} = \left\{\boldsymbol{\ell} = (\ell_1,\dots,\ell_q) \in \{1,\dots,n\}^q \;:\; \sum_{j=1}^q \ell_j \leq n\right\} ;$$

ℓ_j corresponds to $n_j - n_{j-1}$. As in the proof of Lemma 2.7, for all $\ell \in \mathbb{N}$ and $a \in \mathbb{Z}^2$, we define operators $Q_{\ell,a}$ and $\widetilde{Q}_{\ell,a,b,N,n}$ acting on $\mathcal{B}$ by:

$$\begin{aligned}Q_{\ell,a}(G)(x) &:= P^\ell\left(\mathbf{1}_{\{S_\ell F(x)=a\}}G\right)(x) , \\ \widetilde{Q}_{\ell,a,b,N,n}(G)(x) &:= h_n(x,a)^N Q_{\ell,a-b}(G)(x) .\end{aligned}$$

Using $\mathbb{E}_\mu[\cdot] = \mathbb{E}_\mu[P^{\ell_1+\dots+\ell_q}(\cdot)]$ and using repeatedly $P^k(G\circ T^k\cdot H) = G\cdot P^k(H)$, we obtain

$$A_{n;q;\mathbf{N}} = \sum_{\mathbf{a}\in(\mathbb{Z}^2)^q} \sum_{\boldsymbol{\ell}\in E_{q,n}} \mathbb{E}_\mu\left[\widetilde{Q}_{\ell_q,a_q,a_{q-1},N_q,n}\cdots\widetilde{Q}_{\ell_1,a_1,0,N_1,n}(\mathbf{1})\right] . \tag{3.4}$$

We further split the operators $Q_{\ell,a}$:

$$Q_{\ell,a} = Q^{(0)}_{\ell,a} + Q^{(1)}_{\ell,a} \text{ with } Q^{(0)}_{\ell,a} := \Phi(0)\frac{\mathbb{E}_\mu[\cdot]}{\ell} . \tag{3.5}$$

We assume without loss of generality that $\eta = \varkappa/4 \leq 1$. Note that

$$\left\|Q^{(1)}_{\ell,a}\right\|_{\mathcal{L}(\mathcal{B},\mathcal{B})} = O((1+|a|^{2\eta})\ell^{-1-\eta}) \tag{3.6}$$

by Hypothesis (2.2) and using the fact that $|\Phi(x)-\Phi(0)| \leq \mathfrak{c}\min(x^2,1) \leq \mathfrak{c}x^{2\eta}$ for some $\mathfrak{c}>0$. Thus, for all $N \geq 1$,

$$\left\| h_n(\cdot,a)^N Q^{(1)}_{\ell,a-b} \right\|_{\mathcal{L}(\mathcal{B},\mathcal{B})} \leq C \frac{1+|a-b|^{\frac{\varkappa}{2}}}{\mathfrak{a}_n^{\frac{N}{2}}\,\ell^{1+\eta}} \left(\frac{\mathbf{1}_0(a)}{\mathfrak{a}_n^{\frac{N}{2}}} + \|f(\cdot,a)\times\cdot\|^N_{\mathcal{L}(\mathcal{B},\mathcal{B})} \right).$$

We introduce these operators $Q^{(0)}_{\ell,a}$ and $Q^{(1)}_{\ell,a}$ into (3.4), creating new data we need to track: the index of the operator we use at each point in the weighted path. Fix n, q and $\mathbf{N}$. Given $\boldsymbol{\varepsilon}=(\varepsilon_1,\dots,\varepsilon_q)\in\{0,1\}^q$ and $s\in\mathbb{Z}^2$, write:

$$B^{\boldsymbol{\varepsilon}}_{s,\boldsymbol{\ell},\mathbf{N}}(G) := \sum_{\substack{a_0,\dots,a_q\in\mathbb{Z}^2\\ a_0=s}} h_n(\cdot,a_q)^{N_q} Q^{(\varepsilon_q)}_{\ell_q,a_q-a_{q-1}} \cdots h_n(\cdot,a_1)^{N_1}\, Q^{(\varepsilon_1)}_{\ell_1,a_1-a_0}(G)\,,$$

$$b^{\boldsymbol{\varepsilon}}_{s,\boldsymbol{\ell},\mathbf{N}}(G) := \sum_{\mathbf{a}\in(\mathbb{Z}^2)^q} \mathbb{E}_\mu\left[h_n(\cdot,a_q)^{N_q} Q^{(\varepsilon_q)}_{\ell_q,a_q-a_{q-1}} \dots h_n(\cdot,a_1)^{N_1}\, Q^{(\varepsilon_1)}_{\ell_1,a_1-s}(G)\right] = \mathbb{E}_\mu\left[B^{\boldsymbol{\varepsilon}}_{s,\boldsymbol{\ell},\mathbf{N}}(G)\right],$$

$$A^{\boldsymbol{\varepsilon}}_{n;q;\mathbf{N}} := \sum_{\boldsymbol{\ell}\in E_{q,n}} b^{\boldsymbol{\varepsilon}}_{0,\boldsymbol{\ell},\mathbf{N}}(\mathbf{1})\,,$$

so that:

$$A_{n;q;\mathbf{N}} = \sum_{\boldsymbol{\varepsilon}\in\{0,1\}^q} A^{\boldsymbol{\varepsilon}}_{n;q;\mathbf{N}} = \sum_{\boldsymbol{\varepsilon}\in\{0,1\}^q}\sum_{\boldsymbol{\ell}\in E_{q,n}} b^{\boldsymbol{\varepsilon}}_{0,\boldsymbol{\ell},\mathbf{N}}(\mathbf{1}).$$

The goal is now to understand which combinatorial data $(\mathbf{N},\boldsymbol{\varepsilon})$ is negligible as $n\to+\infty$, and which represent the majority of the m-th moment. The following properties are directly implied by the definitions.

Properties 3.1. *Consider a single linear form $b^{\boldsymbol{\varepsilon}}_{s,\boldsymbol{\ell},\mathbf{N}}$. For all $1\leq i\leq q$, the terms on the right side of $Q^{(\varepsilon_i)}_{\ell_i,a_i-a_{i-1}}$ depend only on $a_1,\dots,a_{i-1}$, and the terms on its left side only depend on $a_i,\dots,a_q$. Hence:*

(I) Since $Q^{(0)}_{\ell,a}$ does not depend on a, the value of $b^{(0,\boldsymbol{\varepsilon})}_{s,(\ell_0,\boldsymbol{\ell}),(N_0,\mathbf{N})}$ does not depend on s. Without loss of generality, we shall choose s to be 0 when $\varepsilon_1=0$.

(II) $b^{(0)}_{s,(\ell),(N)}(\cdot) = \frac{\Phi(0)}{\ell}\sum_{a\in\mathbb{Z}^2}\mathbb{E}_\mu[h_n(\cdot,a)^N]\mathbb{E}_\mu[\cdot]$ for all ℓ, $N\geq 1$, so:

$$b^{(0)}_{s,(\ell),(1)}(\cdot) = \frac{\Phi(0)\alpha}{\ell\mathfrak{a}_n}\mathbb{E}_\mu[\cdot]\,,$$

$$b^{(0)}_{s,(\ell),(2)}(\cdot) = \frac{\Phi(0)\beta^2}{\ell\mathfrak{a}_n}\left(\sum_{a\in\mathbb{Z}^2}\mathbb{E}_\mu[f(\cdot,a)^2]\mathbb{E}_\mu[\cdot] + O\left(\frac{1}{\sqrt{\mathfrak{a}_n}}\right)\right).$$

(III) $b^{(\boldsymbol{\varepsilon},0,\boldsymbol{\varepsilon}')}_{s,(\boldsymbol{\ell},\ell_0,\boldsymbol{\ell}'),(\mathbf{N},N_0,\mathbf{N}')} = \mathbb{E}_\mu[B^{(0,\boldsymbol{\varepsilon}')}_{0,(\ell_0,\boldsymbol{\ell}'),(N_0,\mathbf{N}')}(\mathbf{1})]\mathbb{E}_\mu[B^{\boldsymbol{\varepsilon}}_{s,\boldsymbol{\ell},\mathbf{N}}(\cdot)]$, i.e.

$$b^{(\boldsymbol{\varepsilon},0,\boldsymbol{\varepsilon}')}_{s,(\boldsymbol{\ell},\ell_0,\boldsymbol{\ell}'),(\mathbf{N},N_0,\mathbf{N}')}(\cdot) = b^{(0,\boldsymbol{\varepsilon}')}_{0,(\ell_0,\boldsymbol{\ell}'),(N_0,\mathbf{N}')}(\mathbf{1})\, b^{\boldsymbol{\varepsilon}}_{s,\boldsymbol{\ell},\mathbf{N}}(\cdot)\,.$$

(IV) In particular, $b^{(\boldsymbol{\varepsilon},0)}_{s,(\boldsymbol{\ell},\ell_0),(\mathbf{N},1)} = \frac{\Phi(0)\alpha}{\ell\mathfrak{a}_n} b^{\boldsymbol{\varepsilon}}_{s,\boldsymbol{\ell},\mathbf{N}}(\cdot)$, and:

$$\begin{aligned} b^{(\boldsymbol{\varepsilon},0,0,\boldsymbol{\varepsilon}')}_{s,(\boldsymbol{\ell},\ell_0,\ell_0',\boldsymbol{\ell}'),(\mathbf{N},1,N_0',\mathbf{N}')} &= b^{(0,\boldsymbol{\varepsilon}')}_{0,(\ell_0',\boldsymbol{\ell}'),(N_0',\mathbf{N}')}(\mathbf{1})\, b^{(0)}_{0,(\ell_0),(1)}(\mathbf{1})\, b^{\boldsymbol{\varepsilon}}_{s,\boldsymbol{\ell},\mathbf{N}}(\cdot) \\ &= \frac{\Phi(0)\alpha}{\ell_0\mathfrak{a}_n} b^{(0,\boldsymbol{\varepsilon}')}_{0,(\ell_0',\boldsymbol{\ell}'),(N_0',\mathbf{N}')}(\mathbf{1})\, b^{\boldsymbol{\varepsilon}}_{s,\boldsymbol{\ell},\mathbf{N}}(\cdot)\,. \end{aligned}$$

(V)

$$b^{(0,1,\dots,1)}_{s,(\ell_1,\dots,\ell_q),(N_1,N_2,\dots,N_q)}(\mathbf{1}) = \frac{\Phi(0)}{\ell_1}\sum_{a_1\in\mathbb{Z}^2} b^{(1,\dots,1)}_{a_1,(\ell_2,\dots,\ell_q),(N_2,\dots,N_q)}(h_n(\cdot,a_1)^{N_1}).$$

In particular, we can estimate the coefficients corresponding to $\boldsymbol{\varepsilon}=(0,1)$ and $\mathbf{N}=(1,1)$, which will play an important role later on.

Lemma 3.2. *Under the hypotheses of Theorem 2.1 and with the notations of its proof,*

$$b^{(0,1)}_{s,(\ell,\ell'),(1,1)}(\mathbf{1}) = \frac{\Phi(0)\beta^2}{\mathfrak{a}_n \ell} \int_{\widetilde{A}} f \circ \widetilde{T}^{\ell'} \cdot f \, \mathrm{d}\widetilde{\mu} + O\left(\frac{1}{\ell \ell'^{1+\eta} \mathfrak{a}_n^{\frac{3}{2}}} \right). \tag{3.7}$$

Proof of Lemma 3.2. Applying Point (V) of Properties 3.1,

$$b^{(0,1)}_{s,(\ell,\ell'),(1,1)}(\mathbf{1}) = \frac{\Phi(0)}{\ell} \sum_{a,b\in\mathbb{Z}^2} \mathbb{E}_\mu \left[h_n(\cdot,b) Q^{(1)}_{\ell',b-a}(h_n(\cdot,a)) \right].$$

Recall that $h_n = \frac{\alpha}{\mathfrak{a}_n} g + \frac{\beta}{\sqrt{\mathfrak{a}_n}} f$ with $g(x,a) = \mathbf{1}_0(a)$. By Equation (3.6), $\left\| Q^{(1)}_{\ell,a} \right\|_{\mathcal{L}(\mathcal{B},\mathcal{B})} = O((1+|a|^{2\eta})\ell^{-1-\eta})$, therefore

$$\begin{aligned}
\frac{\Phi(0)}{\ell} \sum_{a,b\in\mathbb{Z}^2} \mathbb{E}_\mu \left[\frac{\alpha}{\mathfrak{a}_n} g(\cdot,b) Q^{(1)}_{\ell',b-a}(h_n(\cdot,a)) \right] &= O\left(\frac{1}{\ell\, \ell'^{1+\eta}\, \mathfrak{a}_n} \sum_{a\in\mathbb{Z}^2} (1+|a|^{2\eta}) \left\| h_n(\cdot,a) \right\|_{\mathcal{B}} \right) \\
&= O\left(\frac{1}{\ell\, \ell'^{1+\eta}\, \mathfrak{a}_n} \sum_{a\in\mathbb{Z}^2} (1+|a|^{2\eta}) \left(\frac{\mathbf{1}_0(a)}{\mathfrak{a}_n} + \frac{\|f(\cdot,a)\|_{\mathcal{B}}}{\sqrt{\mathfrak{a}_n}} \right) \right) \\
&= O\left(\frac{1}{\ell\, \ell'^{1+\eta}\, \mathfrak{a}_n^{\frac{3}{2}}} \right),
\end{aligned} \tag{3.8}$$

since $\sum_{a\in\mathbb{Z}^2} (1+|a|^{2\eta}) \|f(\cdot,a)\|_{\mathcal{B}} < \infty$. In the same way,

$$\begin{aligned}
&\frac{\Phi(0)}{\ell} \sum_{a,b\in\mathbb{Z}^2} \mathbb{E}_\mu \left[\frac{\beta}{\sqrt{\mathfrak{a}_n}} f(\cdot,b) Q^{(1)}_{\ell',b-a}(h_n(\cdot,a)) \right] \\
&= \frac{\Phi(0)\beta^2}{\mathfrak{a}_n \ell} \sum_{a,b\in\mathbb{Z}^2} \mathbb{E}_\mu \left[f(\cdot,b) Q^{(1)}_{\ell',b-a}(f(\cdot,a)) \right] + \frac{\Phi(0)\alpha\beta}{\ell \mathfrak{a}_n^{\frac{3}{2}}} \sum_{b\in\mathbb{Z}^2} \mathbb{E}_\mu \left[f(\cdot,b) Q^{(1)}_{\ell',b}(\mathbf{1}) \right] \\
&= \frac{\Phi(0)\beta^2}{\mathfrak{a}_n \ell} \sum_{a,b\in\mathbb{Z}^2} \mathbb{E}_\mu \left[f(\cdot,b) Q_{\ell',b-a}(f(\cdot,a)) \right] + \frac{\Phi(0)\alpha\beta}{\ell \mathfrak{a}_n^{\frac{3}{2}}} \sum_{b\in\mathbb{Z}^2} \|f(\cdot,b) \times \cdot\|_{\mathcal{L}(\mathcal{B},\mathcal{B})} O\left(\frac{1+|b|^{2\eta}}{\ell'^{1+\eta}} \right) \\
&= \frac{\Phi(0)\beta^2}{\mathfrak{a}_n \ell} \int_{\widetilde{A}} f \circ \widetilde{T}^{\ell'} \cdot f \, \mathrm{d}\widetilde{\mu} + O\left(\frac{1}{\ell\, \ell'^{1+\eta}\, \mathfrak{a}_n^{\frac{3}{2}}} \right),
\end{aligned}$$

where we used the fact that $\sum_{a,b\in\mathbb{Z}^2} \mathbb{E}_\mu \left[f(\cdot,b) Q^{(0)}_{\ell',b-a}(f(\cdot,a)) \right] = \frac{\Phi(0)}{\ell'} \left(\int_{\widetilde{A}} f \, \mathrm{d}\widetilde{\mu} \right)^2 = 0$. The claim follows from this, combined with Equation (3.8). □

Given a sequence $\boldsymbol{\varepsilon} \in \{0,1\}^q$, we can iterate Point (III) of Properties 3.1 to cut $b^{\boldsymbol{\varepsilon}}_{s,\boldsymbol{\ell},\mathbf{N}}$ into smaller pieces, for which 0 may only appear at the beginning of the associated sequences of indices, and then use Point (V) to transform the heading $\varepsilon_i = 0$. Let $m_1 < m_2 < \cdots < m_K$ be the indices $i \in \{1,\ldots,q\}$ such that $\varepsilon_i = 0$. We use the conventions that $m_{K+1} := q+1$ and $\varepsilon_{q+1} := 0$, that $b^{\boldsymbol{\varepsilon}}_{s,\boldsymbol{\ell},\mathbf{N}} \equiv 1$ if $q = 0$, and that an empty product is also equal to 1. Then:

$$\begin{aligned}
b^{\boldsymbol{\varepsilon}}_{s,\boldsymbol{\ell},\mathbf{N}}(\mathbf{1}) &= b^{(1,\ldots,1)}_{s,(\ell_1,\ldots,\ell_{m_1-1}),(N_1,\ldots,N_{m_1-1})}(\mathbf{1}) \prod_{i=1}^{K} b^{(0,1,\ldots,1)}_{0,(\ell_{m_i},\ldots,\ell_{m_{i+1}-1}),(N_{m_i},\ldots,N_{m_{i+1}-1})}(\mathbf{1}) \\
&= (\Phi(0))^K b^{(1,\ldots,1)}_{s,(\ell_1,\ldots,\ell_{m_1-1}),(N_1,\ldots,N_{m_1-1})}(\mathbf{1}) \\
&\qquad \times \prod_{i=1}^{K} \frac{1}{\ell_{m_i}} \sum_{a\in\mathbb{Z}^d} b^{(1,\ldots,1)}_{a,(\ell_{m_i+1},\ldots,\ell_{m_{i+1}-1}),(N_{m_i+1},\ldots,N_{m_{i+1}-1})}(h_n(\cdot,a)^{N_{m_i}}).
\end{aligned}$$

We sum over $\boldsymbol{\ell} \in E_{q,n}$, and get:

$$\left|A^{\varepsilon}_{n,q,\mathbf{N}}\right| \leq \sum_{\boldsymbol{\ell}\in\{1,\dots,n\}^q} \left|b^{\varepsilon}_{0,\boldsymbol{\ell},\mathbf{N}}(\mathbf{1})\right|$$

$$\leq (\Phi(0))^K \left(\sum_{\substack{(\ell_1,\dots,\ell_{m_1-1})\\ \in\{1,\dots,n\}^{m_1-1}}} \left| b^{(1,\dots,1)}_{0,(\ell_1,\dots,\ell_{m_1-1}),(N_1,\dots,N_{m_1-1})}(\mathbf{1})\right| \right) \tag{3.9}$$

$$\times \prod_{i=1}^{K} \left(\sum_{\substack{(\ell_{m_i},\dots,\ell_{m_{i+1}-1})\\ \in\{1,\dots,n\}^{m_{i+1}-m_i}}} \frac{1}{\ell_{m_i}} \left| \sum_{a\in\mathbb{Z}^d} b^{(1,\dots,1)}_{a,(\ell_{m_i+1},\dots,\ell_{m_{i+1}-1}),(N_{m_i+1},\dots,N_{m_{i+1}-1})}(h_n(\cdot,a)^{N_{m_i}}) \right| \right). \tag{3.10}$$

Now, let us bound the terms (3.9) and (3.10); our goal is to find conditions on the combinatorial data ensuring that these terms are negligible. Starting with (3.9),

$$\left\| b^{(1,\dots,1)}_{0,(\ell_1,\dots,\ell_{m_1-1}),(N_1,\dots,N_{m_1-1})}\right\|_{\mathcal{B}^*}$$

$$= O\left(\frac{(\ell_1\dots\ell_{m_1-1})^{-1-\eta}}{\mathfrak{a}_n^{\frac{N_1+\dots+N_{m_1-1}}{2}}} \sum_{a_1,\dots,a_{m_1-1}\in\mathbb{Z}^2} \prod_{j=1}^{m_1-1} (1+|a_j-a_{j-1}|^{2\eta}) \left(\mathbf{1}_0(a_j) + \|f(\cdot,a_j)\times\cdot\|^{N_j}_{\mathcal{L}(\mathcal{B},\mathcal{B})}\right)\right)$$

$$= O\left(\frac{(\ell_1\dots\ell_{m_1-1})^{-1-\eta}}{\mathfrak{a}_n^{\frac{N_1+\dots+N_{m_1-1}}{2}}} \prod_{j=1}^{m_1-1} \sum_{a\in\mathbb{Z}^2} (1+|a|^{4\eta}) \left(\mathbf{1}_0(a) + \|f(\cdot,a)\times\cdot\|^{N_j}_{\mathcal{L}(\mathcal{B},\mathcal{B})}\right)\right).$$

Therefore, (3.9) is bounded, and converges to 0 as $n\to+\infty$ if $m_1\neq 1$. Focusing now on (3.10),

$$\left\| \sum_{a\in\mathbb{Z}^d} b^{(1,\dots,1)}_{a,(\ell_{m_i+1},\dots,\ell_{m_{i+1}-1}),(N_{m_i+1},\dots,N_{m_{i+1}-1})}(h_n(\cdot,a)^{N_{m_i}}) \right\|_{\mathcal{B}^*}$$

$$= O\Bigg(\frac{(\ell_{m_i+1}\dots\ell_{m_{i+1}-1})^{-1-\eta}}{\mathfrak{a}_n^{\frac{N_{m_i}+\dots+N_{m_{i+1}-1}}{2}}} \sum_{a_0,\dots,a_{m_{i+1}-m_i-1}\in\mathbb{Z}^2} \prod_{j=1}^{m_{i+1}-m_i-1} (1+|a_j-a_{j-1}|^{2\eta})$$

$$\times \prod_{k=0}^{m_{i+1}-m_i-1} \left(\mathbf{1}_0(a_k) + \|f(\cdot,a_k)\times\cdot\|^{N_{m_i+k}}_{\mathcal{L}(\mathcal{B},\mathcal{B})}\right)\Bigg)$$

$$= O\left(\frac{(\ell_{m_i+1}\dots\ell_{m_{i+1}-1})^{-1-\eta}}{\mathfrak{a}_n^{\frac{N_{m_i}+\dots+N_{m_{i+1}-1}}{2}}} \prod_{j=m_i}^{m_{i+1}-1} \sum_{a\in\mathbb{Z}^2} (1+|a|^{4\eta}) \left(\mathbf{1}_0(a) + \|f(\cdot,a)\times\cdot\|^{N_j}_{\mathcal{L}(\mathcal{B},\mathcal{B})}\right)\right).$$

Therefore the i-th term appearing in (3.10) is in

$$O\left(\left(\sum_{\ell_{m_i}\in\{1,\dots,n\}} \frac{1}{\ell_{m_i}} \right) \mathfrak{a}_n^{-\frac{N_{m_i}+\dots+N_{m_{i+1}-1}}{2}} \right) = O\left(\mathfrak{a}_n^{1-\frac{N_{m_i}+\dots+N_{m_{i+1}-1}}{2}} \right). \tag{3.11}$$

If $m_{i+1}=m_i+1$, then the i-th term in (3.10) is bounded by Point (II) of Properties 3.1. If $m_{i+1}\geq m_i+2$, then $N_{m_i}+\dots+N_{m_{i+1}-1}\geq 2$, so the i-th term in (3.10) is still bounded by Equation (3.11). Furthermore, if $N_{m_i}+\dots+N_{m_{i+1}-1}\geq 3$ for some i, then the i-th term converges to 0, and thus $A^{\varepsilon}_{n,q,\mathbf{N}}$ converges to 0 as $n\to+\infty$ by Equation (3.11). Hence, $A^{\varepsilon}_{n,q,\mathbf{N}}$ may not converge to 0 only if $m_1=1$ and $N_{m_i}+\dots+N_{m_{i+1}-1}\leq 2$ for all i. To sum up, if:

$$m_1 = 1 \text{ and for all } 1\leq i\leq K, \text{ either } \begin{cases} m_{i+1} &= m_i+1 \quad \text{and} \quad N_{m_i} = 1 \\ m_{i+1} &= m_i+1 \quad \text{and} \quad N_{m_i} = 2 \\ m_{i+1} &= m_i+2 \quad \text{and} \quad N_{m_i} = N_{m_i+1} = 1 \end{cases}, \tag{3.12}$$

then $(A^{\boldsymbol{\varepsilon}}_{n,q,\mathbf{N}})_{n\ge 0}$ is bounded; otherwise, $(A^{\boldsymbol{\varepsilon}}_{n,q,\mathbf{N}})_{n\ge 0}$ converges to 0. In particular, $\mathbb{E}_\mu\left[\mathcal{Z}_n(h_n)^m\right]$ is bounded, and we only need to take into account the data $(\mathbf{N},\boldsymbol{\varepsilon})$ satisfying Condition (3.12), which can be rewritten:

- $N_i \in \{1,2\}$;
- $N_i = 2 \ \Rightarrow\ \varepsilon_i = 0$;
- $\varepsilon_i = 1 \ \Rightarrow\ i \ge 2,\ N_i = N_{i-1} = 1,\ \varepsilon_{i-1} = 0$.

We shall call such couples $(\mathbf{N},\boldsymbol{\varepsilon})$ *admissible*. Given $1 \le q \le m$, let $\mathcal{G}(q)$ be the set of admissible $(\mathbf{N},\boldsymbol{\varepsilon}) = ((N_1,\dots,N_q),(\varepsilon_1,\dots,\varepsilon_q)) \in \{1,2\}^q \times \{0,1\}^q$. For $(\mathbf{N},\boldsymbol{\varepsilon}) \in \mathcal{G}(q)$, we set:

- $\mathcal{J}_2 := \{i \in \{1,...,q\} \ :\ \varepsilon_i = 0, N_i = 2\}$;
- $\mathcal{J}_1 := \{i \in \{1,...,q\} \ :\ (\varepsilon_i,\varepsilon_{i+1}) = (0,0), N_i = 1\}$;
- $\mathcal{J}_{1,1} := \{i \in \{1,...,q-1\} \ :\ (\varepsilon_i,\varepsilon_{i+1}) = (0,1), (N_i,N_{i+1}) = (1,1)\}$,

recalling the convention $\varepsilon_{q+1} = 0$.

For instance, the data $\mathbf{N} = (1,1,1,2,1,1,2,2,1,1)$, $\boldsymbol{\varepsilon} = (0,0,1,0,0,1,0,0,0,0)$ is admissible, as it can be decomposed in blocs as follows:

$\mathbf{N}$	1	1	1	2	1	1	2	2	1	1
$\boldsymbol{\varepsilon}$	0	0	1	0	0	1	0	0	0	0

For this example, $\mathcal{J}_2 = \{4,7,8\}$, $\mathcal{J}_1 = \{1,9,10\}$ and $\mathcal{J}_{1,1} = \{2,5\}$.

Then:

$$b^{\boldsymbol{\varepsilon}}_{0;\boldsymbol{\ell},\mathbf{N}}(\mathbf{1}) = \left(\prod_{i\in\mathcal{J}_2} b^{(0)}_{(\ell_i),(2)}(\mathbf{1})\right)\left(\prod_{i\in\mathcal{J}_1} b^{(0)}_{(\ell_i),(1)}(\mathbf{1})\right)\left(\prod_{i\in\mathcal{J}_{1,1}} b^{(0,1)}_{(\ell_i,\ell_{i+1}),(1,1)}(\mathbf{1})\right).$$

Note that $m = 2|\mathcal{J}_2| + 2|\mathcal{J}_{1,1}| + |\mathcal{J}_1|$ while $q = |\mathcal{J}_2| + 2|\mathcal{J}_{1,1}| + |\mathcal{J}_1|$; in particular, $|\mathcal{J}_2| = m - q$. Due to Point (II) in Properties 3.1 and Lemma 3.2, we obtain:

$$\begin{aligned}
A^{\boldsymbol{\varepsilon}}_{n;q;\mathbf{N}} &= \sum_{\boldsymbol{\ell}\in E_{q,n}} \left(\prod_{i\in\mathcal{J}_2} \frac{\Phi(0)\beta^2\sum_{a\in\mathbb{Z}^2}\mathbb{E}_\mu[f(\cdot,a)^2]}{\ell_i\mathfrak{a}_n}\right)\left(\prod_{i\in\mathcal{J}_1}\frac{\Phi(0)\alpha}{\ell_i\mathfrak{a}_n}\right) \\
&\quad\times\left(\prod_{i\in\mathcal{J}_{1,1}}\frac{\Phi(0)\beta^2}{\ell_i\mathfrak{a}_n}\int_{\widetilde{A}} f\circ\widetilde{T}^{\ell_i}\cdot f\,\mathrm{d}\widetilde{\mu}\right) + o(1) \\
&= \left(\frac{\Phi(0)}{\mathfrak{a}_n}\right)^{\frac{m+|\mathcal{J}_1|}{2}} \beta^{m-|\mathcal{J}_1|}\alpha^{|\mathcal{J}_1|}\sum_{\boldsymbol{\ell}\in E_{q,n}}\left(\prod_{i\in\mathcal{J}_2}\frac{\sum_{a\in\mathbb{Z}^2}\mathbb{E}_\mu[f(\cdot,a)^2]}{\ell_i}\right) \\
&\quad\times\left(\prod_{i\in\mathcal{J}_1}\frac{1}{\ell_i}\right)\left(\prod_{i\in\mathcal{J}_{1,1}}\frac{1}{\ell_i}\int_{\widetilde{A}} f\circ\widetilde{T}^{\ell_i}\cdot f\,\mathrm{d}\widetilde{\mu}\right) + o(1) \\
&= \left(\frac{\Phi(0)}{\mathfrak{a}_n}\right)^{\frac{m+|\mathcal{J}_1|}{2}} \beta^{m-|\mathcal{J}_1|}\alpha^{|\mathcal{J}_1|}\left(\int_{\widetilde{A}} f^2\,\mathrm{d}\widetilde{\mu}\right)^{|\mathcal{J}_2|} \\
&\quad\times\sum_{\ell_1,\dots,\ell_{|\mathcal{J}_{1,1}|}\ge 1}\left[\left(\prod_{i\in\mathcal{J}_{1,1}}\int_{\widetilde{A}} f\circ\widetilde{T}^{\ell_i}\cdot f\,\mathrm{d}\widetilde{\mu}\right)\sum_{\boldsymbol{\ell}'\in E_{q-|\mathcal{J}_{1,1}|,n-\sum_{i=1}^{|\mathcal{J}_{1,1}|}\ell_i}}\prod_{i=1}^{q-|\mathcal{J}_{1,1}|}\frac{1}{\ell'_i}\right] + o(1)
\end{aligned} \tag{3.13}$$

Due to [22, Lemma 3.7], for all $\ell_1,\dots,\ell_{|\mathcal{J}_{1,1}|} \ge 1$, as $n\to+\infty$,

$$\sum_{\boldsymbol{\ell}'\in E_{q-|\mathcal{J}_{1,1}|,n-\sum_{i=1}^{|\mathcal{J}_{1,1}|}\ell_i}}\prod_{i=1}^{q-|\mathcal{J}_{1,1}|}{\ell'_i}^{-1} \sim \mathfrak{a}_n^{q-|\mathcal{J}_{1,1}|} = \mathfrak{a}_n^{\frac{m+|\mathcal{J}_1|}{2}}.$$

Hence, by the dominated convergence theorem,

$$A^{\varepsilon}_{n;q;\mathbf{N}} = \Phi(0)^{\frac{m+|\mathcal{J}_1|}{2}} \beta^{m-|\mathcal{J}_1|}\alpha^{|\mathcal{J}_1|} \left(\int_{\widetilde{A}} f^2 \,\mathrm{d}\widetilde{\mu}\right)^{|\mathcal{J}_2|} \left(\sum_{\ell\geq 1}\int_{\widetilde{A}} f\circ \widetilde{T}^{\ell}\cdot f \,\mathrm{d}\widetilde{\mu}\right)^{|\mathcal{J}_{1,1}|} + o(1)\,.$$

If $(\mathbf{N},\boldsymbol{\varepsilon})$ is admissible, then $c_{\mathbf{N}} = 2^{-|\mathcal{J}_2|}m!$. Applying Equation (3.3), we obtain

$$\mathbb{E}_\mu\left[\mathcal{Z}_n(h_n)^m\right] = \sum_{q=1}^m \sum_{(\mathbf{N},\boldsymbol{\varepsilon})\in\mathcal{G}(q)} c_{\mathbf{N}} A^{\varepsilon}_{n;q;\mathbf{N}} + o(1) = m!\sum_{q=1}^m 2^{-|\mathcal{J}_2|}\sum_{(\mathbf{N},\boldsymbol{\varepsilon})\in\mathcal{G}(q)} A^{\varepsilon}_{n;q;\mathbf{N}} + o(1)\,.$$

Let $r := 2|\mathcal{J}_{1,1}| + 2|\mathcal{J}_2|$ and $s := |\mathcal{J}_2|$. Note that r is even, $s\leq r/2$ and $r\leq m$. We split the later sum depending on the value of r, and then depending on the value of $s = m-q$. Note that, once r and s are fixed, the number of admissible $(\mathbf{N},\boldsymbol{\varepsilon})$ such that $r = 2|\mathcal{J}_{1,1}| + 2|\mathcal{J}_2|$ and $s = |\mathcal{J}_2|$ is $\binom{m-r/2}{r/2}\cdot\binom{r/2}{s}$. We get:

$$\begin{aligned}
&\lim_{n\to+\infty}\mathbb{E}_\mu\left[\mathcal{Z}_n(h_n)^m\right]\\
&= m!\sum_{\substack{0\leq r\leq m\\ r\in 2\mathbb{Z}}}\sum_{s=0}^{r/2} 2^{-s}\sum_{(\mathbf{N},\boldsymbol{\varepsilon})\in\mathcal{G}(m-s)} \Phi(0)^{m-r/2}\beta^r\alpha^{m-r}\left(\int_{\widetilde{A}} f^2\,\mathrm{d}\widetilde{\mu}\right)^s\left(\sum_{\ell\geq 1}\int_{\widetilde{A}} f\circ\widetilde{T}^{\ell}\cdot f\,\mathrm{d}\widetilde{\mu}\right)^{r/2-s}\\
&= m!\sum_{\substack{0\leq r\leq m\\ r\in 2\mathbb{Z}}}\binom{m-r/2}{r/2}\Phi(0)^{m-r/2}\beta^r\alpha^{m-r}\sum_{s=0}^{r/2}\binom{r/2}{s}2^{-s}\left(\int_{\widetilde{A}} f^2\,\mathrm{d}\widetilde{\mu}\right)^s\left(\sum_{\ell\geq 1}\int_{\widetilde{A}} f\circ\widetilde{T}^{\ell}\cdot f\,\mathrm{d}\widetilde{\mu}\right)^{r/2-s}\\
&= m!\sum_{\substack{0\leq r\leq m\\ r\in 2\mathbb{Z}}}\binom{m-r/2}{r/2}\Phi(0)^{m-r/2}\beta^r\alpha^{m-r}\left(\frac{\widetilde{\sigma}^2(f)}{2}\right)^{r/2}\\
&= \sum_{\substack{0\leq r\leq m\\ r\in 2\mathbb{Z}}}\binom{m}{r}\alpha^{m-r}\left[(m-r/2)!\Phi(0)^{m-r/2}\right]\frac{r!}{2^{r/2}(r/2)!}\left(\beta\widetilde{\sigma}(f)\right)^r\\
&= \sum_{r=0}^m\binom{m}{r}\alpha^{m-r}\mathbb{E}\left[(\Phi(0)\mathcal{E})^{m-r/2}\right]\mathbb{E}\left[(\beta\widetilde{\sigma}(f)\mathcal{N})^r\right]\\
&= \mathbb{E}\left[\left(\alpha\Phi(0)\mathcal{E}+\beta\sqrt{\Phi(0)\mathcal{E}}\,\widetilde{\sigma}(f)\mathcal{N}\right)^m\right],
\end{aligned}$$

where $\mathcal{E}$ has a standard exponential distribution, $\mathcal{N}$ a standard Gaussian distribution, and $\mathcal{E}$, $\mathcal{N}$ are independent. This finishes the proof of Theorem 2.1 for $t=1$. □

3.2. **Functional convergence.** We finish the proof of Theorem 2.1, by extending the distributional limit theorem (for $t=1$) to a functional limit theorem. This is made easier by the fact that, in dimension 2, the local time at step n is of the order of $\ln(n)$, which has slow variation.

End of the proof of Theorem 2.1. A crucial observation is given by the next lemma:

Lemma 3.3. *Under Hypothesis 2.2, there exists $C>0$ such that for every $f:\widetilde{A}\to\mathbb{R}$, for every $0<T_1<T_2$ and every $n\geq T_1^{-1}$,*

$$\left\|\sup_{t\in(T_1,T_2)}\left|\widetilde{S}_{nt}f-\widetilde{S}_{nT_1}f\right|\right\|_{\mathbb{L}^1(\widetilde{T}_*(\mu\otimes\delta_0))} \leq C\sum_{a\in\mathbb{Z}^2}\|f(\cdot,a)\|_{\mathbb{L}^{p^*}(A,\mu)}\log\frac{\lceil nT_2\rceil}{\lfloor nT_1\rfloor}\,. \tag{3.14}$$

Proof. Assume first that $p = 1$. Let $c_a := \|f(\cdot, a)\|_\infty$ and set $h_0(x, a) := c_a$. Using Hypothesis 2.2, there exists a constant $C > 0$ such that:

$$\left\|\widetilde{S}_k h_0 - \widetilde{S}_j h_0\right\|_{\mathbb{L}^1(\widetilde{T}_*(\mu\otimes\delta_0))} \leq \sum_{a\in\mathbb{Z}^2} c_a \sum_{m=j+1}^{k} \mu(S_m F = a)$$

$$\leq \sum_{a\in\mathbb{Z}^2} c_a \sum_{m=j+1}^{k} \mathbb{E}_\mu\left[Q_{m,a}(\mathbf{1})\right] \leq C \sum_{a\in\mathbb{Z}^2} c_a \sum_{m=j+1}^{k} \frac{1}{m}.$$

Since $|f| \leq h_0$, for every $n \geq T_1^{-1}$.

$$\left\|\sup_{t\in(T_1,T_2)}\left|\widetilde{S}_{nt} f - \widetilde{S}_{nT_1} f\right|\right\|_{\mathbb{L}^1(\widetilde{T}_*(\mu\otimes\delta_0))} \leq \left\|\widetilde{S}_{\lceil nT_2\rceil} h_0 - \widetilde{S}_{\lfloor nT_1\rfloor} h_0\right\|_{\mathbb{L}^1(\mu\otimes\delta_0)}$$

$$\leq C \sum_{a\in\mathbb{Z}^2} c_a \log \frac{\lceil nT_2\rceil}{\lfloor nT_1\rfloor}.$$

When $p > 1$, using again Hypothesis 2.2, we get:

$$\left\|\sup_{t\in(T_1,T_2)}\left|\widetilde{S}_{nt} f - \widetilde{S}_{nT_1} f\right|\right\|_{\mathbb{L}^1(\widetilde{T}_*(\mu\otimes\delta_0))} \leq \sum_{a\in\mathbb{Z}^2} \sum_{\ell=\lfloor nT_1\rfloor}^{\lceil nT_2\rceil} \left\|f(\cdot, a) P^\ell(\mathbf{1}_{S_\ell F = a})\right\|_{\mathbb{L}^1(A,\mu)}$$

$$\leq C' \sum_{a\in\mathbb{Z}^2} \|f(\cdot, a)\|_{\mathbb{L}^{p^*}(A,\mu)} \sum_{\ell=\lfloor nT_1\rfloor}^{\lceil nT_2\rceil} \|Q_{\ell,a}(\mathbf{1})\|_{\mathcal{B}}$$

$$\leq C \left(\sum_{a\in\mathbb{Z}^2} \|f(\cdot, a)\|_{\mathbb{L}^{p^*}(A,\mu)}\right) \sum_{\ell=\lfloor nT_1\rfloor}^{\lceil nT_2\rceil} \frac{1}{\ell}$$

□

Lemma 3.3 implies that

$$\left(\sup_{t\in[T_1,T_2]}\left|\frac{\widetilde{S}_{nt} g_0 - \widetilde{S}_{nT_1} g_0}{\ln(n)}\right|, \sup_{t\in[T_1,T_2]}\left|\frac{\widetilde{S}_{nt} f - \widetilde{S}_{nT_1} f}{\sqrt{\ln(n)}}\right|\right)$$

converges in probability to $(0, 0)$ with respect to $\mu \otimes \delta_0$. Hence, as n goes to $+\infty$,

$$\left(\frac{\widetilde{S}_{nt} g_0}{\ln(n)}, \frac{\widetilde{S}_{nt} f}{\sqrt{\ln(n)}}\right)_{t\in[T_1,T_2]} \longrightarrow \left(\Phi(0)\mathcal{E}, \widetilde{\sigma}(f)\sqrt{\Phi(0)\mathcal{E}}\,\mathcal{N}\right)_{t\in[T_1,T_2]}, \tag{3.15}$$

where the convergence is in distribution in $\mathcal{C}([T_1, T_2], \mathbb{R})$ with respect to $\mu \otimes \delta_0$.

Hence, the conclusion of Theorem 2.1 holds for f and g_0, and where the convergence in distribution is with respect to $\mu \otimes \delta_0$. By [31, Theorem 1], the convergence in distribution actually holds with respect to any absolutely continuous probability measure. Finally, let us take any $g \in \mathbb{L}^1(\widetilde{A}, \widetilde{\mu})$. Since the system $(\widetilde{A}, \widetilde{\mu}, \widetilde{T})$ is conservative and ergodic, Hopf's ergodic theorem ensures that, $\widetilde{\mu}$-almost everywhere, $\widetilde{S}_t g \sim \int_{\widetilde{A}} g \, \mathrm{d}\widetilde{\mu} \cdot (\widetilde{S}_t g_0)$, so the convergence in distribution of Equation (3.15) also holds for g. □

4. Limit theorem for flows

We now focus on the results for suspension flows over maps with good spectral properties.

4.1. General theorem for suspension semiflows.

We begin by deducing Theorem 2.4 from Theorem 2.1.

Proof of Theorem 2.4. Let ϕ be as in the hypotheses of Theorem 2.4. Take $\psi(x, a, u) := \tau(x)^{-1}\mathbf{1}_0(a)$ and $\mu_0 := \tau^{-1}(x) \, \mathrm{d}\mu(x) \otimes \delta_0(a) \otimes \, \mathrm{d}u \in \mathcal{P}(\widetilde{M})$. Let $0 < s_1 < s_2$.

From the transformation to the flow

Let $\theta : \widetilde{\mathcal{M}} \to \mathbb{R}$. Recall that we defined $G(\theta)(x,a) = \int_0^{\tau(x)} \theta(x,a,t)\, \mathrm{d}t$. Assume that:

$$\sum_{a\in\mathbb{Z}^2} \|G(|\theta|)(\cdot,a)\|_{\mathbb{L}^{p^*}(A,\mu)} < +\infty,$$

a condition satisfied by both the functions ϕ and ψ.

Recall that we set $n_t(x) = \max\{n \geq 0 : S_n\tau(x) \leq t\}$. let $N_t(x) := n_t(x) + \frac{t-\sum_{k=0}^{n_t(x)-1} \tau\circ T^k(x)}{\tau(T^{n_t(x)}(x))}$, so that $\widetilde{S}_{N_t}\tau = t$. Then, for all $(x,a,u) \in \widetilde{\mathcal{M}}$,

$$\left|\widetilde{S}_t\theta(x,a,u) - \widetilde{S}_{N_t(x)}G(\theta)(x,a)\right| \leq G(|\theta|)(x,a) + \widetilde{S}_{n_{t+u}(x)-n_t(x)+1}G_u(|\theta|)(\widetilde{T}^{n_t(x)}(x,a))\,. \tag{4.1}$$

It is straightforward that $G(|\theta|)(x,a)/\sqrt{\ln(t)} \to 0$ as $t \to +\infty$. We need to control the last term in Equation (4.1).

Since $(\widetilde{A},\widetilde{\mu},\widetilde{T})$ is ergodic, so is (A,μ,T), and thus, by Birkhoff's ergodic theorem, $\lim_{n\to+\infty} n^{-1}S_n\tau = \int_A \tau\, \mathrm{d}\mu$ almost surely for μ. Since $S_{n_t}\tau \leq t < S_{n_t+1}\tau$, we conclude that, μ-almost surely, $n_t \sim \frac{t}{\int_A \tau\, \mathrm{d}\mu}$ as t goes to $+\infty$. Therefore, for $\widetilde{\nu}$-almost every $(x,a,u) \in \widetilde{\mathcal{M}}$, there exists $t_0 = t_0(x,u) \geq 0$ such that

$$\frac{t}{2\int_A \tau\, \mathrm{d}\mu} \leq n_t(x) \leq n_{t+u}(x) + 1 \leq \frac{2t}{\int_A \tau\, \mathrm{d}\mu}$$

for every $t \geq t_0$. Then, on $\{t_0 \leq s_1 t\}$,

$$\sup_{s\in[s_1,s_2]} \widetilde{S}_{n_{ts+u}(x)-n_{ts}(x)+1}G(|\theta|)(\widetilde{T}^{n_{ts}(x)}(x,a)) \leq \sup_{\frac{ts_1}{2\int_A\tau\,\mathrm{d}\mu}\leq n\leq n+m\leq\frac{2ts_2}{\int_A\tau\,\mathrm{d}\mu}} \widetilde{S}_mG(|\theta|)(\widetilde{T}^n(x,a)).$$

By Lemma 3.3,

$$\left\|\mathbf{1}_{\{t_0\leq s_1t\}} \sup_{s\in[s_1,s_2]} \widetilde{S}_{n_{ts+u}(x)-n_{ts}(x)+1}G(|\theta|)(\widetilde{T}^{n_{ts}(x)}(x,a))\right\|_{\mathbb{L}^1(\widetilde{M},\mu_0)} \leq C\left(\sum_{a\in\mathbb{Z}^2}\|G(|\theta|)(\cdot,a)\|_{\mathbb{L}^{p^*}(A,\mu)}\right)\ln\left(\frac{4s_2}{s_1}\right).$$

Hence, the random variable

$$\mathbf{1}_{\{t_0\leq s_1t\}}\frac{\sup_{s\in[s_1,s_2]} G(|\theta|)(\widetilde{T}^{n_{ts+u}(x)}(x,a))}{\sqrt{\ln(t)}}$$

converges to 0 in probability on $(\widetilde{M},\mu_0)$, while the random variable

$$\mathbf{1}_{\{t_0> s_1t\}}\frac{\sup_{s\in[s_1,s_2]} G(|\theta|)(\widetilde{T}^{n_{ts+u}(x)}(x,a))}{\sqrt{\ln(t)}}$$

converges to 0 almost surely on $(\widetilde{M},\mu_0)$.

Applying the above discussion to the functions ψ and ϕ respectively, the convergence in distribution in $\mathcal{C}([s_1,s_2],\mathbb{R})$, with respect to μ_0, of

$$\left(\frac{\widetilde{S}_{ts}\psi}{\ln(t)}, \frac{\widetilde{S}_{ts}\phi}{\sqrt{\ln(t)}}\right)_{s\in[s_1,s_2]}$$

is equivalent to the convergence in distribution in $\mathcal{C}([s_1,s_2],\mathbb{R})$, with respect to μ_0, of

$$(x,a,v) \mapsto \left(\frac{\widetilde{S}_{N_{ts}(x)}G(\psi)(x,a)}{\ln(t)}, \frac{\widetilde{S}_{N_{ts}(x)}G(\phi)(x,a)}{\sqrt{\ln(t)}}\right)_{s\in[s_1,s_2]}.$$

Since this last process depends only on x (recall that $a = 0$ almost surely under μ_0), this is equivalent to the convergence in distribution of the process

$$x \mapsto \left(\frac{\widetilde{S}_{N_{ts}(x)}G(\psi)(x,0)}{\ln(t)}, \frac{\widetilde{S}_{N_{ts}(x)}G(\phi)(x,0)}{\sqrt{\ln(t)}}\right)_{s\in[s_1,s_2]}$$

with respect to $\mu \otimes \delta_0$.

A time change

It remains to prove the convergence in distribution of $\left(\frac{\widetilde{S}_{N_{ts}(\cdot)}G(\psi)(\cdot,0)}{\ln(t)}, \frac{\widetilde{S}_{N_{ts}(\cdot)}G(\phi)(\cdot,0)}{\sqrt{\ln(t)}}\right)_{s\in[s_1,s_2]}$ in $\mathcal{C}([s_1,s_2],\mathbb{R})$ with respect to μ. The main idea is that this process is a time change (by N_t) of a discrete-time process, for which we can apply Theorem 2.1.

We set $T_1 := \frac{s_1}{2\int_A \tau \, \mathrm{d}\mu}$ and $T_2 := \frac{2s_2}{\int_A \tau \, \mathrm{d}\mu}$. By Theorem 2.1, as t goes to $+\infty$,

$$\left(\frac{\widetilde{S}_{\lfloor t\rfloor s'}G(\psi)}{\ln(t)}, \frac{\widetilde{S}_{\lfloor t\rfloor s'}G(\phi)}{\sqrt{\ln(t)}}\right)_{s'\in[T_0,T_1]} \to \left(\int_{\widetilde{A}} g \, \mathrm{d}\widetilde{\mu}\, \Phi(0)\mathcal{E}, \, \widetilde{\sigma}(f)\sqrt{\Phi(0)\mathcal{E}}\,\mathcal{N}\right)_{s'\in[T_0,T_1]}, \tag{4.2}$$

where the convergence is in distribution in $\mathcal{C}([T_1,T_2],\mathbb{R})$ with respect to $\mu\otimes\delta_0$.

Since $N_t(\cdot) \sim \frac{t}{\int_A \tau \, \mathrm{d}\mu}$ as $t\to+\infty$ almost surely for μ,

$$\lim_{t\to+\infty} \sup_{s\in[s_1,s_2]} \left| \frac{N_{ts}(\cdot)}{\lfloor t\rfloor} - \frac{s}{\int_A \tau \, \mathrm{d}\widetilde{\nu}} \right| \to 0$$

μ-almost surely. Thus, still μ-almost surely:

$$\lim_{t\to+\infty} \sup_{s\in[s_1,s_2]} \left| h_{t,s}(\cdot) - \frac{s}{\int_A \tau \, \mathrm{d}\widetilde{\nu}} \right| \to 0\,, \tag{4.3}$$

with:

$$h_{t,s}(x) = \begin{cases} T_1 & \text{if } \frac{N_{ts}(x)}{\lfloor t\rfloor} \leq T_1 \\ \frac{N_{ts}(x)}{\lfloor t\rfloor} & \text{if } T_1 \leq \frac{N_{ts}(x)}{\lfloor t\rfloor} \leq T_2 \\ T_2 & \text{if } T_1 \leq \frac{N_{ts}(x)}{\lfloor t\rfloor} \end{cases}.$$

Observe that $h_{t,s}$ takes its values in $[T_1,T_2]$ and is continuous in s. Therefore, by composition of Equations (4.3) and (4.2),

$$\left(\frac{\widetilde{S}_{N_{ts}}G(\psi)}{\ln(t)}, \frac{\widetilde{S}_{N_{ts}}G(\phi)}{\sqrt{\ln(t)}}\right)_{s\in[s_1,s_2]} = \left(\frac{\widetilde{S}_{\lfloor t\rfloor h_{t,s}}G(\psi)}{\ln(t)}, \frac{\widetilde{S}_{\lfloor t\rfloor h_{t,s}}G(\phi)}{\sqrt{\ln(t)}}\right)_{s\in[s_1,s_2]}$$

converges in distribution in $\mathcal{C}([s_1,s_2],\mathbb{R})$, as t goes to $+\infty$, to $\left(\int_{\widetilde{A}} G(\psi)\, \mathrm{d}\widetilde{\mu}\, \Phi(0)\mathcal{E}, \, \widetilde{\sigma}(G(\phi))\sqrt{\Phi(0)\mathcal{E}}\,\mathcal{N}\right)_{s\in[s_1,s_2]}$.

Moreover $\int_{\widetilde{A}} G(\theta)\, \mathrm{d}\widetilde{\mu} = \int_{\widetilde{\mathcal{M}}} \theta \, \mathrm{d}\widetilde{\nu}$ for $\theta = \psi$, ψ by definition of $\widetilde{\nu}$ and of $G(\theta)$, and

$$\widetilde{\sigma}^2(G(\phi)) = \int_{\widetilde{A}} G(\phi)^2 \, \mathrm{d}\widetilde{\mu} + 2\sum_{k\geq 1} \int_{\widetilde{A}} G(\phi)\cdot G(\phi)\circ \widetilde{T}^k \, \mathrm{d}\widetilde{\mu}.$$

This finishes the proof of Theorem 2.4 for $\psi(x,a,u) := \tau(x)^{-1}\mathbf{1}_0(a)$ and $\mu_0 := \tau^{-1}(x)\, \mathrm{d}\mu(x)\otimes\delta_0(a)\otimes \, \mathrm{d}u \in \mathcal{P}(\widetilde{M})$. The general case follows from the same ideas as in the proof of Theorem 2.1: [31, Theorem 1] extends the result to any probability measure absolutely continuous with respect to $\widetilde{\nu}$, while Hopf's ergodic theorem extends it to any $\psi \in \mathbb{L}^1(\widetilde{M},\widetilde{\nu})$.

□

4.2. **Proof for finite horizon Lorentz gases.** We now derive an application to Lorentz gases, that is Corollary 2.5, from Theorem 2.4.

Proof of Corollary 2.5. There exists $c>0$ such that $(\widetilde{\mathcal{M}}, c\widetilde{\nu}, (\widetilde{Y}_t)_t)$ can be represented as a flow as in Theorem 2.4, with (A,μ,T) the corresponding Sinai billiard and τ the length of the free flight until the next collision. Let us write $\mathcal{C}_p$ for the set of configurations in $\widetilde{\mathcal{M}}$ whose last reflection is on an obstacle corresponding to $A\times\{a\}$. Since τ is uniformly bounded, the condition on ϕ ensures that

$$\sum_{a\in\mathbb{Z}^2} \left\|\phi_{|\mathcal{C}_a}\right\|_\eta < +\infty\,.$$

Again here $\widetilde{T}$ is the billiard transformation in the $\mathbb{Z}^2$-periodic billiard domain. Let x, y in the same continuity domain of $\widetilde{T}$. Then there exists K such that

$$\begin{aligned}
|G(\phi)(x) - G(\phi)(y)| &= \left| \int_0^{\tau(x)} \phi(\widetilde{Y}_s(x,a)) \, \mathrm{d}s - \int_0^{\tau(y)} \phi(\widetilde{Y}_s(y,a)) \, \mathrm{d}s \right| \\
&\leq \int_0^{\min\{\tau(x),\tau(y)\}} \left| \phi(\widetilde{Y}_s(x,a)) - \phi(\widetilde{Y}_s(y,a)) \right| \, \mathrm{d}s + |\tau(x) - \tau(y)| \left\| \phi_{\mathcal{C}_p} \right\|_\infty \\
&\leq \|\tau\|_\infty \left\| \phi_{|\mathcal{C}_p} \right\|_\eta \max_{0 \leq s \leq \min\{\tau(x),\tau(y)\}} d(\widetilde{Y}_s(x,a), \widetilde{Y}_s(y,a))^\eta \\
&\quad + \|\tau\|_{\frac{1}{2}} \left\| \phi_{|\mathcal{C}_a} \right\|_\infty d(x,y)^{\frac{1}{2}}
\end{aligned}$$

since τ is $\frac{1}{2}$-Hölder continuous on each continuity component of T. Since $(x,s) \mapsto \widetilde{Y}_s(x,0)$ is differentiable on $\{(x,s) \in A \times [0,+\infty) \, : \, s \leq \tau(x)\}$, we conclude that $f : (x,a) \mapsto \int_0^{\tau(x)} \phi(x,a,s) \, \mathrm{d}s$ satisfies the assumptions of Corollary 2.2 with η replaced by $\min\{\eta, 1/2\}$.

The assumption on the system can be checked as in the proof of Corollary 2.2: [8, Theorem 3.17] ensures that Hypothesis 2.6 is satisfied with $p = 1$, and [9, Lemma 5.3] ensures that $\|G(\phi)(\cdot,a)\times\|_{\mathcal{L}(\mathcal{B},\mathcal{B})} \leq C \|G(\phi)(\cdot,a)\|_\eta$. All is left is to apply Theorem 2.4. □

5. Limit theorems via induction

We now prove Proposition 2.3 using induced systems as in [27, 29]. The strategy, in a nutshell, is as follows. In the present article, up to now, we worked with suspensions flows over an ergodic $\mathbb{Z}^2$-extension of a dynamical system (A,μ,T), where the extension was given by a jump function $F : A \to \mathbb{Z}^2$ and the roof function $\widetilde{\tau} : (x,a) \mapsto \tau(x)$. The system (A,μ,T) was a billiard map, and the suspension flow the Lorentz gas.

In [27, 29], the setting is very similar, with the difference that (A,μ,T) has to be a Gibbs-Markov map (see e.g. [1, 4.6] for an introduction to these systems, which are Markov maps with a big image property). Using the symbolic coding of Axiom A flows by Bowen [2], a statement very close to that of Theorem 2.4 was obtained for geodesic flows in negative curvature [29, Proposition 6.12]. The case of Sinai billiards is more complex, as one has to use Young towers [30] to make them fit the setting of Gibbs-Markov maps.

5.1. Young towers and Lorentz gas.

To simplify our argument, we shall work with the discrete-time Lorentz gas (i.e. $\mathbb{Z}^2$-periodic billiard system). In order to emphasize the parallel constructions, we keep using the notations (A,μ,T) and τ in this section, although we stress that they do not correspond to the billiard map and the free path length respectively, but to an underlying Gibbs-Markov map and to the height of the Young tower. Using a Young tower, there exist:

- a Gibbs-Markov map (A,μ,T) with Markov partition Γ,
- a function[2] $\tau : A \to \mathbb{N}_+$ constant on each element of Γ, with $\mu(\tau \geq n) \leq C_\varepsilon e^{-\varepsilon n}$ for some ε, $C_\varepsilon > 0$, and a tower $(A_\tau, \mu_\tau, T_\tau)$ over (A,μ,T) with roof function τ,
- a hyperbolic map (A_Y, μ_Y, T_Y), where each point in A_Y has two coordinates (x_u, x_s) (the base of the Young tower, which has a box structure indexed by Γ, the coordinate x_u is the coordinate along the unstable manifold, and x_s along the stable manifold; we write Γ_Y for the corresponding partition of A_Y),
- a function $\tau_Y : A_Y \to \mathbb{N}_+$ depending only on x_u, and a tower $(A_{Y,\tau}, \mu_{Y,\tau}, T_{Y,\tau})$ over (A_Y, μ_Y, T_Y) with roof function τ_Y,
- a factor map $\pi_Y : A_Y \to A$ such that $\tau_Y = \tau \circ \pi_Y$, which lifts to a factor map on the towers: abusing notations, $\pi_Y(x_u, x_s, k) = (\pi_Y(x_u,x_s), k) \in A_\tau$ for all $(x_u, x_s, k) \in A_{Y,\tau}$,
- a factor map π from $A_{Y,\tau}$ to the Sinai billiard table.

These objects behave well when one works with $\mathbb{Z}^2$-extensions. Let F_L be the function describing the jumps for the discrete-time Lorentz gas (i.e. F_L is the function denoted F in Subsection 2.2) and $F_Y(x_u, x_s) =$

[2] This function τ is the time of the next Markovian return to the inducing set; it is not the free path length, as it used to be in Subsection 2.2.

$\sum_{k=0}^{\tau_Y(x_u)-1} F_L \circ \pi(x_u, x_s, k)$. By the construction of the Young tower, $F_L \circ \pi$ depends only on x_u, and thus quotients through π_Y to yield $F : A \to \mathbb{Z}^2$, which is constant on each element of Γ.

Let $(\widetilde{A}_{Y,\tau}, \widetilde{\mu}_{Y,\tau}, \widetilde{T}_{Y,\tau})$ be the system defined by:

- $\widetilde{A}_{Y,\tau} = A_{Y,\tau} \times \mathbb{Z}^2$,
- $\widetilde{\mu}_{Y,\tau} = \sum_{a \in \mathbb{Z}^2} \mu_{Y,\tau} \otimes \delta_a$,
- $\widetilde{T}_{Y,\tau}(x_u, x_s, k, a) = (x_u, x_s, k+1, a)$ if $k < \tau_Y(x_u) - 1$, and otherwise $\widetilde{T}_{Y,\tau}(x_u, x_s, \tau(x_u) - 1, a) = (T_Y(x_u, x_s), 0, a + F_Y(x_u))$.

In the same way, define $(\widetilde{A}_\tau, \widetilde{\mu}_\tau, \widetilde{T}_\tau)$ using the system $(A_\tau, \mu_\tau, T_\tau)$ and the function F. Then there exist two factor maps $\widetilde{\pi}$ and $\widetilde{\pi}_Y$ from $(\widetilde{A}_{Y,\tau}, \widetilde{\mu}_{Y,\tau}, \widetilde{T}_{Y,\tau})$, descending to the discrete-time Lorentz gas (i.e. the $\mathbb{Z}^2$-periodic billiard system $\left(\widetilde{A}, \widetilde{\mu}, \widetilde{T}\right)$ defined in Subsection 2.2) and to $(\widetilde{A}_\tau, \widetilde{\mu}_\tau, \widetilde{T}_\tau)$ respectively. This construction is summed up in the following diagram:

$$\begin{array}{ccccc}
(\widetilde{A}_\tau, \widetilde{\mu}_\tau, \widetilde{T}_\tau) & \xleftarrow[\widetilde{\pi}_Y]{} & (\widetilde{A}_{Y,\tau}, \widetilde{\mu}_{Y,\tau}, \widetilde{T}_{Y,\tau}) & \xrightarrow{\widetilde{\pi}} & \begin{pmatrix}\text{collision map for}\\ \text{the Lorentz gas}\end{pmatrix} \\
\downarrow & & \downarrow & & \downarrow \\
(A_\tau, \mu_\tau, T_\tau) & \xleftarrow[\pi_Y]{} & (A_{Y,\tau}, \mu_{Y,\tau}, T_{Y,\tau}) & \xrightarrow{\pi} & \begin{pmatrix}\text{collision map for}\\ \text{the Sinai billiard}\end{pmatrix}
\end{array}$$

In the diagram above, all the downward arrows consist in forgetting the $\mathbb{Z}^2$-coordinate, all the horizontal arrows are measure-preserving, and $\widetilde{\pi}_Y$ (but not $\widetilde{\pi}$) acts trivially on the $\mathbb{Z}^2$-coordinate.

We shall also write, for $x \in A$:

$$\varphi(x) := \inf\{n \geq 1 : \ S_n^T F(x) = 0\},$$

$$\widetilde{\varphi}(x) := \sum_{k=0}^{\varphi(x)-1} \tau \circ T^k(x),$$

so that φ is the first return time to $A \times \{0\}$ for the underlying $\mathbb{Z}^2$-extension of a Gibbs-Markov map, and $\widetilde{\varphi}$ the first return time to $A \times \{0\} \times \{0\}$ for $\widetilde{T}_\tau$. Then the map $\widetilde{T}_0 := \widetilde{T}_\tau^{\widetilde{\varphi}}$ acts on $A \times \{0\} \simeq A$, and $(A, \mu, \widetilde{T}_0)$ is a measure-preserving ergodic Gibbs-Markov map for some refined partition Γ_0. In the same way, we define $\widetilde{T}_{Y,0} := \widetilde{T}_{Y,\tau}^{\widetilde{\varphi}}$.

Given an observable f defined on the state space of the Lorentz gas ($\mathbb{Z}^2$-periodic billiard map), we define the sum of f along an excursion, either until it comes back to the base of the Young tower or to the basis of the cell 0 in $\widetilde{A}_{Y,\tau}$. For $(x_u, x_s) \in A_Y$ and $a \in \mathbb{Z}^2$, let:

$$\begin{aligned}
G_{Y,\tau}(f)(x_u, x_s, a) &:= \sum_{k=0}^{\tau_Y(x_u)-1} f(x_u, x_s, k, a) \\
G_{Y,\varphi}(f)(x_u, x_s) &:= \sum_{k=0}^{\widetilde{\varphi}(x_u)-1} f \circ \widetilde{T}_{Y,\tau}^k(x, 0, 0) \\
&= \sum_{k=0}^{\varphi(x_u)-1} G_{Y,\tau}(f)(T_Y^k(x_u, x_s), S_k^{T_Y} F_Y(x_u)),
\end{aligned}$$

and define in the same way $G_\tau(f) : A \times \mathbb{Z}^2 \to \mathbb{C}$ and $G_\varphi(f) : A \to \mathbb{C}$ for functions f defined on $\widetilde{A}_\tau$.

5.2. **Proof of Proposition 2.3.** The general strategy, close to that of [29, Proposition 6.12], is as follows:

- Take a function f defined on the state space of the discrete-time Lorentz gas, uniformly η-Hölder on the continuity components of the billiard map, with integral zero and such that $\sum_{a \in \mathbb{Z}^2} (1 + \ln_+ |a|)^{\frac{1}{2}+\varkappa} \|f(\cdot, a)\|_\infty < +\infty$ for some $\varkappa > 0$. Lift it to a function $f \circ \widetilde{\pi}$ defined on $(\widetilde{A}_{Y,\tau}, \widetilde{\mu}_{Y,\tau}, \widetilde{T}_{Y,\tau})$.
- Add a bounded coboundary $u \circ T - u$ to get $f_+ \circ \widetilde{\pi}_Y = f \circ \widetilde{\pi} + u \circ T - u$ (independent from x_s and thus going to the quotient through $\widetilde{\pi}_Y$, so that we only need to work with the Gibbs-Markov extension).

- Check that $G_{Y,\tau}(f \circ \widetilde{\pi})$ satisfies some integrability conditions, then apply [22, Lemma 4.16] and [22, Lemma 2.7] to show that $G_\varphi(f_+)$ is also integrable enough (the precise conditions shall be described later).
- Apply a version of [29, Corollary 6.10], together with [28, Remark 4.6], which states:

Proposition 5.1 ([29])**.** *Let $(\widetilde{A}_\tau, \widetilde{\mu}_\tau, \widetilde{T}_\tau)$ be an ergodic and recurrent Markov $\mathbb{Z}^2$-extension of a Gibbs-Markov map (A, Γ, μ, T), of roof function τ and of step function $F : A \to \mathbb{Z}^2$. Assume that it is aperiodic, that τ and F belong to $\mathbb{L}^2(A, \mu_A)$, and that $\sum_{\gamma\in\Gamma} \mu(\gamma)|\tau|_{\mathrm{Lip}(\gamma)}$ is finite. Under these hypotheses, the covariance matrix $\Sigma^2(F)$ is positive definite, where, for all u and v in $\mathbb{R}^2$:*

$$\left(u, \Sigma^2(F)v\right) = \lim_{N\to+\infty} \frac{1}{N} \int_A \left(\sum_{i=0}^{N-1} F \circ T^i, u\right) \left(\sum_{i=0}^{N-1} F \circ T^i, v\right) \mathrm{d}\mu.$$

Let f_+ be a real-valued, measurable function from $\widetilde{A}_\tau$ to $\mathbb{R}$. Assume that:

- $\sup_{0\leq n\leq\widetilde{\varphi}(x)} \left|\sum_{k=0}^{n-1} f_+ \circ \widetilde{T}_\tau^k(\cdot, 0, 0)\right| \in \mathbb{L}^q(A, \mu)$ *for some* $q > 2$,
- $\int_A G_\varphi(f_+) \,\mathrm{d}\mu = 0$,
- $\sum_{\gamma\in\Gamma_0} \mu(\gamma) \sup_{a\in\mathbb{Z}^d} |G_\tau(f_+)(\cdot, a)|_{\mathrm{Lip}(\gamma)}$ *is finite.*

Then, for any probability measure $\widetilde{\nu}$ absolutely continuous with respect to $\widetilde{\mu}_\tau$:

$$\left(\frac{2\pi\sqrt{\det(\Sigma^2(F))}}{\ln(n)}\right)^{\frac{1}{2}} \sum_{k=0}^{n-1} f_+ \circ \widetilde{T}^k \to \sigma(f_+)L,$$

where the convergence is in distribution when the left-hand side is seen as a random variable from $(\widetilde{A}_\tau, \widetilde{\nu})$ to $\mathbb{R}$, where L follows a centered Laplace distribution of variance 1*, and:*

$$\sigma^2(f_+) = \int_A G_\varphi(f_+)^2 \,\mathrm{d}\mu + 2\sum_{n=1}^{+\infty} \int_A G_\varphi(f_+) \cdot G_\varphi(f_+) \circ \widetilde{T}_0^n \,\mathrm{d}\mu,$$

where the limit is taken in the Cesàro sense.

Proof of Proposition 2.3. Let us go through the assumptions of Proposition 5.1 for the Young towers associated with Sinai billiards.

General assumptions on the system

The bidimensional Lorentz gas is ergodic and recurrent for the Liouville measure; by construction, so is $(\widetilde{A}_{Y,\tau}, \widetilde{\mu}_{Y,\tau}, \widetilde{T}_{Y,\tau})$. As a factor map, $(\widetilde{A}_\tau, \widetilde{\mu}_\tau, \widetilde{T}_\tau)$ is then also ergodic and recurrent.

The theorem stays true if one drops the hypothesis of aperiodicity on the extension; the full reduction can be found in the proof of Proposition 2.11 in [22].

The roof function τ_Y has an exponential tail, and as such belongs to $\mathbb{L}^{2+\varepsilon}(A, \mu)$. Since the billiard has finite horizon, the function F_L is uniformly bounded. Hence, the size of the jumps $F_Y = S_{\tau_Y}(F_L \circ \pi)$ is in $O(\tau_Y)$, and thus also belongs to $\mathbb{L}^{2+\varepsilon}(A, \mu)$. By construction of the Young towers, τ_Y is constant on the elements of the Markov partition. All these properties goes through the quotient to τ, and in particular $\sum_{\gamma\in\Gamma} \mu(\gamma)|\tau|_{\mathrm{Lip}(\gamma)} = 0$.

Defining a coboundary

Let f be an observable of the collision section for the Lorentz gas which is uniformly η-Hölder on the continuity sets of the billiard map. The space $A_{\tau,Y}$ has a box structure, with, by Young's construction, a distinguished piece of unstable manifold on the basis A_Y. Let us choose the coordinates (x_u, x_s) so that this piece of unstable manifold is $\{x_s = 0\}$. Then we get a map:

$$p_+ : \left\{ \begin{array}{lll} \widetilde{A}_{Y,\tau} & \to & \{(x_u, 0, k, a) :\ 0 \leq k < \tau(x_u),\ a \in \mathbb{Z}^2\} \\ (x_u, x_s, k, a) & \mapsto & (x_u, 0, k, a) \end{array} \right. ,$$

The space $A_{Y,\tau}$ is also endowed with a distance $d_{Y,\tau}$ satisfying the properties (P3) and (P4a) in [30], namely, there exist constants $C > 0$ and $\alpha \in (0, 1)$ such that, for all x_u, x_s, x_u', x_s':

- $d_{Y,\tau}(T_{Y,\tau}^n(x_u, x_s, 0), T_{Y,\tau}^n(x_u, x_s', 0)) \leq C\alpha^n$ (contraction along stable leaves),

- $d_{Y,\tau}(T_{Y,\tau}^n(x_u,x_s,0),T_{Y,\tau}^n(x_u',x_s,0)) \leq C\alpha^{s_0(x_u,x_u')-n}$ for $0 \leq n < s_0(x_u,x_u')$ (backward contraction along unstable leaves),

where s_0 is a separation time. In addition, up to working with some power of $d_{Y,\tau}$, we may assume that $f\circ\widetilde{\pi}(\cdot,\cdot,\cdot,a)$ is Lipschitz for $d_{Y,\tau}$ uniformly in $a\in\mathbb{Z}^2$.

The function $f\circ\widetilde{\pi}$ is defined on $A_{Y,\tau}\times\mathbb{Z}^2$. To get a function defined on $A_\tau\times\mathbb{Z}^2$, we use a classical trick by Bowen [3], also used in the proof of [29, Proposition 6.12]. While we shall not repeat the computations, let us ouline the main arguments. Define:

$$u(x_u,x_s,k,a) := \sum_{n=0}^{+\infty}\left[f\circ\widetilde{\pi}\circ\widetilde{T}_{Y,\tau}^n(x_u,x_s,k,a) - f\circ\widetilde{\pi}\circ\widetilde{T}_{Y,\tau}^n\circ p_+(x_u,x_s,k,a)\right].$$

The function u is zero on $\{x_s=0\}$, and the contraction along stable leaves implies that u is bounded. The function $f_+ := f\circ\widetilde{\pi}+u\circ\widetilde{T}_{Y,\tau}-u$ also does not depend on the x_s coordinate. Abusing notations, we may see f_+ as defined on $\widetilde{A}_\tau$. Finally, there exists a constant C' such that, for all x_u, x_u' in the same element of Γ, for all x_s and all $a\in\mathbb{Z}^2$:

$$|u(x_u,x_s,0,a)-u(x_u',x_s,0,a)| \leq C'\alpha^{\frac{s_0(x_u,x_u')}{2}}. \tag{5.1}$$

Since $S_nf = S_nf_+\circ\widetilde{\pi}+u\circ\widetilde{T}_{Y,\tau}^n-u$ and u is bounded, it is enough to prove the convergence in distribution

$$\left(\frac{2\pi\sqrt{\det(\Sigma^2(F))}}{\ln(n)}\right)^{\frac{1}{2}}\sum_{k=0}^{n-1} f_+\circ\widetilde{T}_\tau^k \to \sigma(f_+)\sqrt{\mathcal{E}}\mathcal{N},$$

with respect to the probability distribution $\mu\otimes\delta_0\otimes\delta_0\in\mathcal{P}(\widetilde{A}_\tau)$. The convergence

$$\left(\frac{2\pi\sqrt{\det(\Sigma^2(F))}}{\ln(n)}\right)^{\frac{1}{2}}\sum_{k=0}^{n-1} f\circ\widetilde{\pi}\circ\widetilde{T}_{Y,\tau}^k \to \sigma(f_+)\sqrt{\mathcal{E}}\mathcal{N},$$

with respect to $\mu_Y\otimes\delta_0\otimes\delta_0\in\mathcal{P}(\widetilde{A}_{Y,\tau})$ then follows, and the convergence with respect to any absolutely continuous probability measure on $\widetilde{A}_{Y,\tau}$ follows from [31, Theorem 1]. In addition, since $f_+-f\circ\widetilde{\pi}$ is a bounded coboundary and adding a bounded coboundary does not change the asymptotic variance in the central limit theorem,

$$\sigma(f_+) = \int_{A_Y} G_{Y,\varphi}(f\circ\widetilde{\pi})^2\,\mathrm{d}\mu_Y + 2\sum_{n=1}^{+\infty}\int_{A_Y} G_{Y,\varphi}(f\circ\widetilde{\pi})\cdot G_{Y,\varphi}(f\circ\widetilde{\pi})\circ\widetilde{T}_{Y,0}^n\,\mathrm{d}\mu_Y. \tag{5.2}$$

All is left is to check the integrability and regularity assumptions on f_+.

Integrability of f_+

We start with the first condition on f_+ in Proposition 5.1, which is the hardest. Since

$$\begin{aligned}\sup_{0\leq n\leq\widetilde{\varphi}(x)}\left|\sum_{k=0}^{n-1} f_+\circ\widetilde{T}_\tau^k(\cdot,0,0)\right| &= \sup_{0\leq n\leq\widetilde{\varphi}(x)}\left|\sum_{k=0}^{n-1}(f\circ\widetilde{\pi}+u\circ\widetilde{T}_{Y,\tau}-u)\circ\widetilde{T}_{Y,\tau}^k(\cdot,0,0)\right| \\ &\leq \sup_{0\leq n\leq\widetilde{\varphi}(x)}\left|\sum_{k=0}^{n-1} f\circ\widetilde{\pi}\circ\widetilde{T}_{Y,\tau}^k(\cdot,0,0)\right| + 2\left\|u\right\|_\infty \\ &\leq G_{Y,\varphi}(|f\circ\widetilde{\pi}|)+2\left\|u\right\|_\infty,\end{aligned}$$

it is enough to check that $G_{Y,\varphi}(|f\circ\widetilde{\pi}|)\in\mathbb{L}^q(A_Y,\mu_Y)$ for some $q>2$. For $(x_u,a)\in A\times\mathbb{Z}^2$, let:

$$\overline{f}(x_u,k,a) := \sup_{x_s}|f\circ\widetilde{\pi}|(x_u,x_s,k,a).$$

Then $G_{Y,\varphi}(|f\circ\widetilde{\pi}|)(x_u,x_s)\leq G_\varphi(\overline{f})(x_u)$, so it is enough to check that $G_\varphi(\overline{f})\in\mathbb{L}^q(A_Y,\mu_Y)$ for some $q>2$.

For all $a\in\mathbb{Z}^2\setminus\{0\}$, let

$$N_a(x_u) := G_\varphi(\mathbf{1}_{(A\times\{0\}\times\{a\})})$$

be the number of times an excursion from $A \times \{0\} \times \{0\}$ hits the basis of the Young tower at $A \times \{0\} \times \{a\}$ before going back to $A \times \{0\} \times \{0\}$. Let $A_a := \{N_a \neq 0\} \subset A$, and

$$\mu_a := \mu(A_a)^{-1} \widetilde{T}_{A \times \{0,a\} *} \mu_{|A_a} \otimes \delta_0 \,,$$

where $\widetilde{T}_{A \times \{0,a\}}$ is the map induced by $\widetilde{T}$ on $A \times \{0, a\}$. In other words, $\mu_a \in \mathcal{P}(A \times \{a\}) \simeq \mathcal{P}(A)$ is the distribution of a point at which a trajectory starting from $A \times \{0\}$ enters $A \times \{a\}$, conditioned by the fact that this trajectory enters $A \times \{a\}$ before going back to $A \times \{0\}$. Then the distribution of $N_a - 1$ for $\mu(\cdot|A_a)$ is the distribution of the first non-negative hitting time φ_{-a} of A_{-a} for μ_a.

By [22, Lemma 4.8], the densities $\mathrm{d}\mu_a/\mathrm{d}\mu$ are in $\mathbb{L}^\infty(A, \mu)$ and uniformly bounded in a. We apply [22, Lemma 4.16] to the family of measures $(\mu_a)_{a \in \mathbb{Z}^2 \setminus \{0\}}$ and the function $\mathbf{1}_A$. Note that $\alpha(a) = \mu(A_a) = \mu(A_{-a})$ in the cited article. Hence, for all $q \in (2, \infty)$, there exists a constant $C > 0$ such that, for all $a \in \mathbb{Z}^2$:

$$\begin{aligned}
\left\| G_\varphi(\overline{f} \mathbf{1}_{A_\tau \times \{a\}}) \right\|_{\mathbb{L}^q(A,\mu)} &= \mu(A_a)^{\frac{1}{q}} \left\| G_\varphi(\overline{f} \mathbf{1}_{A_\tau \times \{a\}}) \right\|_{\mathbb{L}^q(A,\mu(\cdot|A_a))} \\
&= \mu(A_a)^{\frac{1}{q}} \left\| \sum_{k=0}^{\varphi_{-a}(x)} G_\tau(\overline{f})(\widetilde{T}_0^k(x), a) \right\|_{\mathbb{L}^q(A,\mu_a)} \\
&\leq C \mu(A_a)^{\frac{1}{q}-1} \left\| G_\tau(\overline{f})(\cdot, a) \right\|_{\mathbb{L}^q(A,\mu)} .
\end{aligned}$$

By [22, Corollary 2.9] and [22, Proposition 2.6], with $\alpha = d = 2$ and $L \equiv 1$,

$$\mu(A_a) = \Theta\left(\frac{1}{1 + \ln_+ |a|} \right) .$$

Hence, up to taking a larger constant C,

$$\left\| G_\varphi(\overline{f}) \right\|_{\mathbb{L}^q(A,\mu)} \leq \sum_{a \in \mathbb{Z}^2} \left\| G_\varphi(\overline{f} \mathbf{1}_{A_\tau \times \{a\}}) \right\|_{\mathbb{L}^q(A,\mu)} \leq C \sum_{a \in \mathbb{Z}^2} (1 + \ln_+ |a|)^{1-\frac{1}{q}} \left\| G_\tau(\overline{f})(\cdot, a) \right\|_{\mathbb{L}^q(A,\mu)} . \tag{5.3}$$

In addition, focusing on a single term $\left\| G_\tau(\overline{f})(\cdot, a) \right\|_{\mathbb{L}^q(A,\mu)}$, we get:

$$\begin{aligned}
\left\| G_\tau(\overline{f})(\cdot, a) \right\|_{\mathbb{L}^q(A,\mu)} &\leq \left\| \sum_{r \geq 1} \mathbf{1}_{\{\tau = r\}} \sum_{k=0}^{r-1} \| f(\cdot, a + S_k F) \|_{\mathbb{L}^\infty(\{\tau = r\})} \right\|_{\mathbb{L}^q(A,\mu)} \\
&= \left\| \sum_{r \geq 1} \mathbf{1}_{\tau = r} \sum_{k=0}^{r-1} \sum_{a' \in \mathbb{Z}^2} \left\| f(\cdot, a') \right\|_\infty \mathbf{1}_{a' = a + S_k F} \right\|_{\mathbb{L}^q(A,\mu)} \\
&\leq \sum_{a' \in \mathbb{Z}^2} \left\| f(\cdot, a') \right\|_\infty \sum_{r \geq 1} \sum_{k=0}^{r-1} \mu(\tau = r, \, S_k F = a' - a)^{\frac{1}{q}} .
\end{aligned} \tag{5.4}$$

Set $h_q(a) := C(1 + \ln_+ |a|)^{1-\frac{1}{q}}$ and $g_q(a) := \sum_{r \geq 1} \sum_{k=0}^{r-1} \mu(\tau = r, \, S_k F = a)^{\frac{1}{q}}$. Equations (5.3) and (5.4) together imply that:

$$\left\| G_\varphi(\overline{f}) \right\|_{\mathbb{L}^q(A,\mu)} \leq \sum_{a \in \mathbb{Z}^2} (h_q * g_q)(a) \, \| f(\cdot, a) \|_\infty \, . \tag{5.5}$$

If $S_k F = a$ with $k \leq r - 1$, then $r \geq k \geq |a| / \|F\|_\infty$. Since $\mu(\tau \geq k) \leq C_\varepsilon e^{-\varepsilon k}$, there exists a constant $C'(q, \varepsilon)$ such that:

$$g_q(a) \leq \sum_{r \geq |a| / \|F\|_\infty} r \, \mu(\tau = r) \leq C'(q, \varepsilon) e^{-\frac{\varepsilon |a|}{2q \|F\|_\infty}} .$$

All is left is to estimate $h_q * g_q$. Let $a \in \mathbb{Z}^2 \setminus \{0\}$. We split $\mathbb{Z}^2$ into rings:

$$A_n(a) = \{a' \in \mathbb{Z}^2 : \; e^n |a| \leq |a'| < e^{n+1} |a|\},$$

with $n \geq 1$, and a central disk $A_0(a)$. We have $\mathrm{Card}(A_n(a)) = \Theta(e^{2n}|a|^2)$ and, for all $a' \in A_n(a)$,

$$\begin{cases} h_q(a') & \leq & h_q(a) + C(1-q^{-1})(n+1), \\ g_q(a-a') & \leq & C'(q,\varepsilon) e^{-\frac{\varepsilon(e^n-1)|a|}{2q\|F\|_\infty}}. \end{cases}$$

Summing over all $a' \in \mathbb{Z}^2$ yields, for some constant $C' > 0$:

$$\begin{aligned} h_q * g_q(a) &= \sum_{n=0}^{+\infty} \sum_{a' \in A_n(a)} h_q(a') g_q(a-a') \\ &\leq \sum_{a' \in \mathbb{Z}^2} h_q(a) g_q(a-a') + C(1-q^{-1}) \sum_{a' \in A_0(a)} g_q(a-a') + C(1-q^{-1}) \sum_{n=1}^{+\infty} (n+1) \sum_{a' \in A_n(a)} g_q(a-a') \\ &\leq [h_q(a) + C(1-q^{-1})] \, \|g_q\|_{\ell^1(\mathbb{Z}^2)} + C' \sum_{n=1}^{+\infty} (n+1) e^{2n} |a|^2 e^{-\frac{\varepsilon(e^n-1)|a|}{2q\|F\|_\infty}}. \end{aligned} \tag{5.6}$$

The sum in Equation (5.6) is finite for all a. Each term in the sum converges to 0 as a goes to infinity (and thus is bounded). In addition, the function $u \mapsto u^2 e^{-\frac{\varepsilon(e^n-1)u}{2q\|F\|_\infty}}$ is decreasing on $[(4q\|F\|_\infty)/(e^n-1), +\infty)$, and thus on $[1, +\infty)$ for all large enough n. Hence, for all large enough n and all $a \in \mathbb{Z}^2 \setminus \{0\}$,

$$(n+1) e^{2n} |a|^2 e^{-\frac{\varepsilon(e^n-1)|a|}{2q\|F\|_\infty}} \leq (n+1) e^{2n} e^{-\frac{\varepsilon(e^n-1)}{2q\|F\|_\infty}},$$

which is summable in n. Hence the sum is bounded in a. Since h_q is bounded from below, we finally get $h_q * g_q = O(h_q)$.

Let $\varkappa > 0$, and f be such that $\sup_{a \in \mathbb{Z}^2} (1 + \ln_+ |a|)^{\frac{1}{2}+\varkappa} \|f(\cdot, a)\|_\infty < +\infty$. Without loss of generality, we assume that $\varkappa < 1/2$. Taking $q = \frac{2}{1-2\varkappa}$, by Equation (5.5), the function $G_\varphi(\overline{f})$ belongs to $\mathbb{L}^q(A, \mu)$.

Remaining conditions on f_+

Let us focus on the last two conditions for $G_\varphi(f)$. Since f is integrable and has integral zero, so does $f \circ \widetilde{\pi}$. By Kac's formula, $G_{Y,\varphi}(f)$ is integrable and:

$$\int_{A_Y} G_{Y,\varphi}(f) \, \mathrm{d}\mu_Y = \int_{\widetilde{A}_{\tau,Y}} f \, \mathrm{d}\widetilde{\mu}_{Y,\tau} = 0.$$

Since $G_\varphi(f_+) - G_{Y,\varphi}(f)$ is a bounded coboundary, $G_\varphi(f_+)$ also has integral zero.

Finally, let us check the regularity condition on $G_\varphi(f_+)$. Summing the identity $f_+ := f \circ \widetilde{\pi} + u \circ \widetilde{T}_{Y,\tau} - u$ on the height of the tower $A_{Y,\tau}$ yields, for all $x_u \in A$ and $a \in \mathbb{Z}^2$,

$$G_\tau(f_+)(x_u, a) = G_\tau(f \circ \widetilde{\pi})(x_u, 0, a) + u(T_Y(x_u, 0), 0, a + F(x_u)) - u(x_u, 0, 0, a).$$

The space A can be endowed with a metric α^s, where s is the separation time for the Gibbs-Markov map (A, μ, T) and $\alpha \in (0,1)$ is close enough to 1. As $s \leq s_0$, we have $\alpha^{s_0} \leq \alpha^s$, and $\alpha^{s_0-\tau} \leq \alpha^{-1}\alpha^s$ if $s_0 \geq \tau$ (so on each element of the partition Γ). Given x_u, x_u' in the same element of Γ,

$$\begin{aligned} |G_\tau(f \circ \widetilde{\pi})(x_u, 0, a) - G_\tau(f \circ \widetilde{\pi})(x_u', 0, a)| &\leq C |f \circ \widetilde{\pi}|_{\mathrm{Lip}(d_{Y,\tau})} \sum_{k=0}^{\tau(x_u)-1} \alpha^{s_0(x_u, x_u')-k} \\ &\leq \frac{C\alpha}{1-\alpha} |f \circ \widetilde{\pi}|_{\mathrm{Lip}(d_{Y,\tau})} \alpha^{s_0(x_u, x_u') - \tau(x_u)} \\ &\leq \frac{C}{1-\alpha} |f \circ \widetilde{\pi}|_{\mathrm{Lip}(d_{Y,\tau})} \alpha^{s(x_u, x_u')}, \end{aligned}$$

so the function $x_u \mapsto G_\tau(f \circ \widetilde{\pi})(x_u, 0, a)$ is Lipschitz for the distance α^s on each element of Γ, uniformly in $a \in \mathbb{Z}^2$ and in Γ.

By Equation (5.1), the function u is uniformly 1/2-Hölder for the distance α^{s_0} (and thus for the distance α^s) on each unstable leaf in $A_Y \times \mathbb{Z}^2$. Up to increasing the value of α, we may assume that u is actually Lipschitz. Since applying T_Y multiplies α^s by at most α^{-1}, the function $x_u \mapsto u(T_Y(x_u, 0), 0, a + F(x_u))$ is also Lipschitz for the distance α^s on each element of Γ, uniformly in $a \in \mathbb{Z}^2$ and in Γ. Hence, f_+ is also

Lipschitz for the distance α^s on each element of Γ, uniformly in $a \in \mathbb{Z}^2$ and in Γ, and thus $G_\varphi(f_+)$ satisfies the regularity condition of Proposition 5.1 by [29, Lemma 6.5]. □

Remark 5.2 (Infinite horizon billiards)**.** *Young towers are still available for infinite horizon Lorentz gases* [6]*, although the height of the tower only has a polynomial tail:* $\mu(\tau \geq n) = O(n^{-2})$*. In the finite horizon case, we used the facts that jumps in the billiard are bounded and that the tails of* τ *decay exponentially to control* g_q*; both fail in the infinite horizon setting.*

Moreover it is not known whereas a spectral local limit theorem analogous to Condition (2.2) *holds (with* ℓ *replaced by* $\ell \log \ell$*) in the infinite horizon setting. Using Young towers, in* [25]*, Szász and Varjú proved estimates analogous to our Hypothesis 2.6 under a weaker form, namely with* $\mathcal{L}(\mathcal{B}, \mathcal{B})$ *replaced by* $\mathcal{L}(\mathcal{B}, \mathbb{L}^1(\mu))$*. However, Condition* (2.2) *with* $\mathcal{L}(\mathcal{B}, \mathcal{B})$ *being replaced by* $\mathcal{L}(\mathcal{B}, \mathbb{L}^1(\mu))$ *would not be enough to adapt our proof of Theorem 2.1, because we use iterations of operators* $Q_{\ell,a} : h \mapsto P^\ell \left(\mathbf{1}_{\{S_\ell F = a\}} h\right)$*.*

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