RATIO ERGODIC THEOREMS: FROM HOPF TO BIRKHOFF AND KINGMAN

HANS HENRIK RUGH AND DAMIEN THOMINE

ABSTRACT. Hopf's ratio ergodic theorem has an inherent symmetry which we exploit to provide a simplification of standard proofs of Hopf's and Birkhoff's ergodic theorems. We also present a ratio ergodic theorem for conservative transformations on a σ-finite measure space, generalizing Kingman's ergodic theorem for subadditive sequences and generalizing previous results by Akcoglu and Sucheston.

1. INTRODUCTION AND STATEMENT OF RESULTS

Birkhoff's pointwise ergodic theorem [3] is a key tool in ergodic theory. It admits many notable generalizations, including Hopf's ratio ergodic theorem [5], Kingman's subadditive ergodic theorem [10], and more recently Karlsson-Ledrappier and Gouëzel-Karlsson theorems on cocycles of isometries [8, 4]. Since the work of Kamae [6] and Katznelson and Weiss [9], there exist very short and easy proofs of Birkhoff's ergodic theorem. These proofs have also been adapted to Hopf's and Kingman's ergodic theorems ([7] and [9, 11], respectively).

In this article, we provide a proof of Hopf's and Kingman's ergodic theorem, in the context of conservative transformations preserving a σ-finite measure. We follow the argument of Katznelson and Weiss [9] but add a noticeable twist: the statement of the ratio ergodic theorem has a natural symmetry which is not present in Birkhoff's ergodic theorem, a symmetry which can be leveraged to simplify proofs. This makes, in our opinion, Hopf's theorem more fundamental, with Birkhoff's theorem now appearing as a corollary (the inverse point of view is given in e.g. [12], where Hopf's theorem is deduced from Birkhoff's by inducing).

As for the Kingman ratio ergodic theorem on a σ-finite measure space, a similar result was obtained by Akcoglu and Suchestom [2] under an additional integrability assumption. Our result does not make this assumption and the proof is significantly simpler (in our opinion). As the reader may note there are no significant complications coming from working with σ-finite measures, but some parts may be simplified quite a lot if one assumes ergodicity.

Definition 1.1. Consider a σ-finite measure space $(X, \mathcal{B}, \mu)$ and a measure preserving transformation $T : X \to X$. The transformation T is said to be *conservative* if:
$$\forall A \in \mathcal{B} : \quad \mu(A) > 0 \quad \Rightarrow \exists n \geq 1 : \mu(A \cap T^{-1}A) > 0. \tag{1}$$
A subset $A \subset X$ is *T-invariant* if $T^{-1}A = A$, and a function $f : X \to \mathbb{R}$ is *T-invariant* if $f \circ T = f$. The transformation T is *ergodic* (for μ) if for any measurable T-invariant subset A, either $\mu(A) = 0$ or $\mu(X \setminus A) = 0$.

Unless stated otherwise, we make throughout the standard assumption that $(X, \mathcal{B}, \mu, T)$ is a conservative measure-preserving transformation on a σ-finite measure space. Most of the time, ergodicity shall not be assumed.

Given any $f : X \to \mathbb{R}$, we write $S_n f := \sum_{k=0}^{n-1} f \circ T^k$ for the Birkhoff sums. Our first goal is to give a proof of the following well-known:

Theorem 1.2 (Hopf's ratio ergodic theorem)**.** *Let $(X, \mathcal{B}, \mu, T)$ be a conservative, measure preserving transformation on a σ-finite measure space. Let $f, g \in \mathbb{L}^1(X, \mu)$ with $g > 0$ almost everywhere. Then the following limit exists μ-almost everywhere:*

$$h := \lim_{n \to \infty} \frac{S_n f}{S_n g}. \tag{2}$$

The function h is finite μ-almost everywhere, T-invariant, and for any T-invariant subset $A \in \mathcal{B}$:

$$\int_A f \, \mathrm{d}\mu = \int_A hg \, \mathrm{d}\mu. \tag{3}$$

Note that, when $f > 0$ as well, there is a natural symmetry between f and g, which we will exploit in our proof. This symmetry is lost in Birkhoff's version, where $g \equiv 1$.

We will proceed to prove a ratio version of Kingman's theorem for subadditive sequences. Recall that a sequence $(a_n)_{n \geq 1}$ of measurable functions is said to be subadditive (with respect to T) if for $n, m \geq 1$, we have μ-almost everywhere:

$$a_{n+m} \leq a_n + a_m \circ T^n. \tag{4}$$

Theorem 1.3 (A Kingman ratio ergodic theorem)**.** *Let $(X, \mathcal{B}, \mu, T)$ be a conservative, measure preserving transformation on a σ-finite measure space. Let $(a_n)_{n \geq 1}$ be a sub-additive measurable sequence of functions with values in $[-\infty, +\infty]$, and with $(a_1)^+ \in \mathbb{L}^1(X, \mu)$. Let $g \in \mathbb{L}^1(X, \mu)$ with $g > 0$ almost everywhere. Then the following limit exists μ-almost everywhere:*

$$h := \lim_{n \to +\infty} \frac{a_n}{S_n g} \in [-\infty, +\infty).$$

If $A \in \mathcal{B}$ is a T-invariant set, then

$$\inf_n \frac{1}{n} \int_A a_n \, \mathrm{d}\mu = \lim_n \frac{1}{n} \int_A a_n \, \mathrm{d}\mu = \int_A hg \, \mathrm{d}\mu \in [-\infty, +\infty).$$

For a similar version of this theorem, but under an additional uniform integrability condition on a_n, cf. [2]. To our knowledge the theorem is new in the stated generality.

The remainder of this article is organized as follows. In Section 2 we prove some classical lemmas about conservative dynamical systems, and the main lemma (Lemma 2.3, which slightly generalizes the main theorem of [9]). In Section 3 we prove Hopf's ratio ergodic theorem, and in Section 4 the above Kingman ratio ergodic theorem (whose proof uses Hopf's ergodic theorem).

2. Main lemmas

First, and so that our proofs will be essentially self-contained, let us state and prove some consequences of conservativity. For details the reader may consult e.g. [1, Chap 1]. Conservativity is an a priori mild recurrence condition which is equivalent to the following seemingly stronger recurrence condition. Let $(X, \mathcal{B}, \mu, T)$ be a conservative, measure-preserving dynamical system. Then, given any measurable subset A, almost everywhere on A,

$$\sum_{n \geq 0} \mathbf{1}_A \circ T^n = +\infty. \tag{5}$$

In other words, almost every point in A returns infinitely often to A. A consequence is the following:

Lemma 2.1. *Let $(X, \mathcal{B}, \mu, T)$ be a measure-preserving and conservative transformation. Let $f : X \to \mathbb{R}_+$ be measurable. Then, μ-almost everywhere on $\{f > 0\}$:*

$$\lim_{n \to +\infty} S_n f = +\infty. \tag{6}$$

Proof. For $n \geq 1$, let $A_n := \{f \geq \frac{1}{n}\}$. Let $\Omega_n := A_n \cap \bigcap_{m \geq 0} \bigcup_{k \leq m} T^{-k} A_n$ be the set of points of A_n which return to A_n infinitely many times.

Since f is positive and takes value at least $1/n$ on A_n, Equation (6) follows for all $x \in \Omega_n$. Let $\Omega_\infty := \bigcup_{n \geq 1} \Omega_n$. Then Equation (6) holds on Ω_∞. Since $(X, \mathcal{B}, \mu, T)$ is assumed to be conservative, $\mu(\Omega_n \Delta A_n) = 0$, so that $\mu(\Omega_\infty \Delta \bigcup_{n \geq 1} A_n) = 0$. But, $\bigcup_{n \geq 1} A_n = \{f > 0\}$, so Ω_∞ has full measure in $\{f > 0\}$. $\square$

Let $(a_n)_{n \geq 1}$ be a super-additive sequence of functions and $g \in \mathbb{L}^1(A, \mu)$. We define for every $x \in X$ the following lower and upper limits:

$$0 \quad \leq \quad \underline{h}(x) = \liminf_{n \to \infty} \frac{a_n(x)}{S_n g(x)} \quad \leq \quad \overline{h}(x) = \limsup_{n \to \infty} \frac{a_n(x)}{S_n g(x)} \quad \leq \quad +\infty. \tag{7}$$

Both $\underline{h}$ and $\overline{h}$ are measurable and, in fact, a.e. T-invariant:

Lemma 2.2. *Let $(X, \mathcal{B}, \mu, T)$ be a measure-preserving and conservative transformation. Let $(a_n)_{n \geq 1}$ be a super-additive sequence of functions, with $a_1 \geq 0$ a.e.. Let $g \in \mathbb{L}^1(A, \mu; \mathbb{R}_+^*)$. Then $\underline{h} \circ T = \underline{h}$ and $\overline{h} \circ T = \overline{h}$ almost everywhere.*

Proof. We prove the result for $\underline{h}$; the proof for $\overline{h}$ is essentially the same. Let (a_n) and g be as in the lemma. Then:

$$\frac{a_{n+1}}{S_{n+1} g} \geq \frac{a_1 + a_n \circ T}{g + (S_n g) \circ T}$$

By Lemma 2.1, $\lim_{n \to +\infty} S_n g = +\infty$ almost everywhere, whence, taking the liminf,

$$\underline{h} \geq \underline{h} \circ T.$$

The function $\eta = \underline{h}/(1 + \underline{h})$ takes values in $[0, 1]$ and we have $A := \{\underline{h} > \underline{h} \circ T\} = \{\eta > \eta \circ T\}$. By Lemma 2.1, the map $\phi := \eta - \eta \circ T$ verifies $\lim_{n \to +\infty} S_n \phi = +\infty$ almost everywhere on A. But as $S_n \phi = \eta - \eta \circ T^{n+1} \in [-1, 1]$ everywhere, we must have $\mu(A) = 0$.

$\square$

We can now state and prove our main lemma.

Lemma 2.3. *Let $(X, \mathcal{B}, \mu, T)$ be a measure-preserving and conservative transformation. Let $(a_n)_{n \geq 1}$ be a super-additive sequence of functions, with $a_1 \geq 0$ almost everywhere. Let $g \in \mathbb{L}^1(A, \mu; \mathbb{R}_+^*)$.*

Then $\overline{h}$ is T-invariant, and for all T-invariant $A \in \mathcal{B}$,

$$\liminf_{n \to +\infty} \frac{1}{n} \int_A a_n \, d\mu \geq \int_A \overline{h} g \, d\mu. \tag{8}$$

Proof. By Lemma 2.2, $\overline{h}$ is T-invariant, but with values a priori in $[0, +\infty]$. For $\varepsilon > 0$, we set

$$\overline{h}_\varepsilon := \frac{\overline{h}}{1 + \varepsilon \overline{h}},$$

with the convention that $\overline{h}_\varepsilon(x) = 1/\varepsilon$ when $\overline{h}(x) = +\infty$. Let:

$$n_\varepsilon := \inf \left\{ k \geq 1 : a_k \geq S_k(\overline{h}_\varepsilon g) \right\}.$$

Then $n_\varepsilon(x) = 1$ whenever $\overline{h}(x) = 0$. When $\overline{h}(x) > 0$, we have $\overline{h}_\varepsilon(x) < \overline{h}(x)$, so $n_\varepsilon(x)$ is finite by the definition of the limsup. Introduce a time-cutoff $L \geq 1$ and denote $E = E_{\varepsilon,L} := \{n_\varepsilon \leq L\}$. We then set:

$$\varphi(x) = \varphi_{\varepsilon,L}(x) := \begin{cases} 1 & \text{if } x \notin E \\ n_\varepsilon(x) & \text{if } x \in E \end{cases}.$$

One verifies that for every $x \in X$:

$$a_{\varphi(x)} \geq S_{\varphi(x)}(g\overline{h}_\varepsilon \mathbf{1}_E).$$

When $x \in E$ this is true by the very definition of $n_\varepsilon(x)$, while for $x \notin E$ it holds because the right hand side vanishes.

Define a sequence of stopping times:

$$\begin{cases} \tau_0(x) & = 0, \\ \tau_{k+1}(x) & = \tau_k(x) + \varphi\left(T^{\tau_k(x)}x\right), \quad k \geq 0. \end{cases}$$

Note that $1 \leq \tau_{k+1} - \tau_k \leq L$ for every $k \geq 0$, and for all $x \in X$:

$$a_{\tau_k(x)}(x) \geq \sum_{j=0}^{k-1} a_{\varphi(x)} \circ T^{\tau_j}(x) \geq \sum_{j=0}^{k-1} S_{\varphi(x)}(\overline{h}_\varepsilon g \mathbf{1}_E) \circ T^{\tau_j}(x) = S_{\tau_k(x)}(g\overline{h}_\varepsilon \mathbf{1}_E)(x).$$

Let $N \geq 1$ and $x \in X$. There exists $k \geq 1$ such that $N < \tau_k(x) \leq N + L$. Then:

$$a_{N+L}(x) \geq a_{\tau_k(x)}(x) + \sum_{i=\tau_k(x)+1}^{N+L} a_1 \circ T^i \geq S_{\tau_k(x)}(\overline{h}_\varepsilon g \mathbf{1}_E)(x) \geq S_N(g\overline{h}_\varepsilon \mathbf{1}_E)(x),$$

which allows us to get rid of the intermediate stopping times. Take now a T-invariant set $A \in \mathcal{B}$ and integrate the above inequality over A. By T-invariance:

$$\frac{1}{N+L} \int_A a_{N+L} \, \mathrm{d}\mu \geq \frac{1}{N+L} \int_A S_N(\overline{h}_\varepsilon g \mathbf{1}_E) \, \mathrm{d}\mu = \frac{N}{N+L} \int_A \overline{h}_\varepsilon g \mathbf{1}_E \, \mathrm{d}\mu.$$

Letting $N \to +\infty$, we conclude that:

$$\liminf_{n \to +\infty} \frac{1}{n} \int_A a_n \, \mathrm{d}\mu \geq \int_A \overline{h}_\varepsilon g \mathbf{1}_E \, \mathrm{d}\mu.$$

Letting $L \to +\infty$ and finally $\varepsilon \to 0$, we obtain by monotone convergence:

$$\liminf_{n \to +\infty} \frac{1}{n} \int_A a_n \, \mathrm{d}\mu \geq \int_A \overline{h}g \, \mathrm{d}\mu.$$

$\square$

3. Proof of Hopf's Ratio ergodic theorem

We are now ready to prove to prove Hopf's ergodic theorem. Lemma 2.3 only provides a upper bound on $\overline{h}$. In most proofs using these techniques, the lower bound on $\underline{h}$ follows by repeating the same argument, with a modified stopping time (and some handwaving). Here we notice that the symmetry of Hopf's ratio ergodic theorem provides us with a shortcut:

Proof of Theorem 1.2. Let $f, g \in \mathbb{L}^1(X, \mu; \mathbb{R}_+^*)$. By Lemma 2.3, with $a_n = S_n f$, for any T-invariant measurable A:

$$\int_A f \, \mathrm{d}\mu \geq \int_A \overline{h}g \, \mathrm{d}\mu.$$

In particular, taking $A = X$, we see that $\overline{h}g \in \mathbb{L}^1(X, \mu)$.

We now use the symmetry, and apply Lemma 2.3 with g and f. Since

$$\limsup_{n \to +\infty} (S_n g)/(S_n f) = \underline{h}^{-1},$$

we get $\underline{h}^{-1}f \in \mathbb{L}^1(X,\mu)$, and in particular $+\infty > \overline{h} \geq \underline{h} > 0$ almost everywhere.

We now apply Lemma 2.3 again, with $\overline{h}g$ and f. Since

$$\limsup_{n \to +\infty} \frac{S_n(\overline{h}g)}{S_n f} = \frac{\overline{h}}{\underline{h}},$$

we get that, for any T-invariant measurable A,

$$\int_A f \, \mathrm{d}\mu \geq \int_A \overline{h}g \, \mathrm{d}\mu \geq \int_A f \frac{\overline{h}}{\underline{h}} \, \mathrm{d}\mu.$$

As $f > 0$ a.e. and the integral is finite we conclude that $\overline{h} = \underline{h} =: h$ almost everywhere.

Let us now turn towards the proof of Theorem 1.2 without positivity assumption on f. Since μ is σ-finite, we can write $f = f_+ - f_-$, with f_+ and f_- in $\mathbb{L}^1(X,\mu;\mathbb{R}_+^*)$. Then, μ-almost everywhere:

$$\lim_{n \to +\infty} \frac{S_n f}{S_n g} = \lim_{n \to \infty} \frac{S_n f_+}{S_n g} - \frac{S_n f_-}{S_n g} = h_+ - h_- =: h.$$

In addition, $\int_A f \, \mathrm{d}\mu = \int_A (f_+ - f_-) \, \mathrm{d}\mu = \int_A (h_+ - h_-)g \, \mathrm{d}\mu = \int_A hg \, \mathrm{d}\mu.$ $\qquad \square$

Remark 3.1. The proof of Theorem 1.2 itself can be significantly shortened if one assumes that $(X, \mathcal{B}, \mu, T)$ is ergodic: since $\overline{h}$ and $\underline{h}$ are then constant, applying Lemma 2.3 to the pairs (f,g) and (g,f) yields directly:

$$\underline{h} \geq \frac{\int_X f \, \mathrm{d}\mu}{\int_X g \, \mathrm{d}\mu} \geq \overline{h}.$$

There is, to our knowledge, not much gain to be had in the proof of Lemma 2.3.

In the ergodic case, the statement of Hopf's ergodic theorem can be simplified.

Corollary 3.2 (Hopf's theorem, ergodic version)**.** *Let $(X, \mathcal{B}, \mu, T)$ be a measure-preserving, conservative and ergodic transformation. Let f, $g \in \mathbb{L}^1(A, \mu)$ with $\int_X g \, \mathrm{d}\mu \neq 0$. Then μ-almost everywhere:*

$$\lim_{n \to +\infty} \frac{S_n f}{S_n g} = \frac{\int_X f \, \mathrm{d}\mu}{\int_X g \, \mathrm{d}\mu}. \tag{9}$$

Proof. We decompose as above $f = f_+ - f_-$, with $f_\pm$ integrable and positive. By the Theorem 1.2, μ-almost everywhere,

$$\lim_{n \to +\infty} \frac{S_n g}{S_n f_\pm} = k_\pm.$$

By ergodicity, the $k_\pm$ are constant and then non-zero, since $\int_X g \, \mathrm{d}\mu = k_\pm \int_X f_\pm \, \mathrm{d}\mu \neq 0$. Thus, almost everywhere:

$$\lim_{n \to \infty} \frac{S_n f}{S_n g} = \lim_{n \to \infty} \frac{S_n f_+}{S_n g} - \frac{S_n f_-}{S_n g} = \frac{1}{k_+} - \frac{1}{k_-} = \frac{\int_X (f_+ - f_-) \, \mathrm{d}\mu}{\int_X g \, \mathrm{d}\mu} = \frac{\int_X f \, \mathrm{d}\mu}{\int_X g \, \mathrm{d}\mu}. \quad \square$$

As a special case, we may also consider when μ is a probability measure and $g \equiv 1$ (thus integrable). T is automatically conservative by Poincaré recurrence theorem. From Theorem 1.2 we deduce:

Corollary 3.3 (Birkhoff's Ergodic Theorem)**.** *Let $(X, \mathcal{B}, \mu, T)$ be a measure preserving transformation on a probability space. Let $f \in \mathbb{L}^1(X, \mu)$. Then the following limit exists μ-almost everywhere:*

$$f^* := \lim_{n \to +\infty} \frac{1}{n} S_n f. \tag{10}$$

f^* is T-invariant (up to a set of measure 0), and for any T-invariant measurable subset A:

$$\int_A f \, \mathrm{d}\mu = \int_A f^* \, \mathrm{d}\mu. \tag{11}$$

4. KINGMAN, σ-FINITE VERSION

We proceed here with a ratio version of Kingman's theorem for non-negative super-additive sequences, from which Theorem 1.3 shall follow easily.

Proposition 4.1. *Let $(X, \mathcal{B}, \mu, T)$ be a measure-preserving and conservative transformation. Let $(a_n)_{n \geq 1}$ be a super-additive sequence of functions, with $a_1 \geq 0$ almost everywhere. Let $g \in \mathbb{L}^1(A, \mu; \mathbb{R}_+^*)$.*

Then the following limit exists μ-almost everywhere:

$$h := \lim_{n \to +\infty} \frac{a_n}{S_n g} \in [0, +\infty].$$

In addition, for any T-invariant measurable set A,

$$\sup_{n \in \mathbb{N}} \frac{1}{n} \int_A a_n \, \mathrm{d}\mu = \lim_{n \to +\infty} \frac{1}{n} \int_A a_n \, \mathrm{d}\mu = \int_A hg \, \mathrm{d}\mu \in [0, +\infty].$$

Proof. Let A be any T-invariant measurable set. By Lemma 2.3, we know that:

$$\liminf_{n \to +\infty} \frac{1}{n} \int_A a_n \, \mathrm{d}\mu \geq \int_A \overline{h} g \, \mathrm{d}\mu.$$

We want to prove the converse inequality (inverting the direction of the inequality, and the $\liminf$ and $\limsup$). Let $K < \sup_{n \geq 1} \frac{1}{n} \int_A a_n \, \mathrm{d}\mu$. Then we can find $k \geq 1$ such that $\frac{1}{k} \int_A a_k \, \mathrm{d}\mu > K$. For $M > 0$, let $f_{k,M} := \min\{a_k, MS_k g, Mk\}/k$. By the monotone convergence theorem, there exists $M > 0$ such that:

$$\int_A f_{k,M} \, \mathrm{d}\mu > K.$$

Let $n \geq 2k$, and let q, r be such that $n = qk + r$ and $0 \leq r < k$. Then:

$$a_n = \sum_{i=0}^{k-1} \frac{a_n}{k} \geq \sum_{i=0}^{k-1} \frac{1}{k} \left(a_i + \sum_{j=0}^{q-2} a_k \circ T^{i+jk} + a_{n-i-(q-1)k} \circ T^{i+(q-1)k} \right)$$

$$\geq \sum_{i=0}^{(q-1)k-1} \frac{a_k}{k} \circ T^i = S_{(q-1)k}(a_k/k) \geq S_{n-2k} f_{k,M}.$$

Let $\underline{h}_{k,M} := \liminf_{n \to +\infty} S_n f_{k,M}/S_n g$. Note that $|S_n f_{k,M} - S_{n-2k} f_{k,M}| \leq 2kM$, so that $\underline{h}_{k,M} = \liminf_{n \to +\infty} S_{n-2k} f_{k,M}/S_n g$ by Lemma 2.1. Hence, $\underline{h} \geq \underline{h}_{k,M}$. By Hopf's theorem (cf. Theorem 1.2),

$$K \leq \int_A f_{k,M} \, \mathrm{d}\mu = \int_A \underline{h}_{k,M} g \, \mathrm{d}\mu \leq \int_A \underline{h} g \, \mathrm{d}\mu.$$

Since this is true for all $K < \sup_{n \geq 1} \frac{1}{n} \int_A a_n \, \mathrm{d}\mu$, we finally get:

$$\limsup_{n \to +\infty} \frac{1}{n} \int_A a_n \, \mathrm{d}\mu \leq \sup_{n \geq 1} \frac{1}{n} \int_A a_n \, \mathrm{d}\mu \leq \int_A \underline{h} g \, \mathrm{d}\mu,$$

whence the sequence $(\frac{1}{n} \int_A a_n \, \mathrm{d}\mu)_{n \geq 1}$ converges to its supremum, and:

$$\int_A \overline{h} g \, \mathrm{d}\mu = \int_A \underline{h} g \, \mathrm{d}\mu. \tag{12}$$

All is left is to prove that $\overline{h} = \underline{h}$ almost everywhere. For $M \geq 0$, take $A := \{\underline{h} \leq M\}$. Then $\underline{h} g$ is integrable on A, and since $g > 0$ a.e. Equation (12) implies $\underline{h} = \overline{h}$

almost everywhere on A. Since this is true for all $M \geq 0$, we get that $\underline{h} = \overline{h}$ almost everywhere on $\{\underline{h} < +\infty\}$, and obviously $\underline{h} = \overline{h}$ on $\{\underline{h} = +\infty\}$. $\qquad\square$

Let us finish the proof of Theorem 1.3.

Proof of Theorem 1.3. Up to taking the opposite sequences, we work with super-additive sequences. Let $(a_n)_{n \geq 1}$ be a super-additive sequence, and g a positive and integrable function. Write $a_1 = a_1^+ - a_1^-$ and $b_n := a_n + S_n a_1^-$. Then $(b_n)_{n \geq 1}$ and g satisfy the hypotheses of Proposition 4.1, and so do $(S_n a_1^-)_{n \geq 1}$ and g. The (almost everywhere) limits and integrals concerning $(S_n a_1^-)_{n \geq 1}$ and g are finite, so we can subtract them from the limits and integrals concerning $(b_n)_{n \geq 1}$ and g. $\qquad\square$

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Hans Henrik Rugh, Laboratoire de Mathématiques d'Orsay, Université Paris-Sud, CNRS, Université Paris-Saclay, 91405 Orsay Cedex, France. Email: hans-henrik.rugh@math.u-psud.fr

Damien Thomine, Laboratoire de Mathématiques d'Orsay, Université Paris-Sud, CNRS, Université Paris-Saclay, 91405 Orsay Cedex, France. Email: damien.thomine@math.u-psud.fr