Local time and first return time for periodic semi-flows

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Abstract

We study $\mathbb{Z}^d$-periodic semi-flows, which are versions in continuous time of $\mathbb{Z}^d$-extensions of dynamical systems. These systems are defined by an underlying dynamical system, a step time (the time to wait before the system makes a move), and a step function (the displacement in $\mathbb{Z}^d$ at each step). We are interested in two statistics related to these semi-flows: the local time, i.e. the time spent in some subset, and the first return time to the origin. We get some partial results under spectral conditions on the transfer operator of the underlying dynamical system. If the underlying dynamics is Gibbs-Markov, and under additional constraints on the step time and step function, we get distributional asymptotics for the local time, and an equivalent of the tail of the first return time.

We wish to investigate the recurrence properties of $\mathbb{Z}^d$-extensions of suspension flows over dynamical systems. These objects share some properties with random walks, with two important caveats. The steps are not independent, as the randomness is generated by an underlying dynamical system. In addition, instead of taking steps at each unit of time, the process waits some deterministic time before taking each step, and the sequence of step times is generated by the same underlying dynamical system. Such systems arise naturally when one studies the properties of the geodesic flow on a $\mathbb{Z}^d$-periodic billiard or a $\mathbb{Z}^d$-periodic surface of negative curvature.

The quantities we are interested in are the first return time to and the time spent in some subset.

Let us introduce the systems with which we will work. Let d be a non-negative integer. We denote by (A, μ_A, T_A) an ergodic dynamical system preserving the probability measure μ_A, by r a positive measurable function on A (the *step time*), which may be infinite on a set of null measure, and by F a $\mathbb{Z}^d$-valued measurable function on A (the *step function*).

The *suspension flow with base* (A, μ_A, T_A) *and roof function* r is a semi-flow $(\Omega, \mu, (g_t)_{t\geq 0})$ on a measured space constructed in the following way. On $A \times \mathbb{R}_+$ we define the product measure $\widetilde{\mu} := \mu_A \otimes \mathrm{Leb}$ and a semi-flow $(\widetilde{g}_t)_{t\geq 0}$ by translation along the second coordinate (i.e. $\widetilde{g}_t(x,s) = (x, t+s)$). The measure $\widetilde{\mu}$ is invariant under the action of the semi-flow. Let $\sim$ be the equivalence relation generated by $(x, t + r(x)) \sim (T_A(x), t)$ for all x and all non-negative t. Then $\Omega = (A \times \mathbb{R}_+)_{/\sim}$, the measure μ is defined by restriction to a fundamental domain, and $(g_t)_{t\geq 0}$ is the image of $(\widetilde{g}_t)_{t\geq 0}$ under the canonical projection. The space Ω can also be seen as the space $A \times [0, r] := \{(x,t) \in A \times \mathbb{R}_+ : t \in [0, r(x)]\}$ once the end points $(x, r(x))$ and $(T_A(x), 0)$ have been glued together. We will often use the identification $\Omega \simeq A \times [0, r)$. Since r is finite almost everywhere, the suspension flow with base (A, μ_A, T_A) and roof function r is ergodic. Moreover, the measure μ is $(g_t)_{t\geq 0}$-invariant.

Let $d \geq 0$. The $\mathbb{Z}^d$-*extension with step function* F *of a suspension flow with base* (A, μ_A, T_A) *and step time* r is a semi-flow $(\Omega, \mu, (g_t)_{t\geq 0})$ on a measured space constructed in the following way. We denote by $\mathrm{Leb}_{\mathbb{R}_+}$ the Lebesgue measure on $\mathbb{R}_+$, and by $\mathrm{Leb}_{\mathbb{Z}^d}$ the counting measure on $\mathbb{Z}^d$. On $A \times \mathbb{Z}^d \times \mathbb{R}_+$ we define the product measure $\widetilde{\mu} := \mu_A \otimes \mathrm{Leb}_{\mathbb{Z}^d} \otimes \mathrm{Leb}_{\mathbb{R}_+}$ and a semi-flow $(\widetilde{g}_t)_{t\geq 0}$ by translation along the third coordinate. We then do the same construction as above, but with the equivalence relation generated by $(x, q, t + r(x)) \sim (T_A(x), q + F(x), t)$ for all points x where r is finite and all non-negative t. We will use the identification $\Omega \simeq A \times \mathbb{Z}^d \times [0, r)$.

On a $\mathbb{Z}^d$-extension of a suspension flow with base (A, μ_A, T_A) we use the afore-mentioned identification to define an origin by $A_0 := A \times \{0\} \times \{0\} \simeq A$. For any point $x \in A$ we define the first return time φ in A_0 by:

$$\varphi(x) := \inf\{t > 0 : \ g_t(x, 0, 0) \in A_0\},$$

and the local time at the origin by putting, for all $t \geq 0$:

$$\xi_t(x) := \mathrm{Card}\{s \in (0, t] : \ g_s(x, 0, 0) \in A_0\}.$$

By Birkhoff's ergodic theorem, the local time is finite for all positive times μ_A-almost everywhere.

The goal of this article is to get quantitative results on the recurrence properties of the semi-flow to the origin. We aim to answer two questions: What do the tails of the first return time look like? What is the asymptotic distributional behavior of the local time?

One possible way to prove limit theorems for these systems is to look up the extensive literature for discrete time systems (which, in our setting, is equivalent to taking $r \equiv 1$), and from there deduce limit theorems for semi-flows. For instance, [2] gives a local central limit theorem for $\mathbb{Z}^d$-extensions of Gibbs-Markov maps, and the series of articles [18] [19] [20] [8] give the local central limit theorem for planar Lorentz gases with finite and infinite horizon respectively. Some of these results can be transposed to continuous time systems.

The problem with this method is twofold. First, it is inelegant: most of the algebra which works with discrete time systems can be transposed to semi-flows, so that the passage *via* discrete time systems can look like an unnecessary step. Secondly, in order to go from discrete time systems to continuous time systems, we need to control the deviation between the sequence $(\sum_{k=0}^{n-1} r \circ T_A^k)_{n \geq 0}$ and the sequence of its averages $(n\mathbb{E}(r))_{n \geq 0}$, which leads to some quite artificial restrictions on the integrability of r. In particular, some arguments may fail when the step time r is not square integrable [21, Lemma 6.10], as is the case for infinite horizon Lorentz gases, and there is little hope for any such argument to succeed if r is not integrable. These restrictions can be bypassed by avoiding completely discrete time systems.

We use a combination of spectral methods and Tauberian theorems, akin to the methods used to count closed geodesics on periodic manifolds of negative curvature [14] [17] [16] [3] [4]; twisted transfer operators have already been used in different ways to prove statistical properties of semi-flows [7]. The result we get are new even in the case of $\mathbb{Z}^0$-extensions, that is, suspension semi-flows.

Our article is divided in two sections. Section 1 provides some background on spectral theory. Then, we work with abstract spectral assumptions on the transfer operator of the system (A, μ_A, T_A), and relate the tail of the first return time and the distributional limit of the local time to the spectral properties of perturbed transfer operators.

However, some heavy computations are needed to make use of the results of the first section. They are done in Section 2, where we do not work with general dynamical systems, but with Gibbs-Markov maps. We get some precise asymptotics of the perturbed main eigenvalue of the transfer operator, and then apply the results of the first section.

In addition, one appendix includes standard material on the theory of functions with regular variation. We also give a table which sums up the result of our work under varying assumptions on the dimension, the step function, and the step time.

In the remainder of this article, we use the notation $S_n f := \sum_{k=0}^{n-1} f \circ T^k$, and the notation $S_n^T f$ if there is some ambiguity on the transformation T we use.

1 Statistical properties of $\mathbb{Z}^d$-periodic semi-flows

The goal of this first section is to understand the tails of the first return time and the distributional asymptotics of the local time at the origin, under abstract conditions on the induced system. We shall also prove a renewal lemma which related the perturbed transfer operator of the induced system to the perturbed transfer operator of the system (A, μ_A, T_A).

We will mainly need two different kinds of tools. The first set is composed of general considerations of spectral theory, while the second set deals with regular variation and Tauberian theorems. The later set is standard and summed up in the appendix; we shall now introduce the former.

At some points we will make remarks about Gibbs-Markov maps; this class of dynamical systems will be defined later, in Section 2.

1.1 Spectral theory

Throughout this article, $\mathcal{B}$ will stand for a complex Banach space of functions defined on A which are defined μ_A-almost everywhere. Moreover, we assume that constant functions belong to $\mathcal{B}$, that $\mathcal{B} \subset \mathbb{L}^1$, and that $T_A : A \to A$ has at most countably many branches.

Definition of the transfer operator

We define the *Ruelle-Perron-Frobenius* or *transfer operator* $\mathcal{L}$ on $\mathbb{L}^1$ as the dual of the composition by T_A:

$$\int_A f \cdot g \circ T_A \, \mathrm{d}\mu_A = \int_A \mathcal{L}f \cdot g \, \mathrm{d}\mu_A \quad \forall f \in \mathbb{L}^1, \ \forall g \in \mathbb{L}^\infty.$$

Let g be the non-negative function defined almost everywhere by:

$$g := \frac{\mathrm{d}\mu_A}{\mathrm{d}\mu_A \circ T_A}.$$

Then, by the change of variable formula, for all f in $\mathbb{L}^1$, for almost every x in A:

$$\mathcal{L}f(x) = \sum_{y \in T_A^{-1}(\{x\})} g(y) f(y).$$

We will assume that $\mathcal{L}$ acts continuously on $\mathcal{B} \subset \mathbb{L}^1$. Since the measure μ_A is assumed to be invariant, these definitions yield $\mathcal{L}1 \equiv 1$, and 1 is an eigenvalue of $\mathcal{L}$ corresponding to constant eigenfunctions. Moreover, by Jensen's inequality, $\mathcal{L}$ is a contraction on any $\mathbb{L}^q$ spaces for q in $[1, \infty]$; since $\mathcal{B} \subset \mathbb{L}^1$, the transfer operator acting on $\mathcal{B}$ cannot have any eigenvalue of modulus larger than 1.

Perturbation of the transfer operator

The transfer operator can be modified in the following way. For any potential $\phi : A \to \mathbb{C}$ such that $\Re(\phi)$ is bounded from above, for any function f in $\mathcal{B}$, let:

$$\mathcal{L}_\phi f(x) := \sum_{y \in T_A^{-1}(\{x\})} e^{\phi(y)} g(y) f(y) = \mathcal{L}(e^\phi f)(x).$$

Let $\mathbb{T}^d \simeq (\mathbb{R}/2\pi\mathbb{Z})^d$ be the d-dimensional torus. Let $r : A \to \mathbb{R}_+$ and $F : A \to \mathbb{Z}^d$. In this article, we are especially interested in the two-parameters family of perturbed operators $\left(\mathcal{L}_{-sr+i\langle w, F\rangle}\right)_{s \geq 0, w \in \mathbb{T}^d}$, so we will use the shorthand:

$$\mathcal{L}_{s,w} := \mathcal{L}_{-sr+i\langle w, F\rangle}.$$

If it exists and is unique, we will denote by $\rho_{s,w}$ the eigenvalue of maximal modulus of $\mathcal{L}_{s,w}$. If in addition the eigenspace corresponding to $\rho_{s,w}$ is not included in the hyperplane of functions with zero average, then we denote by $f_{s,w}$ the unique eigenfunction of $\mathcal{L}_{s,w}$ corresponding to the eigenvalue $\rho_{s,w}$ and whose integral is 1.

If the essential spectral radius of $\mathcal{L}$ is strictly smaller than 1 and the dynamical system is weakly mixing, then 1 is an isolated eigenvalue of multiplicity 1 and every other eigenvalue has modulus strictly smaller than 1. In this case, by [13, Part IV.3.5], for any other operator $\mathcal{L}'$ close enough to $\mathcal{L}$ in operator norm, $\mathcal{L}'$ has a single eigenvalue of maximal modulus, which has multiplicity 1 and is close to 1.

For most of the article, it will be very convenient to assume that the family of operators $(\mathcal{L}_{s,w})$ is continuous for small values of s and w. Among others, it will imply that the main eigenvalue $\rho_{s,w}$ and the eigenfunction $f_{s,w}$ are well-defined and are continuous functions of s and w for small values of s and w [13, Part IV.3.5].

Hypothesis 1.1 (Continuity of the perturbation)**.**

There exist a neighborhood U of 0 in $\mathbb{R}_+$ and a neighborhood V of 0 in $\mathbb{T}^d$ such that the two parameters family of operators $(\mathcal{L}_{s,w})_{(s,w)\in U\times V}$ depends continuously (in the operator norm) on (s,w).

In addition, 1 is a simple, isolated eigenvalue of $\mathcal{L}$.

Induced system

Let us consider a $\mathbb{Z}^d$-extension of a suspension flow with base (A,μ_A,T_A), step time r and step function F, denoted by $(\Omega,\mu,(g_t)_{t\geq 0})$. Assume that this dynamical system is ergodic and conservative. Then the first return time to the origin φ is finite almost everywhere. Let us define $\overline{\varphi}$ as the first return time to the origin for the $\mathbb{Z}^d$-extension of a suspension flow with base (A,μ_A,T_A), step time 1 and step function F. The function $\overline{\varphi}$ counts the number of steps until the first return.

For μ_A-almost every x in A, let us define $\widetilde{T}x\in A$ by:

$$g_{\varphi(x)}(x,0,0)=(\widetilde{T}x,0,0).$$

Then $\widetilde{T}$ is an ergodic transformation which preserves the measure μ_A. Moreover, T_A^n has countably many branches for each $n\geq 0$, so that $\widetilde{T}$ also has countably many branches. The transfer operator $\widetilde{\mathcal{L}}$ of $\widetilde{T}$ is well-defined; for all $f\in\mathbb{L}^1$:

$$\widetilde{\mathcal{L}}f(x)=\sum_{y\in\widetilde{T}^{-1}(\{x\})}\widetilde{g}(y)f(y),$$

where:

$$\widetilde{g}(x)=\prod_{k=0}^{\overline{\varphi}(x)-1}g(T_A^kx)=e^{S_{\overline{\varphi}(x)}^{T_A}\ln g(x)}.$$

As we did above, we define a family of operators $(\widetilde{\mathcal{L}}_s)$ which act on $\mathbb{L}^1(A,\mu_A)$ by $\widetilde{\mathcal{L}}_sf=\widetilde{\mathcal{L}}(e^{-s\varphi}f)$. We warn the reader that while we assume that $\mathcal{L}$ acts nicely on the Banach space $\mathcal{B}$, that might not be the case of $\widetilde{\mathcal{L}}$.

With the notion of induced system, we can describe another set of hypotheses. While they may seem obscure for now, they will be very convenient later on; the doubtful reader may have a look at Lemma 1.8 to see how the expression involved may appear.

Hypothesis 1.2.

There exist a parameter $\beta\in[0,1]$ and a real-valued function G, defined on a neighborhood of 0, such that:

- $\lim_{0^+} G = +\infty$;
- G *has regular variation of index* $-\beta$ *at* 0;
- *for every function* f *in* $\mathcal{B}$:

$$\sum_{n=0}^{+\infty} \widetilde{\mathcal{L}}_s^n f = G(s) \left(\int_A f \, \mathrm{d}\mu_A + o(1) \, \|f\|_{\mathcal{B}} \right) \quad \textit{in } \mathcal{B} \textit{ when } s \textit{ goes to } 0,$$

where the $o(1)$ *term is negligible in* $\mathcal{B}$ *uniformly in* f.

The notion of "regular variation" is presented in the appendix.

The interest of Hypothesis 1.2 is double. First, it has interesting consequences: it is possible to compute the tail of the first return time (Proposition 1.3) and the distributional asymptotics of the local time (Lemma 1.5, Proposition 1.6 and Corollary 1.7) if this hypothesis is satisfied. Then, it is also possible to check that this hypothesis holds, and to compute a function G, if we understand the operators $\mathcal{L}_{s,w}$ well enough (Lemma 1.8) - which we will do for Gibbs-Markov maps in Section 2.

In the case of $\mathbb{Z}^0$-extensions, this hypothesis can be checked with only Lemma 2.3, which requires the continuity of $s \to \mathcal{L}_s$ and the density of $\mathcal{B}$ in $\mathbb{L}^1(A, \mu_A)$. The distributional asymptotics of the local time are then much easier to get.

Let us show that, under Hypothesis 1.1, for all small enough $s > 0$, for all non-negative $f \in \mathcal{B}$, almost everywhere,

$$\left(\mathrm{Id} - \widetilde{\mathcal{L}}_s\right) \sum_{n=0}^{+\infty} \widetilde{\mathcal{L}}_s^n f = f. \tag{1.1}$$

Since $\mathcal{B} \subset \mathbb{L}^1$, the expression $\sum_{n=0}^{+\infty} \widetilde{\mathcal{L}}_s^n f$ is the integrable sum of non-negative functions. Hence, the sequence of partial sums $(\sum_{k=0}^{n-1} \widetilde{\mathcal{L}}_s^k f)_{n \geq 0}$ converges in $\mathbb{L}^1$, and $\widetilde{\mathcal{L}}_s^n f$ converges to 0 in $\mathbb{L}^1$. Since the operator $\widetilde{\mathcal{L}}_s$ acts continuously on $\mathbb{L}^1$ for all $s \geq 0$, for all non-negative function $f \in \mathcal{B}$,

$$\begin{aligned}
\left(\mathrm{Id} - \widetilde{\mathcal{L}}_s\right) \sum_{n=0}^{+\infty} \widetilde{\mathcal{L}}_s^n f &= \left(\mathrm{Id} - \widetilde{\mathcal{L}}_s\right) \lim_{N \to +\infty} \sum_{n=0}^{N-1} \widetilde{\mathcal{L}}_s^n f \\
&= \lim_{N \to +\infty} \left(\mathrm{Id} - \widetilde{\mathcal{L}}_s\right) \sum_{n=0}^{N-1} \widetilde{\mathcal{L}}_s^n f \\
&= \lim_{N \to +\infty} \left(\mathrm{Id} - \widetilde{\mathcal{L}}_s^N\right) f \\
&= f,
\end{aligned}$$

where the limits are taken in $\mathbb{L}^1$.

1.2 First return times

Our first use of Hypothesis 1.2 is to get an equivalent for the tail of the first return time at the origin φ. We will succeed first by finding an equivalent of $\int_0^{+\infty} \mathbb{P}(\varphi > t) e^{-st} \, \mathrm{d}t$ when s goes to 0, and then by using a Tauberian theorem.

Proposition 1.3 (Tails for the first return time)**.**
Under Hypothesis 1.2, assume that $\mathcal{B} \subset \mathbb{L}^\infty(A, \mu_A)$ *and that* $\beta \in [0, 1)$. *Then:*

$$\mathbb{P}_{\mu_A}(\varphi > t) \sim \frac{1}{\Gamma(1-\beta) G(1/t)} \quad \textit{as } t \textit{ goes to } +\infty. \tag{1.2}$$

Proof.

By Fubini's theorem, for all $s > 0$:

$$\int_0^{+\infty} \mathbb{P}_{\mu_A}(\varphi > t) e^{-st} \, \mathrm{d}t = \int_A \int_0^{\varphi(x)} e^{-st} \, \mathrm{d}t \, \mathrm{d}\mu_A(x) = \frac{1 - \int_A e^{-s\varphi} \, \mathrm{d}\mu_A}{s}.$$

Let β and G be as stated in Hypothesis 1.2. Then, by Equation (1.1):

$$\begin{aligned} 1 &= \int_A \left(\mathrm{Id} - \widetilde{\mathcal{L}}_s\right) \sum_{n=0}^{+\infty} \widetilde{\mathcal{L}}_s^n 1 \, \mathrm{d}\mu_A \\ &= \int_A \left(\mathrm{Id} - \widetilde{\mathcal{L}}_s\right) G(s)(1 + o(1)) \, \mathrm{d}\mu_A \\ &= G(s) \int_A (1 - e^{-s\varphi})(1 + o(1)) \, \mathrm{d}\mu_A. \end{aligned} \tag{1.3}$$

Since the $\mathcal{B}$ norm controls the supremum norm, the function $o(1)$ decays uniformly, and:

$$\left| \int_A (1 - e^{-s\varphi}) o(1) \, \mathrm{d}\mu_A \right| \leq \int_A |1 - e^{-s\varphi}| \, \mathrm{d}\mu_A \, \|o(1)\|_{\mathbb{L}^\infty} = \int_A 1 - e^{-s\varphi} \, \mathrm{d}\mu_A \cdot o(1).$$

Inserting this inequality into Equation (1.3), we finally get:

$$1 = G(s) \int_A 1 - e^{-s\varphi} \, \mathrm{d}\mu_A (1 + o(1)),$$

so that:

$$\int_0^{+\infty} \mathbb{P}(\varphi > t) e^{-st} \, \mathrm{d}t \sim \frac{1}{sG(s)}.$$

The function G has regular variation of index $-\beta \in (-1, 0]$ at 0, so the function $s \to (sG(s))^{-1}$ has regular variation of index $\beta - 1 < 0$ at 0. By Karamata's Tauberian theorem (Theorem A.2):

$$\mathbb{P}(\varphi > t) \sim \frac{1}{\Gamma(1-\beta) G(1/t)}.$$

□

1.3 Local time

Our second application consists in distributional asymptotics for the local time ξ_t. Once suitably renormalized, the law of the local time will converge to particular cases of Mittag-Leffler distributions, which we define now.

Definition 1.4 (Mittag-Leffler distribution)**.**

Let β be in $[0, 1]$. A real-valued, nonnegative random variable Y_β is said to have a normalized Mittag-Leffler distribution of order β if, for all z in $\mathbb{C}$ (or all z in the open unit disc of $\mathbb{C}$ if $\beta = 0$):

$$\mathbb{E}(e^{zY_\beta}) = \sum_{n=0}^{+\infty} \frac{\Gamma(1+\beta)^n}{\Gamma(1+n\beta)} z^n. \tag{1.4}$$

The definition is such that $\mathbb{E}(Y_\beta) = 1$ whenever Y_β has a normalized distribution. Mittag-Leffler distributions arise very naturally when one deals with the local time of stochastic processes, such as random walks or Brownian motion, and some classes of dynamical systems endowed with an infinite measure [1, Corollary 3.7.3].

We will now compute asymptotics for the moments of ξ_t, which will in turn reveal its limit distribution.

Lemma 1.5 (Moments of the local time)**.**

Assume that Hypothesis 1.2 hold. Let p be a non-negative integer. Let ν be a probability measure on A_0, which is absolutely continuous with respect to μ_A and whose density lies in $\mathcal{B}$. Then:

$$\mathbb{E}_\nu(\xi_t^p) \sim \frac{p! G(1/t)^p}{\Gamma(1+p\beta)} \quad \textit{as } t \textit{ goes to } +\infty. \tag{1.5}$$

Proof.

We put $h := \mathrm{d}\nu/\mathrm{d}\mu_A$. For $p = 0$, the result is trivial, so we assume that p is positive. We want to get an equivalent when s vanishes of:

$$\int_0^{+\infty} e^{-st} \, \mathrm{d}\mathbb{E}_\nu(\xi_t^p) = \int_A \int_0^{+\infty} e^{-st} \, \mathrm{d}\xi_t^p(x) h(x) \, \mathrm{d}\mu_A(x).$$

Indeed, with the help of Karamata's Tauberian theorem [6, Theorem 1.7.1], this will give the desired asymptotics for $\mathbb{E}_\nu(\xi_t^p)$.

At its nth return, the function ξ_t^p increases from $(n-1)^p$ to n^p. Hence, for any x in A,

$$\int_0^{+\infty} e^{-st} \, \mathrm{d}\xi_t^p(x) = \sum_{n=1}^{+\infty} (n^p - (n-1)^p) e^{-sS_n^{\widetilde{T}}\varphi(x)}.$$

With a touch of Fubini-Tonelli theorem, we force the transfer operators to appear:

$$\begin{aligned}
\int_0^{+\infty} e^{-st} \, \mathrm{d}\mathbb{E}_\nu(\xi_t^p) &= \sum_{n=1}^{+\infty} (n^p - (n-1)^p) \int_A e^{-sS_n^{\widetilde{T}}\varphi(x)} h(x) \, \mathrm{d}\mu_A(x) \\
&= \sum_{n=1}^{+\infty} (n^p - (n-1)^p) \int_A \widetilde{\mathcal{L}}^n e^{-sS_n^{\widetilde{T}}\varphi} h(x) \, \mathrm{d}\mu_A(x) \\
&= \sum_{n=1}^{+\infty} (n^p - (n-1)^p) \int_A \sum_{y:\ \widetilde{T}^n y = x} \widetilde{g}^{(n)}(y) e^{-sS_n^{\widetilde{T}}\varphi(y)} h(y) \, \mathrm{d}\mu_A(x) \\
&= \sum_{n=1}^{+\infty} (n^p - (n-1)^p) \int_A \widetilde{\mathcal{L}}_s^n h \, \mathrm{d}\mu_A.
\end{aligned}$$

Some basic manipulations on Taylor series yield:

$$\begin{aligned}
\sum_{n=1}^{+\infty} (n^p - (n-1)^p) X^n &= (1-X) \sum_{n=1}^{+\infty} n^p X^n \\
&= \sum_{k=0}^{p} \frac{A(p,k)}{(1-X)^k},
\end{aligned}$$

where the $A(p,k)$ are integers such that $A(p,p) = p!$ (they are related to Eulerian numbers). Thus,

$$\int_0^{+\infty} e^{-st} \, \mathrm{d}\mathbb{E}_\nu(\xi_t^p) = \int_A \sum_{k=0}^{p} A(p,k) \left(\sum_{n=0}^{+\infty} \widetilde{\mathcal{L}}_s^n \right)^k h \, \mathrm{d}\mu_A.$$

Hypothesis 1.2 implies that $\left(\sum_{n=0}^{+\infty} \widetilde{\mathcal{L}}_s^n \right)^k h = G(s)^k (1 + o(1))$. Since $A(p,p) = p!$, the leading term in this expression is $p! G(s)^p$:

$$\int_0^{+\infty} e^{-st} \, \mathrm{d}\mathbb{E}_\nu(\xi_t^p) \sim p! G(s)^p \quad \text{as } s \text{ goes to } 0.$$

The function G^p has regular variation of index $-\beta p \in (-p, 0]$. We conclude by using Karamata's Tauberian theorem (Theorem A.2). $\square$

Given the limit of the moments, we deduce the asymptotic distribution of the local time.

Proposition 1.6 (Asymptotics of the local time)**.**
Assume that Hypothesis 1.2 hold. Let p be a non-negative integer. Let ν be a probability measure on A_0, which is absolutely continuous with respect to μ_A and whose density lies in $\mathcal{B}$. Then, under the distribution ν:

$$\frac{\Gamma(1+\beta)\xi_t}{G(1/t)} \to Y_\beta \quad \textit{as } t \textit{ goes to } +\infty, \tag{1.6}$$

where the convergence is in distribution and Y_β has a normalized Mittag-Leffler distribution of order β.

Proof.
The characteristic function for the normalized Mittag-Leffler distributions has a positive radius of convergence at 0. Hence, Mittag-Leffler distributions are characterized by their moments. By [5, Theorem 30.2], if a stochastic process $(X_t)_{t\geq 0}$ is such that all the moments of X_t exist and converge to the corresponding moments of a random variable with a Mittag-Leffler distribution Y_β, then X_t converges in distribution to Y_β.

Let p be a non-negative integer. By Lemma 1.5,

$$\mathbb{E}_\nu\left(\left(\frac{\Gamma(1+\beta)\xi_t}{G(1/t)}\right)^p\right) \to \frac{p!\Gamma(1+\beta)^p}{\Gamma(1+p\beta)} = \mathbb{E}(Y_\beta^p) \text{ as } t \text{ goes to } +\infty,$$

where Y_β has a normalized Mittag-Leffler distribution of order β. $\square$

Proposition 1.6 can be strengthened to account for more diverse probability measures and observables.

Corollary 1.7 (Asymptotics of the Birkhoff sums)**.**
Assume that Hypothesis 1.2 holds, that $\mathcal{B}$ is dense in $\mathbb{L}^1(A, \mu_A)$, and that the $\mathbb{Z}^d$-periodic semi-flow $(\Omega, \mu, (g_t))$ with base (A, μ_A, T_A), step time r and step function F is ergodic and conservative.

Then, for any function $f \in \mathbb{L}^1(\Omega, \mu)$, for any probability measure $\nu \ll \mu$,

$$\frac{\Gamma(1+\beta)}{G(1/t)} \int_0^t f \circ g_s \,\mathrm{d}s \to \int_\Omega f \,\mathrm{d}\mu \cdot Y_\beta \quad \textit{as } t \textit{ goes to } +\infty, \tag{1.7}$$

where the convergence is in distribution when the starting point is chosen with the probability measure ν, and Y_β has a normalized Mittag-Leffler distribution of order β.

Proof.
Let ν be a probability measure on A_0, with $\nu \ll \mu_A$ and $\mathrm{d}\nu/\mathrm{d}\mu_A \in \mathcal{B}$. Then, by Proposition 1.6,

$$\frac{\Gamma(1+\beta)}{G(1/t)}\xi_t \to Y_\beta \text{ as } t \text{ goes to } +\infty.$$

Since $\mathcal{B}$ is dense in $\mathbb{L}^1(A, \mu_A)$, the convergence above holds for all probability measures $\nu \ll \mu$.

Let ν be a probability measure on Ω, with $\nu \ll \mu$. Let $h := \mathrm{d}\nu/\mathrm{d}\mu$. The $\mathbb{Z}^d$-periodic semi-flow is a factor of a suspension semiflow: there is a measure-preserving map $\Pi : (\Omega', \mu') \to (\Omega, \mu)$, where Ω' is the suspension semiflow of base $(A_0, \mu_A, \widetilde{T})$ and height function φ. We can define the pullback $\Pi^*\nu$ of ν as the probability measure on Ω' whose density with respect to μ' is Π^*h.

Let $p^- : \Omega' \to A_0$ be the projection on the first coordinate, and $p^+ := \widetilde{T} \circ p^-$. We define two probability measure ν^- and ν^+ on A_0 by $\nu^- := p^-_* \Pi^* \nu$ and $\nu^- := p^+_* \Pi^* \nu$. By Fubini's theorem, these probability measures are absolutely continuous with respect to μ_A. We can see ν^- and ν^+ as probability measures on Ω.

The measures ν, ν^- and ν^+ are naturally coupled. If $\mathcal{Y}$ is a random variable with values in Ω' and with distribution $\Pi^*\nu$, then $(\Pi(\mathcal{Y}), p_-(\mathcal{Y}), p_+(\mathcal{Y}))$ is a realization of this coupling. Let ξ_t, ξ_t^- and ξ_t^+ be the random variable $\xi_t(y)$ when y is chosen under ν, ν^- and ν_+ respectively. Then ξ_t, ξ_t^- and ξ_t^+ are also coupled. In addition, this coupling satisfies:

$$\xi_t^- \leq \xi_t \leq \xi_t^+ + 1. \tag{1.8}$$

The random variable ξ_t^- is the local time under the distribution $\nu^- \ll \mu_A$ and ξ_t^+ is the local time under the distribution $\nu^+ \ll \mu_A$. But we have shown that, once renormalized by $\Gamma(1+\beta)/G(1/t)$, both converge in distribution to the same random variable Y_β. All we have to apply is a stochastic version of the squeeze theorem. The distribution functions of $\Gamma(1+\beta)\xi_t^-/G(1/t)$ and $\Gamma(1+\beta)\xi_t^+/G(1/t)$ both converge pointwise to the distribution function of Y_β at any continuity point of the later. By Equation (1.8), the distribution function of $\Gamma(1+\beta)\xi_t/G(1/t)$ is in-between, so it also converges pointwise to the distribution function of Y_β at any continuity point of the later. Hence, $\Gamma(1+\beta)\xi_t/G(1/t)$ converges in distribution to Y_β.

We have shown the convergence in distribution of the local time for any absolutely continuous starting distribution. We still need to extend the convergence to the Birkhoff integrals of any integrable function. Note that $\Omega_0 := \{(x, 0, t) : \ x \in A_0, \ t \in [0, r(x))\}$ is a well-defined subset of Ω. For all $y \in \Omega_0$, there is a unique pair $(x(y), t(y))$ such that $y = (x(y), 0, t(y))$ and $t(y) \in [0, r(x(y)))$. We define:

$$f_0 : \left\{ \begin{array}{lcl} \Omega & \to & \mathbb{R}_+; \\ y & \mapsto & 0 \text{ if } y \notin \Omega_0, \text{ otherwise } \frac{1}{r(x(y))}. \end{array} \right.$$

Then, for all $t \geq 0$ and almost all $y \in \Omega$,

$$\xi_t(y) - 1 \leq \int_0^t f_0 \circ g_s(y) \, \mathrm{d}s \leq \xi_t(y) + 1.$$

Hence, for any probability distribution $\nu \ll \mu$, if y is chosen under the distribution ν,

$$\frac{\Gamma(1+\beta)}{G(1/t)} \int_0^t f_0 \circ g_s(y) \, \mathrm{d}s \to Y_\beta \ \text{ as } t \text{ goes to } +\infty. \tag{1.9}$$

Hopf's ergodic theorem ([11, IV.14], individueller Ergodensatz bei Strömungen) is used to conclude. $\square$

The convergence in distribution to a given random variable for any starting probability distribution is called "strong convergence in distribution". The idea of having a result valid for all $f \in \mathbb{L}^1(\Omega, \mu)$ instead of only the local time is the same as the idea behind Hopf's ergodic theorem: if a limit in distribution holds for some integrable function with non-zero integral, then it holds for all integrable functions.

1.4 A renewal equation

The gist of this article is that we can control $\widetilde{\mathcal{L}}_s$ for small values of s if we know the behavior of $\mathcal{L}_{s,w}$ for small values of s and w. The main tool is the following lemma, which we state under fairly general hypotheses.

Lemma 1.8 (Renewal equation)**.**

Assume that the spectral radius of $\mathcal{L}_{s,0}$ is strictly smaller than 1 for all positive s close enough to 0. Then, for all positive s close enough to 0, for every non-negative function f in $\mathcal{B}$:

$$\sum_{n=0}^{+\infty} \widetilde{\mathcal{L}}_s^n f = \frac{1}{(2\pi)^d} \int_{\mathbb{T}^d} \sum_{k=0}^{+\infty} \mathcal{L}_{s,w}^k f \, \mathrm{d}w, \tag{1.10}$$

where the function series converge in $\mathbb{L}^1(A, \mu_A)$.

This equation has been known - with different formulations, and for different operators - since at least the work of M. Pollicott and R. Sharp [16], and has been used in subsequent articles [4] [3]. It still holds if one replaces the transfer operators by the Ruelle operators, without the Jacobian; it is then related to the "twisted Poincaré series".

Proof.

Let f be a non-negative function in $\mathcal{B}$. The function $\sum_{n=0}^{+\infty} \widetilde{\mathcal{L}}_s^n f$ is well-defined on A, although it can take the value $+\infty$. The core of Equation (1.10) is the fact that $\varphi = S_n^{T_A} r$, when n is the number of steps it takes for the flow to return to $A \times \{0\}$, and that $\ln \widetilde{g} = S_n^{T_A} \ln g$ with the same n. Hence, for any x in A:

$$\begin{aligned}
\sum_{n=0}^{+\infty} \widetilde{\mathcal{L}}_s^n f(x) &= \sum_{n=0}^{+\infty} \sum_{\{y:\ \widetilde{T}^n(y)=x\}} e^{-s S_n^{\widetilde{T}} \varphi(y)} e^{S_n^{\widetilde{T}} \ln \widetilde{g}(y)} f(y) \\
&= \sum_{k=0}^{+\infty} \sum_{\{y:\ T_A^k(y)=x\}} e^{-s S_k^{T_A} r(y)} e^{S_k^{T_A} \ln g(y)} 1_{\{0\}}(S_k^{T_A} F(y)) f(y).
\end{aligned}$$

The next step is to use Fourier analysis, and more specifically the Fourier inversion formula, to get rid of the function $1_{\{0\}}$. Note that $\widehat{1_{\{0\}}} \equiv 1$ on $\mathbb{T}^d$.

$$\begin{aligned}
\sum_{n=0}^{+\infty} \widetilde{\mathcal{L}}_s^n f(x) &= \frac{1}{(2\pi)^d} \sum_{k=0}^{+\infty} \sum_{\{y:\ T_A^k(y)=x\}} \int_{\mathbb{T}^d} e^{-s S_k^{T_A} r(y)} e^{i\langle w, S_k^{T_A} F(y)\rangle} e^{S_k^{T_A} \ln g(y)} f(y) \, \mathrm{d}w \\
&= \frac{1}{(2\pi)^d} \int_{\mathbb{T}^d} \sum_{k=0}^{+\infty} \mathcal{L}_{s,w}^k f(x) \, \mathrm{d}w.
\end{aligned}$$

The inversion of the sum and the integral is an application of Fubini-Lebesgue Theorem. Indeed:

$$\frac{1}{(2\pi)^d} \sum_{k=0}^{+\infty} \sum_{\{y:\ T_A^k(y)=x\}} \int_{\mathbb{T}^d} \left| e^{-s S_k^{T_A} r(y)} e^{i\langle w, S_k^{T_A} F(y)\rangle} e^{S_k^{T_A} \ln g(y)} f(y) \right| \mathrm{d}w = \sum_{k=0}^{+\infty} \mathcal{L}_{s,0}^k f(x). \tag{1.11}$$

Using the assumption that the spectral radius of $\mathcal{L}_{s,0}$ is smaller than 1, the sequence $\left\| \mathcal{L}_{s,0}^k \right\|_{\mathcal{B} \to \mathcal{B}}$ decays at an exponential speed and the right-hand side is well defined for almost every x and in $\mathbb{L}^1$. □

Remark 1.9 (Renewal equation for other potentials)**.**

Let h be a measurable, complex-valued, bounded function on A. Then one can define other Perron-Frobenius operators by:

$$\mathcal{L}_{h,s,w} f(x) = \sum_{y \in T_A^{-1}(\{x\})} e^{h(y) - s r(y) + i\langle w, F\rangle(y)} f(y).$$

If $h = \ln g$, this is the transfer operator. Let $\widetilde{h}(x) := S_{\widetilde{\varphi}(x)}^{T_A} h(x)$ for almost all $x \in A$, and let:

$$\widetilde{\mathcal{L}}_{\widetilde{h},s} f(x) = \sum_{y \in \widetilde{T}^{-1}(\{x\})} e^{\widetilde{h}(y) - s\varphi(y)} f(y).$$

Under suitable assumptions, some variant of the renewal equation above is still valid, and relates $\widetilde{\mathcal{L}}_{\widetilde{h},s}$ and $\mathcal{L}_{h,s,w}$.

For any given $u \in [0,1]$, one interesting case is when $h = u \ln g$. Then $\widetilde{h} = u \ln \widetilde{g}$. The articles [16], [4] and [3] use these operators with $u = 0$ to count closed geodesics on periodic manifolds of negative curvature; in this article, we are taking $u = 1$.

Remark 1.10 (Use of the renewal equation)**.**

Informally, Equation (1.10) *can be read as:*

$$\left(\mathrm{Id} - \widetilde{\mathcal{L}}_s\right)^{-1} = \frac{1}{(2\pi)^d} \int_{\mathbb{T}^d} (\mathrm{Id} - \mathcal{L}_{s,w})^{-1} \, \mathrm{d}w, \tag{1.12}$$

so that:

$$\frac{1}{1 - \widetilde{\rho}_s} \sim \frac{1}{(2\pi)^d} \int_{\mathbb{T}^d} \frac{1}{1 - \rho_{s,w}} \, \mathrm{d}w, \tag{1.13}$$

where the growth of the integral as s goes to zero is due to small values of w. Thus, informally, Equation (1.10) *relates the behavior of the main eigenvalue $\widetilde{\rho}_s$ (which may not be defined) of $\widetilde{\mathcal{L}}_s$ for s close to zero to the behavior of $\rho_{s,w}$ for s and w close to zero.*

2 Computations for Gibbs-Markov maps

To go further, we work with systems for which the Hypothesis 1.1 is satisfied. We choose to work with Gibbs-Markov maps, whose transfer operators have a very nice behavior. In addition, Gibbs-Markov maps are symbolic, and we can work explicitly with this encoding.

2.1 Gibbs-Markov maps

For one of our core lemmas, Lemma 2.4, we shall need very precise assumptions on the behavior of the perturbed transfer operators. Instead of presenting a page-long list of hypotheses, we choose to state this lemma, and its consequences, in the context of Gibbs-Markov transformations. We recall the definition of these transformations and some properties of their transfer operators. In short, they are Markov maps with bounded distortion and the big image property. This class of transformations includes the subshifts of finite type together with a Gibbs measure.

Definition 2.1 (Gibbs-Markov maps)**.**

Let (A, d) be a measurable, metric, bounded Polish space, endowed with a probability measure μ_A. A non-singular, measurable map $T_A : A \mapsto A$ is said to be a Markov map if μ_A is T_A-invariant and if there exists a countable partition π of A in sets of positive measure such that:

- *for all a in π, the image of a by T_A is a union of elements of π (up to a set of null measure);*
- *for all a in π, the map T_A is an isomorphism from a onto its image;*
- *the completion for μ_A of the σ-algebra $\bigvee_{n \in \mathbb{N}} T_A^{-n} \pi$ is the Borel σ-algebra on A.*

The full data defining a Markov map is $(A, \pi, d_A, \mu_A, T_A)$, but we shall often omit some of the objects if there is no ambiguity. A Markov map is said to be Gibbs-Markov if it also has the following properties:

- $\inf_{a \in \pi} \mu(T_A a) > 0$ *(large image property);*
- *it is locally uniformly expanding: there exists $\lambda > 1$ such that, for all a in π and x, y in a, we have $d_A(T_A x, T_A y) \geq \lambda d(x, y)$;*
- *it has a Lipschitz distortion: there exists a constant C such that, for all a in π, for almost every x and y in a:*
$$|g(x) - g(y)| \leq C d_A(T_A x, T_A y) g(x). \tag{2.1}$$

A Gibbs-Markov map is said to be mixing if, for any Borel sets B and C,

$$\lim_{n \to +\infty} \mu_A(B \cap T_A^{-n} C) = \mu_A(B) \mu_A(C).$$

For any points x and y, let us denote by $s(x, y)$ the time of separation of x and y for the partition π and the transformation T_A, i.e., the smallest time $n \geq 0$ at which the points $T_A^n x$ and $T_A^n y$ do not belong to the same element of the partition π. Then, for any $\kappa > 1$, one can define a metric d_κ on A by $d_\kappa(x, y) := \kappa^{-s(x,y)}$. The dynamical system $(A, d_\kappa, \mu_A, T_A)$ is also Gibbs-Markov if κ belongs to $(1, \lambda]$. With this canonical choice of a distance, a Gibbs-Markov map is entirely defined by the data $(A, \pi, \kappa, T_A, \mu_A)$. For all $\theta \in (0, 1]$, a θ-Hölder function for the initial metric d is Lipschitz for the metric $d_\lambda^\theta = d_{\lambda^\theta}$. Hence, any result stated for Lipschitz functions actually holds for Hölder functions.

Banach spaces

For any subset ω of A, we denote by $|\cdot|_{\mathrm{Lip}_d(\omega)}$ the Lipschitz semi-norm on ω: for any function from ω to a metric space (E, d'), it is defined by:

$$|f|_{\mathrm{Lip}_d(\omega)} := \inf \left\{ C > 0 : \ \forall x \in \omega, \ \forall y \in \omega, \ d'(f(x), f(y)) \leq C d(x, y) \right\}.$$

If there is no ambiguity on the metric, we may denote the semi-norm by $|\cdot|_{\mathrm{Lip}(\omega)}$.

Let Lip_d^∞ be the set of functions f from A to $\mathbb{C}$ such that $\|f\|_{\mathrm{Lip}_d^\infty} := \|f\|_{\mathbb{L}^\infty} + \sup_{a \in \pi} |f|_{\mathrm{Lip}_d(a)}$ is finite. This is a Banach space, on which the transfer operator acts continuously. If the Gibbs-Markov map is mixing, the transfer operator acting on Lip_d^∞ has a spectral gap: it is quasicompact, 1 is the unique eigenvalue of modulus 1 and is simple, and the modulus of any other eigenvalue is less than 1.

A spectral lemma for Gibbs-Markov maps

For any $\kappa > 1$, let $|\cdot|_{\mathrm{Lip}_\kappa(a)}$ be the Lipschitz semi-norm on a for the distance d_κ, and let $\mathrm{Lip}_\kappa^\infty$ be the space Lip^∞ for the distance d_κ on A.

We will be interested in the following property, which is a generalization of [9, Lemma 3.5] and [9, Corollary 3.6]:

Lemma 2.2.

Let $(A, \pi, \lambda, \mu_A, T_A)$ be a mixing Gibbs-Markov map. Let d be a non-negative integer. Let r and F be measurable functions on A, such that r takes its values in $\mathbb{R}_+$ and F in $\mathbb{R}^d$. Assume that $\sum_{a \in \pi} \mu_A(a) |r|_{\mathrm{Lip}(a)}$ is finite, that F is $\sigma(\pi)$-measurable, and that $F \in \mathbb{L}^q(A, \mu_A)$ for some $q > 1$. Then there exist $\kappa \in (1, \lambda]$ and a constant $C > 0$ such that:

- *The two parameters family of operators $(\mathcal{L}_{s,w})_{(s,w) \in \mathbb{R}_+ \times \mathbb{R}^d}$ is continuous for the operator topology when acting on $\mathrm{Lip}_\kappa^\infty$;*

- *For all $s \geq 0$,*

$$\|\mathcal{L} - \mathcal{L}_{s,0}\|_{\mathrm{Lip}_\kappa^\infty \to \mathrm{Lip}_\kappa^\infty} \leq C \int_A 1 - e^{-sr} \, \mathrm{d}\mu_A;$$

- *For all small enough s and all $w \in \mathbb{T}^d$,*

$$\|\mathcal{L}_{s,w} - \mathcal{L}_{s,0}\|_{\mathrm{Lip}_\kappa^\infty \to \mathrm{Lip}_\kappa^\infty} \leq C \|w\| .$$

Proof.

The first point is true if we assume that $\sum_{a\in\pi} \mu_A(a) |r|^2_{\mathrm{Lip}(a)}$ is finite, as per the remark after Lemma 4.1.2 in [10]. However, according to the same remark, the condition $\sum_{a\in\pi} \mu_A(a) |r|_{\mathrm{Lip}(a)} < +\infty$ is sufficient if we take $\kappa \leq \sqrt{\lambda}$.

The second point can be shown with small modifications in the proofs of Lemma 4.1.1 and Lemma 4.1.2 in [10]. We actually get:

$$\|\mathcal{L} - \mathcal{L}_{s,0}\|_{\mathrm{Lip}_\kappa^\infty \to \mathrm{Lip}_\kappa^\infty} \leq C \max \left\{ s, \int_A 1 - e^{-sr} \, \mathrm{d}\mu_A \right\},$$

but if r is not equal to 0 then $s = O(\int_A 1 - e^{-sr} \, \mathrm{d}\mu_A)$.

The proof of the third point is more involved. Let $\eta \in (0,1]$. In the following, we take $\kappa := \lambda^\eta$. For every $(s,w) \in \mathbb{R}_+ \times \mathbb{T}^d$ and $a \in \pi$, let us define a function $\psi_{s,w}$ on A by $\psi_{s,w}(x) = e^{-sr(y)+i\langle w,F\rangle(y)}$ whenever there exists $y \in a$ such that $T_A y = x$, and $\psi_{s,w}(x) = 0$ otherwise. Let us also define an operator $M_{a,s,w}$ on $\mathrm{Lip}_\kappa^\infty$ by:

$$M_{a,s,w} f(x) := \sum_{y:\ T_A y = x} 1_a(y) e^{-sr(y)+i\langle w,F\rangle(y)} g(y) f(y).$$

Let us fix $s \geq 0$. By the proof of Lemma 4.1.2 in [10], there exists a constant C independent from a and such that, for all $w \in \mathbb{T}^d$:

$$\|M_{a,s,w} - M_{a,s,0}\|_{\mathrm{Lip}_\kappa^\infty \to \mathrm{Lip}_\kappa^\infty} \leq C \mu_A(a) \|\psi_{s,w} - \psi_{s,0}\|_{\mathrm{Lip}_\kappa^\infty} .$$

Let x and x' be in the same cylinder $a' \in \pi$. A Gibbs-Markov map is onto from a cylinder to a union of cylinders, so that either both x and x' have pre-images in a, or neither do. Assume that both have a pre-image (the other case is trivial), and let us denote them by y and y' respectively. Since F is assumed to be $\sigma(\pi)$-measurable, $F(y) = F(y')$, whence:

$$\begin{aligned}
|\psi_{s,w}(x) \ & -\psi_{s,0}(x) - \psi_{s,w}(x') + \psi_{s,0}(x')| \\
&= \left| e^{-sr(y)+i\langle w,F\rangle(y)} - e^{-sr(y)+i\langle 0,F\rangle(y)} - e^{-sr(y')+i\langle w,F\rangle(y')} + e^{-sr(y')+i\langle 0,F\rangle(y')} \right| \\
&= \left| 1 - e^{i\langle w,F\rangle(y)} \right| \left| e^{-sr(y)} - e^{-sr(y')} \right| \\
&\leq C s^\eta |r(y) - r(y')|^\eta \frac{1}{\mu_A(a)} \int_a \left| 1 - e^{i\langle w,F\rangle} \right| \, \mathrm{d}\mu_A.
\end{aligned}$$

Which gives a upper bound on the Lipschitz semi-norm of $\psi_{s,w} - \psi_{s,0}$:

$$\sup_{a'\in\pi} |\psi_{s,w} - \psi_{s,0}|_{\mathrm{Lip}_\kappa(a')} \leq C s^\eta |r|^\eta_{\mathrm{Lip}_\kappa(a)} \frac{1}{\mu_A(a)} \int_a |\langle w,F\rangle| \, \mathrm{d}\mu_A.$$

It remains to estimate $\|\psi_{s,w}-\psi_{s,0}\|_{\mathbb{L}^\infty}$. But, for every $a' \in \pi$ and $y \in a$,

$$\|\psi_{s,w}-\psi_{s,0}\|_{\mathbb{L}^\infty(a')} \leq \left|1-e^{i\langle w,F\rangle(y)}\right| \leq \frac{1}{\mu_A(a)}\int_a |\langle w,F\rangle| \, \mathrm{d}\mu_A.$$

Finally, we get:

$$\|M_{a,s,w}-M_{a,s,0}\|_{\mathrm{Lip}_\kappa^\infty \to \mathrm{Lip}_\kappa^\infty} \leq C\left(1+s^\eta |r|_{\mathrm{Lip}_\kappa(a)}^\eta\right)\int_a |\langle w,F\rangle| \, \mathrm{d}\mu_A.$$

By taking the sum on all $a \in \pi$, this yields:

$$\begin{aligned}\|\mathcal{L}_{s,w}-\mathcal{L}_{s,0}\|_{\mathrm{Lip}_\kappa^\infty \to \mathrm{Lip}_\kappa^\infty} &\leq C\|w\|\sum_{a\in\pi}\left(1+s^\eta |r|_{\mathrm{Lip}_\kappa(a)}^\eta\right)\int_a \|F\| \, \mathrm{d}\mu_A \\ &= C\|w\|\left(\int_A \|F\| \, \mathrm{d}\mu_A + s^\eta \sum_{a\in\pi}|r|_{\mathrm{Lip}_\kappa(a)}^\eta \int_a \|F\| \, \mathrm{d}\mu_A\right).\end{aligned}$$

By Hölder's inequality,

$$\begin{aligned}&\sum_{a\in\pi}\mu_A(a)|r|_{\mathrm{Lip}_\kappa(a)}^\eta \frac{1}{\mu_A(a)}\int_a \|F\| \, \mathrm{d}\mu_A \\ &\leq \left(\sum_{a\in\pi}\mu_A(a)|r|_{\mathrm{Lip}_\kappa(a)}^{\frac{q\eta}{q-1}}\right)^{1-\frac{1}{q}}\left(\sum_{a\in\pi}\mu_A(a)\left(\frac{1}{\mu_A(a)}\int_a \|F\| \, \mathrm{d}\mu_A\right)^q\right)^{\frac{1}{q}} \\ &= \left(\sum_{a\in\pi}\mu_A(a)|r|_{\mathrm{Lip}_\kappa(a)}^{\frac{q\eta}{q-1}}\right)^{1-\frac{1}{q}} \|\|F\|\|_{\mathbb{L}^q}.\end{aligned}$$

By choosing $\eta = 1-1/q$, we ensure that the bound above is finite, so that $\|\mathcal{L}_{s,0}-\mathcal{L}_{s,w}\|_{\mathrm{Lip}_\kappa^\infty \to \mathrm{Lip}_\kappa^\infty} = O(\|w\|)$ uniformly on compact subsets of $\mathbb{R}_+$. □

Gibbs-Markov maps and induction

Gibbs-Markov maps also enjoy some nice properties when it comes to $\mathbb{Z}^d$-extensions. Let us take an ergodic Gibbs-Markov map (A,π,d_A,μ_A,T_A), and let F be $\sigma(\pi)$-measurable. By [1, Proposition 4.6.2], if the $\mathbb{Z}^d$-extension is ergodic and conservative, then the measure-preserving transformation $(A,\mu_A,\widetilde{T})$ can be endowed with a Gibbs-Markov structure. Hence, the operator $\widetilde{\mathcal{L}}$ acts nicely on some Banach space, which is in general strictly larger than $\mathrm{Lip}_{d_A}^\infty$: it is quasicompact, and 1 is a simple eigenvalue.

By [21, Lemma 6.6], if r is smooth for the Gibbs-Markov structure associated with T_A (i.e. $\sum_{a\in\pi}\mu_A(a)\,|r|_{\mathrm{Lip}(a)}$ is finite), then φ is smooth for the Gibbs-Markov structure associated with $\widetilde{T}$. By Lemma 2.2, this implies that the one parameter family of operators $(\widetilde{\mathcal{L}}_s)_{s\geq 0}$ is continuous. Hence, we can define a main eigenvalue $\widetilde{\rho}_s$ so that it is continuous for small values of s, and equal to 1 if $s=0$.

We need not use these additional properties of Gibbs-Markov maps, but they may make some of the algebra more intuitive; for instance, these properties allow us to make a rigorous use of the operators $(\mathrm{Id}-\widetilde{\mathcal{L}}_s)^{-1}$. This makes it possible to simplify, in this setting, the statements and the proofs of Lemma 1.8, Proposition 1.3 and Lemma 1.5. Assume that the base (A,μ_A,T_A) is Gibbs-Markov, the step function F is $\sigma(\pi)$-measurable, the $\mathbb{Z}^d$-extension is ergodic and conservative, and the function r is smooth. Then the Equations (1.12) and (1.13) are true, that is:

$$\left(\mathrm{Id}-\widetilde{\mathcal{L}}_s\right)^{-1} = \frac{1}{(2\pi)^d}\int_{\mathbb{T}^d}(\mathrm{Id}-\mathcal{L}_{s,w})^{-1} \, \mathrm{d}w,$$

and the renewal equation (1.13) gives the asymptotic behavior of $(1-\tilde{\rho}_s)^{-1}$ when s converges to 0.

The algebraic manipulations in the proof of Proposition 1.3 become:

$$\int_0^{+\infty} \mathbb{P}_{\mu_A}(\varphi > t)e^{-st} \, \mathrm{d}t = \frac{1}{s}\int_A \left(\mathrm{Id} - \widetilde{\mathcal{L}}_s\right) 1 \, \mathrm{d}\mu_A \sim \frac{1-\tilde{\rho}_s}{s}.$$

Finally, the algebraic manipulations in the proof of Lemma 1.5 become:

$$\int_0^{+\infty} e^{-st} \, \mathrm{d}\mathbb{E}_\nu(\xi_t^p) = \int_A \sum_{k=0}^{p} A(p,k) \left(\mathrm{Id} - \widetilde{\mathcal{L}}_s\right)^{-k} h \, \mathrm{d}\mu_A \sim p! \int_A \left(\mathrm{Id} - \widetilde{\mathcal{L}}_s\right)^{-p} h \, \mathrm{d}\mu_A \sim \frac{p!}{(1-\tilde{\rho}_s)^p}.$$

2.2 Integrability conditions on the step function and step time

We define functions H, I and P on a neighborhood of 0 in $\mathbb{R}_+$, $\mathbb{T}^d$ and $\mathbb{R}_+ \times \mathbb{T}^d$ respectively. The functions H and I are related to r and F respectively. These functions will be very convenient later on, as they appear naturally when one studies the behavior of the perturbed main eigenvalue $\rho_{s,w}$ for small values of s and w. They will also appear when we will give the result of our computations at the end of the article.

We will assume that the function r satisfies one of the three following assumptions:

- r is integrable. Then, we put $H(s) := s\int_A r \, \mathrm{d}\mu_A$;
- r is not integrable but $\mathbb{P}_{\mu_A}(r > t)$ has regular variation of index -1. Then, we put $H(s) := s\int_0^{1/s} \mathbb{P}_{\mu_A}(r > t) \, \mathrm{d}t$;
- the function $\mathbb{P}_{\mu_A}(r > t)$ has regular variation of index $-\beta \in (-1, 0]$. Then, we put $H(s) := \Gamma(1-\beta)\mathbb{P}_{\mu_A}(r > s^{-1})$.

In all three cases, the function H can be extended by continuity in 0 by $H(0) := 0$. This function is defined so that $H(s) \sim \mathbb{E}(1-e^{-sr}) = s\int_{\mathbb{R}_+} e^{-st}\mathbb{P}(r > t) \, \mathrm{d}t$ in 0.

We will also assume that the function F satisfies one of the two following assumptions:

- F is in $\mathbb{L}^2(A, \mu_A)$ and satisfies the assumptions of a Central Limit Theorem, i.e. $N^{-1/2}S_N F$ converges in distribution to a Gaussian random variable of non-degenerate covariance operator S. Then, $I(w) := \left\|\sqrt{S}w\right\|^2/2$, and $p := 2$;
- F is not in $\mathbb{L}^2(A, \mu_A)$, and there exist a non-negative function J with regular variation of order $p \in (1, 2]$ at 0 and an automorphism M of $\mathbb{R}^d$ such that $\mathbb{E}(1 - e^{i\langle w, F\rangle}) \sim J(\|Mw\|)$ at 0. Then, we put $I(w) := J(\|Mw\|)$.

Both assumptions imply not only that F is integrable and centered, but also that there exists a $q > 1$ such that $F \in \mathbb{L}^q$, and that F is in the basin of attraction of a symmetric Lévy stable distribution of index $p \in (1, 2]$. We shall not study the case of step functions in the basin of attraction of a Cauchy random variable (for which Lemma 2.2 does not apply), or in the basin of attraction of a non symmetric Lévy stable distribution (for which our method works equally well, but whose study would involve even heavier notations).

In dimension one, the second case is satisfied if F has regularly varying tails. More precisely, if the function F is not in $\mathbb{L}^2$, and $\mathbb{P}_{\mu_A}(\|F\| > x)$ has regular variation of index -2, then $I(w) := w^2/2 \cdot \int_A F^2 1_{\{-|w|^{-1}, |w|^{-1}\}}(F) \, \mathrm{d}\mu_A$ [12, Theorem 2.6.2]. If $\mathbb{P}_{\mu_A}(F > x)$ and $\mathbb{P}_{\mu_A}(F < -x)$ are equivalent and have regular variation of index $-\gamma \in (-2, -1)$, then $I(w) := \mathbb{P}_{\mu_A}(F > |w|^{-1})$ [12, Theorem 2.6.5].

In dimension one, we will always assume that $M = \mathrm{Id}$.

Let $m > 0$ be such that $1/J$ is well-defined and locally integrable on $(0, m)$. Then we will put $\widetilde{J}(x) = \int_x^m t/J(t)\,\mathrm{d}t$ for all $x \in (0, m)$. For instance, if $J(x) \sim x^2/2$, then $\widetilde{J}(x) \sim -2\ln(x)$ at 0. The function $\widetilde{J}$ will appear in computations involving $\mathbb{Z}^2$-periodic semi-flows.

Finally, we put $P(s, w) := H(s) + I(w)$ whenever this function is well defined. Under good joint integrability conditions on r and F, we shall see later (Lemma 2.4) that $\rho_{s,w} = 1 - P(s, w) + o(P(s, w)) \sim (1 - H(s))(1 - I(w)) + o(P(s, w))$ for small values of s and w. These joint integrability conditions ensure that, heuristically, the effects of the step time and step function are independent, so that $\rho_{s,w}$ can be approximated by the function $1 - P(s, w)$ which depends simply on s and w.

2.3 Perturbation of the main eigenvalue

As we have seen in the first section, we need a very precise control on the main eigenvalue of $\mathcal{L}_{s,w}$ to get a limit theorem for the local times of $\mathbb{Z}^d$-extensions of Gibbs-Markov semi-flows. This is achieved by three lemmas. The first lemma gives an asymptotic of the main eigenvalue of $\mathcal{L}_{s,0}$, under abstract assumptions on the system. The second lemma upgrades this to an asymptotic of the main eigenvalue of $\mathcal{L}_{s,w}$ for s and w close to 0, and the third lemma deals with values of w far from 0; for these, we will assume the dynamical system to be Gibbs-Markov.

Lemma 2.3.

Assume that Hypothesis 1.1 holds, that $\mathcal{B} \subset \mathbb{L}^\infty(A, \mu_A)$, and that r satisfies one of the hypotheses of Subsection 2.2. Then the function $s \to \rho_{s,0}$ is continuous on a neighborhood of 0 in $\mathbb{R}_+^*$, and:

$$1 - \rho_{s,0} \sim H(s). \tag{2.2}$$

If in addition the system (A, μ_A, T_A) can be given a Gibbs-Markov structure with respect to which $\sum_{a\in\pi} \mu_A(a)|r|_{\mathrm{Lip}(a)}$ is finite, then:

$$\|\mathcal{L} - \mathcal{L}_{s,0}\|_{\mathcal{B}\to\mathcal{B}} = O(H(s)), \tag{2.3}$$

and:

$$\|1 - f_{s,0}\|_{\mathcal{B}} = O(H(s)). \tag{2.4}$$

Proof.

We begin by proving Equation (2.2). Let s be close to 0. By definition,

$$\rho_{s,0} f_{s,0} = \mathcal{L}\left(e^{-sr} f_{s,0}\right)$$

which gives by integrating over A:

$$\rho_{s,0} = \int_A e^{-sr} f_{s,0}\,\mathrm{d}\mu_A.$$

The family of eigenfunctions $(f_{s,0})$ is continuous in the $\mathcal{B}$ norm [13, Theorem 3.16], which controls the $\mathbb{L}^\infty$ norm, so that:

$$1 - \rho_{s,0} = \int_A (1 - e^{-sr}) f_{s,0}\,\mathrm{d}\mu_A = \left(\int_A 1 - e^{-sr}\,\mathrm{d}\mu_A\right)(1 + o(1)).$$

Now, we rewrite the above expression:

$$\frac{1}{s}\int_A 1 - e^{-sr}\,\mathrm{d}\mu_A = \int_A \int_0^{r(x)} e^{-st}\,\mathrm{d}t\,\mathrm{d}\mu_A(x) = \int_0^{+\infty} e^{-st}\mathbb{P}_{\mu_A}(r > t)\,\mathrm{d}t.$$

If r is integrable, the integral converges monotonically to $\int_A r \, \mathrm{d}\mu_A$, and we are done. If $\mathbb{P}_{\mu_A}(r > t)$ has regular variation of index $-\beta \in (-1, 0]$, then we use the Tauberian Theorem A.2. If $\mathbb{P}_{\mu_A}(r > t)$ has regular variation of index -1, then we use the Tauberian Theorem A.3.

Since we have got an equivalent of $\int_A 1 - e^{-sr} \, \mathrm{d}\mu_A$, the Equations (2.3) and (2.4) follow by Lemma 2.2. $\square$

Now, we turn to an asymptotic development of the main eigenvalue $\rho_{s,w}$.

Lemma 2.4.
Let (A, π, d, μ_A, T) be a mixing Gibbs-Markov map. Let $d \geq 0$. Let F and r be such that F is $\sigma(\pi)$-measurable, $\sum_{a \in \pi} \mu_A(a) |r|_{\mathrm{Lip}(a)}$ is finite, and each of r and F satisfies one the assumptions of Subsection 2.2.

Assume that either $\mathbb{P}(\|F\| > t) = o(t^{-2})$, or $r \in \mathbb{L}^q$ for some $q > 1$. Then, for the family of operators $(\mathcal{L}_{s,w})$ acting on Lip^∞*:*

$$\rho_{s,w} = 1 - P(s,w) + o(P(s,w)). \tag{2.5}$$

Note that the condition $\mathbb{P}(\|F\| > t) = o(t^{-2})$ is always satisfied if F is in $\mathbb{L}^2$.

Proof.
The proof is split into four parts, with subparts corresponding to the two different cases in the lemma. The last two parts deal specifically with the case $F \in \mathbb{L}^2$.

Let us start from the following expression for the main eigenvalue of $\mathcal{L}_{s,w}$ on a neighborhood of 0:

$$\begin{aligned} \rho_{s,w} &= \int_A e^{-sr + i\langle w, F \rangle} f_{s,w} \, \mathrm{d}\mu_A \\ &= \int_A e^{-sr + i\langle w, F \rangle} \, \mathrm{d}\mu_A + \int_A \left(e^{-sr + i\langle w, F \rangle} - 1 \right) \left(f_{s,w} - 1 \right) \, \mathrm{d}\mu_A. \end{aligned} \tag{2.6}$$

We will use the notation:

$$Q(s,w) := \int_A e^{-sr + i\langle w, F \rangle} \, \mathrm{d}\mu_A.$$

In the first part of the proof, we compute asymptotics for the function Q.

Study of the function Q.

We decompose Q into more manageable parts:

$$Q(s,w) = \int_A e^{i\langle w, F \rangle} \, \mathrm{d}\mu_A + \int_A (e^{-sr} - 1) \, \mathrm{d}\mu_A + \int_A \left(e^{-sr} - 1 \right) \left(e^{i\langle w, F \rangle} - 1 \right) \, \mathrm{d}\mu_A. \tag{2.7}$$

Moreover, let $\overline{I} := I$ if F is not in $\mathbb{L}^2$, and $\overline{I}(w) = \int_A \langle w, F \rangle^2 \, \mathrm{d}\mu_A$ otherwise. Let $\overline{P}(s,w) := H(s) + \overline{I}(w)$.

We want to prove that the cross term in Equation (2.7) is negligible. This is where the non-independence of the hypotheses on r and on F appears. First, note that:

$$\left| \int_A \left(e^{-sr} - 1 \right) \left(e^{i\langle w, F \rangle} - 1 \right) \, \mathrm{d}\mu_A \right| \leq \|w\| \int_A \|F\| \left(1 - e^{-sr} \right) \, \mathrm{d}\mu_A.$$

Now let us split the proof in two parts, corresponding to the two sets of assumptions on r and F.

First case: Assume that $\mathbb{P}(\|F\| > t) = o(t^{-2})$. Let $\varepsilon > 0$. Let $m \geq 0$ be such that, for all $t > m$,

$$\mathbb{P}(\|F\| > t) \leq \frac{\varepsilon}{t^2}.$$

Let $t > m$. We use the upper bound:

$$\|F\| \left(1 - e^{-sr}\right) \leq t \left(1 - e^{-sr}\right) + (\|F\| - t) 1_{\|F\| \geq t},$$

whence:

$$\begin{aligned}
\int_A \|F\| \left(1 - e^{-sr}\right) \, \mathrm{d}\mu_A &\leq t \int_A \left(1 - e^{-sr}\right) \, \mathrm{d}\mu_A + \int_t^{+\infty} \mathbb{P}(\|F\| > s) \, \mathrm{d}s \\
&\leq t \int_A \left(1 - e^{-sr}\right) \, \mathrm{d}\mu_A + \frac{\varepsilon}{t}.
\end{aligned}$$

For all s small enough, let us choose:

$$t := \sqrt{\frac{\varepsilon}{\int_A \left(1 - e^{-sr}\right) \, \mathrm{d}\mu_A}} > m.$$

Then, we get for all s small enough:

$$\int_A \|F\| \left(1 - e^{-sr}\right) \, \mathrm{d}\mu_A \leq 2 \sqrt{\varepsilon \int_A \left(1 - e^{-sr}\right) \, \mathrm{d}\mu_A}.$$

As this property is true for all $\varepsilon > 0$, we have finally:

$$\|w\| \int_A \|F\| \left(1 - e^{-sr}\right) \, \mathrm{d}\mu_A = o(\|w\| \sqrt{H(s)}) = o(P(s, w)).$$

Second case: If $r \in \mathbb{L}^q$ for some $q > 1$ and F is in the domain of attraction of a symmetric stable law of parameter $p \in (1, 2]$, then for all $\varepsilon \in [0, q-1]$:

$$\int_A \left(1 - e^{-sr}\right)^{1+\varepsilon} \, \mathrm{d}\mu_A \leq \int_A (sr)^{1+\varepsilon} \, \mathrm{d}\mu_A = s^{1+\varepsilon} \|r\|_{\mathbb{L}^{1+\varepsilon}}^{1+\varepsilon} = O(s^{1+\varepsilon}).$$

Hence, for all $\kappa \in (0, p-1)$, for all $\varepsilon \in [0, \min(q-1, (p-1)^{-1})]$:

$$\begin{aligned}
\int_A \|F\| \left(1 - e^{-sr}\right) \, \mathrm{d}\mu_A &\leq \|F\|_{\mathbb{L}^{p-\kappa}} \left(\int_A \left(1 - e^{-sr}\right)^{1 + \frac{1}{p-1-\kappa}} \, \mathrm{d}\mu_A \right)^{1 - \frac{1}{p-\kappa}} \\
&\leq \|F\|_{\mathbb{L}^{p-\kappa}} \left(\int_A \left(1 - e^{-sr}\right)^{1+\varepsilon} \, \mathrm{d}\mu_A \right)^{1 - \frac{1}{p-\kappa}} \\
&= O\left(s^{(1+\varepsilon)\left(1 - \frac{1}{p-\kappa}\right)} \right).
\end{aligned}$$

Let $\delta > 0$. By Young's inequality,

$$\|w\| \int_A \|F\| \left(1 - e^{-sr}\right) \, \mathrm{d}\mu_A = O\left(\|w\|^{p+\delta} + s^{(1+\varepsilon)\left(1 - \frac{\kappa+\delta}{(p+\delta-1)(p-\kappa)}\right)} \right).$$

If κ and δ are chosen close enough to 0, then ε can be chosen so that the exponent of s is larger than 1. Moreover, $\|w\|^{p+\delta} = o(I(w))$ for all $\delta > 0$. Thus:

$$\|w\| \int_A \|F\| \left(1 - e^{-sr}\right) \, \mathrm{d}\mu_A = o(P(s, w)),$$

and we are done.

In all cases, we have proved that $Q(s,w) = 1 - \overline{P}(s,w) + o(P(s,w))$.

Study of a second order term.

We now study the last member of Equation (2.6), that is:

$$\begin{aligned}
\int_A &\left(e^{-sr+i\langle w,F\rangle} - 1\right)(f_{s,w} - 1)\ \mathrm{d}\mu_A \\
&= \int_A e^{-sr}\left(e^{i\langle w,F\rangle} - 1\right)(f_{s,w} - 1)\ \mathrm{d}\mu_A + \int_A \left(e^{-sr} - 1\right)(f_{s,w} - 1)\ \mathrm{d}\mu_A \\
&= \int_A e^{-sr}\left(e^{i\langle w,F\rangle} - 1\right)(f_{s,w} - f_{s,0})\ \mathrm{d}\mu_A + \int_A e^{-sr}\left(e^{i\langle w,F\rangle} - 1\right)(f_{s,0} - 1)\ \mathrm{d}\mu_A \\
&\quad + \int_A \left(e^{-sr} - 1\right)(f_{s,w} - f_{s,0})\ \mathrm{d}\mu_A + \int_A \left(e^{-sr} - 1\right)(f_{s,0} - 1)\ \mathrm{d}\mu_A.
\end{aligned}$$

Since F is integrable and $\mathrm{Lip}^\infty \subset \mathbb{L}^\infty$, by Lemma 2.3,

$$\int_A e^{-sr}\left(e^{i\langle w,F\rangle} - 1\right)(f_{s,0} - 1)\ \mathrm{d}\mu_A = O(H(s)\,\|w\|) = o(P(s,w)).$$

Using directly Lemma 2.2, we get $\|f_{s,w} - f_{s,0}\|_{\mathcal{B}} = O(\|w\|)$ uniformly in s, so that:

$$\int_A \left(e^{-sr} - 1\right)(f_{s,w} - f_{s,0})\ \mathrm{d}\mu_A = O(H(s)\,\|w\|) = o(P(s,w));$$

finally:

$$\int_A \left(e^{-sr} - 1\right)(f_{s,0} - 1)\ \mathrm{d}\mu_A = O(H(s)^2) = o(P(s,w)).$$

Hence, we only need to control the first term of the decomposition. It is a $O(\|w\|^2)$ term. If F does not belong to $\mathbb{L}^2$, then it is negligible with respect to $I(w)$, and we are done. For the remainder of the proof, we assume that $F \in \mathbb{L}^2$. We then need more precise asymptotics, and thus a more precise control on $f_{s,w} - f_{s,0}$. This is the goal of the next part of the proof.

Control of $f_{s,w} - f_{s,0}$.

Our goal in this part of the proof is to study the precise behavior of $f_{s,w} - f_{s,0}$. The $O(\|w\|)$ bound is no longer sharp enough for our purposes. We will first differentiate the function $w \mapsto f_{s,w} - f_{s,0}$ in 0, and then control the behavior of this derivative when s converges to 0. Notice that :

$$\begin{aligned}
\rho_{s,w} - \rho_{s,0} &= \int_A \mathcal{L}_{s,w} f_{s,w} - \mathcal{L}_{s,0} f_{s,0}\ \mathrm{d}\mu_A \\
&= \int_A \mathcal{L}_{s,0}\,(f_{s,w} - f_{s,0})\ \mathrm{d}\mu_A + \int_A (\mathcal{L}_{s,w} - \mathcal{L}_{s,0})\, f_{s,0}\ \mathrm{d}\mu_A \\
&\quad + \int_A (\mathcal{L}_{s,w} - \mathcal{L}_{s,0})\,(f_{s,w} - f_{s,0})\ \mathrm{d}\mu_A \\
&= \int_A e^{-sr}\,(f_{s,w} - f_{s,0})\ \mathrm{d}\mu_A + \int_A e^{-sr}\left(e^{i\langle w,F\rangle} - 1\right) f_{s,0}\ \mathrm{d}\mu_A + O(\|w\|^2).
\end{aligned}$$

We denote by $\widetilde{P}_{s,0}$ the rank one operator on Lip^∞ defined, for all h in $\mathcal{B}$, by:

$$\widetilde{P}_{s,0} h := \int_A e^{-sr} h\ \mathrm{d}\mu_A \cdot f_{s,0}.$$

Then:

$$\begin{aligned}\rho_{s,0}\left(f_{s,w}-f_{s,0}\right) &= \mathcal{L}_{s,0}\left(f_{s,w}-f_{s,0}\right)+\left(\mathcal{L}_{s,w}-\mathcal{L}_{s,0}\right) f_{s,0} \\ &\quad +\left(\rho_{s,0}-\rho_{s,w}\right) f_{s,w}+\left(\mathcal{L}_{s,w}-\mathcal{L}_{s,0}\right)\left(f_{s,w}-f_{s,0}\right) \\ &= \mathcal{L}_{s,0}\left(f_{s,w}-f_{s,0}\right)+\mathcal{L}_{s,0}\left[\left(e^{i\langle w,F\rangle}-1\right) f_{s,0}\right] \\ &\quad +\left(\rho_{s,0}-\rho_{s,w}\right) f_{s,0}+O(\|w\|^2) \\ &= \mathcal{L}_{s,0}\left(f_{s,w}-f_{s,0}\right)+\mathcal{L}_{s,0}\left[\left(e^{i\langle w,F\rangle}-1\right) f_{s,0}\right] \\ &\quad -\widetilde{P}_{s,0}\left(f_{s,w}-f_{s,0}\right)-\widetilde{P}_{s,0}\left(\left(e^{i\langle w,F\rangle}-1\right) f_{s,0}\right)+O(\|w\|^2),\end{aligned}$$

so that:

$$\left(\rho_{s,0}\operatorname{Id}-\mathcal{L}_{s,0}+\widetilde{P}_{s,0}\right)\left(f_{s,w}-f_{s,0}\right)=\left(\mathcal{L}_{s,0}-\widetilde{P}_{s,0}\right)\left[\left(e^{i\langle w,F\rangle}-1\right) f_{s,0}\right]+O(\|w\|^2). \tag{2.8}$$

The function $(s,w)\mapsto\sum_{a\in\pi}\mu_A(a)|\langle w,F\rangle f_{s,0}|_{\mathrm{Lip}(a)}$ is finite for all small enough $s\geq 0$ and $w\in\mathbb{T}^d$. In addition, for all $s\geq 0$, the operator $\mathcal{L}_{s,0}-\widetilde{P}_{s,0}$ is continuous from the Banach space Lip_d^1 of integrable functions f such that $\sum_{a\in\pi}\mu_A(a)|f|_{\mathrm{Lip}(a)}$ is finite to Lip_d^∞. Finally, the operator $\widetilde{P}_{0,0}$ is the eigenprojection corresponding to the eigenvalue 1 of $\mathcal{L}$; hence, the operator $\operatorname{Id}-\mathcal{L}+\widetilde{P}_{0,0}$ is invertible. The family of operators $\left(\rho_{s,0}\operatorname{Id}-\mathcal{L}_{s,0}+\widetilde{P}_{s,0}\right)_{s\geq 0}$ is continuous, so these operators are also invertible if s is small enough. Thus, we can define:

$$h_{s,w}:=\left(\rho_{s,0}\operatorname{Id}-\mathcal{L}_{s,0}+\widetilde{P}_{s,0}\right)^{-1}\left(\mathcal{L}_{s,0}-\widetilde{P}_{s,0}\right)\left(\langle w,F\rangle f_{s,0}\right). \tag{2.9}$$

Thanks to Equation (2.8),

$$f_{s,w}-f_{s,0}=h_{s,w}+\left(\rho_{s,0}\operatorname{Id}-\mathcal{L}_{s,0}+\widetilde{P}_{s,0}\right)^{-1}\left(\mathcal{L}_{s,0}-\widetilde{P}_{s,0}\right)\left(\left(e^{i\langle w,F\rangle}-1-\langle w,F\rangle\right) f_{s,0}\right)+O(\|w\|^2).$$

But $\left(e^{i\langle w,F\rangle}-1-\langle w,F\rangle\right) f_{s,0}=O(\|w\|^2)$ in Lip_d^1. Finally, we have $f_{s,w}-f_{s,0}=h_{s,w}+O(\|w\|^2)$, where the $O(\|w\|^2)$ expression is uniform in s.

We still need to prove that the function $(s,w)\mapsto h_{s,w}$ is well-behaved on a neighborhood of 0. We compute:

$$\begin{aligned}\left(\mathcal{L}_{s,0}-\widetilde{P}_{s,0}\right)\left(\langle w,F\rangle f_{s,0}\right) &= \left(\mathcal{L}-\widetilde{P}_{0,0}\right)\left(e^{-sr}\langle w,F\rangle f_{s,0}\right)+\int_A e^{-sr}\langle w,F\rangle f_{s,0}\,\mathrm{d}\mu_A(1-f_{s,0}) \\ &= \left(\mathcal{L}-\widetilde{P}_{0,0}\right)\left(e^{-sr}\langle w,F\rangle f_{s,0}\right)+o(H(s)),\end{aligned}$$

and the operator $\mathcal{L}-\widetilde{P}_{0,0}$ is continuous from Lip_d^1 to Lip_d^∞. Since $e^{-sr}\langle w,F\rangle f_{s,0}$ converges to $\langle w,F\rangle$ in Lip_d^1, the right-hand side of Equation (2.9) converges to $\mathcal{L}(\langle w,F\rangle)$ in Lip^∞. Finally,

$$h_{s,w}-h_{0,w}=O(\|w\|)o_s(1),$$

whence:

$$h_{0,w}=(\operatorname{Id}-\mathcal{L})^{-1}\mathcal{L}\left(\langle w,F\rangle\right)\in\mathrm{Lip}^\infty.$$

End of the proof.

By the previous part, for w and s close to 0, we have $e^{i\langle w,F\rangle}-1=\langle w,F\rangle+o(\|w\|)$ in $\mathbb{L}^2$, and $f_{s,w}-f_{s,0}=h_{s,w}+o(\|w\|)$. Therefore:

$$\begin{aligned}\int_A e^{-sr}\left(e^{i\langle w,F\rangle}-1\right)(f_{s,w}-f_{s,0})\ \mathrm{d}\mu_A\\ &=\int_A e^{-sr}\langle w,F\rangle h_{s,w}\ \mathrm{d}\mu_A+o(\|w\|^2)\\ &=\int_A \langle w,F\rangle h_{s,w}\ \mathrm{d}\mu_A+\int_A\left(e^{-sr}-1\right)\langle w,F\rangle h_{s,w}\ \mathrm{d}\mu_A+o(\|w\|^2).\end{aligned}$$

Note that the family of functions $(h_{s,w})_{s\geq 0}$ is uniformly bounded on a neighborhood of 0. The study of $\int_A (e^{-sr}-1)\langle w,F\rangle h_{s,w}\ \mathrm{d}\mu_A$ can be done as in the first part of the proof. In addition,

$$\int_A \langle w,F\rangle\left(h_{s,w}-h_{0,w}\right)\ \mathrm{d}\mu_A=O(\|w\|^2)o(1).$$

Finally, we get:

$$\int_A e^{-sr}\left(e^{i\langle w,F\rangle}-1\right)(f_{s,w}-f_{s,0})\ \mathrm{d}\mu_A=\int_A\langle w,F\rangle\left(\mathrm{Id}-\mathcal{L}\right)^{-1}\mathcal{L}\left(\langle w,F\rangle\right)\ \mathrm{d}\mu_A+o(P(s,w)).$$

Since $(w,Sw)=\mathbb{E}(\langle w,F\rangle\left(\mathrm{Id}-\mathcal{L}\right)^{-1}\mathcal{L}\left(\langle w,F\rangle\right))$ is the covariance in the Central Limit Theorem [10, Theorem 4.1.4], the expression we find is $P(s,w)-H(s)$, and:

$$\rho_{s,w}=1-P(s,w)+o(P(s,w)).\qquad\square$$

Remark 2.5.
If r and F are independent, then the proof goes through even if we assume that both r and F are heavy-tailed. If r and F are not independent, the situation is more complicated.

Now that we have a control of the function $\rho_{s,w}$ for small values of s and w, we need a control of the norm of $\mathcal{L}_{s,w}$ for small values of s and larger values of w.

Lemma 2.6.
Assume the hypotheses of Lemma 2.4, and that F is not the sum of a measurable coboundary and a function which takes its values in some translate of a proper sub-lattice of $\mathbb{Z}^d$.

Then, for all $(s,w)\neq(0,0)$, the spectral radius of $\mathcal{L}_{s,w}$ is strictly smaller than 1.

Proof.
First, note that for all $s\geq 0$ and all $w\in\mathbb{T}^d$, the essential spectral radius of $\mathcal{L}_{s,w}$ is strictly smaller than 1 [10, Corollary 4.1.3]. Thus we are able to work with eigenvalues and eigenfunctions.

Assume that $s>0$. Let h be any non-zero eigenfunction of $\mathcal{L}_{s,w}$, and ρ the associated eigenvalue. Then, on $\{h\neq 0\}$,

$$|\rho h|=|\mathcal{L}_{s,w}h|=\left|\mathcal{L}(e^{-sr+i\langle w,F\rangle}h)\right|\leq\mathcal{L}(e^{-sr}|h|)<\mathcal{L}|h|.$$

Hence, $|\rho||h|<\mathcal{L}|h|$ on $\{h\neq 0\}$. By integrating, we get $|\rho|<1$.

We now assume that $s=0$ and $w\neq 0$. Assume that the spectrum of $\mathcal{L}_{0,w}$ contains a point $\rho_{0,w}$ of modulus greater than or equal to 1. Since the essential spectral radius of $\mathcal{L}_{0,w}$ is strictly smaller than 1, such a point is an eigenvalue of $\mathcal{L}_{0,w}$; let $f_{0,w}$ be one of the corresponding eigenfunctions.

By the triangular inequality,

$$\|\mathcal{L}_{0,w} f_{0,w}\|_{\mathbb{L}^1} = |\rho_{0,w}| \, \|f_{0,w}\|_{\mathbb{L}^1} \leq \|\mathcal{L}|f_{0,w}|\|_{\mathbb{L}^1} = \|f_{0,w}\|_{\mathbb{L}^1} .$$

The operator $\mathcal{L}$ is a contraction on $\mathbb{L}^1$, so $|\rho_{0,w}| = 1$. Then,

$$|f_{0,w}| = |\rho_{0,w} f_{0,w}| = |\mathcal{L}_{0,w} f_{0,w}| \leq \mathcal{L}|f_{0,w}|.$$

Since $\|f_{0,w}\|_{\mathbb{L}^1} = \|\mathcal{L}|f_{0,w}|\|_{\mathbb{L}^1}$, this last inequality is actually an equality. Hence, $|f_{0,w}| \in \mathbb{L}^1$ is a non-trivial eigenfunction of the operator $\mathcal{L}$, corresponding to the eigenvalue 1. The Gibbs-Markov dynamical system is mixing, so the only eigenfunctions corresponding to the eigenvalue 1 are constant. So the function $|f_{0,w}|$ must be constant. Without loss of generality, we can assume that $|f_{0,w}| \equiv 1$.

Let $\theta_{0,w} : A \to \mathbb{T}^1$ be the phase of $f_{0,w}$, i.e., be such that $f_{0,w} = e^{i\theta_{0,w}}$. Let $R_{0,w} \in \mathbb{T}^1$ be such that $\rho_{0,w} = e^{iR_{0,w}}$. Since $|f_{0,w}| = |\mathcal{L}_{0,w} f_{0,w}|$, for all x, the quantity $\theta_{0,w} + \langle w, F \rangle$ must be constant on $\{y : T_A y = x\}$. Hence:

$$\theta_{0,w} \circ T - \theta_{0,w} = R_{0,w} + \langle w, F \rangle \ [2\pi]. \tag{2.10}$$

For all $n \in \mathbb{Z}$, by multiplying this equation by n, one gets:

$$\mathcal{L}_{0,nw} f_{0,w}^n = \rho_{0,w}^n f_{0,w}^n.$$

The vector space Lip^∞ together with the multiplication of functions is an algebra, so $f_{0,w}^n$ also belongs to Lip^∞ and is non-zero. Hence, $\rho_{0,w}^n$ is an eigenvalue of $\mathcal{L}_{0,nw}$. If w has some coordinate which is rationally independent from π, then the sequence $(nw)_{n\in\mathbb{Z}}$ takes values which are not 0, but are arbitrarily close to 0. This contradicts the fact that, by Lemma 2.4, if w' is close enough to 0 but non-zero, then the eigenvalues of $\mathcal{L}_{0,w'}$ all have a modulus strictly smaller than 1.

Hence, all the coordinates of w are a rational multiple of π. Let $q > 0$ be such that $qw = 0$. Then Equation (2.10) yields:

$$q\theta_{0,w} \circ T - q\theta_{0,w} = qR_{0,w} + \langle qw, F \rangle \ [2\pi q],$$

and up to adding a coboundary, the function F takes its values in a translate of a proper sub-lattice of $\mathbb{Z}^d$. This is in contradiction with the hypotheses of the lemma. $\square$

2.4 On the transfer operator of the induced system

With the next proposition, we take the asymptotic development of the main eigenvalue, which was the object of the previous section, and the renewal equation, to get a function G as in Hypothesis 1.2. This requires some further restrictions on the dimension and on the function F, since the $\mathbb{Z}^d$-extension must be conservative.

We will denote by I^* and J^* the generalized inverses of respectively I and J, which were defined in Subsection 2.2.

Proposition 2.7 (Asymptotics for the main eigenvalue).

Let $(A, \pi, d_A, \mu_A, T_A)$ be a mixing Gibbs-Markov map. Let $d \in \{1, 2\}$. Let F and r be such that F is $\sigma(\pi)$-measurable and $\sum_{a\in\pi} \mu_A(a) |r|_{\mathrm{Lip}(a)}$ is finite, each of r and F satisfy one the assumptions of Subsection 2.2, and modulo a coboundary F does not take its values in a translate of a proper sub-lattice of $\mathbb{Z}^d$. Assume that $\mathbb{P}(\|F\| > t) = o(t^{-2})$ or that $r \in \mathbb{L}^q$ for some $q > 1$.

If $d = 1$, for all $f \in \mathrm{Lip}^\infty$,

$$\frac{1}{2\pi} \int_{\mathbb{T}} (\mathrm{Id} - \mathcal{L}_{s,w})^{-1} \ \mathrm{d}w f = \frac{1}{p \sin(\pi/p)} \frac{I^*(H(s))}{H(s)} \left(\int_A f \ \mathrm{d}\mu_A + o(1) \|f\|_{\mathcal{B}} \right) + O(1) \|f\|_{\mathcal{B}} . \tag{2.11}$$

If $d = 2$, for all $f \in \mathrm{Lip}^\infty$,

$$\frac{1}{(2\pi)^2}\int_{\mathbb{T}^2} (\mathrm{Id} - \mathcal{L}_{s,w})^{-1} \, \mathrm{d}w f = \frac{1}{2\pi \det(M)} \widetilde{J}(J^*(H(s))) \left(\int_A f \, \mathrm{d}\mu_A + o(1) \|f\|_{\mathcal{B}} \right) + O(1) \|f\|_{\mathcal{B}} . \tag{2.12}$$

In both cases, the $o(1)$ and $O(1)$ terms are respectively uniformly negligible and uniformly bounded on Lip^∞.

Proof.

Let $V = [0, s_0) \times U$ be any neighborhood of the origin in $\mathbb{R}_+ \times \mathbb{T}^d$ small enough that the main eigenvalue $\rho_{s,w}$ is well-defined and continuous for $(s, w) \in V$. For $(s, w) \in V$, we denote by $Q_{s,w}$ the eigenprojection corresponding to the main eigenvalue $\rho_{s,w}$ of $\mathcal{L}_{s,w}$.

Up to taking a smaller neighborhood of 0, the family of operators $(\mathrm{Id} - \mathcal{L}_{s,w})^{-1}$ restricted to the kernel of $Q_{s,w}$ is uniformly bounded for $(s, w) \in V$. By Lemma 2.6, outside of V, the operators $(\mathrm{Id} - \mathcal{L}_{s,w})^{-1}$ are also uniformly bounded. Hence, for all $f \in \mathrm{Lip}^\infty$,

$$\begin{aligned}
\frac{1}{(2\pi)^d}\int_{\mathbb{T}^d} (\mathrm{Id} - \mathcal{L}_{s,w})^{-1} \, \mathrm{d}w f &= \frac{1}{(2\pi)^d}\int_U (\mathrm{Id} - \mathcal{L}_{s,w})^{-1} Q_{s,w} f \, \mathrm{d}w + O(1) \|f\|_{\mathcal{B}} \\
&= \frac{1}{(2\pi)^d}\int_U \frac{1}{1-\rho_{s,w}} Q_{s,w} f \, \mathrm{d}w + O(1) \|f\|_{\mathcal{B}} \\
&= \frac{1}{(2\pi)^d}\int_U \frac{1}{1-\rho_{s,w}} Q_{0,w} f \, \mathrm{d}w \, (1 + o(1) \|f\|_{\mathcal{B}}) + O(1) \|f\|_{\mathcal{B}} .
\end{aligned}$$

But $Q_{0,w}$ converges to μ_A as w goes to 0, whence:

$$\left| \int_U \frac{1}{1-\rho_{s,w}} Q_{0,w} f \, \mathrm{d}w - \int_U \frac{1}{1-\rho_{s,w}} \, \mathrm{d}w \int_A f \, \mathrm{d}\mu_A \right| \leq \varepsilon_1(\mathrm{Diam}(U)) \left| \int_U \frac{1}{1-\rho_{s,w}} \, \mathrm{d}w \right| \|f\|_{\mathcal{B}} , \tag{2.13}$$

where $\lim_0 \varepsilon_1 = 0$.

Let us define for $s \in [0, s_0)$:

$$G(s) := \frac{1}{(2\pi)^d}\int_U \frac{1}{1-\rho_{s,w}} \, \mathrm{d}w.$$

Up to choosing a smaller neighborhood, the function $h(s, w) := 1 - \rho_{s,w} - P(s, w)$ is well defined on V. By Lemma 2.4,

$$\lim_{(s,w)\to 0} \frac{h(s, w)}{P(s, w)} = 0.$$

Let $\delta \in (0, 1)$. We can choose positive s_0 and ε such that, if U is the open set $\{w \in \mathbb{R}^d : \ I(w) < \varepsilon\}$, then:

$$\sup_V \frac{|h(s, w)|}{P(s, w)} \leq \delta.$$

Then, on V:

$$\left| \frac{1}{1-\rho_{s,w}} - \frac{1}{P(s, w)} \right| \leq \frac{\delta}{1-\delta} \frac{1}{P(s, w)}.$$

For all $s \in [0, s_0)$, by integrating on U, we get:

$$\left| G(s) - \frac{1}{(2\pi)^d}\int_U \frac{1}{P(s, w)} \, \mathrm{d}w \right| \leq \frac{\delta}{1-\delta} \frac{1}{(2\pi)^d}\int_U \frac{1}{P(s, w)} \, \mathrm{d}w. \tag{2.14}$$

If $G(s)$ goes to $+\infty$ as s goes to 0, then the contribution from outside of any neighborhood of 0 is negligible. In other words, the asymptotic behavior of G is a local property of $\mathcal{L}_{s,w}$ at 0. Hence, any value of ε yields the same asymptotics for $(2\pi)^{-d}\int_U P(s,w)^{-1}\,\mathrm{d}w$. Since ε can then be taken as small as one wishes, if $\lim_{s\to 0}(2\pi)^{-d}\int_U P(s,w)^{-1}\,\mathrm{d}w = +\infty$ then:

$$G(s) \sim \frac{1}{(2\pi)^d}\int_U \frac{1}{P(s,w)}\,\mathrm{d}w. \tag{2.15}$$

By the same argument, if $G(s)$ goes to $+\infty$ as s goes to 0, then U can be chosen as small as one whishes, and a the $\varepsilon_1(\mathrm{Diam}(U))$ term of Equation (2.13) can be made as small as one whishes, and:

$$\frac{1}{(2\pi)^d}\int_{\mathbb{T}^d}(\mathrm{Id}-\mathcal{L}_{s,w})^{-1}\,\mathrm{d}wf = \frac{1}{(2\pi)^d}\int_U \frac{1}{P(s,w)}\,\mathrm{d}w\left(\int_A f\,\mathrm{d}\mu_A + o(1)\,\|f\|_{\mathcal{B}}\right).$$

In dimension 1

Let $\varepsilon, s > 0$. Without loss of generality, we can assume that I is continuous and increasing, so that I^* is the true inverse of I on a neighborhood of 0. Let $\gamma := I^*(H(s))$. Then:

$$\begin{aligned}
\frac{1}{2\pi}\int_U \frac{1}{P(s,w)}\,\mathrm{d}w &= \frac{1}{2\pi}\int_{B(0,I^*(\varepsilon))}\frac{1}{H(s)+I(w)}\,\mathrm{d}w\\
&= \frac{1}{\pi H(s)}\int_0^{I^*(\varepsilon)}\frac{1}{1+\frac{I(w)}{I(\gamma)}}\,\mathrm{d}w\\
&= \frac{\gamma}{\pi H(s)}\int_0^{\frac{I^*(\varepsilon)}{\gamma}}\frac{1}{1+\frac{I(\gamma v)}{I(\gamma)}}\,\mathrm{d}v.
\end{aligned}$$

Note that $\gamma v \leq I^*(\varepsilon)$ in this integral. Let $\kappa \in (0, p-1)$. By Potter's theorem [6, Theorem 1.5.6], if ε and s are small enough, $I(\gamma v)/I(\gamma) \geq \min(v^{p-\kappa}, v^{p+\kappa})/2$, and:

$$\frac{1}{1+\frac{I(\gamma v)}{I(\gamma)}}\mathbf{1}_{\gamma v\leq I^*(\varepsilon)} \leq \frac{1}{1+\frac{1}{2}\min(v^{p-\kappa}, v^{p+\kappa})}.$$

By the dominated convergence theorem,

$$\lim_{\gamma\to 0}\int_0^{\frac{I^*(\varepsilon)}{\gamma}}\frac{1}{1+\frac{I(\gamma v)}{I(\gamma)}}\,\mathrm{d}v = \int_0^{+\infty}\frac{1}{1+x^p}\,\mathrm{d}x = \frac{1}{\mathrm{sinc}\,(\pi/p)},$$

where the last equality comes from functional equations involving the beta function, the gamma function, and the sinus. Finally:

$$\frac{1}{2\pi}\int_U \frac{1}{P(s,w)}\,\mathrm{d}w \sim \frac{1}{p\sin(\pi/p)}\frac{I^*(H(s))}{H(s)}.$$

Note that $\lim_{s\to 0} H(s) = 0$. Moreover, the function I has regular variation of index strictly greater than 1, so I^* has regular variation of index strictly smaller than 1. Hence, $\lim_{s\to 0} I^*(H(s))/H(s) = +\infty$, so that:

$$G(s) \sim \frac{1}{p\sin(\pi/p)}\frac{I^*(H(s))}{H(s)}. \tag{2.16}$$

In dimension 2

We recall that we can assume that, by the hypotheses or by Lemma 2.4, there exists a function J with regular variation and an automorphism M of $\mathbb{R}^2$ such that $I(w) := J(\|Mw\|)$. If F is in $\mathbb{L}^2$, then M is just the square root of the covariance matrix S in the Central Limit Theorem for the Birkhoff sums of F and $J(v) = v^2/2$. Let $\gamma := J^*(H(s))$.

$$\begin{aligned}\frac{1}{(2\pi)^2}\int_U \frac{1}{P(s,w)}\,\mathrm{d}w &= \frac{1}{(2\pi)^2}\int_{M^{-1}B(0,J^*(\varepsilon))} \frac{1}{H(s)+J(\|Mw\|)}\,\mathrm{d}\,\mathrm{Leb}(w) \\ &= \frac{1}{2\pi\det(M)}\int_0^{J^*(\varepsilon)} \frac{v'}{H(s)+J(v')}\,\mathrm{d}v' \\ &= \frac{1}{2\pi\det(M)H(s)}\int_0^{J^*(\varepsilon)} \frac{v'}{1+\frac{J(v')}{J(\gamma)}}\,\mathrm{d}v' \\ &= \frac{(J^*(H(s)))^2}{2\pi\det(M)H(s)}\int_0^{\frac{J^*(\varepsilon)}{\gamma}} \frac{v}{1+\frac{J(\gamma v)}{J(\gamma)}}\,\mathrm{d}v.\end{aligned}$$

The function J^* has regular variation of index $1/p$, so the function $x \mapsto J^*(x)^2/x$ has regular variation of index $2/p-1$. If $p<2$, then $J^*(H(s))^2/H(s)$ converges to 0 when s converges to 0, so the integral above is bounded in s. Hence, the conclusion of Proposition 2.7 is trivially true, as the $O(1)$ term is non-negligible.

If F is in $\mathbb{L}^2$, then we can take $J(v) = v^2/2$, so that $J^*(x) = \sqrt{2x}$. The computations can be done explicitly, and we get:

$$\frac{1}{(2\pi)^2}\int_U \frac{1}{P(s,w)}\,\mathrm{d}w \sim -\frac{1}{2\pi\det(M)}\ln((J^*(H(s)))^2) \sim \frac{1}{2\pi\det(M)}\widetilde{J}(J^*(H(s))).$$

Let us assume that $p=2$, and that F is not in $\mathbb{L}^2$. Let $N>0$. By the Uniform Convergence Theorem [6, Theorem 1.2.1], $J(\gamma v)/J(\gamma)$ converges uniformly to v^2 on $[0,N]$, whence, as s vanishes:

$$\int_0^{\frac{J^*(\varepsilon)}{\gamma}} \frac{v}{1+\frac{J(\gamma v)}{J(\gamma)}}\,\mathrm{d}v = \frac{\ln(1+N^2)}{2}(1+o(1)) + \frac{H(s)}{(J^*(H(s)))^2}\int_{NJ^*(H(s))}^{J^*(\varepsilon)} \frac{v}{H(s)+J(v)}\,\mathrm{d}v.$$

Since F is not in $\mathbb{L}^2$, we have $J(x) \gg x^2$ for small x, so $J^*(x) \ll \sqrt{x}$ for small x. The function $x/J^*(x)^2$ goes to $+\infty$ as x vanishes, so that:

$$\frac{1}{(2\pi)^2}\int_U \frac{1}{P(s,w)}\,\mathrm{d}w = \frac{1}{2\pi\det(M)}\int_{NJ^*(H(s))}^{J^*(\varepsilon)} \frac{v}{H(s)+J(v)}\,\mathrm{d}v(1+o(1)).$$

Note that $J(NJ^*(H(s))) \sim \sqrt{N}H(s)$ as s goes to 0. Let $\varepsilon'>0$. We can choose N large enough that, for all s small enough, $\varepsilon' J(v) \geq H(s)$ whenever $v \geq NJ^*(H(s))$. Then:

$$\left|\frac{1}{(2\pi)^2}\int_U \frac{1}{P(s,w)}\,\mathrm{d}w - \frac{1}{2\pi\det(M)}\int_{NJ^*(H(s))}^{J^*(\varepsilon)} \frac{v}{J(v)}\,\mathrm{d}v\right| \leq \varepsilon' \int_{NJ^*(H(s))}^{J^*(\varepsilon)} \frac{v}{J(v)}\,\mathrm{d}v(1+o(1)).$$

If the function $v/J(v)$ is not integrable on a neighborhood of 0, then, by the same argument we used to prove the equivalence (2.15),

$$\frac{1}{(2\pi)^2}\int_U \frac{1}{P(s,w)}\,\mathrm{d}w \sim \frac{1}{2\pi\det(M)}\widetilde{J}(NJ^*(H(s))) - \widetilde{J}(J^*(\varepsilon)) \sim \frac{1}{2\pi\det(M)}\widetilde{J}(J^*(H(s))). \tag{2.17}$$

□

Note that, if F is in $\mathbb{L}^2$, Equation (2.17) becomes:

$$\frac{1}{(2\pi)^2}\int_U \frac{1}{P(s,w)}\,\mathrm{d}w \sim -\frac{1}{2\pi\sqrt{\det(S)}}\ln(H(s)).$$

A Regular variation

We used a few times in this article a Tauberian theorem. We recall some basics of Karamata's theory of functions with regular variation, which provides a suitable framework for our work.

Definition A.1 (Regular variation)**.**

Let β be a real number. A real-valued measurable function ψ defined on a neighborhood of $+\infty$ in $\mathbb{R}_+$ is said to have regular variation of index β at $+\infty$ if, for all $y > 0$:

$$\lim_{x \to +\infty} \frac{\psi(xy)}{\psi(x)} = y^\beta.$$

A real-valued measurable function ψ defined on a neighborhood of 0 in $\mathbb{R}_+$ is said to have regular variation of index β at 0 if $\psi(1/x)$ has regular variation of index $-\beta$ at $+\infty$.

A function with regular variation of index 0 is also called a function with slow variation.

We give a version of Karamata's Tauberian theorem [6, Theorem 1.7.1], [15, Theorem 8.1]. We separate the cases of regular variation of index $\beta < 1$ and regular variation of index 1, because of the failure of Karamata's theorem in the latter situation [6, Proposition 1.5.8].

Theorem A.2 (Karamata's Tauberian theorem, $\beta < 1$)**.**

Let $\beta < 1$. Let $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ be a positive and nondecreasing function. Then there is equivalence between:

- *ψ has regular variation of index β at infinity;*
- *$\int_{\mathbb{R}_+} \psi(t)^{-1} e^{-st} \,\mathrm{d}t$ has regular variation of index $\beta - 1$ at 0.*

If any of these conditions is satisfied, then in addition, for s close to 0,

$$\int_{\mathbb{R}_+} \frac{1}{\psi(t)} e^{-st} \,\mathrm{d}t \sim \frac{\Gamma(1-\beta)}{s\psi(1/s)}.$$

Theorem A.3 (Karamata's Tauberian theorem, $\beta = 1$)**.**

Let $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ be a positive and nondecreasing function with regular variation of index 1 at infinity. Then:

$$\int_{\mathbb{R}_+} \frac{1}{\psi(t)} e^{-st} \,\mathrm{d}t \sim \int_0^{1/s} \frac{1}{\psi(t)} \,\mathrm{d}t,$$

where the equivalent is for s close to 0, and the right-end side has slow variation at 0.

Proof of Theorems A.2 and A.3.

Let $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ be a positive and nondecreasing function with regular variation of index $\beta \leq 1$ at infinity. For all $t \geq 0$, let:

$$\Psi(t) := \int_0^t \frac{1}{\psi(u)} \,\mathrm{d}u,$$

so that:

$$\int_{\mathbb{R}_+} \frac{1}{\psi(t)} e^{-st} \,\mathrm{d}t = \int_{\mathbb{R}_+} e^{-st} \,\mathrm{d}\Psi(t).$$

By [6, Proposition 1.5.8] (for $\beta < 1$) or [6, Proposition 1.5.9a] (for $\beta = 1$), the function Ψ has regular variation of index $1 - \beta \geq 0$ at infinity. By [6, Theorem 1.7.1], for s close to 0,

$$\int_{\mathbb{R}_+} \frac{1}{\psi(t)} e^{-st} \,\mathrm{d}t \sim \Gamma(2-\beta)\Psi(1/s) = \Gamma(2-\beta) \int_0^{1/s} \frac{1}{\psi(t)} \,\mathrm{d}t.$$

In particular, for $\beta = 1$, we have proved Theorem A.3. For now on, we assume that $\beta < 1$.

By [6, Proposition 1.5.8], at infinity,

$$\Psi(t) \sim \frac{t}{(1-\beta)\psi(t)},$$

whence, for s close to 0,

$$\int_{\mathbb{R}_+} \frac{1}{\psi(t)} e^{-st} \, \mathrm{d}t \sim \frac{\Gamma(2-\beta)}{s(1-\beta)\psi(1/s)} = \frac{\Gamma(1-\beta)}{s\psi(1/s)}.$$

In particular, $\int_{\mathbb{R}_+} \psi(t)^{-1} e^{-st} \, \mathrm{d}t$ has regular variation of index $\beta - 1$ at 0.

It remains to prove the converse implication of Theorem A.2. Let $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ be a positive and nondecreasing function. Assume that $\int_{\mathbb{R}_+} \psi(t)^{-1} e^{-st} \, \mathrm{d}t$ has regular variation of index $\beta - 1 < 0$ at 0. Let Ψ be defined as previously. By [6, Theorem 1.7.1], for s close to 0,

$$\Psi(1/s) \sim \frac{1}{\Gamma(2-\beta)} \int_{\mathbb{R}_+} \frac{1}{\psi(t)} e^{-st} \, \mathrm{d}t,$$

and the function Ψ has regular variation of index $1-\beta$ at infinity. By the Monotone Density Theorem [6, Theorem 1.7.2],

$$\frac{1}{\psi(t)} \sim \frac{(\beta-1)\Psi(t)}{t}$$

and ψ has regular variation of index β at infinity. □

When we apply Theorem A.2 or Theorem A.3, the function ψ will stand for the inverse of the tail of a random variable, typically:

$$\psi(x) = \frac{1}{\mathbb{P}_{\mu_A}(\varphi > x)}.$$

B Results

The result of our computations is summed up in the table below. Here is the way to use this table:

- the base of the $\mathbb{Z}^d$-extension of the suspension semi-flow is assumed to be a mixing Gibbs-Markov map;
- the step function F is assumed to be $\sigma(\pi)$-measurable and non-degenerate;
- the step time r is assumed to be almost surely positive and such that $\sum_{a \in \pi} \mu_A(a) |r|_{\mathrm{Lip}(a)}$ is finite;
- the first column, d, is the dimension of the random walk. The case $d = 0$ corresponds to suspension semi-flows.
- for all β in $[0, 1]$, saying that the "assumption on r" is "RV of index β" means that the function $\mu_A(r > t)^{-1}$ has regular variation of index β at infinity. Slow variation corresponds to $\beta = 0$.
- for all p in $(1, 2]$, saying that the "assumption on F" is "RV of index p" means that the function J has regular variation of index p at 0.

- the "index of variation" is a parameter $\gamma \in [0, 1]$ such that $\mathbb{P}_{\mu_A}(\varphi > t)^{-1}$ has regular variation of index γ at infinity.

- the "renormalization function" is a function a defined on a neighborhood of infinity in $\mathbb{R}_+$, non-decreasing, diverging, and with regular variation of index γ given by the column "index of variation". It is such that $a(t)^{-1} \int_0^t f \circ g_s \, \mathrm{d}s$ converges strongly in distribution to $\int_\Omega f \, \mathrm{d}\mu \cdot Y_\gamma$, where Y_γ has a standard Mittag-Leffler distribution of index γ and $f \in \mathbb{L}^1(\Omega, \mu)$.

To reduce the number of cases, we do not give the formulas when $\mathbb{P}(\|F\| > t) = o(t^{-2})$ but $F \notin \mathbb{L}^2$ and $r \notin \mathbb{L}^q$ for all $q > 1$.

If $d = 0$, then the abstract assumptions of Lemma 2.3 and the additional assumption that $\mathcal{B}$ is dense in $\mathbb{L}^1(A, \mu_A)$ are actually sufficient to prove the convergence in distribution of the Birkhoff integral (see Lemma 2.3 and Corollary 1.7).

Table 1: Summary of the results

d	Assumption on r	Assumption on F	Equivalent of $\mathbb{P}_{\mu_A}(\varphi > t)$	Index of variation	Renormalization function
0	RV of index 1	-	$\mathbb{P}_{\mu_A}(r > t)$	1	$\frac{t}{\int_0^t \mathbb{P}_{\mu_A}(r > s)\,\mathrm{d}s}$
0	RV of index $\beta \in [0,1)$	-	$\mathbb{P}_{\mu_A}(r > t)$	β	$\frac{\mathrm{sinc}(\beta\pi)}{\mathbb{P}_{\mu_A}(r > t)}$
1	$r \in \mathbb{L}^1$	$F \in \mathbb{L}^2$	$\sqrt{\frac{2\int_A r\,\mathrm{d}\mu_A \det S}{\pi}}\frac{1}{\sqrt{t}}$	$1/2$	$\sqrt{\frac{2}{\pi\int_A r\,\mathrm{d}\mu_A \det S}}\sqrt{t}$
1	RV of index 1	$F \in \mathbb{L}^2$	$\sqrt{\frac{2\det S}{\pi}}\sqrt{\frac{\int_0^t \mathbb{P}_{\mu_A}(r > s)\,\mathrm{d}s}{t}}$	$1/2$	$\sqrt{\frac{2}{\pi\det S}}\sqrt{\frac{t}{\int_0^t \mathbb{P}_{\mu_A}(r > s)\,\mathrm{d}s}}$
1	RV of index $\beta \in [0,1)$	$F \in \mathbb{L}^2$	$\frac{\sqrt{\Gamma(1-\beta)}}{\Gamma\left(1-\frac{\beta}{2}\right)}\sqrt{2\det S \cdot \mathbb{P}_{\mu_A}(r > t)}$	$\beta/2$	$\frac{1}{\Gamma\left(1+\frac{\beta}{2}\right)\sqrt{2\Gamma(1-\beta)\det S}}\frac{1}{\sqrt{\mathbb{P}_{\mu_A}(r > t)}}$
1	$r \in \mathbb{L}^q$, $q > 1$	RV of index $p \in (1,2]$	$\frac{p\sin(\pi/p)\left(\int_A r\,\mathrm{d}\mu_A\right)^{1-\frac{1}{p}}}{\Gamma(1/p)}\frac{1}{tI^*(1/t)}$	$1 - 1/p$	$\frac{\Gamma(1/p)tI^*(1/t)}{(p-1)\pi\left(\int_A r\,\mathrm{d}\mu_A\right)^{1-\frac{1}{p}}}$
2	$r \in \mathbb{L}^1$ or RV of index $\beta \in (0,1]$	$F \in \mathbb{L}^2$	$\frac{2\pi\sqrt{\det S}}{\beta\ln(t)}$	0	$\frac{\beta\ln(t)}{2\pi\sqrt{\det S}}$
2	Slow variation	$F \in \mathbb{L}^2$	$\frac{2\pi\sqrt{\det S}}{\lvert\ln(\mathbb{P}_{\mu_A}(r > t))\rvert}$	0	$\frac{\lvert\ln(\mathbb{P}_{\mu_A}(r > t))\rvert}{2\pi\sqrt{\det S}}$
2	$r \in \mathbb{L}^q$, $q > 1$	RV of index 2 and $\widetilde{J}$ diverges	$\frac{2\pi\det(M)}{\widetilde{J}(J^*(1/t))}$	0	$\frac{\widetilde{J}(J^*(1/t))}{2\pi\det(M)}$

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