KODAIRA EMBEDDING THEOREM

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ABSTRACT. The aim of this report is to prove Kodaira embedding theorem:

Theorem 0.1 (Kodaira Embedding Theorem)**.** *A compact Kähler manifold endowed with a positive line bundle admits a projective embedding.*

The main idea is recasting local problems in global ones, with the help of a surgery technique called "blowing up", which means namely replacing a point of a complex manifold with a hypersurface. Despite the growth of complexity of the underlying complex manifold, one is then able to employ a codimension one machinery to tackle the problem. In fact Kodaira-Akizuki-Nakano vanishing theorem yields the result, which in turn is a clever combination of Kähler identities.

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1. Ampleness of a line bundle

Let X be a complex manifold and $\xi : L \longrightarrow X$ holomorphic line bundle.

Definition 1.1. L has **no base points** or L is **spanned** if for all $x \in X$ there exits a section of L, $s \in H^0(X, L)$, such that $s(x) \neq 0$.

Remark. Let $\mathcal{U} = \{U_\alpha\}$ cover of open subsets of X trivializing the line bundle L and $\varphi_\alpha : \xi^{-1}(U_\alpha) \longrightarrow U_\alpha \times \mathbb{C}$. A section $s \in H^0(X, L)$ can be described as a collection of sections $s_\alpha := \varphi_\alpha \circ s \in H^0(U_\alpha, L|_{U_\alpha}) = \mathcal{O}(U_\alpha)$ satisfying the cocycle condition $s_\alpha(x) = g_{\alpha\beta}(x)s_\beta(x)$ in $U_\alpha \cap U_\beta$, where $g_{\alpha\beta} \in \mathcal{O}^*(U_\alpha \cap U_\beta)$ are transition functions of the line bundle L relative to the cover $\mathcal{U}$. Hence, the vanishing of a section is independent of the trivialization φ_α and the condition $s(x) \neq 0$ is thus meaningful.

Given a spanned line bundle, we can define a morphism

$$\begin{aligned} i_L : X &\longrightarrow \mathbb{P}\big(H^0(X, L)\big)^* \\ x &\longmapsto H_x, \end{aligned}$$

where H_x is the hyperplane in $\mathbb{P}\big(H^0(X, L)\big)$ consisting of sections of the line bundle L vanishing at x.

We can describe the morphism i_L more explicitly as follow. Choose a basis $s_0, \ldots, s_n$ of $H^0(X, L)$. In the notation of the remark, $s_i = (s_{i,\alpha})$ with $s_{i,\alpha} \in \mathcal{O}(U_\alpha)$ such that $s_{i,\alpha} = g_{\alpha\beta}s_{i,\beta}$, for $i = 0, \ldots, n$. Under the identification $\mathbb{P}(H^0(X, L))^* \cong \mathbb{P}^n$ induced by the choice of the basis, the map is given by

$$i_L(x) = [s_{0,\alpha}(x) : \ldots : s_{n,\alpha}(x)].$$

(1) The map is independent of the trivialization. Indeed,

$$\begin{aligned} [s_{0,\alpha}(x) : \ldots : s_{n,\alpha}(x)] &= [g_{\alpha\beta}(x)s_{0,\beta}(x) : \ldots : g_{\alpha\beta}(x)s_{n,\beta}(x)] \\ &= [s_{0,\beta}(x) : \ldots : s_{n,\beta}(x)], \end{aligned}$$

since $g_{\alpha\beta}(x) \neq 0$.

(2) The map is well-defined since the line bundle L is spanned and then $(s_{0,\alpha}(x), \ldots, s_{n,\alpha}(x)) \neq (0, \ldots, 0)$.

(3) i_L is holomorphic. In the affine open coordinate subsets of $\mathbb{P}^n$, $V_i = \{[z_0 : \ldots : z_n] \in \mathbb{P}^n |\, z_i \neq 0\}$, the map is described by

$$\begin{aligned} i_L^{-1}(V_i) &\longrightarrow \mathbb{C}^n \\ x &\longmapsto \left(\frac{s_{0,\alpha}(x)}{s_{i,\alpha}(x)}, \ldots, \widehat{s_{i,\alpha}(x)}, \ldots, \frac{s_{n,\alpha}(x)}{s_{i,\alpha}(x)} \right), \end{aligned}$$

and each $\frac{s_{j,\alpha}}{s_{i,\alpha}}$ is a holomorphic map outside the zero locus of $s_{i,\alpha}$ and in particular in $i_L^{-1}(V_i)$ (independent of the trivialization φ_α).

(4) The map is independent of the basis $s_0, \ldots, s_n$ of $H^0(X, L)$ up to projective transformation.

(5) The pullback of the hyperplane section defined by the equation $\sum_{i=0}^n a_i z_i = 0$ is the divisor $\mathrm{div}(s) = \mathrm{div}(\sum_{i=0}^n a_i s_i) = L$. Hence,

$$i_L^*\big(\mathcal{O}_{\mathbb{P}^n}(1)\big) = L$$
$$i_L^* H^0\big(\mathbb{P}^n, \mathcal{O}(1)\big) = H^0(X, L).$$

Let X be a compact complex manifold

Definition 1.2. A line bundle L is **very ample** if $i_L : X \longrightarrow \mathbb{P}^n$ is an embedding.

Given a section $s \in H^0(X, L)$ of a very ample divisor, the divisor $D = \mathrm{div}(s)$ is a hyperplane section under a projective embedding.

The interest of this definition relies on the fact that a compact complex manifold endowed with a very ample line bundle enjoys the properties of a submanifold of a projective space.

Example 1.3. $\mathcal{O}_{\mathbb{P}^n}(1)$ is very ample by definition.

We report the argument provided by Robert Lazarsfeld [LAZ] to introduce the concept of ampleness besides that one of very ampleness:

> [Very ampleness] turns out to be rather difficult to work with technically: already on curves it can be quite subtle to decide whether or not a given divisor is very ample. It is found to be much more convenient to focus instead on the condition that some positive multiple of D is very ample; in this case D is ample. This definition leads to a very satisfying theory, which was largely worked out in the fifties and in the sixties. The fundamental conclusion is that on a projective variety, amplitude can be characterized geometrically (which we take as the definition), cohomologically (theorem Cartan-Serre-Grothendieck) or numerically (Nakai-Moishezon-Kleiman criterion).

Definition 1.4. L is **ample** if there exists $m > 0$ such that $L^{\otimes m}$ is very ample.

Remark. A divisor D is very ample or ample if its corresponding line bundle $\mathcal{O}_X(D)$ is so.

Remark. A power of an ample divisor may have enough sections to define a projective embedding, but in general the divisor itself is not very ample. For instance, let X be a Riemann surface of genus 1. One can show that a divisor of degree 3 is very ample (proposition 9.1). By Riemann-Roch theorem for curves, $\dim H^0(X,D) = \dim H^1(X,D) + \deg(D)+1-g = 3$, thus X is a hypersurface in $\mathbb{P}^2$ and, since $\deg(i_D(X)) = \deg(D)$, X can be realized as a smooth cubic in $\mathbb{P}^2$. All the hyperplane divisors are equivalent, in particular $D \sim 3P$ where P is a flex of the cubic. Hence, $3P$ is very ample but P is not (although it is by definition ample). Indeed, again by Riemann-Roch, $\dim H^0(X,P) = \dim H^1(X,P) + \deg(P) + 1 - g = 1$, hence i_P is not an embedding.

2. Holomorphic hermitian line bundles

Let (X,ω) be a compact Kähler manifold. Let (L,h) a holomorphic line bundle on X endowed with the hermitian metric h. We denote $D = D' + D''$ its Chern connection, $\Theta(D) \in \Lambda^{1,1}T^*_X$ its curvature form[1] and $c_1(L) = \left[\frac{i}{2\pi}\Theta(D)\right]$ its first Chern class.

Let $\mathcal{U} = \{U_\alpha\}$ be a cover of open subsets of X trivializing the line bundle L and $\varphi_\alpha : \xi^{-1}(U_\alpha) \longrightarrow U_\alpha \times \mathbb{C}$. A hermitian metric h on L can be described as a collection of smooth (real) function $h_\alpha \in \mathcal{C}^\infty(U_\alpha)$, satisfying the cocycle condition $h_\alpha(x) = |g_{\alpha\beta}(x)|^2 h_\beta(x)$ in $U_\alpha \cap U_\beta$, where $g_{\alpha\beta} \in \mathcal{O}^*(U_\alpha \cap U_\beta)$ are the transition functions of the line bundle L relative to the cover $\mathcal{U}$. Then, more explicitly,

$$D' \cong_{\varphi_\alpha} \partial + \partial \log(h_\alpha) \wedge \cdot, \qquad D'' = \overline{\partial}, \qquad \Theta(D) = \overline{\partial}\partial \log(h_\alpha).$$

Notice that $\Theta(D)$ is independent of the trivialization φ_α. Indeed,

$$\overline{\partial}\partial \log(h_\alpha) = \overline{\partial}\partial \log(|g_{\alpha\beta}|^2 h_\beta) = \overline{\partial}\partial \log|g_{\alpha\beta}|^2 + \overline{\partial}\partial \log(h_\beta) = \overline{\partial}\partial \log(h_\beta),$$

since the function $\log|g_{\alpha\beta}|^2$ is pluriharmonic. Hence,

$$c_1(L) = \left[\frac{i}{2\pi}\overline{\partial}\partial \log(h)\right].$$

Equivalently, if we define the differential operator $d^c = \frac{i}{4\pi}(\overline{\partial} - \partial)$,

$$c_1(L) = \left[-dd^c \log(h)\right].$$

[1] If there is no ambiguity, we will simply denote $\Theta(L)$ the curvature form of the Chern connection of a hermitian line bundle (L,h).

3. Positivity of a line bundle

Definition 3.1. A real $(1,1)$-form ω is **positive** if for all non zero v in the real tangent space of X

$$\omega(v, Jv) > 0$$

where J is the complex structure of X.

Definition 3.2. A line bundle L is **positive** if there exists a metric on L with positive curvature form.

The positivity of a line bundle of a compact Kähler manifold is a topological property.

Theorem 3.3. *A line bundle L is positive if and only if its first Chern class may be represented by a positive form in $H^2_{dR}(X)$.*

Proof. If L is positive, the statement holds because, even if $c_1(L) = \left[\frac{i}{2\pi}\Theta(D)\right]$, the first Chern class of a line bundle does not depend on the connection the line bundle is endowed with.

Indeed, in the notation of the previous remark, given any two hermitian metric h and h' on L with curvature form respectively Θ and Θ', the quotient $\frac{h'(z)}{h(z)} := \frac{h'_\alpha(z)}{h'_\alpha(z)}$ is independent of the trivialization φ_α and thus it is a well defined positive function e^ρ for some real smooth function ρ. The formula $h' = e^\rho h$ yields

$$\Theta' = \overline{\partial}\partial\rho + \Theta.$$

In particular,

$$\left[\frac{i}{2\pi}\Theta'\right] = \left[\frac{i}{2\pi}\Theta\right].$$

Conversely, let $\frac{i}{2\pi}\vartheta$ be a real positive $(1,1)$-form representing $c_1(L)$ in $H^2_{dR}(X)$ and Θ the curvature form of the Chern connection of any hermitian metric h on L. By $\overline{\partial}\partial$-lemma[2] the equation

$$\vartheta = \overline{\partial}\partial\rho + \Theta$$

can be solved for a real smooth function ρ. It means that the hermitian metric $e^\rho h$ on L will have curvature ϑ. □

[2]For a proof of $\overline{\partial}\partial$-lemma we refer the interested reader to Corollary 3.2.10, [HYB].

4. Positivity of the hyperplane bundle

The basic example of a positive line bundle is the hyperplane bundle $\mathcal{O}_{\mathbb{P}^n}(1)$. The tautological bundle $\mathcal{O}_{\mathbb{P}^n}(-1)$, the dual of the hyperplane bundle, is the bundle whose fibre over $[z_0 : \ldots : z_n] \in \mathbb{P}^n$ is the complex line in $\mathbb{C}^n \setminus \{0\}$ through $(z_0, \ldots, z_n)$.

The standard hermitian metric in $\mathbb{C}^n$ induces by restriction a hermitian metric on the tautological bundle. In the standard coordinates of $\mathbb{C}^n$, $|(z_0, \ldots, z_n)|^2 = \sum_{i=0}^n |z_i|^2$. In the trivialization

$$\begin{aligned} \varphi_\alpha : \mathcal{O}_{\mathbb{P}^n}(-1)_{[z_0:\ldots:z_n]} &\longrightarrow [z_0 : \ldots : z_n] \times \mathbb{C} \\ (z_0, \ldots, z_n) &\longmapsto ([z_0 : \ldots : z_n], z_\alpha), \end{aligned}$$

with $\alpha = 0, \ldots, n$, the hermitian metric on the tautological bundle can be described by the collection of smooth (real) functions

$$h_\alpha = \frac{1}{|z_\alpha|^2} \sum_{i=0}^n |z_i|^2.$$

The curvature form Θ^* in $\mathcal{O}_{\mathbb{P}^n}(-1)$ is then

$$\Theta^* = \overline{\partial}\partial \log \left(\frac{1}{|z_\alpha|^2} \sum_{i=0}^n |z_i|^2 \right),$$

or more intrinsically,

$$\Theta^* = \overline{\partial}\partial \log \left(\sum_{i=0}^n |z_i|^2 \right).$$

The curvature form Θ of the dual metric in $\mathcal{O}_{\mathbb{P}^n}(1)$ is $-\Theta^*$. Hence,

$$c_1(\mathcal{O}_{\mathbb{P}^n}(1)) = -\frac{i}{2\pi} \overline{\partial}\partial \log \left(\sum_{i=0}^n |z_i|^2 \right) = dd^c \log \left(\sum_{i=0}^n |z_i|^2 \right),$$

which is just the fundamental $(1,1)$-form associated to the Fubini-Study metric in $\mathbb{P}^n$ and hence positive.

In particular, any ample line bundle L can be endowed with a hermitian metric with positive curvature. Indeed, if $i_{L^{\otimes m}}$ is a projective embedding, the pullback of a positive hermitian metric on $\mathcal{O}_{\mathbb{P}^n}(1)$ gives rise to a positive hermitian metric on $L^{\otimes m}$ and its m-th root gives a positive metric on L. Conversely, Kodaira embedding theorem grants that any positive line bundle is ample.

5. Blowing up

Blowing up is a surgery tool which allows to replace a point with a divisor *blowing up* (i.e. magnifying) the local geometry of a neighbourhood of complex manifold.

Let U be a neighbourhood of 0 in $\mathbb{C}^n$ with local coordinate $z_1, \dots, z_n$. Define

$$\begin{aligned}\tilde{U} &= \{(z,l) \in U \times \mathbb{P}^{n-1} |\, z_i l_j = z_j l_i \text{ for all } i,j = 0, \dots, n\} \\ &= \{(z,l) \in U \times \mathbb{P}^{n-1} |\, \operatorname{rk} \begin{pmatrix} z_1 & \dots & z_n \\ l_1 & \dots & l_n \end{pmatrix} \leq 1\} \\ &= \{(z,l) \in U \times \mathbb{P}^{n-1} |\, z = (z_1, \dots z_n) \in l = [l_1 : \dots : l_n] \text{ complex line}\}\end{aligned}$$

and the map

$$\begin{aligned}\pi : \ \tilde{U} &\longrightarrow U \\ (z,l) &\longmapsto z,\end{aligned}$$

such that

(1) $\pi|_{\tilde{U}\setminus\pi^{-1}(0)} : \tilde{U} \setminus \pi^{-1}(0) \longrightarrow U \setminus \{0\}$ is a biholomorphism;
(2) $E := \pi^{-1}(0) \cong \mathbb{P}^{n-1}$, called exceptional divisor.

Morally, $\tilde{U}$ consists of lines through the origin of $\mathbb{C}^n$ made disjoint. We replace a point with the directions pointing out of 0.

We can repeat the same construction for a neighbourhood of a point x of a complex manifold X of dimension n. Moreover, exploiting the fact that away from the exceptional divisor the map π is a biholomorphism, we can glue $\tilde{U}$ and $X \setminus \{x\}$ to obtain a complex compact manifold called **blowing up** or blowup of X at x.

Remark. The construction is independent of the choice of coordinates. Choose $z' = (z'_1, \dots, z'_n) = \big(f_1(z), \dots, f_n(z)\big)$ coordinates of U centred at x. Then the isomorphism

$$f : \tilde{U} \setminus E \longrightarrow \tilde{U}' \setminus E'$$

may be extended by setting $f(0,l) = (0,l')$, where

$$l'_j = \sum \frac{\partial f_j}{\partial z_i}(0) l_i.$$

In particular, the identification

$$\begin{aligned}E &\longrightarrow \mathbb{P}\left(T_{1,0}(X)_x\right) \\ (0,l) &\longmapsto \left[\sum l_i \frac{\partial}{\partial z_i}\right]\end{aligned}$$

is independent of the choice of the coordinates. This identification formalizes the previous informal remark: *we replace a point with the directions pointing out of* 0.

We describe the complex structure of a blowup providing explicit charts. In terms of coordinate $z_1, \dots, z_n$ in an open coordinate U of x, we have denoted $\tilde{U} = \{(z, l) \in U \times \mathbb{P}^{n-1} |\, z_i l_j = z_j l_i \text{ for all } i, j = 0, \dots, n\}$ and in addition we set $\tilde{U}_i = \tilde{U} \setminus \{(l_i = 0)\}$.

We endow $\tilde{U}_i$ with coordinates

$$z(i)_j = \begin{cases} \frac{z_j}{z_i} = \frac{l_j}{l_i} & j \neq i; \\ z_i & j = i. \end{cases}$$

Hence, locally

(1) $\pi|_{U_i} : (z(i)_1, \dots, z(i)_n) \longrightarrow (z_i z(i)_1, \dots, z_i, \dots, z_i z(i)_n)$;
(2) $E|_{U_i} = (z(i)_i) = (z_i)$;
(3) $(\tilde{U}_i, \varphi_i)$ is an open coordinate subset with the charts φ_i given by

$$\begin{aligned} \varphi_i : \ \tilde{U}_i &\longrightarrow \mathbb{C}^n \\ (z, l) &\longmapsto \left(\frac{z_1}{z_i}, \dots, z_i, \dots, \frac{z_n}{z_i} \right) = (z(i)_1, \dots, z(i)_i, \dots, z(n)_i). \end{aligned}$$

Without loss of generality suppose $i < j$. Then, the change of coordinates are given by

$$\begin{aligned} &\varphi_j \circ \varphi_i^{-1}|_{U_j \cap U_i}(z(i)_1, \dots, z(i)_i, \dots, z(i)_j, \dots, z(i)_n) = \\ &\qquad = \left(\frac{z(i)_1}{z(i)_j} \dots, \frac{1}{z(j)_i}, \dots, z(i)_i z(i)_j, \dots, \frac{z(i)_n}{z(i)_j} \right). \end{aligned}$$

Since $E|_{U_i} = (z_i)$, the transition functions of the line bundle $\mathcal{O}_{\tilde{X}}(E)$ are given by

$$g_{ij} = z(j)_i = \frac{z_i}{z_j} = \frac{l_i}{l_j} \qquad \text{in } \tilde{U}_i \cap \tilde{U}_j$$

and so we can realize $\mathcal{O}_{\tilde{U}}(E)$ by identifying the fibre in (z, l) with the complex line in $\mathbb{C}^n$ passing through $(l_1, \dots, l_n)$,

$$\mathcal{O}_{\tilde{U}}(E)|_{(z,l)} = \{(\lambda l_1, \dots, \lambda l_2) |\, \lambda \in \mathbb{C}\}. \tag{1}$$

In particular, the line bundle $\mathcal{O}_E(E)$ is just the tautological bundle $\mathcal{O}_{\mathbb{P}^{n-1}}(-1)$. Through the identification of E with $\mathbb{P}(T_{1,0}(X)_x)$, we obtain

$$H^0(E, -E) \cong T^{1,0}(X)_x.$$

Holomorphic functions vanishing at x in X correspond via the map π to holomorphic section of the line bundle $\mathcal{O}_{\tilde{X}}(-E)$. Hence, the differential map $H^0(U,\mathcal{I}_x) \longrightarrow T^{1,0}(U)_x$ which sends $f \in \mathcal{O}(U)$ to $d_x f$ is induced by the restriction map $\mathcal{O}_{\tilde{U}}(-E) \longrightarrow \mathcal{O}_{\tilde{E}}(-E) \longrightarrow 0$. Equivalently, the following diagram commutes:

$$\begin{array}{ccc} H^0(\tilde{U},-E) & \xrightarrow{|_E} & H^0(E,-E) \\ \pi^* \uparrow & & \| \\ H^0(U,\mathcal{I}_x) & \xrightarrow{d_x} & T^{1,0}(U)_x \end{array}$$

More precisely, after extending in series $f \in H^0(U,\mathcal{I}_x)$

$$f = \sum \frac{\partial f}{\partial z_j} z_j + O(z),$$

in the open coordinate subset $(\tilde{U}_i, \varphi_i)$ the map $\pi^* f \in H^0(\tilde{U}, -E)$ can be described by

$$\pi^* f = z_i \left(\sum \frac{\partial f}{\partial z_j} z(i)_j + O(z_i) \right).$$

It means that the previous diagram commutes:

$$\begin{array}{ccc} \sum \frac{\partial f}{\partial z_j} z(i)_j + O(z_i) & \xrightarrow{|_E} & \sum \frac{\partial f}{\partial z_j} l_j \\ \pi^* \uparrow & & \| \\ f & \xrightarrow{d_x} & \sum \frac{\partial f}{\partial z_j} l_j \end{array}$$

With Griffiths and Harris' words [GH],

> This correspondence reflects a basic aspect of the local analytic character of blowups: the infinitesimal behaviour of functions, maps, or differential forms at the point x of X is transformed into global phenomena on $\tilde{X}$.

6. Positivity of a line bundle on a blowing up

In the following we will display some properties of blowing up that can be exploit to prove Kodaira embedding theorem.

First we discuss positivity of the line bundle $\mathcal{O}_X(E)$. We construct a hermitian metric h on $\mathcal{O}_X(E)$:

(1) Let h_1 be the metric on $\mathcal{O}_{\tilde{U}}(E)$ restriction of the standard metric in $\mathbb{C}^n$ onto the complex line in $\mathbb{C}^n$ passing through $(l_1, \dots, l_n)$ (cfr. identification (1)).

(2) Let h_2 be the metric on $\mathcal{O}_{\tilde{X}\setminus E}(E)$ such that $h_2(\sigma) \equiv 1$, where $\sigma \in H^0(\tilde{X}, E)$ is a global section of $\mathcal{O}_{\tilde{X}}(E)$ with $(\sigma) = E$ (in the notation above $\sigma = (z_i)$).

(3) For $\epsilon > 0$, $U_\epsilon := \{z \in U | \, \|z\| < \epsilon\}$ and $\tilde{U}_\epsilon := \pi^{-1}(U_\epsilon)$. Let ρ_1, ρ_2 be a partion of unity relative to the cover $\{\tilde{U}_{2\epsilon}, \tilde{X} \setminus \tilde{U}_\epsilon\}$ of $\tilde{X}$ and h be a global hermitian metric given by

$$h = \rho_1 h_1 + \rho_2 h_2.$$

We will compute the positivity of the first Chern class of the hermitian line bundle (E, h).

(1) On $\tilde{X} \setminus \tilde{U}_{2\epsilon}$, $\rho_2 \equiv 1$ so $h(\sigma) \equiv 1$, i.e. in the trivialization above $h_\alpha |\sigma_\alpha|^2 = 1$, and

$$c_1(E) = -dd^c \log \frac{1}{|\sigma|^2} = 0$$

since $\log \frac{1}{|\sigma|^2}$ is a harmonic function.

(2) On $\tilde{X} \setminus \tilde{U}_{2\epsilon}$, $\rho_2 \equiv 0$ and denote

$$\begin{aligned} \pi' : \tilde{U} &\longrightarrow \mathbb{P}^{n-1} \\ (z, l) &\longmapsto l. \end{aligned}$$

Then

$$c_1(E) = -dd^c \log \|z\|^2 = -(\pi')^* \omega_{FS},$$

i.e. the pullback $(\pi')^*\omega_{FS}$ of the fundamental $(1,1)$-form associated to the Fubini-Study metric under the map π'. Hence, $c_1(E)$ is semi-positive on $\tilde{U}_\epsilon \setminus E$.

(3) On E, $-c_1(E)|_E = \omega > 0$ by continuity from the previous remark or since $h_1|_E$ is the hermitian metric induced by the standard metric in $\mathbb{C}^n$ (section 4).

To sum up,

$$c_1(-E) = \begin{cases} 0 & \text{on } \tilde{X} \setminus \tilde{U}_{2\epsilon}; \\ \geq 0 & \text{on } \tilde{U}_\epsilon; \\ > 0 & \text{on } T_{1,0}(E)_x \subset T_{1,0}(\tilde{X})_x \quad \forall x \in E. \end{cases}$$

Let (L, h_L) a hermitian positive line bundle on $\tilde{X}$. Then

$$c_1(\pi^* L) = \pi^* c_1(L).$$

For any $x \in E$ and $v \in T(\tilde{X})_x$

$$c_1(\pi^* L)(v, \overline{v}) = c_1(L)(\pi_* v, \overline{\pi_* v}) \geq 0$$

and equality holds if and only if $\pi^* v = 0$. Hence,

$$c_1(\pi^* L) = \begin{cases} \geq 0 & \text{everywhere;} \\ > 0 & \text{on } \tilde{X} \setminus E; \\ > 0 & \text{on } T_{1,0}(\tilde{X})_x / T_{1,0}(E)_x \quad \forall x \in E. \end{cases}$$

Finally, $c_1(\pi^* L^k \otimes (-E)) = kc_1(\pi^* L) - c_1(E)$ is positive on $\tilde{U}_\epsilon$ and on $\tilde{X} \setminus \tilde{U}_{2\epsilon}$ for ϵ small enough. Since $\tilde{U}_{2\epsilon} \setminus \tilde{U}_\epsilon$ is relatively compact, $-c_1(E)$ is bounded below and $c_1(\pi^* L)$ is strictly positive, then for k large enough $\pi^* L^k \otimes (-E)$ is a positive line bundle on $\tilde{X}$.

Therefore,

Proposition 6.1. *If L is a positive line bundle on a compact complex line bundle X, for any multiple nE of the exceptional divisor there exists $k > 0$ such that $L^k - nE$ is a positive line bundle on the blowing up $\tilde{X}$ (at a point).*

7. Canonical line bundle on a blowing up

Proposition 7.1. $K_{\tilde{X}} = \pi^* K_X + (n-1)E$.

Proof. We will just prove the statement in the case X admits a meromorphic n-form α (in the general case one has to compute explicitly the transition function of the canonical bundle). In terms of coordinate $z_1, \ldots, z_n$ in an open coordinate U of x, meromorphic n-form α can be expressed as

$$\alpha = \frac{f}{g} dz_1 \wedge \cdots \wedge dz_n,$$

where $f, g \in \mathcal{O}(U)$.

In the open neighbourhood $\tilde{U}_i$, the map π is given by

$$\pi|_U : (z(i)_1, \ldots, z(i)_n) \longrightarrow (z_i z(i)_1, \ldots, z_i, \ldots, z_i z(i)_n)$$

and

$$\begin{aligned} \pi^* \alpha &= \pi^* \left(\frac{f}{g} \right) d(z_i z(i)_1) \wedge \cdots \wedge d(z_i) \wedge \cdots \wedge d(z_i z(i)_n) \\ &= \pi^* \left(\frac{f}{g} \right) z_i^{n-1} d(z(i)_1) \wedge \cdots \wedge d(z_i) \wedge \cdots \wedge d(z(i)_n). \end{aligned}$$

Writing $E := \pi^{-1}(x)$ the exceptional divisor, we obtain $\mathrm{div}(\pi^* \alpha) = \pi^* \mathrm{div}(\alpha) + (n-1)E$. Away from E, $\mathrm{div}(\pi^* \alpha) = \pi^* \mathrm{div}(\alpha)$ since $\pi|_{\tilde{U} \setminus E}$ is a biholomorphism. The two arguments together yields the result. □

8. Kodaira-Akizuki-Nakano vanishing theorem

Let (X, ω) be a Kähler manifold. Let (L, h) a holomorphic line bundle on X endowed with the hermitian metric h and $\Theta(L) \in \Lambda^{1,1}T^*_X$ the curvature form of the Chern connection of the hermitian line bundle (L, h). Let $\Delta' := D'D'^* + D'^*D'$ and $\Delta'' := D''D''^* + D''^*D''$ be the (complex) Laplacian operators, $L := \omega \wedge \cdot$ be the Lefschetz operator and $\Lambda := L^*$ its adjoint.

Theorem 8.1 (Bochner-Kodaira-Nakano identity)**.**

$$\Delta'' = \Delta' + [i\Theta(L), \Lambda].$$

Proof. Kähler identities (for vector bundles) yield $D''^* = -i[\Lambda, D']$. Hence,

$$\Delta'' = [D'', D''^*] = -i[D'', [\Lambda, D']].$$

Finally, graded Jacobi identity[3] implies

$$[D'', [\Lambda, D']] = [\Lambda, [D', D'']] + [D', [D'', \Lambda]] = [\Lambda, \Theta(L)] + i[D', D'^*],$$

since $[D', D''] = D^2 = \Theta(L)$. □

Suppose that X is a compact Kähler manifold.

For any $u \in C^\infty(X, \Lambda^{p,q}T^*X \otimes L)$,

$$\int_X h(\Delta' u, u) dV = \|D'u\|^2 + \|D'^*u\|^2 \geq 0$$

$$\int_X h(\Delta'' u, u) dV = \|D''u\|^2 + \|D''^*u\|^2 \geq 0.$$

The previous relations combined with Bochner-Kodaira-Nakano identity yield

$$\|D''u\|^2 + \|D''^*u\|^2 \geq \int_X h([i\Theta(L), \Lambda]u, u)\, dV.$$

If u is $\Delta''-$harmonic,

$$0 \geq \int_X h([i\Theta(L), \Lambda]u, u)\, dV.$$

If the operator $h([i\Theta(L), \Lambda]\cdot, \cdot)$ is positive on each fibre of $\Lambda^{p,q}T^*X \otimes L$, then $u \equiv 0$ and $H^{p,q}(X, L) \cong \mathcal{H}^{p,q}_{\Delta''}(X, L) = 0$. Therefore, a positivity

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[3]Let A and B be endomorphisms of the graded module $C^\infty(\Lambda^{\cdot,\cdot}T^*X \otimes L)$ of degree respectively a and b. The graded commutator is defined as

$$[A, B] = AB - (-1)^{ab}BA.$$

If C is another endomorphism of degree c, then the graded Jacobi identity holds:

$$(-1)^{ca}[A, [B, C]] + (-1)^{ab}[B, [C, A]] + (-1)^{bc}[C, [A, B]] = 0.$$

assertion on the operator $[i\Theta(L), \Lambda]$ yields to vanishing theorems for cohomology.

Suppose for instance that $i\Theta(L)$ is a (real) positive $(1,1)$-form. We can endow X with the Kähler metric $\omega := i\Theta(L)$. Since

$$\left\{ \frac{i}{2\pi}\Theta(L), \Lambda, (\deg - n)\,\mathrm{Id} \right\}$$

is an $\mathfrak{sl}_2$-triplet,

$$h([i\Theta(L), \Lambda]u, u) = (p + q - n)|u|^2.$$

Therefore,

Theorem 8.2 (Kodaira-Akizuki-Nakano vanishing theorem)**.** *If L is a positive line bundle on a complex compact manifold X, then*

$$H^{p,q}(X, L) = H^q(X, \Omega_X^p \otimes L) = 0 \qquad \textit{for } p + q > n.$$

Corollary 1. *If L is a positive line bundle on a complex compact manifold X, then $H^q(X, K_X + L) = 0$ for $q > 0$.*

9. Cohomological characterization of very ampleness

To answer to the question whether a line bundle is very ample or not, we will recast the property of being an embedding for i_L in cohomological term, as follows:

(1) i_L is a well-defined **morphism** if L is spanned, i.e. for all $x \in X$ there exists a section of L, $s \in H^0(X, L)$, such that $s(x) \neq 0$, or equivalently the map

$$H^0(X, L) \longrightarrow L_x$$

is surjective. Notice that this map is sited in the long exact sequence induced by the short exact sequence of sheaves[4]

$$0 \longrightarrow L \otimes \mathcal{I}_x \longrightarrow L \longrightarrow L_x \longrightarrow 0.$$

where $\mathcal{I}_x \in \mathcal{O}_X$ is the ideal sheaf of holomorphic functions vanishing at x and L_x the skyscraper sheaf centred in x with global sections[5] the fiber of the line bundle L over x.

---

[4]Recall that any short exact sequence of sheaves

$$0 \to \mathcal{E} \to \mathcal{F} \to \mathcal{G} \to 0$$

induces a long exact sequence in cohomology

$$0 \to H^0(X, \mathcal{E}) \to H^0(X, \mathcal{F}) \to H^0(X, \mathcal{G}) \to H^1(X, \mathcal{E}) \to \cdots .$$

[5]In the following we will use indistinctly the same notation for the skyscraper sheaf and for the set of its global sections.

(2) i_L is **injective**. This is the case if for all $x, y \in X$, $x \neq y$, there exists a section of L, $s \in H^0(X, L)$, vanishing at x but not at y (cfr. the intrinsic definition of the morphism $i_L : X \longrightarrow \mathbb{P}(H^0(X, L))^*)$, or equivalently the map

$$H^0(X, L) \longrightarrow L_x \oplus L_y$$

is surjective. Notice that this map is sited in the long exact sequence induced by the short exact sequence

$$0 \longrightarrow L \otimes \mathcal{I}_{x,y} \longrightarrow L \longrightarrow L_x \oplus L_y \longrightarrow 0.$$

(3) i_L is an **immersion**. We need to check the injectivity of $d_x i_L$ at any $x \in X$. Complete a basis $s_1, \dots s_n$ of the hyperplane of sections in $H^0(X, L)$ vanishing at x, identified with $H^0(X, L \otimes \mathcal{I}_x) \subset H^0(X, L)$, to a basis $s_0, s_1, \dots s_n$ of $H^0(X, L)$ (so that $s_0(x) \neq 0$ since L is spanned). In an open neighbourhood of x, the map i_L is given by

$$x \longmapsto \left(\frac{s_1(x)}{s_0(x)}, \dots, \frac{s_n(x)}{s_0(x)} \right).$$

Hence, $d_x i_L$ is injective if and only if $d(\frac{s_1}{s_0}), \dots, d(\frac{s_n}{s_0})$ span the holomorphic cotangent space $T^{1,0}(X)_x$. Equivalently, we require that the map

$$d_x : H^0(X, L \otimes \mathcal{I}_x) \longrightarrow L_x \otimes T^{1,0}(X)_x \cong \mathrm{End}\big(T(X), L\big)_x$$
$$s_x \longmapsto d_x(s_\alpha),$$

is surjective. Notice that the map is well-defined since independent of the trivialization

$$d_x(s_\alpha) = d_x(g_{\alpha\beta} s_\beta) = g_{\alpha\beta}(x) d_x(s_\beta),$$

as $s_\beta(x) = 0$. Again, this map is sited in the long exact sequence induced by the short exact sequence

$$0 \longrightarrow L \otimes \mathcal{I}_x^2 \longrightarrow L \otimes \mathcal{I}_x \longrightarrow L_x \otimes T^{1,0}(X)_x \longrightarrow 0.$$

Indeed, $\mathcal{I}_x / \mathcal{I}_x^2 \cong T^{1,0}(X)_x$.

To prove that the previous maps are surjective, it would suffice to prove

$$H^1(X, L \otimes \mathcal{I}_x^2) = H^1(X, L \otimes \mathcal{I}_x) = 0.$$

Let X be a Riemann surface of genus g.

Proposition 9.1. *If D is a divisor of degree $\geq 2g + 1$, then the line bundle $\mathcal{O}_X(D)$ is very ample.*

Proof. By Kodaira-Akizuki-Nakano vanishing theorem

$H^1(X, L \otimes \mathcal{I}_x^2) = H^1(X, L - 2[x]) = H^1(X, K_X + (L - 2[x] - K_X)) = 0.$

In fact, notice that $\deg(L - 2[x] - K_X) = \deg(L) - 2 - 2g + 2 \geq 1$: the divisor $L - 2[x] - K_X$ is, then, positive, since its first Chern class is a multiple of the fundamental class of X, which is positive (recall $H^{1,1}(X, \mathbb{Z}) = H^2(X, \mathbb{Z}) \cong \mathbb{Z}$).

Analogously, $H^1(X, L \otimes \mathcal{I}_x) = 0$. □

However, unless X is a Riemann surface, the sheaves $\mathcal{I}_x^2$ and $\mathcal{I}_x$ are not invertible, which prevents us to exploit Kodaira-Akizuki-Nakano vanishing theorem. Then, one would replace x with a divisor by blowing up X at x.

10. Kodaira Embedding Theorem: a proof

Theorem 10.1 (Kodaira Embedding Theorem)**.** *If L is a compact Kähler manifold, a line bundle L is positive if and only if it is ample.*

Proof. As we discuss in section 4, the difficult implication is proving that a positive line bundle is ample, i.e. we need to prove that there exist $k > 0$ such that

(1) the restriction map
$$H^0(X, L^k) \longrightarrow L_x^k \oplus L_y^k \tag{2}$$
is surjective for any $x, y \in X$, $x \neq y$;

(2) the differential map
$$d_x : H^0(X, L^k \otimes \mathcal{I}_x) \longrightarrow L_x^k \otimes T^{1,0}(X)_x \tag{3}$$
is surjective for any $x \in X$.

Let $\pi : \tilde{X} \longrightarrow X$ be the blowing up of X at $x, y \in X$, $x \neq y$ with $E_x := \pi^{-1}(x)$ and $E_y = \pi^{-1}(y)$ exceptional divisors and $E = E_x + E_y$ - if X is a Riemann surface, $\pi = id_X$ and $E = \{x, y\}$. Consider the following commutative diagram:

$$\begin{array}{ccccc}
H^0(\tilde{X}, \pi^* L^k) & \longrightarrow & H^0(E, \pi^* L^k) & \longrightarrow & H^1(\tilde{X}, \pi^* L^k - E) \\
\pi^* \big\uparrow & & \big\| & & \\
H^0(X, L^k) & \longrightarrow & L_x^k \oplus L_y^k & &
\end{array}$$

(1) $\pi^* L^k$ is trivial along E_x and E_y, i.e.
$$\pi^* L^k|_{E_x} \cong E_x \times L_x^k \qquad \pi^* L^k|_{E_y} \cong E_y \times L_y^k,$$
so that
$$H^0(E, \pi^* L^k) = L_x^k \oplus L_y^k.$$

(2) $\pi^* : H^0(X, L^k) \longrightarrow H^0(\tilde{X}, \pi^* L^k)$ is an isomorphism. Since π is a biholomorphism away from E, π^* is injective. By Hartogs' theorem, any holomorphic section of $\pi^* L^k$ on $\tilde{X} \setminus \{E_x, E_y\} \cong X \setminus \{x, y\}$ extends to a holomorphic section of L^k on the whole X. Hence, π^* is surjective.

(3) Notice that the restriction map $H^0(\tilde{X}, \pi^* L^k) \longrightarrow H^0(E, \pi^* L^k)$ is sited in the long exact sequence induced by the short exact sequence

$$0 \longrightarrow \mathcal{O}_{\tilde{X}}(\pi^* L^k - E) \longrightarrow \mathcal{O}_{\tilde{X}}(\pi^* L^k) \longrightarrow \mathcal{O}_E(\pi^* L^k) \longrightarrow 0.$$

Hence, surjectivity of the map (2) follows from the vanishing of $H^1(\tilde{X}, \pi^* L^k - E)$ by Kodaira-Akizuki-Nakano vanishing theorem.

Indeed, by proposition 7.1,

$$\begin{aligned} \pi^* L^k - E &= \pi^* L^k - E + K_{\tilde{X}} - K_{\tilde{X}} \\ &= \pi^* L^k - E + K_{\tilde{X}} - \pi^* K_X - (n-1)E \\ &= K_{\tilde{X}} + (\pi^* L^{k_1} - nE) + \pi^*(L^{k_2} - K_X) \end{aligned}$$

for some $k > k_1 + k_2$ suitably chosen such that both the line bundle $\pi^* L^{k_1} - nE$ and $-K_X + L^{k_2}$ are positive (cfr. proposition 6.1 and 7.1). In particular, $(\pi^* L^{k_1} - nE) + \pi^*(L^{k_2} - K_X)$ is positive since product of a positive line bundle and a semipositive line bundle. Finally, Kodaira-Akizuki-Nakano vanishing theorem applies.

Similarly, one can prove surjectivity of the differential map (3) Let $\pi : \tilde{X} \longrightarrow X$ blowing up of X at $x \in X$, with $E := \pi^{-1}(x)$ exceptional divisors and E. Consider the following commutative diagram

$$\begin{array}{ccccc} H^0(\tilde{X}, \pi^* L^k - E) & \longrightarrow & H^0(E, \pi^* L^k - E) & \longrightarrow & H^1(\tilde{X}, \pi^* L^k - 2E) \\ \pi^* \uparrow & & \| & & \\ H^0(X, L^k \otimes \mathcal{I}_x) & \longrightarrow & L^k_x \otimes T^{1,0}(X)_x & & \end{array}$$

(1) Since $\pi^* L^k$ is trivial along E,

$$H^0(E, \pi^* L^k - E) = L^k_x \otimes H^0(E, -E) \cong L^k_x \otimes T^{1,0}(X)_x,$$

where the first identity holds since by dimensional reasons the injection $H^0(E, \pi^* L^k) \otimes H^0(E, -E) \hookrightarrow H^0(E, \pi^* L^k - E)$ is a bijection.

(2) $\pi^* : H^0(X, L^k \otimes \mathcal{I}_x) \longrightarrow H^0(\tilde{X}, \pi^* L^k - E)$ is an isomorphism. Indeed, holomorphic sections of the line bundle L on X vanishing at x are in bijective correspondence with holomorphic sections of the line bundle $\pi^* L^k$ on $\tilde{X}$ vanishing along E.

(3) Notice that the restriction map $H^0(\tilde{X}, \pi^*L^k - E) \longrightarrow H^0(E, \pi^*L^k - E)$ is sited in the long exact sequence induced by the short exact sequence

$$0 \longrightarrow \mathcal{O}_{\tilde{X}}(\pi^*L^k - 2E) \longrightarrow \mathcal{O}_{\tilde{X}}(\pi^*L^k - E) \longrightarrow \mathcal{O}_E(\pi^*L^k - E) \longrightarrow 0.$$

Hence, surjectivity of the map (3) follows from the vanishing of $H^1(\tilde{X}, \pi^*L^k - 2E)$ for some $k > 0$ by Kodaira-Akizuki-Nakano vanishing theorem as before.

To conclude we exhibit a compactness argument to check that the choice of k is independent of the choice of $x, y \in X$ (cfr. [NOG]). More precisely, we have established the existence of a suitable $k = k(x) > 0$ such that i_{L^k} is defined at $x \in X$ and it separates tangents in x (i.e. $d_x i_{L^k}$ is injective). The same is true in a neighbourhood U_x of x. Since X is compact, X is covered by finitely many neighbourhoods U_x, with $x \in X$, and there exists a common k_0, sufficiently large, such that $i_{L^{k_0}}$ is a holomorphic immersion on the whole X.

Consider the product

$$i_{L^k} \times i_{L^k} : X \times X \longrightarrow \mathbb{P}^n \times \mathbb{P}^n.$$

Since i_{L^k} is an immersion, it is injective in a neighbourhood W of the diagonal $\{(x, y) \in X \times X | x = y\}$. For each $(x, y) \in X \times X \setminus W$ there exists a $k = k(x, y)$ such that $i_{L^k}(x) \neq i_{L^k}(y)$. However, since $X \times X \setminus W$ is compact, there exists a common k_0 such that $i_{L^{k_0}}$ is an embedding.

□

11. Applications of Kodaira Embedding Theorem

Corollary 2. *A compact complex manifold X is a projective algebraic submanifold if and only if it has a closed positive $(1,1)$-form ω whose cohomology class $[\omega]$ is rational.*

Proof. A multiple of $[\omega]$ is an integer cohomology class. By Lefschetz $(1,1)$-theorem, there exists a line bundle L with first Chern class $c_1(L) = k[\omega]$. Since the form is positive, the line bundle L is positive. By Kodaira embedding theorem X is a projective submanifold, and by Chow's theorem (section 12) it is algebraic. □

Equivalently, the projectivity of a compact Kähler manifold can be read off the position of the Kähler cone $K_X \subset H^{1,1}(X) \cap H^2(X, \mathbb{R})$ with respect to the lattice $H^2(X, \mathbb{Z})$.

Definition 11.1. The Kähler cone $K_X \subset H^{1,1}(X) \cap H^2(X, \mathbb{R})$ is the cone in $H^2(X, \mathbb{R})$ generated by Kähler classes, i.e. cohomology classes which can be represented by a closed real positive $(1,1)$-form.

Lemma 11.2. *The Kähler cone is an open convex cone in $H^{1,1}(X) \cap H^2(X, \mathbb{R})$.*

Proof. Since $t\alpha + (1-t)\beta$ is a Kähler class for $t \in [0,1]$ and for any $\alpha, \beta \in H^{1,1}(X) \cap H^2(X,\mathbb{R})$, the Kähler cone is convex.

Choose a basis $\{\beta_i\}$ of $H^{1,1}(X) \cap H^2(X,\mathbb{R})$. Then, the open neighbourhoods of the Kähler class α

$$P_n = \left\{\alpha + \sum t_i\beta_i | \, 0 < t_i < \frac{1}{n}\right\}$$

are contained in the Kähler cone for n large enough. It suffices n greater than the ratio between the maximum value attained by the elements of the basis β_i on the unit sphere subbundle of $T_{1,0}(X)$ (with respect to any hermitian metric) and the minimum attained by α on the same unit sphere subbundle. Hence, the Kähler cone is open. □

Corollary 3. *A compact complex manifold X is a projective algebraic submanifold if and only if $K_X \cap H^2(X,\mathbb{Z}) \neq 0$.*

Proof. X has a closed positive integer $(1,1)$-form. □

Example 11.3. Any compact curve is projective (cfr. proposition 9.1).

Example 11.4. Every compact Kähler manifold with $H^{0,2} = 0$ is projective. In that case, $H^{1,1}(X) = H^2(X,\mathbb{C}) = H^2(X,\mathbb{Z}) \otimes \mathbb{C}$ and, since the Kähler cone, is open it has non-empty intersection with the lattice of integer cohomology classes.

In the following corollaries we exhibits general constructions of projective algebraic submanifolds.

Corollary 4. *If X and Y are projective algebraic submanifolds, $X \times Y$ is a projective algebraic submanifold.*

Proof. If ω_X and ω_Y are closed positive rational $(1,1)$-forms on X and Y respectively, $\pi_X^*\omega_X + \pi_Y^*\omega_Y$ is a closed positive rational $(1,1)$-form on $X \times Y$, where $\pi_X : X \times Y \longrightarrow X$ and $\pi_Y : X \times Y \longrightarrow Y$ are the projection maps. □

Corollary 5. *If X is a projective algebraic submanifold and $\pi : \tilde{X} \longrightarrow X$ is the blowing up of X at $x \in X$, then $\tilde{X}$ is a projective algebraic submanifold.*

Proof. By proposition 6.1 $\tilde{X}$ carries a positive line bundle $\pi^*L^k - E$ for $k >> 0$, with $E := \pi^{-1}(x)$ exceptional divisor. □

Corollary 6. *If $\pi : \tilde{X} \longrightarrow X$ is a finite unbranched covering of a complex compact manifold, $\tilde{X}$ is a projective algebraic submanifold if and only if X is a projective algebraic submanifold.*

Proof. Clearly, if ω is a closed positive rational $(1,1)$-form on X, then $\pi^*\omega$ is a closed positive rational $(1,1)$-form on $\tilde{X}$.

Conversely, we provide a positive $(1,1)$-form $\omega' \in H^{1,1}(X,\mathbb{Q})$ by averaging a positive $(1,1)$-form $\omega' \in H^{1,1}(\tilde{X},\mathbb{Q})$ along the fibre of π. Indeed, we define

$$\omega'(x) = \sum_{y\in\pi^{-1}(x)} (\pi^{-1})^*\omega(y) \in H^{1,1}(\tilde{X},\mathbb{Q}).$$

Notice that π^{-1} is locally well-defined since π is a local diffeomorphism. Moreover, ω' is closed, positive and of type $(1,1)$ since ω is. Finally, $[\omega']$ is also rational. Indeed, since π is a local diffeomorphism (of degree d), for any $\eta \in H^{2n-2}(X,\mathbb{Q})$

$$\int_X \omega' \wedge \eta = \frac{1}{d}\int_{\tilde{X}} \omega \wedge \pi^*\eta \in \mathbb{Q}.$$

□

As an application, we prove that any line bundle on a projective algebraic submanifold arises from a divisor.

Corollary 7. *Let X be a complex compact manifold. If E is a line bundle on X and L a positive line bundle, there exists $k > 0$ such that $L^k \otimes E$ is very ample.*

Proof. Compactness implies that for a suitable k the line bundle $L^k\otimes E$ is positive. Adapt the proof of Kodaira embedding theorem to conclude. □

Corollary 8. *If X is a projective algebraic submanifold, the map from* $\mathrm{Div}(X)$ *to* $\mathrm{Pic}(X)$ *which sends a divisor D to its associated line bundle* $\mathcal{O}_X(D)$ *is surjective.*

Proof. It suffices to prove that any line bundle E on a projective algebraic submanifold has a meromorphic section s so that $L = \mathcal{O}_X(\mathrm{div}(s))$.

Let L be a positive line bundle on X. Then for k large enough both the line bundle $L^k\otimes E$ and L^k are very ample and in particular effective. If $0 \neq s_1 \in H^0(X, L^k \otimes E)$ and $0 \neq s_2 \in H^0(X, L^k)$, then

$$E = \mathrm{div}\left(\frac{s_1}{s_2}\right).$$

□

12. Chow's Theorem

We first recall the following classical results.

Theorem 12.1. $\mathrm{Pic}(\mathbb{P}^n) \cong \mathbb{Z}$ *and it is generated by the hyperplane bundle.*

Let $S := \mathbb{C}[z_0, \ldots, z_n] = \oplus_{d\geq 0} S_d$, where S_d is the set of homogeneous polynomials of degree d.

Theorem 12.2.

$$H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(d)) = \begin{cases} S_d & d \geq 0 \\ 0 & d < 0. \end{cases}$$

The content of the latter theorem is that the line bundle $\mathcal{O}_{\mathbb{P}^n}(d)$ does not carry other holomorphic sections different from the algebraic ones, i.e. homogeneous polynomials of degree d in the projective coordinates $z_0, \ldots, z_n$. The consequence is a sort of rigidity for projective analytic subvarieties, namely irreducible subsets of $\mathbb{P}^n$ which are locally zero locus of a finite family of holomorphic functions.

Definition 12.3. A projective algebraic subvariety is the zero locus of a family of homogeneous polynomials in the projective coordinates $z_0, \ldots, z_n$ in $\mathbb{P}^n$.

Theorem 12.4 (Chow's theorem)**.** *Any projective analytic subvariety X is algebraic.*

Proof. Suppose that Y is a hypersurface or equivalently a prime divisor. By theorem 12.1 the line bundle $\mathcal{O}(Y)$ is of the form $\mathcal{O}_{\mathbb{P}^n}(d)$ for some d, and Y is the zero locus of some holomorphic section σ of $\mathcal{O}(Y)$, i.e. a homogeneous polynomial of degree d. Thus,

$$\mathcal{O}(Y) = \mathrm{div}(\sigma) = \mathrm{div}\left(\sum_{|I|=d} a_I z^I\right)$$

is algebraic.

In general, if $\dim Y = k$, for any $x \in \mathbb{P}^n$ not contained in Y we can find a $(k+1)$-plane such that the projection, say π_x, of Y along a $(n-k-2)$-plane disjoint from x is disjoint from the projection of x. It is sufficient to choose a $(n-k-1)$-plane in $\mathbb{P}^n$ through x missing Y (which exists since otherwise Y would project surjectively onto $\mathbb{P}^{k+1}$).

Since the projection π_x is a closed map, $\pi_x(X)$ is a hypersurface in $\mathbb{P}^{k+1}$, hence it satisfies a homogeneous polynomial F, which separates $\pi_x(Y)$ and $\pi_x(x)$.

Completing a projective coordinate system $z_0, \ldots, z_{k+1}$ of $\mathbb{P}^{k+1}$ to a projective coordinate system $z_0, \ldots, z_{k+1}, \ldots z_n$ of $\mathbb{P}^n$, the homogeneous polynomial $F(z_0, \ldots, z_n) := F(z_0, \ldots, z_{k+1}) \in \mathbb{C}[z_0, \ldots, z_n]$ vanishes on X (on $\pi_x^{-1}(X)$), but not at x (on $\pi_x^{-1}(x)$). $\square$

Appendix: Hirzebruch-Riemann-Roch theorem

The celebrated Hirzebruch-Riemann-Roch theorem expresses the Euler-Poincaré characteristic of a holomorphic vector bundle E on a complex compact manifold X

$$\chi(X, E) = \sum_{i=0}^{\dim_{\mathbb{C}} X} (-1)^i \dim_{\mathbb{C}} H^i(X, E)$$

in terms of the Chern classes of E and X. Combined with various vanishing theorems, it can often effectively determine the dimension of $H^0(X, E)$. This turns out to be important in the study of the geometry of X. For instance, if L is an ample line bundle, for m large enough the line bundle $L^m \otimes K_X^*$ is positive and

$$H^q(X, L^m) = H^q\big(X, K_X \otimes (L^m \otimes K_X^*)\big) = 0, \quad q > 0,$$

by Kodaira-Akizuki-Nakano vanishing theorem. Hence,

$$\chi(X, L^m) = H^0(X, L^m)$$

and the Euler characteristic of E determines the dimension of the projective space in which X can be embedded.

Chern-Weil theory establishes a homomorphism between the ad-invariant k-multilinear symmetric form on $\mathfrak{gl}(r, \mathbb{C})$ and the cohomology $H^{2*}(X, \mathbb{C})$ of X with complex coefficients in even degree.

A k-multilinear form

$$P : \mathfrak{gl}(r, \mathbb{C}) \times \ldots \times \mathfrak{gl}(r, \mathbb{C}) \longrightarrow \mathbb{C}$$

is ad-invariant if for all $G \in \mathrm{GL}(r, \mathbb{C})$

$$P(GB_1G^{-1}, \ldots, GB_kG^{-1}) = P(B_1, \ldots, B_k).$$

We will briefly describe how to associate a cohomology class in even degree to an ad-invariant k-multilinear symmetric form P on $\mathfrak{gl}(r, \mathbb{C})$. Indeed, an ad-invariant k-multilinear symmetric form P on $\mathfrak{gl}(r, \mathbb{C})$ induces a k-multilinear symmetric form

$$P : \Lambda^2(M) \otimes \mathrm{End}(E) \times \ldots \times \Lambda^2(M) \otimes \mathrm{End}(E) \longrightarrow \Lambda^{2n}_{\mathbb{C}}(X)$$

defined by $P(\alpha_1 \otimes t_1, \ldots, \alpha_r \otimes t_r) = \alpha_1 \wedge \cdots \wedge \alpha_r P(t_1, \ldots, t_r)$ and a polarized form $\tilde{P}(\alpha \otimes t) = P(\alpha \otimes t, \ldots, \alpha \otimes t)$. For the time being, the complex vector bundle E does not need to be holomorphic. Let Θ be the curvature of an arbitrary connection on the complex vector bundle E. The following facts hold:

(1) $P(\Theta)$ is a closed form (apply Bianchi identity and ad-invariance);
(2) $[P(\Theta)]$ is a cohomology class independent of the connection chosen.

Example 12.5. In the spirit of Chern-Weil homomorphism, we are led to select families of homogeneous polynomials to define families of cohomology classes, possibly describing some cohomological invariants.

Define the ad-invariant polynomials $\tilde{P}_k$, $\tilde{Q}_k$, $\tilde{R}_k$:

(1) $\det(\mathrm{Id} + tB) = \sum_{k=0} \tilde{P}_k t^k$;
(2) $\mathrm{tr}(\exp(tB)) = \sum_{k=0} \tilde{Q}_k t^k$;
(3) $\det(tB)/\det(\mathrm{Id} - e^{-tB}) = \sum_{k=0} \tilde{R}_k t^k$.

Define the k-th Chern class, the k-th Chern character and the k-th Todd class respectively:

(1) $c_k(E) = \left[\tilde{P}_k\left(\frac{i}{2\pi}\Theta\right)\right] \in H^{2k}(X, \mathbb{C})$;
(2) $ch_k(E) = \left[\tilde{Q}_k\left(\frac{i}{2\pi}\Theta\right)\right] \in H^{2k}(X, \mathbb{C})$;
(3) $td_k(E) = \left[\tilde{R}_k\left(\frac{i}{2\pi}\Theta\right)\right] \in H^{2k}(X, \mathbb{C})$.

Define the total Chern class, the total Chern character and the total Todd class respectively:

(1) $\mathrm{c}(E) = \sum c_k(E) = \left[\det(\mathrm{Id} + \frac{i}{2\pi}\Theta)\right] \in H^{2*}(X, \mathbb{C})$;
(2) $\mathrm{ch}(E) = \sum ch_k(E) = \left[\mathrm{tr}(\exp(\frac{i}{2\pi}\Theta))\right] \in H^{2*}(X, \mathbb{C}))$;
(3) $\mathrm{td}(E) = \sum td_k(E) = \left[\det(\frac{i}{2\pi}\Theta)/\det(\mathrm{Id} - e^{-\frac{i}{2\pi}\Theta})\right] \in H^{2*}(X, \mathbb{C})$.

Define the Chern classes, the Chern characters and the Todd classes of X as the respective classes of its tangent bundle.

Let E be a holomorphic vector bundle on a complex compact manifold X of complex dimension n.

Theorem 12.6 (Hirzebruch-Riemann-Roch theorem)**.**

$$\chi(X, E) = \int_X \mathrm{ch(E)td(X)}.$$

Remark. Notice that ch(E)td(X) is not in general a top degree form. What it is meant by the integral is the evaluation of the top degree component $(\mathrm{ch}(E)\mathrm{td}(X))_{2n} = \sum_{k=0}^n ch_k(E) td_{n-k}(X)$.

If L is an ample line bundle,

$$\begin{aligned}\chi(X, L^m) &= \sum_{i=0}^{n} ch_i(L^m) td_{n-i}(X) \\ &= \sum_{k=0}^{n} \left[\tilde{Q}_k \left(m \frac{i}{2\pi} \Theta(L)\right)\right] td_{n-k}(X) \\ &= \sum_{k=0}^{n} m^k \left(\left[\tilde{Q}_k \left(\frac{i}{2\pi} \Theta(L)\right)\right] td_{n-k}(X)\right).\end{aligned}$$

$\chi(X, L^m)$ is called the Hilbert polynomial of the polarized manifold (X, L), i.e. L is an ample line bundle on X. The leading coefficient of the Hilbert polynomial is $ch_n(L) = c_1(L) \wedge \cdots \wedge c_1(L) \tilde{P}'_n(id)$.

In fact, recall that for any line bundle L, $\mathrm{End}(L) = L^* \otimes L \cong \mathcal{O}_X$, as a consequence of the group structure of $\mathrm{Pic}(X)$ (more concretely, we are left to check the transition functions of those line bundles). Since

$$\mathrm{tr}(e^{t\,\mathrm{id}}) = \mathrm{tr}\left(\sum_{k=0}^{+\infty} \frac{t^k}{k!}\,\mathrm{id}^k\right) = \sum_{k=0}^{+\infty} \frac{t^k}{k!}.$$

We conclude with an easier version of the asymptotic Riemann-Roch theorem.

Theorem 12.7. *In the hypothesis above,*

$$H^0(X, L^m) = \chi(X, L^m) = \frac{(c_1(L)^n)}{n!} m^n + O(m^{n-1}).$$

References

[DBIP] Demailly, Jean-Pierre and Bertini, Jos and Illusie, Luc and Peters, Chris, *Introduction la théorie de Hodge*, Société Mathématiques de France, 1996.

[GH] Griffits, Phillip and Harris, Joseph, *Principles of Algebraic Geometry*, Wiley-Interscience Publication, 1978.

[HYB] Huybrechts, Daniel. *Complex Geometry.* Springer, 2005.

[LAZ] Lazarsfeld, Robert, *Positivity in Algebraic Geometry*, Springer, 2004.

[NOG] Noguchi, Junjiro. *Analytic Function Theory of Several Variables.*

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