Locally homogeneous flows and Anosov representations

Exercises

The lecture notes will be available at

https://www.imo.universite-paris-saclay.fr/~daniel.monclair/teaching.html

Exercise 1. Proximality

Prove the following claims:

1. If $g \in \mathrm{SL}(\mathrm{V})$ is proximal, the attracting fixed point $\ell^+(g) \in \mathbb{P}(\mathrm{V})$ of g and the repelling fixed point $\mathrm{H}^-(g) \in \mathbb{P}(\mathrm{V}^*)$ are transverse.
2. If $g \in \mathrm{SL}(\mathrm{V})$ is proximal and so is g^{-1}, then $\ell^+(g^{-1}) \subset \mathrm{H}^-(g)$.

Exercise 2. The geodesic flow of $\mathbb{H}^2_{\mathbb{R}}$

1. Prove that there is an $\mathrm{SL}_2(\mathbb{R})$-equivariant diffeomorphism between the unit tangent bundle $\mathrm{T}^1\mathbb{H}^2_{\mathbb{R}}$ and the space
$$\mathbb{L} = \left\{[v : \alpha] \in \mathbb{P}\left(\mathbb{R}^2 \oplus (\mathbb{R}^2)^*\right) \middle| \alpha(v) > 0\right\}$$
that conjugates the flow
$$\varphi^t_{\mathbb{L}}([v : \alpha]) = [\mathrm{e}^t v : \mathrm{e}^{-t}\alpha]$$
to a constant speed reparametrisation of the geodesic flow.
2. Prove that for all projective Anosov representation $\rho : \Gamma \to \mathrm{SL}_2(\mathbb{R})$, the domain
$$\widehat{\mathrm{M}}_\rho = \left\{[v : \alpha] \in \mathbb{L} \middle| \forall \eta \in \partial_\infty\Gamma,\ [v] \pitchfork \xi^*(\eta) \text{ or } \xi(\eta) \pitchfork [\alpha]\right\}$$
is equal to the whole space $\mathbb{L}$.

Exercise 3. Convex cocompact subgroups of $\mathrm{Isom}(\mathbb{H}^d_{\mathbb{R}})$

The goal of this exercise is to prove all claims that were made concerning convex cocompact subgroups of $\mathrm{Isom}(\mathbb{H}^d)$ and their geodesic flows.

Limit set, Gromov boundary and topological dynamics

1. State/explain/prove the results mentioned in the lectures for hyperbolic groups acting on their Gromov boundary in the setting of a convex cocompact subgroup $\Gamma < \mathrm{Isom}(\mathbb{H}^d)$ acting on its limit set $\Lambda_\Gamma \subset \partial_\infty \mathbb{H}^d$.

2. Prove that the non wandering set $\mathrm{NW}(\varphi_\Gamma^t)$ of the geodesic flow $\varphi_G^t : \mathsf{M}_\Gamma \to \mathsf{M}_\Gamma$ (where $\mathsf{M}_\Gamma = \Gamma\backslash \mathrm{T}^1\mathbb{H}^d$) has the following description:

$$\mathrm{NW}(\varphi_\Gamma^t) = \Gamma\backslash \left\{ (x,v) \in \mathrm{T}^1\mathbb{H}^d \,\middle|\, \lim_{t\to\pm\infty} \varphi^t(x,v) \in \Lambda_\Gamma \right\}.$$

Global structure of the geodesic flow of $\mathbb{H}^d$

Consider the bilinear form $\langle x, y\rangle = x_1y_1 + \cdots + x_dy_d - x_{d+1}y_{d+1}$ on $\mathbb{R}^{d+1}$ and the hyperboloid model $\mathbb{H}^d \subset \mathbb{R}^{d,1}$ defined as

$$\mathbb{H}^d = \left\{ x \in \mathbb{R}^{d+1} \,\middle|\, \langle x, x\rangle = -1,\ x_{d+1} > 0 \right\},$$

with Riemannian metric g defined on the tangent spaces

$$\mathrm{T}_x\mathbb{H}^d = \left\{ v \in \mathbb{R}^{d+1} \,\middle|\, \langle x, v\rangle = 0 \right\}$$

by the formula $g_x(v, v') = \langle v, v'\rangle$.

3. Describe the unit tangent bundle $\mathrm{T}^1\mathbb{H}^d$ as a submanifold of $\mathbb{R}^{d+1}\times\mathbb{R}^{d+1}$.

4. For $(x,v) \in \mathrm{T}^1\mathbb{H}^d$, describe the tangent space $\mathrm{T}_{(x,v)}\mathbb{H}^d \subset \mathbb{R}^{d+1}\times\mathbb{R}^{d+1}$.

5. Give an explicit formula for the geodesic flow $\varphi^t : \mathrm{T}^1\mathbb{H}^d \to \mathrm{T}^1\mathbb{H}^d$ as well as for the geodesic vector field $\mathcal{Z} : \mathrm{T}^1\mathbb{H}^d \to \mathrm{T}\left(\mathrm{T}^1\mathbb{H}^d\right)$ defined by

$$\mathcal{Z}(x,v) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \varphi^t(x,v) \in \mathrm{T}_{(x,v)}\left(\mathrm{T}^1\mathbb{H}^d\right).$$

6. For $(x,v) \in \mathrm{T}^1\mathbb{H}^d$, we set:

$$\mathrm{E}^s_{(x,v)} = \left\{ (y,-y) \in \mathbb{R}^{d+1}\times\mathbb{R}^{d+1} \,\middle|\, \langle x, y\rangle = \langle v, y\rangle = 0 \right\}$$
$$\mathrm{E}^u_{(x,v)} = \left\{ (y,y) \in \mathbb{R}^{d+1}\times\mathbb{R}^{d+1} \,\middle|\, \langle x, y\rangle = \langle v, y\rangle = 0 \right\}$$

   Check that $\mathrm{T}_{x,v)}\left(\mathrm{T}^1\mathbb{H}^d\right) = \mathrm{E}^s_{(x,v)} \oplus \mathrm{E}^u_{(x,v)} \oplus \mathbb{R}.\mathcal{Z}(x,v)$, and that this splitting is invariant under the geodesic flow.

7. For $(x,v) \in \mathrm{T}^1\mathbb{H}^d$, consider the bilinear form $\widetilde{g}_{(x,v)}$ on $\mathrm{T}_{(x,v)}\mathrm{T}^1\mathbb{H}^d \subset \mathbb{R}^{d+1}\times\mathbb{R}^{d+1}$ defined by

$$\widetilde{g}_{(x,v)}\Big((y_x, y_v), (y'_x, y'_v)\Big) = \langle y_x, y'_x\rangle + \langle y_v, y'_v\rangle + \langle y_x, v\rangle\langle y'_x, v\rangle.$$

   Prove that this defines a Riemannian metric on $\mathrm{T}^1\mathbb{H}^d$ for which the action of $\mathrm{Isom}(\mathbb{H}^d)$ is isometric.

8. Defining norms with the Riemannian metric $\widetilde{g}$, compute the ratio

$$\frac{\left\| \mathrm{d}\varphi^t\big|_{(x,v)}(y_x, y_v)\right\|_{\varphi^t(x,v)}}{\|(y_x, y_v)\|_{(x,v)}}$$

for $(y_x, y_v) \in \mathrm{E}^{\mathrm{s}}_{(x,v)}$ and $(y_x, y_v) \in \mathrm{E}^{\mathrm{u}}_{(x,v)}$.

9. If $\Gamma < \mathrm{Isom}(\mathbb{H}^d)$ is convex cocompact, prove that the geodesic flow on $\Gamma\backslash \mathrm{T}^1\mathbb{H}^d$ is convex cocompact.

Axiom A quotient flow

10. Prove that the geodesic flow of a convex cocompact subgroup of $\mathrm{Isom}(\mathbb{H}^d)$ is Axiom A.

Exercise 4. Real hyperbolic groups

Let $\Gamma < \mathrm{Isom}_\circ(\mathbb{H}^d_{\mathbb{R}})$ be a convex cocompact subgroup, and consider the inclusion $\iota : \mathrm{Isom}_\circ(\mathbb{H}^d_{\mathbb{R}}) = \mathrm{SO}_\circ(d,1) \hookrightarrow \mathrm{SL}_{d+1}(\mathbb{R})$. Using your favorite definition of Anosov representations, prove that $\iota : \Gamma \to \mathrm{SL}_{d+1}(\mathbb{R})$ is projective Anosov.

Exercise 5. Complex hyperbolic groups

Let $\Gamma < \mathrm{Isom}(\mathbb{H}^d_{\mathbb{C}})$ be a convex cocompact subgroup, and consider the adjoint representation $\mathrm{Ad} : \mathrm{Isom}(\mathbb{H}^d_{\mathbb{C}}) = \mathrm{SU}(d,1) \to \mathrm{SL}(\mathfrak{su}(d,1)) = \mathrm{SL}_{d^2+2d}(\mathbb{R})$. Using your favorite definition of Anosov representations, prove that $\mathrm{Ad} : \Gamma \to \mathrm{SL}_{d^2+2d}(\mathbb{R})$ is projective Anosov. What about the real hyperbolic and quaternionic hyperbolic cases?

Exercise 6. The discontinuity domain for Benoist representations

Consider a hyperbolic group Γ and a *Benoist representation* $\rho \in \mathrm{Hom}(\Gamma, \mathrm{SL}(\mathrm{V}))$. This means that ρ is projective Anosov, and preserves a properly convex domain $\Omega_\rho \subset \mathbb{P}(\mathrm{V})$ with $\mathcal{C}^1$ boundary on which Γ acts properly discontinuously and cocompactly. In this case, the limit map $\xi : \partial_\infty\Gamma \to \mathbb{P}(\mathrm{V})$ is a homeomorphism onto $\partial\Omega_\rho$, and for every $\eta \in \partial_\infty\Gamma$ we have the correspondence

$$\mathrm{T}_{\xi(\eta)}\partial\Omega_\rho = \xi^*(\eta).$$

1. Explain the identification between the tangent space $\mathrm{T}_\ell\partial\Omega_\rho$ and an element of $\mathbb{P}(\mathrm{V}^*)$.

2. Describe the subsets $\widehat{\mathrm{K}}_\rho \subset \widehat{\mathrm{M}}_\rho \subset \mathbb{L}$. Draw a picture in an affine chart of $\mathbb{P}(\mathrm{V})$ when $\dim \mathrm{V} = 3$.