Locally homogeneous flows and Anosov representations

Daniel Monclair

July 18, 2025

Introduction

Anosov representations of hyperbolic groups into semisimple Lie groups are often described as generalisations of convex cocompact subgroups of rank one Lie groups. One of the powerful tools in the study of these rank one subgroups is the geodesic flow. The goal of these notes is to discuss possible replacements for the geodesic flow in the study of Anosov representations that share similar dynamical properties.

Definition (Hyperbolic set)

Consider a smooth complete flow $\varphi^t : M \to M$ without fixed points. A subset $K \subset M$ is called *hyperbolic* if it is φ^t-invariant, compact, and there is a continuous splitting into $d\varphi^t$-invariant vector subbundles

$$TM|_K = E^s \oplus E^0 \oplus E^u$$

with the following properties:

- The bundle E^0 is one-dimensional and spanned by the vector field generating φ^t.
- There exists constants $C, a > 0$ such that

$$\forall x \in K\ \forall v \in E^s|_x\ \forall t \geq 0 \quad \left\| d\varphi^t\big|_x v \right\|_{\varphi^t(x)} \leq Ce^{-at} \|v\|_x ,$$

$$\forall x \in K\ \forall v \in E^u|_x\ \forall t \geq 0 \quad \left\| d\varphi^t\big|_x v \right\|_{\varphi^t(x)} \geq \frac{1}{C} e^{at} \|v\|_x .$$

The assumed compactness of a hyperbolic set guarantees that the contraction/dilation requirements do not depend on the choice of a Riemannian metric to define norms (up to changing the constant C). A flow is called *Anosov* if the whole manifold M is a hyperbolic set.

Definition (Periodic and non wandering points)

Consider a continuous flow $\varphi^t : M \to M$.
Its set of *periodic points* is $\mathrm{Per}(\varphi^t) = \left\{ x \in M \,\middle|\, \exists T > 0\ \varphi^T(x) = x \right\}$.
Its *non wandering set* is $\mathrm{NW}(\varphi^t) = \{ x \in M \,|\, \exists x_k \to x,\ t_k \to \infty\ \varphi^{t_k}(x_k) \to x \}$.

Definition (Axiom A flow)

A smooth complete flow $\varphi^t : \mathbf{M} \to \mathbf{M}$ without fixed points satisfies *Smale's axiom A* if $\mathrm{NW}(\varphi^t)$ is a hyperbolic set and $\mathrm{NW}(\varphi^t) = \overline{\mathrm{Per}(\varphi^t)}$.

To illustrate the rank one case, consider a discrete torsion free subgroup $\Gamma < \mathrm{Isom}(\mathbb{H}^d)$ (the reader is encouraged to think of $\mathbb{H}^d$ as the real hyperbolic space for simplicity, but this discussion is also valid for the complex and quarternionic hyperbolic spaces, as well as the Cayley hyperbolic plane). One can then consider the hyperbolic manifold $\mathbf{N}_\Gamma = \Gamma\backslash\mathbb{H}^d$, and its unit tangent bundle $\mathbf{M}_\Gamma = \mathrm{T}^1\mathbf{N}_\Gamma = \Gamma\backslash\mathrm{T}^1\mathbb{H}^d$. It comes equipped with the geodesic flow

$$\varphi_\Gamma^t : \mathbf{M}_\Gamma \to \mathbf{M}_\Gamma\,.$$

Fact

The geodesic flow $\varphi_\Gamma^t : \mathbf{M}_\Gamma \to \mathbf{M}_\Gamma$ satisfies Smale's axiom A if and only if Γ is convex cocompact.

To understand this correspondence between a dynamical notion (Smale's axiom A) and a geometric notion (convex cocompactness), we can relate the non wandering set of the geodesic flow φ_Γ^t to the *limit set* $\Lambda_\Gamma \subset \partial_\infty\mathbb{H}^d$ through the description

$$\mathrm{NW}(\varphi_\Gamma^t) = \Gamma\backslash\left\{(x,v) \in \mathrm{T}^1\mathbb{H}^d \,\middle|\, \lim_{t\to\pm\infty} \varphi^t(x,v) \in \Lambda_\Gamma\right\}.$$

An Anosov representation $\rho \in \mathrm{Hom}(\Gamma, \mathrm{G})$ of a hyperbolic group Γ into a semi-simple Lie group G is defined with respect to a pair of *opposite flag manifolds* $\mathcal{F}^+$ and $\mathcal{F}^-$, and comes with a replacement for the limit set in the form of limit maps

$$\xi^+ : \partial_\infty\Gamma \to \mathcal{F}^+ \quad \text{and} \quad \xi^- : \partial_\infty\Gamma \to \mathcal{F}^-\,.$$

Just as the geodesic flow of $\mathbb{H}^d$ is related to the boundary $\partial_\infty\mathbb{H}^d$ through the orbit projection

$$\mathrm{T}^1\mathbb{H}^d \to \left(\partial_\infty\mathbb{H}^d\right)^{(2)}.$$

Sambarino [Sam14, Sam24] showed that the restriction of the geodesic flow to its non wandering set can be generalised to Anosov representations by studying Gromov flow of Γ, essentially flows with orbit space $\partial_\infty\Gamma^{(2)}$. This strategy produces Hölder flows on compact metric spaces with hyperbolic dynamical properties (called *refraction flows*). The main idea in the work presented in these notes is to stay in the context of smooth dynamics and replace the geodesic flow with a flow on a homogeneous space for G whose orbit space is the *transverse flag space*

$$\mathcal{F}^{\pitchfork} \subset \mathcal{F}^+ \times \mathcal{F}^-$$

consisting of *transverse pairs* (i.e. the only open G-orbit in $\mathcal{F}^+ \times \mathcal{F}^-$). Let us first focus on *projective Anosov representations,* i.e. the case where $\mathrm{G} = \mathrm{SL}(\mathrm{V})$ for some finite dimensional real vector space V, $\mathcal{F}^+ = \mathbb{P}(\mathrm{V})$ and $\mathcal{F}^- = \mathbb{P}(\mathrm{V}^*)$. In this case, the transverse flag space is

$$\mathbb{P}(\mathrm{V}) \overset{\pitchfork}{\times} \mathbb{P}(\mathrm{V}^*) \overset{\text{def}}{=} \left\{\big([v],[\alpha]\big) \in \mathbb{P}(\mathrm{V}) \times \mathbb{P}(\mathrm{V}^*) \,\middle|\, \alpha(v) \neq 0\right\}.$$

When studying projective Anosov representations, we will replace the unit tangle bundle $T^1\mathbb{H}^d$ with the SL(V)-homogeneous space

$$\mathbb{L} \overset{\text{def}}{=} \{[v:\alpha] \in \mathbb{P}(V \oplus V^*) \,|\, \alpha(v) > 0\}\,,$$

and the geodesic flow with the flow $\varphi_{\mathbb{L}}^t : \mathbb{L} \to \mathbb{L}$ defined by

$$\forall [v:\alpha] \in \mathbb{L}\ \forall t \in \mathbb{R} \qquad \varphi_{\mathbb{L}}^t\big([v:\alpha]\big) = [e^t v : e^{-t}\alpha].$$

Remark

There are several ways of defining this flow: $\varphi_{\mathbb{L}}^t\big([v:\alpha]\big) = \big[e^t v : e^{-t}\alpha\big] = \big[e^{2t} v : \alpha\big] = \big[v : e^{-2t}\alpha\big]$.

It is however not direct that one can use this space to produce a quotient flow by a discrete group: the action of SL(V) on $\mathbb{L}$ is not proper, meaning that the quotient $\rho(\Gamma)\backslash\mathbb{L}$ will fail to be a manifold. This leads us to start our dynamical study with the action of a projective Anosov representation on $\mathbb{L}$, with the goal of finding a domain of proper discontinuity.

Theorem A ([DMS25a, Theorem A])

Let Γ be a torsion free hyperbolic group, and $\rho \in \mathrm{Hom}(\Gamma, \mathrm{SL}(V))$ a projective Anosov representation. There exist an open set $\widehat{\mathbf{M}}_\rho \subset \mathbb{L}$ and a closed subset $\widehat{\mathbf{K}}_\rho \subset \widehat{\mathbf{M}}_\rho$ with the following properties:

1. $\widehat{\mathbf{M}}_\rho$ and $\widehat{\mathbf{K}}_\rho$ are invariant under Γ and the flow $\varphi_{\mathbb{L}}^t$.
2. The action of Γ on $\widehat{\mathbf{M}}_\rho$ is free and properly discontinuous, and the action on $\widehat{\mathbf{K}}_\rho$ is cocompact.
3. The flow induced by $\varphi_{\mathbb{L}}^t$ on $\mathbf{M}_\rho \overset{\text{def}}{=} \Gamma\backslash\widehat{\mathbf{M}}_\rho$ satisfies Smale's axiom A, and its non wandering set is $\mathbf{K}_\rho \overset{\text{def}}{=} \Gamma\backslash\widehat{\mathbf{K}}_\rho$.

When studying Anosov representations into an arbitrary semi-simple Lie group G and pair of opposite flag manifolds $\mathcal{F}^\pm$, it is common to use representation theory to reduce the study to the projective Anosov case. We will see that this trick will not suffice in the general setting, and that replacing linear algebraic reasoning with differential geometry will allow for this generalisation.

Here is a basic plan for the five lectures:

1. Proximality and proper discontinuity (construct the manifold $\mathbf{M}_\rho$ in Theorem A).
2. The refraction flow (construct the restriction of the flow to its non wandering set).
3. Axiom A dynamics and consequences (establish the third point in Theorem A and study further dynamical properties).
4. Lie theory (a generalisation of Theorem A to a general semi-simple Lie group).
5. Geometric aspects (including non Riemannian geodesic flows).

Part 1. Proximality and proper discontinuity

The goal of this first lecture is to construct the open subset $\widehat{\mathrm{M}}_\rho$ of Theorem A and prove the proper discontinuity of the action of Γ on $\widehat{\mathrm{M}}_\rho$. We will focus on two key notions: proximality and convergence dynamics.

Notations for Grassmannian manifolds

Throughout this document we will use the notation $\mathrm{Gr}_k(\mathrm{V})$ for the Grassmannian manifold of k-dimensional vector subspaces of a real vector space V. We will consider the projective space $\mathbb{P}(\mathrm{V})$ as $\mathrm{Gr}_1(\mathrm{V})$, and use the notation $[v] \in \mathbb{P}(\mathrm{V})$ for the line spanned by v for any $v \in \mathrm{V} \setminus \{0\}$ (the use of the notation $[v]$ implicitly adds the assumption that $v \neq 0$).

We will repeatedly use the identification between the projective space $\mathbb{P}(\mathrm{V}^*)$ of the dual space $\mathrm{V}^* = \mathrm{Hom}_{\mathbb{R}}(\mathrm{V}, \mathbb{R})$ and the Grassmannian manifold $\mathrm{Gr}_{d-1}(\mathrm{V})$, where $d = \dim \mathrm{V}$, through the natural identification sending the line $[\alpha] \in \mathbb{P}(\mathrm{V}^*)$ in V^* to the hyperplane $\ker \alpha \in \mathrm{Gr}_{d-1}(\mathrm{V})$ in V. To make this identification $\mathrm{GL}(\mathrm{V})$-equivariant, the action of $\mathrm{GL}(\mathrm{V})$ on V^* is given by $g \cdot \alpha = \alpha \circ g^{-1}$.

A line $\ell \in \mathbb{P}(\mathrm{V})$ and a hyperplane $\mathrm{H} \in \mathbb{P}(\mathrm{V}^*)$ are called *transverse* if $\mathrm{V} = \ell \oplus \mathrm{H}$, and use the notation $\ell \pitchfork \mathrm{H}$ to indicate this property. It is equivalent to $\ell \notin \mathbb{P}(\mathrm{H})$, where $\mathbb{P}(\mathrm{H}) \subset \mathbb{P}(\mathrm{V})$ is the set of lines included in H. If $\ell = [v]$ and $\mathrm{H} = [\alpha]$, transversality is also characterised by $\alpha(v) \neq 0$. The set of transverse pairs will be denoted by

$$\mathbb{P}(\mathrm{V}) \overset{\pitchfork}{\times} \mathbb{P}(\mathrm{V}^*) \overset{\mathrm{def}}{=} \left\{ \left([v],[\alpha]\right) \in \mathbb{P}(\mathrm{V}) \times \mathbb{P}(\mathrm{V}^*) \,\middle|\, \alpha(v) \neq 0 \right\}.$$

We will often need to understand convergence in $\mathbb{P}(\mathrm{V})$ in terms of decomposition of vectors in varying direct sums.

Lemma 1.1

Let $(\ell_k, \mathrm{H}_k) \in \mathbb{P}(\mathrm{V}) \times \mathbb{P}(\mathrm{V}^*)$ be a sequence converging to $(\ell, \mathrm{H}) \in \mathbb{P}(\mathrm{V}) \times \mathbb{P}(\mathrm{V}^*)$ such that $\ell \pitchfork \mathrm{H}$, and fix a norm $\|-\|_{\mathrm{V}}$ on V. Consider sequences $v_k \in \ell_k \setminus \{0\}$ and $w_k \in \mathrm{H}_k$.

(1) $\displaystyle\lim_{k\to+\infty} [v_k + w_k] = \ell \iff \lim_{k\to+\infty} \frac{\|w_k\|_{\mathrm{V}}}{\|v_k\|_{\mathrm{V}}} = 0.$

(2) $\displaystyle\lim_{k\to+\infty} \|v_k\|_{\mathrm{V}} = +\infty \implies \lim_{k\to+\infty} \|v_k + w_k\|_{\mathrm{V}} = +\infty.$

Proof. Since the statements do not depend on the choice of the norm $\|-\|_{\mathrm{V}}$, we may assume that it is induced from an inner product $\langle -, - \rangle$ such that $\mathrm{H} = \ell^\perp$. Note that in all statements, using a standard contradiction argument it is enough to prove that the desired limit is obtained after passing to a subsequence. Also, both statements are equivalent when there is a subsequence for which w_k is indentically 0, so we may always choose a subsequence such that $w_k \neq 0$ and such that the limits $\overline{v} = \lim_{k\to+\infty} \frac{v_k}{\|v_k\|_{\mathrm{V}}}$ and $\overline{w} = \lim_{k\to+\infty} \frac{w_k}{\|w_k\|_{\mathrm{V}}}$ exist. Note that $\overline{v} \in \ell$ and $\overline{w} \in \mathrm{H}$.

1. If $\lim_{k\to+\infty} \frac{\|w_k\|_{\mathrm{V}}}{\|v_k\|_{\mathrm{V}}} = 0$, then $[v_k + w_k] = \left[\frac{v_k + w_k}{\|v_k\|_{\mathrm{V}}}\right] = \Bigg[\frac{v_k}{\|v_k\|_{\mathrm{V}}} + \underbrace{\frac{\|w_k\|_{\mathrm{V}}}{\|v_k\|_{\mathrm{V}}} \frac{w_k}{\|w_k\|_{\mathrm{V}}}}_{\to 0}\Bigg] \to [\overline{v}] = \ell.$

   Now assume that $\lim_{k\to+\infty} [v_k + w_k] = \ell$. If by contradiction $\frac{\|w_k\|_{\mathrm{V}}}{\|v_k\|_{\mathrm{V}}}$ does not go to 0, consider a subsequence such that this ratio has a limit $r \in (0, +\infty]$. We then find

$$[v_k + w_k] = \left[\frac{v_k + w_k}{\|w_k\|_V}\right] = \left[\frac{\|v_k\|_V}{\|w_k\|_V}\frac{v_k}{\|v_k\|_V} + \frac{w_k}{\|w_k\|_V}\right] \to \left[\frac{\overline{v}}{r} + \overline{w}\right],$$

which is not equal to ℓ since $\overline{w} \neq 0$, a contradiction.

2. The result is straightforward if w_k is bounded, so we may assume that $w_k \to \infty$. Since $\langle \overline{v}, \overline{w} \rangle = 0$, we find

$$\begin{aligned}\|v_k + w_k\|_V^2 &= \|v_k\|_V^2 + \|w_k\|_V^2 + 2\langle v_k, w_k \rangle \\ &= \|v_k\|_V^2 + \|w_k\|_V^2 + o(\|v_k\|_V \|w_k\|_V) \to +\infty.\end{aligned}$$

This proves that the sequence $\|v_k + w_k\|_V$ has no bounded subsequence, hence $\|v_k + w_k\|_V \to +\infty$.

□

Topological shortcuts

There are some simple tricks that we will use repeatedly in order to prove convergences in compact metric spaces. Let us start with the most simple (and most standard).

Lemma 1.2

Let (x_k) be a sequences in a compact metrisable space X, and X. The following are equivalent:

(1) $\lim_{k \to +\infty} x_k = x$,

(2) For any converging subsequence $x_{k_n} \to x' \in \mathsf{X}$ we must have $x' = x$.

The next result is for sequences of functions. We will say that a sequence of functions f_k convergences *locally uniformly* to a function f if it converges uniformly on all compact subsets.

Lemma 1.3

Let $f_k : \mathsf{X} \to \mathsf{Y}$ be a sequence of functions between two metrisable spaces, and assume that Y is compact. For a continuous function $f : \mathsf{X} \to \mathsf{Y}$, the following are equivalent:

(1) The sequence (f_k) converges locally uniformly to f.

(2) For any converging sequence $x_k \to x \in \mathsf{X}$, we have $f_k(x_k) \to f(x)$.

Proof. The non immediate implication is (2)⇒(1). Assume that (2) is satisfied. The first step is to notice that this condition is also valid for subsequences: if $x_n \to x$ and (k_n) is an increasing sequence of natural numbers then $f_{k_n}(x_n) \to f(x)$. For this purpose consider the sequence (x'_k) defined by $x'_k = x_{k_n}$ for n such that $k_n \leq k < k_{n+1}$, so that $x'_k \to x$ and $f_{k_n}(x_n) = f_{k_n}(x'_{k_n})$ is a subsequence of $f_k(x'_k)$.

Now consider a distance d_{Y} defining the topology of Y, a compact subset $\mathsf{K} \subset \mathsf{X}$ and assume by contradiction that (f_k) does not converge uniformly to f on K. This means there exist some $\varepsilon > 0$, an increasing sequence (k_n) of natural numbers and a sequence $x_n \in \mathsf{K}$ such that $d_{\mathsf{Y}}(f_{k_n}(x_n), f(x_n)) \geq \varepsilon$ for all $n \in \mathbb{N}$. Extracting so that $x_n \to x \in \mathsf{K}$, we find a contradiction thanks to the continuity of f. □

Combining these two results, we get the stronger result that will often be used to prove locally uniform convergences.

Lemma 1.4 (Criterion for locally uniform convergence)

Let $f_k : X \to Y$ be a sequence of functions between two metrisable spaces, and assume that Y is compact. For a continuous function $f : X \to Y$, the following are equivalent:

(1) The sequence (f_k) converges locally uniformly to f.

(2) For any converging sequence $x_k \to x \in X$ such that $f_k(x_k)$ converges to some $y \in Y$, we must have $y = f(x)$.

Note that the limit function f must be required to be continuous.

1.1 Proximality

Notation

Let $g \in \mathrm{GL}(V)$ for some vector space V of dimension $d \in \mathbb{N}$. We denote by

$$\lambda_1(g) \geq \cdots \geq \lambda_d(g)$$

the logarithms of the moduli of the complex eigenvalues of g.

We will treat proximality from three point of views: algebra (conditions of eigenvalues), differential geometry (existence of a contracting fixed point) and topological dynamics (source-sink attraction).

Proposition 1.5 (Definition of proximality)

For any $g \in \mathrm{GL}(V)$, the following are equivalent:

(1) $\lambda_1(g) > \lambda_2(g)$.

(2) The action of g on $\mathbb{P}(V)$ has an attracting fixed point (i.e. a fixed point $\ell \in \mathbb{P}(V)$ such that $\lambda_1(\mathrm{d}g|_\ell) < 0$).

(3) There exists a line $\ell^+ \in \mathbb{P}(V)$ and a hyperplane $H^- \in \mathbb{P}(V^*)$ such that $\ell^+ \notin \mathbb{P}(H^-)$ and $\lim_{k \to +\infty} g^k = \ell^+$ for all $\ell \in \mathbb{P}(V) \setminus \mathbb{P}(H)$.

An element $g \in \mathrm{GL}(V)$ satisfying these conditions is called *proximal*. If g is proximal, it has a unique pair $(\ell^+(g), H^-(g)) \in \mathbb{P}(V) \times \mathbb{P}(V^*)$ satisfying 3. and $\ell^+(g) \in \mathbb{P}(V)$ is the only attractive fixed point of g.

Proof. Let us first assume that $\lambda_1(g) > \lambda_2(g)$, and let $\ell^+ \in \mathbb{P}(V)$ be the eigendirection for the eigenvalue of maximal modulus. The transpose $g^\top \in \mathrm{GL}(V^*)$ (i.e. the map $g^\top : \alpha \mapsto \alpha \circ g$) satisfies $\lambda_1(g^\top) - \lambda_2(g^\top) = \lambda_1(g) - \lambda_2(g)$, so we can also consider the eigendirection $H^- \in \mathbb{P}(V^*)$ of $g^\top$ for its eigenvalue of maximal modulus. Then ℓ^+ and H^- are transverse, and a are fixed points of the action of g on $\mathbb{P}(V)$ and $\mathbb{P}(V^*)$ respectively. In order to understand the derivative of g at its fixed point ℓ^+, decompose g in a basis adapted to the decomposition $V = \ell^+ \oplus H^-$ as

$$g \sim \begin{pmatrix} t & 0 \\ 0 & \widetilde{g} \end{pmatrix},$$

where $\mathrm{Log}|t| = \lambda_1(g)$ and $\widetilde{g} \in \mathrm{GL}(H^-)$. The differential $\mathrm{d}g|_{\ell^+} \in \mathrm{End}(T_{\ell^+}\mathbb{P}(V)) \simeq \mathrm{End}(H^-)$ of the action of g on $\mathbb{P}(V)$ is represented by the matrix $t^{-1}\widetilde{g}$. Since $\lambda_1(\widetilde{g}) = \lambda_2(g)$, we have

$$\lambda_1(\mathrm{d}g|_\ell) = \lambda_2(g) - \lambda_1(g) < 0,$$

and $\ell^+ \in \mathbb{P}(V)$ is an attracting fixed point of g. Since $\lambda_1\left(g|_{H^-}\right) = \lambda_2(g) < \lambda_1(g)$, we can consider a norm $\|-\|$ on V and some number $c \in (0, \lambda_1(g))$ with the property that

$$\forall v \in H^- \quad \|gv\| \leq e^c \|v\|.$$

Now let $\ell \in \mathbb{P}(V) \setminus \mathbb{P}(H^-)$ and consider some vector $v \in \ell \setminus \{0\}$ decomposed into

$$v = v^+ + v^- \in \ell^+ \oplus H^-.$$

Now $\ell \notin \mathbb{P}(H^-)$ means that $v^+ \neq 0$, and we find

$$g^k \cdot \ell = g^k \cdot [v] = \left[\pm e^{k\lambda_1(g)} v^+ + g^k v^-\right] = \left[v^+ \mp e^{-k\lambda_1(g)} g^k v^-\right],$$

which by virtue of

$$\left\|e^{-k\lambda_1(g)} g^k v^-\right\| \leq e^{k(c-\lambda_1(g))} \|v^-\| \to 0$$

implies that $g^k \cdot \ell \to [v^+] = \ell^+$ as $k \to +\infty$.
We have proved that 1.⇒2. and 1.⇒3. Let us now prove that 2.⇒1. If g has an attracting fixed point $\ell \in \mathbb{P}(V)$, then by definition ℓ is an eigendirection of g. Consider any decomposition $V = \ell \oplus H$ and the associate matrix decomposition

$$g \sim \begin{pmatrix} t & * \\ 0 & \widetilde{g} \end{pmatrix}.$$

We now have that $\lambda_1(\mathrm{d}g|_\ell) = \lambda_1(\widetilde{g}) - \mathrm{Log}|t| < 0$, and every eigenvalue of g is either t or an eigenvalue of $\widetilde{g}$, thus proving that $\mathrm{Log}|t| = \lambda_1(g)$ and $\lambda_1(\widetilde{g}) = \lambda_2(g)$, and achieving the proof of $\lambda_1(g) - \lambda_2(g) > 0$ as well as the fact that ℓ is the eigendirection for the eigenvalue of maximal modulus (hence the uniqueness of the attracting fixed point).
We now turn to 3.⇒1. Both ℓ^+ and H^- must be preserved by g, in particular ℓ^+ is an eigendirection of g for some eigenvalue $t \in \mathbb{R}$. The dynamical condition implies that $\lambda_1(g|_{H^-}) < \mathrm{Log}|t|$, thus $\mathrm{Log}|t| = \lambda_1(g)$ and $\lambda_1(g|_{H^-}) = \lambda_2(g)$, showing that $\lambda_1(g) > \lambda_2(g)$ and also that ℓ^+ is the eigendirection of g for its eigenvalue of maximal modulus. Since the only hyperplane transverse to ℓ^+ and invariant by g is the eigendirection in V^* of $g^\top$, we also find the uniqueness of H^-. □

Note that if $g \in \mathrm{GL}(V)$ is proximal, then so if $g^\top \in \mathrm{GL}(V^*)$, and $H^-(g)$ is the attracting fixed point of $g^\top$. The convergence in condition (3) is always locally uniform.

Proposition 1.6 (Proximality and source-sink dynamics)

Let $g \in \mathrm{GL}(\mathrm{V})$ be a proximal element. For any open neighbourhoods $\mathrm{U}^+ \subset \mathbb{P}(\mathrm{V})$ of $\ell^+(g)$ and $\mathrm{U}^- \subset \mathbb{P}(\mathrm{V})$ of $\mathbb{P}(\mathrm{H}^-(g))$, there is some $k_0 \in \mathbb{N}$ such that

$$\forall k \geq k_0 \quad g^k\Big(\mathbb{P}(\mathrm{V}) \setminus \mathrm{U}^-\Big) \subset \mathrm{U}^+ .$$

Proof. If $g \in \mathrm{GL}(\mathrm{V})$ is proximal, then $\lambda_1\Big(g|_{\mathrm{H}^-(g)}\Big) = \lambda_2(g) < \lambda_1(g)$, so we can consider a norm $\|\cdot\|$ on V and some number $c \in (0, \lambda_1(g))$ with the property that

$$\forall v \in \mathrm{H}^-(g) \quad \|gv\| \leq \mathrm{e}^c \|v\| .$$

For any $t \geq 0$, consider the following open subsets

$$\mathrm{U}_t^+ = \Big\{[v_+ + v_-] \in \mathbb{P}(\mathrm{V}) \Big| v_+ \in \ell^+(g),\ v_- \in \mathrm{H}^-(g) \,\&\, \|v_+\| > \mathrm{e}^t \|v_-\|\Big\} ,$$
$$\mathrm{U}_t^- = \Big\{[v_+ + v_-] \in \mathbb{P}(\mathrm{V}) \Big| v_+ \in \ell^+(g),\ v_- \in \mathrm{H}^-(g) \,\&\, \|v_+\| < \mathrm{e}^{-t} \|v_-\|\Big\} .$$

For t large enough, we have $\mathrm{U}_t^+ \subset \mathrm{U}^+$ and $\mathrm{U}_t^- \subset \mathrm{U}^-$, so it is enough to prove the result with the open sets U_t^+ and U_t^-. Now consider $[v] \notin \mathrm{U}_t^-$, that is $v = v_+ + v_-$ with $\|v_+\| \geq e^{-t}\|v_-\|$ (in particular $v_+ \neq 0$). For $k \geq 0$, we have

$$\begin{aligned} \frac{\left\|g^k v_-\right\|}{\left\|g^k v_+\right\|} &\leq \frac{\mathrm{e}^{kc}\|v_-\|}{\mathrm{e}^{k\lambda_1(g)}\|v_+\|} \\ &\leq \mathrm{e}^{t-k(\lambda_1(g)-c)} . \end{aligned}$$

It follows that $g^k\Big(\mathbb{P}(\mathrm{V}) \setminus \mathrm{U}_t^-\Big) \subset \mathrm{U}_t^+$ whenever $k \geq \frac{2t}{\lambda_1(g)-c}$. □

1.2 Hyperbolic groups

Instead of diving deep into definitions of hyperbolic groups, let us focus on the properties that we will use. At the forefront stands the action on the *Gromov boundary* $\partial_\infty\Gamma$, generalising the limit set of a convex cocompact subgroup of $\mathrm{Isom}(\mathbb{H}^d)$. It arises as the boundary in an equivariant compactification $\overline{\Gamma} = \Gamma \sqcup \partial_\infty\Gamma$, meaning that $\overline{\Gamma}$ is a compact metrisable space in which Γ is discrete, open and dense, and on which left multiplication extends to a continuous action of Γ. This action has simple topological dynamics.

Proposition 1.7 (Convergence action of a hyperbolic group on its compactification)

Let Γ be a hyperbolic group.

1. If $\gamma \in \Gamma$ has infinite order, then the limits

$$\gamma^+ = \lim_{k\to+\infty} \gamma^k \quad \text{and} \quad \gamma^- = \lim_{k\to-\infty} \gamma^k$$

exist, are distinct elements of $\partial_\infty \Gamma$ and are the only fixed points of γ on $\Gamma \sqcup \partial_\infty \Gamma$. Moreover, the set of pairs (γ^+, γ^-) for $\gamma \in \Gamma$ of infinite order is dense in $\partial_\infty \Gamma^{(2)}$.

2. If a sequence $\gamma_k \in \Gamma$ satisfies $\gamma_k \to \gamma_+ \in \partial_\infty \Gamma$ and $\gamma_k^{-1} \to \gamma_- \in \partial_\infty \Gamma$, then $\gamma_k \cdot x \to \gamma_+$ for all $x \in \overline{\Gamma} \setminus \{\gamma_-\}$, and the convergence is locally uniform.

We will keep geometric group theoretic requirements at a minimum, and simply state a few properties of sequences in a hyperbolic group. We will regularly be working with sequences $\gamma_k \in \Gamma$ such that the limits $\lim \gamma_k \in \partial_\infty \Gamma$ and $\lim \gamma_k^{-1} \in \partial_\infty \Gamma$ exist and are distinct. The main reason is that this is setting to have some compatibility between the two statements of Proposition 1.7.

Proposition 1.8 (Sequences with distinct boundary limits)

Let Γ be a hyperbolic group, and $\gamma_k \in \Gamma$ a sequence such that $\gamma_+ = \lim_{k\to\infty} \gamma_k \in \partial_\infty \Gamma$ and $\gamma_- = \lim_{k\to\infty} \gamma_k^{-1} \in \partial_\infty \Gamma$ exist. If $\gamma_+ \neq \gamma_-$, then

1. γ_k has infinite order for k large enough,
2. $\lim\limits_{k\to+\infty} \gamma_k^+ = \gamma_+$ and $\lim\limits_{k\to+\infty} \gamma_k^- = \gamma_-$.

If we are in the case where the limits $\gamma_\pm$ are identical, we can use a simple trick to make them different by using the convergence property of the action on $\Gamma \sqcup \partial_\infty \Gamma$.

Lemma 1.9

Let Γ be a hyperbolic group. If $\gamma_k \in \Gamma$ is a sequence such that the limits $\gamma_+ = \lim_{k\to\infty} \gamma_k \in \partial_\infty \Gamma$ and $\gamma_- = \lim_{k\to\infty} \gamma_k^{-1} \in \partial_\infty \Gamma$ exist, then for any $f \in \Gamma$ we have

$$\lim_{k\to+\infty} f\gamma_k = f \cdot \gamma_+ \quad \text{and} \quad \lim_{k\to+\infty} \left(f\gamma_k\right)^{-1} = \gamma_- .$$

Combining Lemma 1.9 and the compactness of $\Gamma \sqcup \partial_\infty \Gamma$, we get the following useful fact:

Proposition 1.10 (Limits of sequences in a hyperbolic group)

Let Γ be a hyperbolic group, and $\delta_N \in \Gamma$ an unbounded sequence. There exist an increasing sequence $N_k \in \mathbb{N}$, an element $f \in \Gamma$ and distinct points $\gamma_+, \gamma_- \in \partial_\infty \Gamma$ such that the sequence $\gamma_k = f\delta_{N_k}$ satisfies $\lim \gamma_k = \gamma_+$ and $\lim \gamma_k^{-1} = \gamma_-$.

1.3 Limit maps and convergence dynamics

Let us postpone the definition of projective Anosov representations, which form an open set of $\mathrm{Hom}(\Gamma, \mathrm{SL}(V))$ for any hyperbolic group Γ, and focus on the main tool in the study of Anosov representations: limit maps.

Definition (Transverse limit maps)

Let Γ be a hyperbolic group, and $\rho \in \mathrm{Hom}(\Gamma, \mathrm{SL}(V))$. A *pair of transverse projective limit maps* is a pair (ξ, ξ^*) of continuous Γ-equivariant maps $\xi : \partial_\infty\Gamma \to \mathbb{P}(V)$ and $\xi^* : \partial_\infty\Gamma \to \mathbb{P}(V^*)$ such that $\xi(\eta_+) \pitchfork \xi^*(\eta_-)$ for every $(\eta_+, \eta_-) \in \partial_\infty\Gamma^{(2)}$.

There are many equivalent characterisations of (projective) Anosov representations that are stated in terms of the existence of a pair of transverse (projective) limit maps with additional dynamical properties.

Definition (Dynamics preserving limit maps)

Let Γ be a hyperbolic group, and $\rho \in \mathrm{Hom}(\Gamma, \mathrm{SL}(V))$.

- A pair of transverse projective limit maps (ξ, ξ^*) is called *dynamics preserving* if for any infinite order element $\gamma \in \Gamma$, $\rho(\gamma)$ is proximal, $\xi(\gamma^+) = \ell^+(\rho(\gamma))$ and $\xi^*(\gamma^-) = H^-(\rho(\gamma))$.
- A pair of transverse projective limit maps (ξ, ξ^*) is called *uniformly dynamics preserving* if for any unbounded sequence $\gamma_k \in \Gamma$ with boundary limit points $\gamma_+ = \lim_{k\to+\infty} \gamma_k \in \partial_\infty\Gamma$ and $\gamma_- = \lim_{k\to+\infty} \gamma_k^{-1} \in \partial_\infty\Gamma$, the actions on $\mathbb{P}(V)$ and $\mathbb{P}(V^*)$ obey the following dynamics:

$$(1)\quad \forall \ell \in \mathbb{P}(V) \qquad \ell \pitchfork \xi^*(\gamma_-) \implies \lim_{k\to+\infty} \rho(\gamma_k)\cdot \ell = \xi(\gamma_+),$$

$$(2)\quad \forall H \in \mathbb{P}(V^*) \qquad \xi(\gamma_-) \pitchfork H \implies \lim_{k\to+\infty} \rho(\gamma_k)\cdot H = \xi^*(\gamma_+),$$

  and the convergences are locally uniform.

Each one of the convergences in this last definition implies the other, and local uniformity is automatic (this is not so straightforward, but will be an easy consequence of the proof of Theorem 1.12 stated below). Keeping the same notations, a uniformly dynamics preserving pair of transverse limit maps also satisfies the following locally uniform convergences:

$$(3)\quad \forall \ell \in \mathbb{P}(V) \qquad \ell \pitchfork \xi^*(\gamma_+) \implies \lim_{k\to+\infty} \rho(\gamma_k^{-1})\cdot \ell = \xi(\gamma_-),$$

$$(4)\quad \forall H \in \mathbb{P}(V^*) \qquad \xi(\gamma_+) \pitchfork H \implies \lim_{k\to+\infty} \rho(\gamma_k^{-1})\cdot H = \xi^*(\gamma_-).$$

Also, thanks to Lemma 1.9 it is enough to have the property for sequences satisfying $\gamma_+ \neq \gamma_-$.

Proposition 1.11 (Properties of dynamics preserving transverse limit maps)

Let Γ be a hyperbolic group, $\rho \in \mathrm{Hom}(\Gamma, \mathrm{SL}(V))$ a representation and (ξ, ξ^*) a pair of transverse projective limit maps.

1. If (ξ, ξ^*) is uniformly dynamics preserving, then it is dynamics preserving.
2. There is at most one pair of dynamics preserving transverse projective limit maps.
3. If (ξ, ξ^*) is dynamics preserving, then $\xi(\eta) \subset \xi^*(\eta)$ for all $\eta \in \partial_\infty \Gamma$.
4. If $\xi(\eta) \subset \xi^*(\eta)$ for all $\eta \in \partial_\infty \Gamma$, then the maps ξ and ξ^* are injective.

Proof.

(1) Apply the definition of uniformly dynamics preserving maps to the sequence γ^k.

(2) It follows from continuity of limit maps and density of attracting fixed points of infinite order elements in $\partial_\infty \Gamma$.

(3) It comes from the fact that when $g \in \mathrm{SL}(V)$ is proximal as well as g^{-1}, then $\ell(g) \subset \mathrm{H}^-(g^{-1})$ (and the same density argument).

(4) Consider $\eta_1, \eta_2 \in \partial_\infty \Gamma$ such that $\xi(\eta_1) = \xi(\eta_2)$. If $\eta_1 \neq \eta_2$, the transversality of the limit maps yields $\xi(\eta_1) \pitchfork \xi^*(\eta_2)$, yet $\xi(\eta_1) = \xi(\eta_2) \subset \xi^*(\eta_2)$, establishing a contradiction. The same proof applies to ξ^*.

□

Theorem 1.12 (Kapovich-Leeb-Porti [KLP17])

Let Γ be a hyperbolic group. A representation $\rho \in \mathrm{Hom}(\Gamma, \mathrm{SL}(V))$ is projective Anosov if and only if it admits a uniformly dynamics preserving pair of transverse limit maps.

A proof of Theorem 1.12 will be given in Part 2.

Remark

A representation possessing a uniformly dynamics preserving pair of transverse limit maps is called an *asymptotic embedding* in [KLP17].

If we want to use the definition of proximality involving eigenvalues, then uniformity should essentially mean that $\lambda_1(\rho(\gamma)) - \lambda_2(\rho(\gamma))$ should go to $+\infty$ as γ escapes to infinity. This cannot be true along any sequence though, unless Γ is virtually cyclic. Indeed, consider an infinite order element $\delta \in \Gamma$ and another element $\beta \in \Gamma$ such that $\beta \cdot \delta^- \neq \delta^-$. Setting $\gamma_k = \delta^k \beta \delta^{-k}$, we find that $\gamma_k \to \delta^+$ (hence $\gamma_k \to \infty$ in Γ) and for any representation $\rho \in \mathrm{Hom}(\Gamma, \mathrm{SL}(V))$ we find $\lambda_i(\rho(\gamma_k)) = \lambda_i(\rho(\beta))$ for any $k \in \mathbb{N}$ and $i \in \{1, \ldots, d\}$. The way around this is to only consider sequences with distinct limit points in $\partial_\infty \Gamma$ (in the example above, we have both $\gamma_k \to \delta^+$ and $\gamma_k^{-1} \to \delta^+$).

Definition (Uniformly proximal)

Let Γ be a hyperbolic group. A representation $\rho \in \mathrm{Hom}(\Gamma, \mathrm{SL}(V))$ is called *uniformly proximal* if for any unbounded sequence $\gamma_k \in \Gamma$ with boundary limit points $\gamma_+ = \lim_{k\to+\infty} \gamma_k \in \partial_\infty\Gamma$ and $\gamma_- = \lim_{k\to+\infty} \gamma_k^{-1} \in \partial_\infty\Gamma$, we have that

$$\gamma_+ \neq \gamma_- \Longrightarrow \lim_{k\to+\infty} \lambda_1(\rho(\gamma_k)) - \lambda_2(\rho(\gamma_k)) = +\infty.$$

Proposition 1.13

Projective Anosov representations are uniformly proximal.

Proof. Since we have seen in 1.11 that the limit maps (ξ, ξ^*) of a projective asymptotic embedding $\rho \in \mathrm{Hom}(\Gamma, \mathrm{SL}(V))$ are dynamics preserving, we only need to show that ρ is uniformly proximal. For this purpose, consider an unbounded sequence $\gamma_k \in \Gamma$ with boundary limits $\gamma_+ = \lim_{k\to+\infty} \gamma_k$ and $\gamma_- = \lim_{k\to+\infty} \gamma_k^{-1}$, and assume that $\gamma_+ \neq \gamma_-$. By a standard contradiction argument, it is enough to show that there is a subsequence for which the limit $\lim_{k\to+\infty} \lambda_1(\rho(\gamma_k)) - \lambda_2(\rho(\gamma_k)) = +\infty$ occurs. Let $\|-\|_V$ be a norm on V, and for every $k \in \mathbb{N}$ consider some unit vectors $v_k^+ \in \xi(\gamma_k^+)$ and $v_k^- \in \xi^*(\gamma_k^-)$, such that $\left\|\rho(\gamma_k)v_k^-\right\|_V \geq e^{\lambda_2(\rho(\gamma_k))}$ (this is possible because $e^{\lambda_2(\rho(\gamma_k))}$ is the spectral radius of the restriction of $\rho(\gamma_k)$ to its invariant subspace $\xi^*(\gamma_k^-)$). Now let $v_k = v_k^+ + v_k^-$, and notice that since $[v_k^+] \to \xi(\gamma^+)$, the sequence $[v_k]$ lies in a compact subset of $\mathbb{P}(V) \setminus \mathbb{P}(\xi^*(\gamma_-))$. Since the limit maps are uniformly dynamics preserving, we find that

$$\lim_{k\to+\infty} \rho(\gamma_k) \cdot [v_k] = \xi(\gamma_+).$$

Now considering the decomposition $\rho(\gamma_k)v_k = \pm e^{\lambda_1(\rho(\gamma_k))} v_k^+ + \rho(\gamma_k)v_k^- \in \xi(\gamma_k^+) \oplus \xi^*(\gamma_k^-)$, the convergence $\rho(\gamma_k) \cdot [v_k] \to \xi(\gamma_+)$ means that

$$\lim_{k\to+\infty} \frac{\left\|\rho(\gamma_k))v_k^-\right\|_V}{\left\|e^{\lambda_1(\rho(\gamma_k))} v_k^+\right\|_V} = 0.$$

We get by the requirement on v_k^- that $\lambda_1(\rho(\gamma_k)) - \lambda_2(\rho(\gamma_k)) \to +\infty$. □

Corollary 1.14

Let Γ be a hyperbolic group, and $\rho \in \mathrm{Hom}(\Gamma, \mathrm{SL}(V))$ a projective Anosov representation. For any unbounded sequence $\gamma_k \in \Gamma$ with boundary limit points $\gamma_+ = \lim_{k\to+\infty} \gamma_k \in \partial_\infty\Gamma$ and $\gamma_- = \lim_{k\to+\infty} \gamma_k^{-1} \in \partial_\infty\Gamma$, we have that

$$\gamma_+ \neq \gamma_- \Longrightarrow \lim_{k\to+\infty} \lambda_1(\gamma_k) = +\infty.$$

Proof. From $\lambda_1 + \cdots + \lambda_d = 0$, one finds $\lambda_1 - \lambda_2 = \lambda_1 + (\lambda_1 + \lambda_3 + \cdots + \lambda_d) \leq d\lambda_1$, so $\lambda_1(\rho(\gamma_k)) \geq (\lambda_1(\rho(\gamma_k)) - \lambda_2(\rho(\gamma_k)))/d$. □

Theorem 1.15 ([GGKW17])

Let Γ be a hyperbolic group. A representation $\rho \in \mathrm{Hom}(\Gamma, \mathrm{SL}(V))$ is projective Anosov if and only if it admits a pair of transverse and dynamics preserving limit maps and is uniformly proximal.

The statement in [GGKW17] may appear different at a first glance, as it involves the *stable word length* $|-|_\infty$ rather that distinct limit points in $\partial_\infty\Gamma$, but it is rather simple to prove that the conditions are equivalent. It is also possible to characterise Anosov representations without invoking limit maps, asking for a quantitative version of uniform proximality.

Theorem 1.16 (Kassel-Potrie [KP22])

Let Γ be a hyperbolic group. A representation $\rho \in \mathrm{Hom}(\Gamma, \mathrm{SL}(V))$ is projective Anosov if and only if there is a constant $c > 0$ such that

$$\forall \gamma \in \Gamma \quad \lambda_1(\rho(\gamma)) - \lambda_2(\rho(\gamma)) \geq c|\gamma|_\infty .$$

The fact that an Anosov representation satisfies this condition is rather straightforward, the difficult part is the converse, and goes through another equivalent characterisation (that does not need limit maps) due to Bochi-Potrie-Sambarino [BPS19] and Kapovich-Leeb-Porti [KLP17].

1.4 Action on the flow space

We finally come back to our flow space

$$\mathbb{L} = \{[v : \alpha] \in \mathbb{P}(V \oplus V^*) \,|\, \alpha(v) > 0\}$$

equipped with the homogeneous flow

$$\varphi^t_{\mathbb{L}}\big([v : \alpha]\big) = [e^t v : e^{-t}\alpha].$$

Note that the action of $\mathrm{SL}(V)$ on $\mathbb{P}(V \oplus V^*)$ is given by $g \cdot [v : \alpha] = [g \cdot v : \alpha \circ g^{-1}]$.

Definition (Proper discontinuity domain)

For a projective Anosov representation $\rho \in \mathrm{Hom}(\Gamma, \mathrm{SL}(V))$, we introduce the following subset:

$$\widehat{M}_\rho \overset{\text{def}}{=} \left\{[v : \alpha] \in \mathbb{L} \,\middle|\, \forall \eta \in \partial_\infty\Gamma,\ [v] \pitchfork \xi^*(\eta) \text{ or } \xi(\eta) \pitchfork [\alpha]\right\}.$$

Theorem 1.17 ([DMS25a, Theorem A.(1)])

Let $\rho \in \mathrm{Hom}(\Gamma, \mathrm{SL}(V))$ be a projective Anosov representation.

1. The subset $\widehat{M}_\rho \subset \mathbb{L}$ is open, non empty, invariant under $\rho(\Gamma)$ and the flow $\varphi^t_{\mathbb{L}}$.
2. The action of Γ on $\widehat{M}_\rho$ is properly discontinuous.
3. If Γ is torsion-free, then the action of Γ on $\widehat{M}_\rho$ is free.

Proof of Theorem 1.17 (1). The invariance under Γ follows from the equivariance of the limit maps, and invariance under the flow $\varphi^t_{\mathbb{L}}$ follows directly from the definition. Fix norms on V and V^*, and consider the function

$$\theta : \begin{cases} \mathbb{P}(V)\times\mathbb{P}(V^*) & \to & \mathbb{R}_{\geq 0} \\ ([v],[\alpha]) & \mapsto & \frac{|\alpha(v)|}{\|v\|\|\alpha\|} \end{cases}$$

Using the continuity of θ and the compactness of $\partial_\infty\Gamma$, rewriting

$$\widehat{M}_\rho = \left\{[v:\alpha]\in\mathbb{L}\,\middle|\,\forall\,\eta\in\partial_\infty\Gamma,\ \theta\Big([v],\xi^*(\eta)\Big)+\theta\Big(\xi(\eta),[\alpha]\Big)\neq 0\right\}$$

shows that $\widehat{M}_\rho$ is open. For non-emptyness we may assume that Γ is infinite (otherwise $\partial_\infty\Gamma = \emptyset$ and $\widehat{M}_\rho = \mathbb{L}$), and consider the set

$$\widehat{K}_\rho \overset{\text{def}}{=} \left\{[v:\alpha]\in\mathbb{L}\,\middle|\,\exists\eta_+\neq\eta_-\in\partial_\infty\Gamma\ \ [v]=\xi(\eta_+),\ [\alpha]=\xi^*(\eta_-)\right\}.$$

It is non empty (since we assume $\partial_\infty\Gamma$ to be non empty, and it cannot consist in a single point), and contained in $\widehat{M}_\rho$. Indeed, consider $[v:\alpha]\in\widehat{K}_\rho$, $\eta_+\neq\eta_-\in\partial_\infty\Gamma$ such that $[v]=\xi(\eta_+)$ and $[\alpha]=\xi^*(\eta_-)$, and $\eta\in\partial_\infty\Gamma$. Let us consider the following possibilities:

- If $\eta\neq\eta_+$, then $[v]\pitchfork\xi^*(\eta)$ by transversality of the limit maps.
- If $\eta=\eta_+$, then $\eta\neq\eta_-$ and $\xi(\eta)\pitchfork[\alpha]$ by transversality of the limit maps.

This proves the inclusion $\widehat{K}_\rho\subset\widehat{M}_\rho$, hence the non vacuity of $\widehat{M}_\rho$. □

Before we prove the proper discontinuity, we need to study the dynamics of a projective Anosov representation $\rho:\Gamma\to SL(V)$ acting on V. With this in mind, it is important to notice that contraction for the action on the projective space translates to expansion for the linear action, as a contracting fixed point in $\mathbb{P}(V)$ is the eigendirection for the largest eigenvalue.

Proposition 1.18 ([DMS25a, Lemma 3.2])

Let $\rho\in\mathrm{Hom}(\Gamma,SL(V))$ be a projective Anosov representation, and let $\gamma_k\in\Gamma$ be a sequence admitting limits $\gamma_+=\lim\gamma_k\in\partial_\infty\Gamma$ and $\gamma_-=\lim\gamma_k^{-1}\in\partial_\infty\Gamma$. For any sequence $v_k\to v\in V\setminus\{0\}$ such that $[v]\pitchfork\xi^*(\gamma_-)$, one has $\rho(\gamma_k)v_k\to\infty$ as $k\to\infty$.

Proof. Let us start by assuming that $\gamma_+\neq\gamma_-$. Let $\gamma_k^\pm\in\partial_\infty\Gamma$ be the attracting and repelling points of γ_k, and can consider the decomposition

$$v_k = v_k^+ + v_k^- \in \xi(\gamma_k^+)\oplus\xi^*(\gamma_k^-).$$

Since $\gamma_\pm=\lim_{k\to+\infty}\gamma_k^\pm$ (Proposition 1.8), we also have $v_k^+\to v_+$ and $v_k^-\to v_-$ where

$$v = v_+ + v_- \in \xi(\gamma_+)\oplus\xi^*(\gamma_-).$$

The assumption that $[v]\pitchfork\xi^*(\gamma_-)$ means that $v_+\neq 0$, hence $\rho(\gamma_k)v_k^+=\pm e^{\lambda_1(\gamma_k)}v_k^+\to\infty$ by Corollary 1.14, and $\rho(\gamma_k)v_k\to\infty$ by Lemma 1.1.
Now assume that $\gamma_+\neq\gamma_-$, and consider some element $f\in\Gamma$ such that $f\cdot\gamma_-\neq\gamma_-{}^a$. Set $\delta_k=\gamma_k f^{-1}$, so that $\delta_k\to\gamma_+$ and $\delta_k^{-1}\to f\cdot\gamma_-\neq\gamma_+$ (Lemma 1.9). Applying the first case to the sequence $w_k=\rho(f)v_k$ that satisfies $w_k\to\rho(f)v$ and

$$[\rho(f)v]=\rho(f)\cdot[v]\pitchfork\rho(f)\cdot\xi^*(\gamma_-)=\xi^*(f\cdot\gamma_-),$$

we find that $\rho(\delta_k)w_k \to \infty$, which is the desired property since

$$\rho(\delta_k)w_k = \rho(\gamma_k f^{-1})\rho(f)v_k = \rho(\gamma_k)v_k.$$

□

[a] In general, given $\eta \in \partial_\infty\Gamma$, the existence of $f \in \Gamma$ such that $f \cdot \eta \neq \eta$ is only guaranteed if Γ is *non elementary*, i.e. not virtually isomorphic to $\mathbb{Z}$. But in the elementary case, $\partial_\infty\Gamma$ consist of two points $\partial_\infty\Gamma = \{\eta^+, \eta^-\}$, and any sequence with $\gamma_k \to \eta^+$ satisfies $\gamma_k^{-1} \to \eta^-$, so the case $\gamma_+ = \gamma_-$ never occurs for elementary hyperbolic groups.

Proof of Theorem 1.17 (2). Contradicting proper discontinuity amounts to assuming the existence of:

- Elements $[v:\alpha], [w:\beta] \in \widehat{\mathbf{M}}_\rho$
- A sequence of elements $[v_k:\alpha_k] \in \widehat{\mathbf{M}}_\Gamma$ such that $[v_k:\alpha_k] \to [v:\alpha] \in \widehat{\mathbf{M}}_\rho$,
- A sequence of elements $\gamma_k \in \Gamma$ such that $\gamma_k \to \infty$ and $\rho(\gamma_k)\cdot[v_k:\alpha_k] \to [w:\beta]$.

The convergence $[v_k:\alpha_k] \to [v:\alpha]$ means that there is a sequence $\lambda_k \in \mathbb{R}\setminus\{0\}$ such that $\lambda_k v_k \to v$ and $\lambda_k\alpha_k \to \alpha$. Replacing v_k by $\lambda_k v_k$ and α_k by $\lambda_k\alpha_k$, we arrange that $v_k \to v$ and $\alpha_k \to \alpha$. Up to a extracting a subsequence, we may assume that the limits $\gamma_+ = \lim \gamma_k \in \partial_\infty\Gamma$ and $\gamma_- = \lim \gamma_k^{-1} \in \partial_\infty\Gamma$ exist.
By virtue of $[v:\alpha] \in \widehat{\mathbf{M}}_\rho$, either $[v] \pitchfork \xi^*(\gamma_-)$ or $[\alpha] \pitchfork \xi(\gamma_-)$. The first condition $[v] \pitchfork \xi^*(\gamma_-)$ implies via Proposition 1.18 that $\rho(\gamma_k)v_k \to \infty$, which is absurd because of the assumption $\rho(\gamma_k)\cdot[v_k:\alpha_k] \to [w:\beta] \in \widehat{\mathbf{M}}_\rho$. Indeed, the latter means that there is a sequence $\mu_k \in \mathbb{R}\setminus\{0\}$ such that $\mu_k\rho(\gamma_k)v_k \to w$ and $\mu_k\alpha_k \circ \rho(\gamma_k^{-1}) \to \beta$, and $\rho(\gamma_k)v_k \to \infty$ implies $\mu_k \to 0$. Now

$$0 < \beta(w) = \lim_{k\to\infty} \left(\mu_k\alpha_k \circ \rho(\gamma_k^{-1})\right)\left(\mu_k\rho(\gamma_k)v_k\right) = \lim_{k\to\infty} \mu_k^2 \underbrace{\alpha_k(v_k)}_{\to\alpha(v)} = 0,$$

a contradiction. For the other condition $[\alpha] \pitchfork \xi(\gamma_-)$, repeating the same argument for the dual representation leads to the analogous absurdity, and therefore the Γ-action on $\widehat{\mathbf{M}}_\rho$ is properly discontinuous. □

Proof of Theorem 1.17 (3). Since Γ acts properly discontinuously on $\widehat{\mathbf{M}}_\rho$, stabilisers of points are finite. Assuming that Γ is torsion free, this means that stabilisers are trivial, i.e. the action is free. □

Part 2. The refraction flow

Consider a hyperbolic group Γ and a projective Anosov representation $\rho \in \mathrm{Hom}(\Gamma, \mathrm{SL}(V))$ with dynamics preserving transverse limit maps (ξ, ξ^*). Recall the closed subset $\widehat{\mathbf{K}}_\rho \subset \mathbb{L}$ introduced in the proof of Theorem 1.17:

Definition (Lifted basic set)

Let Γ be a hyperbolic group, and $\rho \in \mathrm{Hom}(\Gamma, \mathrm{SL}(V))$ a projective Anosov representation. The *lifted basic set* is the subset

$$\widehat{K}_\rho \overset{\text{def}}{=} \left\{ [v:\alpha] \in \mathbb{L} \,\middle|\, \exists (\eta_+, \eta_-) \in \partial_\infty \Gamma^{(2)} \;\; [v] = \xi(\eta_+),\; [\alpha] = \xi^*(\eta_-) \right\}.$$

Note that by construction, the lifted basic set $\widehat{K}_\rho$ is invariant under the flow $\varphi^t_{\mathbb{L}} : \mathbb{L} \to \mathbb{L}$. It can be described in terms of *Hopf coordinates* on $\mathbb{L}$. For this, fix a norm $\|-\|_V$ on V and consider the map

$$\mathcal{H} : \begin{cases} \mathbb{L} & \to & \mathbb{P}(V) \overset{\pitchfork}{\times} \mathbb{P}(V^*) \times \mathbb{R} \\ [v:\alpha] & \mapsto & \left([v], [\alpha], \mathrm{Log} \frac{\|v\|_V}{\sqrt{\alpha(v)}} \right) \end{cases}$$

It is a diffeomorphism mapping $\widehat{K}_\rho$ onto $\Lambda^{\pitchfork}_\rho \times \mathbb{R}$ where $\Lambda^{\pitchfork}_\rho \subset \mathbb{P}(V) \overset{\pitchfork}{\times} \mathbb{P}(V^*)$ is the *transverse limit set*

$$\Lambda^{\pitchfork}_\rho \overset{\text{def}}{=} \left\{ \left(\xi(\eta_+), \xi^*(\eta_-) \right) \,\middle|\, (\eta_+, \eta_-) \in \partial_\infty \Gamma^{(2)} \right\}.$$

This description allows us to notice that $\widehat{K}_\rho$ is closed in $\mathbb{L}$.

Theorem 2.1 (Sambarino)

The action of Γ on $\widehat{K}_\rho$ is properly discontinuous and cocompact.

Remark

We already proved that $\widehat{K}_\rho \subset \widehat{M}_\rho$, so the proper discontinuity is a consequence of Theorem 1.17. However, the proof of cocompactness will give another proof of the proper discontinuity of the action on $\widehat{K}_\rho$.

2.1 Flows and discrete groups

2.1.1 Γ-flows

When we are only interested in qualitative properties of a flow $\varphi^t : X \to X$ (meaning its topological dynamics), it would be natural to want to work with the quotient space $X/\mathbb{R}$ of orbits of the flow φ^t. This is usually a very bad setting as any interesting dynamical property will forbid this quotient space to be Hausdorff. For example, topological transitivity (i.e. the existence of a dense orbit) means that the quotient has a dense point.

The way around this is lifting to a flow $\widehat{\varphi}^t$ on a cover $\widehat{X}$ of X where the flow has trivial dynamics, so that $\widehat{X}/\mathbb{R}$ is well behaved. We will only consider Galois coverings, i.e. $X = \Gamma \backslash \widehat{X}$ where Γ is a countable group acting properly discontinuouly and freely by homeomorphisms on $\widehat{X}$. The flow $\varphi^t : X \to X$ then lifts to $\widehat{\varphi}^t : \widehat{X} \to \widehat{X}$ that commutes with the action of Γ.

Definition (Γ-flow)

Consider a discrete group Γ. A Γ-flow is the data of a continuous flow $\varphi^t : \mathsf{X} \to \mathsf{X}$ on a compact metrisable space, such that $\mathsf{X} = \Gamma \backslash \widehat{\mathsf{X}}$ where Γ acts properly discontinuously on $\widehat{\mathsf{X}}$, and φ^t lifts to a flow $\widehat{\varphi}^t : \widehat{\mathsf{X}} \to \widehat{\mathsf{X}}$ that commutes with Γ and that defines a free and proper action of $\mathbb{R}$ on $\widehat{\mathsf{X}}$. The *orbit space* of a Γ-flow is the quotient $\widehat{\mathsf{X}}/\mathbb{R}$ under the flow $\widehat{\varphi}^t$.

Remark

When introducing a Γ-flow, we will only write $\varphi^t : \mathsf{X} \to \mathsf{X}$ and implicitly fix the covering $\widehat{\mathsf{X}} \to \mathsf{X}$ as well as the lifted flow $\widehat{\varphi}^t : \widehat{\mathsf{X}} \to \widehat{\mathsf{X}}$.

2.1.2 Gromov geodesic flows

Consider a hyperbolic group Γ, and a Γ-flow $\varphi^t : \mathsf{X} \to \mathsf{X}$. Since Γ acts properly discontinuously and cocompactly on $\widehat{\mathsf{X}}$, endowed with the right distance $\widehat{\mathsf{X}}$ is hyperbolic with boundary $\partial_\infty \mathsf{X} \simeq \partial_\infty \Gamma$. Concretely, given sequences $\widehat{x}_k \in \widehat{\mathsf{X}}$ and $\gamma_k \in \Gamma$ such that $\gamma_k^{-1} \cdot \widehat{x}_k$ lies in a compact subset of $\widehat{\mathsf{X}}$, then $\widehat{x}_k$ converges to a point $\eta \in \partial_\infty \Gamma$ if and only if γ_k converges to η.

Definition (Gromov flows)

Let Γ be a hyperbolic group.

- A *Gromov flow* for Γ is a Γ-flow $\varphi^t : \mathsf{X} \to \mathsf{X}$ such that the map

$$\begin{cases} \widehat{\mathsf{X}}/\mathbb{R} & \to & \partial_\infty \Gamma^{(2)} \\ \widehat{x} \bmod \mathbb{R} & \mapsto & \left(\lim\limits_{t \to +\infty} \widehat{\varphi}^t(\widehat{x}) \, , \lim\limits_{t \to -\infty} \widehat{\varphi}^t(\widehat{x}) \right) \end{cases}$$

  is well defined and is a homeomorphism.

- A *coarse Gromov flow* for Γ is a Γ-flow $\varphi^t : \mathsf{X} \to \mathsf{X}$ such that the map

$$\begin{cases} \widehat{\mathsf{X}}/\mathbb{R} & \to & \partial_\infty \Gamma^{(2)} \\ \widehat{x} \bmod \mathbb{R} & \mapsto & \left(\lim\limits_{t \to +\infty} \widehat{\varphi}^t(\widehat{x}) \, , \lim\limits_{t \to -\infty} \widehat{\varphi}^t(\widehat{x}) \right) \end{cases}$$

  is well defined and is continuous, surjective and proper.

Notation

$$\forall \widehat{x} \in \widehat{\mathsf{X}} \qquad \widehat{x}^+ \overset{\text{def}}{=} \lim_{t \to +\infty} \widehat{\varphi}^t(\widehat{x}) \quad \text{and} \quad \widehat{x}^- \overset{\text{def}}{=} \lim_{t \to -\infty} \widehat{\varphi}^t(\widehat{x}).$$

The typical non example is the horocyclic flow of a closed surface Σ of genus $g \geq 2$. It is a Γ-flow where $\Gamma = \pi_1 \Sigma$, but not a coarse Gromov flow (flow lines in the Γ-cover have the same endpoint for $t \to \pm\infty$). Coarse Gromov flows are essentially Γ-flows for which the orbits of $\widehat{\varphi}^t$ are quasi-geodesics. One way of constructing (coarse) Gromov flows is to consider *geometric* actions of Γ on negatively curved spaces. Consider a geodesic metric space $(\mathsf{Y}, \mathsf{d}_\mathsf{Y})$ endowed with an isometric action of Γ that we assume to be faithful, properly discontinuous and cocompact. Then Γ also acts properly discontinuously on the space of geodesics

$$\mathscr{G}(\mathbf{Y}) \overset{\text{def}}{=} \left\{g : \mathbb{R} \to \mathbf{Y} \,\middle|\, \forall t, t' \in \mathbb{R}\ \mathrm{d}_{\mathbf{Y}}(g(t), g(t')) = |t - t'|\right\},$$

and commutes with the *geodesic flow*

$$\widehat{\psi}^t : \begin{cases} \mathscr{G}(\mathbf{Y}) & \to & \mathscr{G}(\mathbf{Y}) \\ g & \mapsto & g(t + -) \end{cases}$$

allowing us to consider the quotient flow

$$\psi^t : \Gamma\backslash\mathscr{G}(\mathbf{Y}) \to \Gamma\backslash\mathscr{G}(\mathbf{Y}).$$

Fact

- If $(\mathbf{Y}, \mathrm{d}_{\mathbf{Y}})$ is δ-hyperbolic for some $\delta > 0$, then $\psi^t : \Gamma\backslash\mathscr{G}(\mathbf{Y}) \to \Gamma\backslash\mathscr{G}(\mathbf{Y})$ is a coarse Gromov flow.
- If $(\mathbf{Y}, \mathrm{d}_{\mathbf{Y}})$ is $\mathrm{CAT}(-\kappa)$ for some $\kappa > 0$, then $\psi^t : \Gamma\backslash\mathscr{G}(\mathbf{Y}) \to \Gamma\backslash\mathscr{G}(\mathbf{Y})$ is a Gromov flow.

In particular, for a convex cocompact subgroup $\Gamma < \mathrm{Isom}(\mathbb{H}^d)$, the restriction of the geodesic flow $\varphi^t_\Gamma : \mathbf{M}_\Gamma \to \mathbf{M}_\Gamma$ (where $\mathbf{M}_\Gamma = \Gamma\backslash \mathrm{T}^1\mathbb{H}^d$) to its non wandering set is a Gromov geodesic flow.

A systematic construction of a coarse Gromov flow is to consider the Cayley graph with respect to a finite generating subset $\mathrm{S} \subset \Gamma$, i.e. for any $\gamma \in \Gamma \setminus \{\mathbf{1}_\Gamma\}$ there are $s_1, \ldots, s_k \in \mathrm{S} \cup \mathrm{S}^{-1}$ such that $\gamma = s_1 \cdots s_k$. The smallest $k \in \mathbb{N}$ such that γ can be decomposed in this way is called the *word length* of γ, denoted by $|\gamma|_{\mathrm{S}}$ (by convention we assume that $\mathbf{1}_\Gamma \notin \mathrm{S}$ and set $|\mathbf{1}_\Gamma| = 0$).

The *Cayley graph* of Γ with respect to S is the graph whose vertices are elements of Γ and two verticles γ_1, γ_2 are joined by an edge if and only if $\gamma_1^{-1}\gamma_2 \in \mathrm{S} \cup \mathrm{S}^{-1}$. When we refer to the Cayley graph $\mathrm{Cay}_{\Gamma,\mathrm{S}}$, we will always consider it as a topological graph, and endow it with the length distance d_{S} such that every edge hast length 1. Note that for vertices $\gamma_1, \gamma_2 \in \mathrm{Cay}_{\Gamma,\mathrm{S}}$, we have $\mathrm{d}_{\mathrm{S}}(\gamma_1, \gamma_2) = \left|\gamma_1^{-1}\gamma_2\right|_{\mathrm{S}}$. The left action of Γ on vertices extends to an isometric action on $\mathrm{Cay}_{\Gamma,\mathrm{S}}$ which is free, properly discontinuous and cocompact.

The fact that the geodesic flow of a Cayley graph provides a coarse Gromov flow, mainly that the limits $\widehat{g}(\pm\infty) \in \partial_\infty\Gamma$ exist, follows from the very definition of $\partial_\infty\Gamma$ (for a more precise statement on the properness of the map to $\partial_\infty\Gamma^{(2)}$, see [CK02, Lemma 7.1]).

As it is still unknown whether all hyperbolic groups admit a geometric action on a $\mathrm{CAT}(-\kappa)$ space for some $\kappa > 0$, construction a Gromov flow is harder. It is however possible: Gromov proposed a construction starting with the geodesic flow of a Cayley graph (or any hyperbolic space on which Γ acts geometrically) and collapsing geodesics with the same endpoints to a single line.

Theorem 2.2 ([Gro87])

Every hyperbolic group has a Gromov flow.

2.2 Suspension bundles and flows

2.2.1 Geometric structures as sections of suspension bundles

A $(\mathrm{G}, \mathbb{X})$-structure is only defined on a manifold $\mathbf{X}$ whose dimension is that of $\mathbb{X}$. If the dimension of $\mathbf{X}$ is smaller (or if $\mathbf{X}$ fails to be a manifold), we could adapt the definition by dropping the requirement that the maps f_i in a $(G,\mathbb{X})$-atlas are local homeomorphisms. One could define a $(G,\mathbb{X})$-*embedding structure* by asking for the maps f_i to be topological embeddings, or even drop any requirement. In this case, the holonomy homomorphism and developing map may fail to be defined. For this reason, the generalisation of a $(G,\mathbb{X})$-structure to a smaller dimensional space X goes through a different description using sections of suspension bundles, which are a different way of considering a holonomy homomorphism and a developing map.

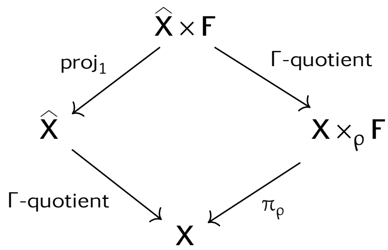
Definition (Suspension bundle)

Consider a metrisable space $\mathsf{X} = \Gamma\backslash\widehat{\mathsf{X}}$ where Γ acts properly discontinuously on $\widehat{\mathsf{X}}$, a manifold F and $\rho \in \mathrm{Hom}(\Gamma, \mathrm{Diff}(\mathsf{F}))$. The *suspension bundle* of ρ is the bundle $\mathsf{X} \times_\rho \mathsf{F} = \Gamma\backslash(\widehat{\mathsf{X}} \times \mathsf{F})$ where Γ acts diagonally on $\widehat{\mathsf{X}} \times \mathsf{F}$, i.e.

$$\forall \gamma \in \Gamma \ \forall (\hat{x}, f) \in \widehat{\mathsf{X}} \times \mathsf{F} \quad \gamma \cdot (\hat{x}, f) = (\gamma \cdot \hat{x}, \rho(\gamma) \cdot f).$$

The projection $\pi_\rho : \mathsf{X} \times_\rho \mathsf{F} \to \mathsf{X}$ sends $\Gamma \cdot (\hat{x}, f) \in \mathsf{X} \times_\rho \mathsf{F}$ to $\Gamma \cdot \hat{x} \in \mathsf{X}$.

The projection $\pi_\rho : \mathsf{X} \times_\rho \mathsf{F} \to \mathsf{X}$ is characterised by commutativity of the diagram (where $\mathrm{proj}_1 : \widehat{\mathsf{X}} \times \mathsf{F} \to \widehat{\mathsf{X}}$ stands for the projection on the first factor):



Notation

For $x \in \mathsf{X}$, we will write $\{x\} \times_\rho \mathsf{F} \overset{\text{def}}{=} \pi_\rho^{-1}(\{x\}) \subset \mathsf{X} \times_\rho \mathsf{F}$ for the fibre over x.

Example (Mapping torus)

Consider $\mathsf{X} = \mathbb{S}^1$, $\widehat{\mathsf{X}} = \mathbb{R}$ and $\Gamma = \mathbb{Z}$. Any $\psi \in \mathrm{Diff}(\mathsf{F})$ induces $\rho_\psi \in \mathrm{Hom}(\mathbb{Z}, \mathrm{Diff}(\mathsf{F}))$ defined by $\rho_\psi(n) = \psi^{-n}$, and the total space of the suspension bundle $\mathbb{S}^1 \times_{\rho_\psi} \mathsf{X}$ is called the *mapping torus* of ψ.

If a map $\hat{\sigma} : \widehat{\mathsf{X}} \to \mathsf{F}$ is ρ-equivariant, i.e.

$$\forall \gamma \in \Gamma \ \forall \hat{x} \in \widehat{\mathsf{X}} \quad \hat{\sigma}(\gamma \cdot \hat{x}) = \rho(\gamma) \cdot \hat{\sigma}(\hat{x}),$$

then the map

$$\sigma : \begin{cases} \mathsf{X} & \to & \mathsf{X} \times_\rho \mathsf{F} \\ \Gamma \cdot \hat{x} & \mapsto & \Gamma \cdot (\hat{x}, \hat{\sigma}(\hat{x})) \end{cases}$$

is a section of the suspension bundle $\pi_\rho : \mathsf{X} \times_\rho \mathsf{F} \to \mathsf{X}$. All sections are obtained in this way.

Remark (Tangent spaces to fibres)

Even if X is not a manifold, the fibres of $\mathsf{X} \times_\rho \mathsf{F}$ inherit a manifold structure diffeomorphic to F, so we can talk about the vertical space $\mathsf{X} \times_\rho \mathrm{T}\mathsf{F}$ at $z \in \mathsf{X} \times_\rho \mathsf{F}$, i.e. the tangent space

$$\{x\} \times_\rho \mathrm{T}_z\mathsf{F} \stackrel{\text{def}}{=} \mathrm{T}_z\big(\{x\} \times_\rho \mathsf{F}\big)$$

of the fibre $\{x\} \times_\rho \mathsf{F}$ over $x = \pi_\rho(z)$. They form a continuous vector bundle $\mathsf{X} \times_\rho \mathrm{T}\mathsf{F} \to \mathsf{X} \times_\rho \mathsf{F}$.

2.2.2 The suspended flow and its vertical derivative

Now consider a Γ-flow $\varphi^t : \mathsf{X} \to \mathsf{X}$ and a group homomorphism $\rho : \Gamma \to \mathrm{Diff}(\mathsf{F})$ for some manifold F. The flow $\varphi^t : \mathsf{X} \to \mathsf{X}$ has a natural lift $\varphi^t_\rho : \mathsf{X} \times_\rho \mathsf{F} \to \mathsf{X} \times_\rho \mathsf{F}$. In the smooth case, it corresponds to parallel transport along flow lines of φ^t for a natural flat connection on $\mathsf{X} \times_\rho \mathsf{F}$, but it can be defined in the continuous setting.

Definition (Suspended flow)

The *suspended flow* is defined by $\Phi^t_\rho : \begin{cases} \mathsf{X} \times_\rho \mathsf{F} & \to & \mathsf{X} \times_\rho \mathsf{F} \\ \Gamma \cdot (\widehat{x}, f) & \mapsto & \Gamma \cdot (\widehat{\varphi}^t(\widehat{x}), f) \end{cases}$

Notation

We will be lead to consider representations $\rho \in \mathrm{Hom}(\Gamma, \mathrm{G})$ for a Lie group G and actions of G on various manifolds F. When there is a possible confusion, we will denote by $\Phi^t_{\rho,\mathsf{F}}$ the suspended flow on $\mathsf{X} \times_\rho \mathsf{F}$.

Now consider a *flow-equivariant* section $\sigma : \mathsf{X} \to \mathsf{X} \times_\rho \mathsf{F}$, i.e. a section satisfying the equivariance condition

$$\forall t \in \mathbb{R} \qquad \sigma \circ \varphi^t = \Phi^t_\rho \circ \sigma.$$

One can define the vector bundle $\mathsf{X} \times_\rho \mathrm{T}_\sigma\mathsf{F} \to \mathsf{X}$ whose fibre over $x \in \mathsf{X}$ is the vertical space $\{x\} \times_\rho \mathrm{T}_{\sigma(x)}\mathsf{F}$, i.e. the tangent space at $\sigma(x)$ to the fibre $\{x\} \times_\rho \mathsf{F}$, that is

$$\forall x \in \mathsf{X} \qquad \mathsf{X} \times_\rho \mathrm{T}_\sigma\mathsf{F}\big|_m = \{x\} \times_\rho \mathrm{T}_{\sigma(x)}\mathsf{F} = \mathrm{T}_{\sigma(x)}\big(\{x\} \times_\rho \mathsf{F}\big).$$

Definition (Vertical flow)

The *vertical flow* of a flow-equivariant section $\sigma : \mathsf{X} \to \mathsf{X} \times_\rho \mathsf{F}$ is the linear lift

$$\mathrm{d}^{\mathrm{V}}_\sigma \Phi^t_\rho : \mathsf{X} \times_\rho \mathrm{T}_\sigma\mathsf{F} \to \mathsf{X} \times_\rho \mathrm{T}_\sigma\mathsf{F}$$

of φ^t mapping the fibre $\{x\} \times_\rho \mathrm{T}_{\sigma(x)}\mathsf{F}$ over $x \in \mathsf{X}$ to the fibre $\{\varphi^t(x)\} \times_\rho \mathrm{T}_{\sigma(\varphi^t(x))}\mathsf{X}$ over $\varphi^t(x)$ as the differential of the restriction of Φ^t_ρ to the fibre $\{x\} \times_\rho \mathsf{F}$.

We can also give a direct definition of $\mathrm{d}^{\mathrm{V}}_\sigma \Phi^t_\rho$ by considering lifts to trivial bundles over $\widehat{\mathsf{X}}$. Lifting $\sigma : \mathsf{X} \to \mathsf{X} \times_\rho \mathsf{F}$ to a ρ-equivariant map $\widehat{\sigma} : \widehat{\mathsf{X}} \to \mathsf{F}$, we can also see $\mathsf{X} \times_\rho \mathrm{T}_\sigma\mathsf{F}$ as $\mathsf{X} \times_\rho \mathrm{T}_\sigma\mathsf{F} = \Gamma \backslash \widehat{\sigma}^{\,*}\mathrm{T}\mathsf{F}$ where the diagonal action of Γ on

$$\widehat{\sigma}^{\,*}\mathsf{TF} = \left\{ (\widehat{x}, v) \in \widehat{\mathsf{X}} \times \mathsf{TF} \,\middle|\, v \in \mathsf{T}_{\widehat{\sigma}(\widehat{x})}\mathsf{F} \right\}$$

is defined by the formula

$$\gamma \cdot (\widehat{x}, v) = \left(\gamma \cdot \widehat{x},\ d\rho(\gamma)\big|_{\widehat{\sigma}(\widehat{x})}\, v \right)$$

thanks to the ρ-equivariance of $\widehat{\sigma}$. The flow $\mathrm{d}^{\mathsf{V}}_{\sigma}\Phi^{t}_{\rho}$ sends $\Gamma \cdot (\widehat{x}, v)$ to $\Gamma \cdot (\widehat{\varphi}^{t}(\widehat{x}), v)$, i.e. is defined by commutativity of the diagram

$$\begin{array}{ccc}
\widehat{\sigma}^{\,*}\mathsf{TF} & \xrightarrow{\ \widehat{\varphi}^{t} \times \mathrm{id}\ } & \widehat{\sigma}^{\,*}\mathsf{TF} \\
\Gamma\text{-quotient}\downarrow & & \downarrow\Gamma\text{-quotient} \\
\mathsf{X} \times_{\rho} \mathsf{T}_{\sigma}\mathsf{F} & \xrightarrow{\ \mathrm{d}^{\mathsf{V}}_{\sigma}\Phi^{t}_{\rho}\ } & \mathsf{X} \times_{\rho} \mathsf{T}_{\sigma}\mathsf{F} \\
\downarrow & & \downarrow \\
\mathsf{X} & \xrightarrow{\ \varphi^{t}\ } & \mathsf{X}
\end{array}$$

This is the construction used by Guichard and Wienhard in [GW12]. In the smooth case (i.e. X is a smooth manifold and the flow $\varphi^{t} : \mathsf{X} \to \mathsf{X}$ is smooth), we can also interpret Φ^{t}_{ρ} as a parallel transport itself, by considering any linear connection ∇ on the vertical bundle $\mathsf{X} \times_{\rho} \mathsf{TF} \to \mathsf{X} \times_{\rho} \mathsf{F}$ of $\mathsf{X} \times_{\rho} \mathsf{F}$, so that $\mathrm{d}^{\mathsf{V}}_{\sigma}\Phi^{t}_{\rho}$ is the parallel transport for $\sigma^{*}\nabla$ on $\mathsf{X} \times_{\rho} \mathsf{T}_{\sigma}\mathsf{F} = \sigma^{*}\left(\mathsf{X} \times_{\rho} \mathsf{TF} \right)$ (it does not depend on the choice of ∇ because of the flow-equivariance of σ). This was Labourie's approach in [Lab06].

2.3 Contracting linear flows

Let $p : \mathsf{E} \to \mathsf{X}$ be a continuous vector bundle over a compact metric space X equipped with a continuous flow $\varphi^{t} : \mathsf{X} \to \mathsf{X}$. A *linear lift* of φ^{t} to E is a flow $\Phi^{t} : \mathsf{E} \to \mathsf{E}$ such that $p \circ \Phi^{t} = \varphi^{t} \circ p$ and each restriction $\Phi^{t}\big|_{x} : \mathsf{E}|_{x} \to \mathsf{E}|_{\varphi^{t}(x)}$ is linear.

Definition (Contracting and dilating linear flows)

A linear lift $\Phi^{t} : \mathsf{E} \to \mathsf{E}$ of a flow $\varphi^{t} : \mathsf{X} \to \mathsf{X}$ is called *contracting* if for some (hence any) continuous Euclidean metric $\|\cdot\|$ on E, there are constants $\mathrm{C}, a > 0$ such that

$$\forall x \in \mathsf{X}\ \forall v \in \mathsf{E}|_{x}\ \forall t \geq 0 \qquad \left\| \Phi^{t}\big|_{x} v \right\|_{\varphi^{t}(x)} \leq \mathrm{C} e^{-at} \|v\|_{x}\,.$$

We say that Φ^{t} is *dilating* if Φ^{-t} is contracting.

2.3.1 Contraction and dominated splittings

Since X is compact, changing the Euclidean metric will only affect the constant C, making this choice irrelevant in the definition. It will be practical to consider metrics for which this constant is 1, often called *adapted metrics*. This is always possible as long as we are ready to lower the constant a.

Lemma 2.3 (Characterisations of contraction of a linear flow)

Let X be a compact metric space, $p : \mathsf{E} \to \mathsf{X}$ a continuous vector bundle and $\Phi^t : \mathsf{E} \to \mathsf{E}$ a linear lift of a continuous flow $\varphi^t : \mathsf{X} \to \mathsf{X}$. The following are equivalent:

1. Φ^t is contracting.
2. There exist a continuous Euclidean metric $\|\cdot\|$ on E and a constant $b > 0$ such that

$$\forall x \in \mathsf{X}\ \forall v \in \mathsf{E}|_x\ \forall t \geq 0 \qquad \left\| \Phi^t\big|_x v \right\|_{\varphi^t(x)} \leq e^{-bt} \|v\|_x .$$

3. $\Phi^t|_x z \to 0$ as $t \to +\infty$ for any $x \in \mathsf{X}$ and $z \in \mathsf{E}|_x$.

Remark

By $\Phi^t|_x z \to 0$, we mean that $\left\|\Phi^t|_x z\right\|_{\varphi^t(x)} \to 0$ for some (hence any) continuous norm on E. This does no mean convergence to a point in E, but that it gets close to the zero section.

Proof. Let us start with 1.⇒2. Let $\|\cdot\|^\circ$ be any Euclidean metric on E, and consider the constants $C, a > 0$ with respect to $\|\cdot\|^\circ$. Consider $t_0 > 0$ such that $Ce^{-at_0/2} \leq 1$, and set

$$\|v\|_x = \int_0^{t_0} e^{\frac{as}{2}} \|\Phi^s|_x v\|^\circ_{\varphi^s(x)}\, ds.$$

For any $t \geq 0$, a simple change of variables yields

$$\begin{aligned} \left\| \Phi^t\big|_x v \right\|_{\varphi^t(x)} &= \int_0^{t_0} e^{\frac{as}{2}} \left\| \Phi^{t+s}\big|_x v \right\|^\circ_{\varphi^{t+s}(x)} ds \\ &= e^{-\frac{at}{2}} \int_t^{t+t_0} e^{\frac{as}{2}} \|\Phi^s|_x v\|^\circ_{\varphi^s(x)}\, ds. \end{aligned}$$

We then find, first by rearranging integration domains then with a change of variables:

$$\begin{aligned} e^{\frac{at}{2}} \left\| \Phi^t\big|_x v \right\|_{\varphi^t(x)} - \|v\|_x &= \int_t^{t+t_0} e^{\frac{as}{2}} \|\Phi^s|_x v\|^\circ_{\varphi^s(x)}\, ds - \int_0^{t_0} e^{\frac{as}{2}} \|\Phi^s|_x v\|^\circ_{\varphi^s(x)}\, ds \\ &= \int_{t_0}^{t+t_0} e^{\frac{as}{2}} \|\Phi^s|_x v\|^\circ_{\varphi^s(x)}\, ds - \int_0^{t} e^{\frac{as}{2}} \|\Phi^s|_x v\|^\circ_{\varphi^s(x)}\, ds \\ &= \int_0^{t} e^{\frac{as}{2}} \left(e^{\frac{at_0}{2}} \left\| \Phi^{t_0+s}\big|_x v \right\|^\circ_{\varphi^{t_0+s}(x)} - \|\Phi^s|_x v\|^\circ_{\varphi^s(x)} \right) ds \end{aligned}$$

Finally the contraction yields

$$e^{\frac{at}{2}} \left\| \Phi^t\big|_x v \right\|_{\varphi^t(x)} - \|v\|_x \leq \int_0^{t} e^{\frac{as}{2}} \left(\underbrace{e^{\frac{at_0}{2}} C e^{-at_0}}_{\leq 1} - 1 \right) \|\Phi^s|_x v\|^\circ_{\varphi^s(x)}\, ds \leq 0.$$

2.⇒3. is immediate, let us prove 3.⇒1. Let $\|-\|$ be a Euclidean metric on E, and use the same notation for the operator norm between fibres. For a fixed $x \in \mathsf{X}$, choosing an orthonormal basis

$(e_1,\dots,e_d)$ of the fibre $\mathsf{E}|_x$ one finds

$$\left\|\Phi^t|_x\right\| \leq \sum_{i=1}^{d} \left\|\Phi^t(e_i)\right\|$$

and deduces that $\left\|\Phi^t|_x\right\| \to 0$ and $t \to +\infty$. In particular, there is some $t_x \geq 1$ such that $\left\|\Phi^{t_x}|_x\right\| < 1/\mathrm{e}$. Consider a neighbourhood $\mathsf{U}_x \subset \mathsf{X}$ of x such that $\left\|\Phi^{t_x}|_{x'}\right\| < 1/\mathrm{e}$ for all $x' \in \mathsf{U}_x$. The compactness of X allows us to consider a finite set $x_1,\dots,x_k \in \mathsf{X}$ such that $\mathsf{X} = \mathsf{U}_{x_1} \cup \dots \cup \mathsf{U}_{x_k}$. Set $\mathsf{T} = \max(t_{x_1},\dots,t_{x_k})$ and

$$\mathsf{C} = \max\left\{\left\|\Phi^t|_x\right\| \,\middle|\, x \in \mathsf{X},\ t \in [0,\mathsf{T}]\right\}.$$

For $x \in \mathsf{X}$, consider the sequence $(i_j)_{j\geq 0}$ constructed inductively so that $m \in \mathsf{U}_{x_{i_0}}$ and for all $j \geq 0$, setting $\tau_j = t_{x_{i_0}} + \dots + t_{x_j}$, we get the following:

$$\varphi^{\tau_j}(x) \in \mathsf{U}_{x_{i_{j+1}}}.$$

Let $t \geq \mathsf{T}$, and let $j \geq 0$ be defined by $\tau_j \leq t < \tau_{j+1}$. Setting $s = \tau_{j+1} - t \in [0,\mathsf{T})$, we find

$$\begin{aligned}\left\|\Phi^t|_x\right\| &\leq \underbrace{\left\|\Phi^{t_{x_{i_0}}}\Big|_x\right\|}_{\leq \frac{1}{\mathrm{e}}} \cdots \underbrace{\left\|\Phi^{t_{x_{i_j}}}\Big|_{\varphi^{\tau_{x_{j-1}}}(x)}\right\|}_{\leq \frac{1}{\mathrm{e}}} \underbrace{\left\|\Phi^s|_{\varphi^{\tau_{x_j}}(x)}\right\|}_{\leq \mathsf{C}} \\ &\leq \mathsf{C}\mathrm{e}^{-(j+1)}.\end{aligned}$$

Using the fact that $t < \tau_{j+1} \leq (j+1)\mathsf{T}$, we find $\left\|\Phi^t|_x\right\| \leq \mathsf{C}\mathrm{e}^{-t/\mathsf{T}}$. □

Remark

A metric such as in point 2. is called an *adapted metric*. We can make the constant b as close to a as we want.

Definition (Dominated splitting)

Let X be a compact metric space, $p : \mathsf{E} \to \mathsf{X}$ a continuous vector bundle and $\Phi^t : \mathsf{E} \to \mathsf{E}$ a linear lift of a continuous flow $\varphi^t : \mathsf{X} \to \mathsf{X}$. Let $\mathsf{E}_1, \mathsf{E}_2$ be two Φ^t-invariant vector subbundles of E. We say that E_1 *dominates* E_2 if for some (hence any) continuous Euclidean metric $\|-\|$ on E, there are constants $a, \mathsf{C} > 0$ such that for all $x \in \mathsf{X}$ and non zero vectors $v_i \in \mathsf{E}_i|_x$, $i = 1,2$, we have

$$\forall t \geq 0 \quad \frac{\left\|\Phi^t(v_1)\right\|_{\varphi^t(x)}}{\|\Phi^t(v_2)\|_{\varphi^t(x)}} \geq \mathsf{C}\mathrm{e}^{at} \frac{\|v_1\|_x}{\|v_2\|_x}.$$

Domination can also be interpreted as contraction on the homomorphism bundle. If $\mathsf{E}_1, \mathsf{E}_2$ are Φ^t-invariant vector subbundles of E, then $\mathrm{Hom}(\mathsf{E}_1, \mathsf{E}_2)$ also carries a linear lift of φ^t defined by fibrewise conjugation by Φ^t, i.e. $\Phi^t_{\mathrm{End}}(\beta) = \Phi^t|_{\mathsf{E}_1} \circ \beta \circ \left(\Phi^t|_{\mathsf{E}_2}\right)^{-1}$.

Lemma 2.4 (Characterisations of dominated splittings)

Let X be a compact metric space, $p : \mathsf{E} \to \mathsf{X}$ a continuous vector bundle and $\Phi^t : \mathsf{E} \to \mathsf{E}$ a linear lift of a continuous flow $\varphi^t : \mathsf{X} \to \mathsf{X}$. Let $\mathsf{E}_1, \mathsf{E}_2$ be two Φ^t-invariant vector subbundles of E. The following are equivalent:

1. E_1 dominates E_2.
2. $\mathrm{Hom}(\mathsf{E}_1, \mathsf{E}_2)$ is contracted.
3. $\mathrm{Hom}(\mathsf{E}_2, \mathsf{E}_1)$ is dilated.

Proof. In this whole proof, we fix a Euclidean norm $\|-\|$ on E, and consider the associated operator norms on homomorphism bundles.

First assume that E_1 dominates E_2. Let $x \in \mathsf{X}$ and $\beta \in \mathrm{Hom}(\mathsf{E}_1|_x, \mathsf{E}_2|_x)$. For $t \geq 0$, choose some $w_1 \in \mathsf{E}_1|_{\varphi^t(x)} \setminus \{0\}$ such that $\left\|\Phi^t(\beta)(w_1)\right\| = \left\|\Phi^t(\beta)\right\| \|w_1\|$ and set $v_1 = \Phi^{-t}(w_1)$.

$$\begin{aligned}
\left\|\Phi^t_{\mathrm{Hom}(\mathsf{E}_1,\mathsf{E}_2)}(\beta)\right\| &= \frac{\left\|\Phi^t(\beta)(w_1))\right\|}{\|w_1\|} \\
&= \frac{\left\|\Phi^t(\beta(v_1))\right\|}{\|\Phi^t(v_1)\|} \\
&\leq \frac{1}{C} e^{-at} \frac{\left\|\beta(v_1)\right\|}{\|v_1\|} \\
&\leq \frac{1}{C} e^{-at} \left\|\beta\right\|,
\end{aligned}$$

thus showing that $\mathrm{Hom}(\mathsf{E}_1, \mathsf{E}_2)$ is contracted. Now assume that $\mathrm{Hom}(\mathsf{E}_1, \mathsf{E}_2)$ is contracted. Consider $x \in \mathsf{X}$, some non zero vectors $v_1 \in \mathsf{E}_1|_x$, $v_2 \in \mathsf{E}_2|_x$, and choose $\beta \in \mathrm{Hom}(\mathsf{E}_1|_x, \mathsf{E}_2|_x)$ such that $\beta(v_1) = v_2$ and $\left\|\beta\right\| = \frac{\|v_2\|}{\|v_1\|}$. Then $\Phi^t_{\mathrm{Hom}(\mathsf{E}_1,\mathsf{E}_2)}(\beta)$ maps $\Phi^t(v_1)$ to $\Phi^t(v_2)$, and we find

$$\frac{\left\|\Phi^t(v_2)\right\|}{\|\Phi^t(v_1)\|} \leq \left\|\Phi^t_{\mathrm{Hom}(\mathsf{E}_1,\mathsf{E}_2)}(\beta)\right\| \leq Ce^{-at}\left\|\beta\right\| = Ce^{-at}\frac{\|v_2\|}{\|v_1\|},$$

thus establishing domination of E_1 over E_2. Similar proofs show that this domination is equivalent to dilation of $\mathrm{Hom}(\mathsf{E}_2, \mathsf{E}_1)$.

□

2.3.2 Topological dynamics on projective bundles and dominated splittings

Let Γ be a countable group, $\varphi^t : \mathsf{X} \to \mathsf{X}$ a Γ-flow and $\rho \in \mathrm{Hom}(\Gamma, \mathrm{SL}(\mathsf{V}))$. A section

$$\sigma : \mathsf{X} \to \mathsf{X} \times_\rho \mathbb{P}(\mathsf{V}) \overset{\pitchfork}{\times} \mathbb{P}(\mathsf{V}^*)$$

can be seen as a pair of sections

$$\sigma_+ : \mathsf{X} \to \mathsf{X} \times_\rho \mathbb{P}(\mathsf{V}), \quad \sigma_- : \mathsf{X} \to \mathsf{X} \times_\rho \mathbb{P}(\mathsf{V}^*)$$

satisfying a fibrewise transversality condition. This transversality refers to the flat vector bundle $\mathsf{X} \times_\rho \mathsf{V}$. Indeed, the sections $\sigma_\pm$ determine vector subbundles Σ_+ and Σ_- of V, and transversality means that

$$X \times_\rho V = \Sigma_+ \oplus \Sigma_- .$$

Proposition 2.5 (Dilation and contraction of vertical derivatives over projective bundles)

Consider a Γ-flow $\varphi^t : X \to X$, a representation $\rho \in \mathrm{Hom}(\Gamma, \mathrm{SL}(V))$ and a continuous section $\sigma = (\sigma_+, \sigma_-) : X \to X \times_\rho \mathbb{P}(V) \overset{\pitchfork}{\times} \mathbb{P}(V^*)$. The following are equivalent:

1. The vertical derivative $d^V_{\sigma_+} \Phi^t_{\rho, \mathbb{P}(V)} : X \times_\rho T_{\sigma_+} \mathbb{P}(V) \to X \times_\rho T_{\sigma_+} \mathbb{P}(V)$ is dilating.
2. The vertical derivative $d^V_{\sigma_-} \Phi^t_{\rho, \mathbb{P}(V^*)} : X \times_\rho T_{\sigma_-} \mathbb{P}(V^*) \to X \times_\rho T_{\sigma_-} \mathbb{P}(V^*)$ is contracting.
3. The bundle Σ_- dominates Σ_+.

Proof. Thanks to Lemma 2.4, the proof will be achieved if we can find bundle automorphisms $\mathrm{Hom}(\Sigma_+, \Sigma_-) \to X \times_\rho T_{\sigma_+} \mathbb{P}(V)$ and $\mathrm{Hom}(\Sigma_-, \Sigma_+) \to X \times_\rho T_{\sigma_-} \mathbb{P}(V^*)$ conjugating the linear lifts of φ^t. This will be done by understanding tangent spaces to Grassmannians. Consider a direct sum decomposition $V = W_1 \oplus W_2$, and let $k = \dim W_1$. We have an explicit isomorphism

$$\begin{cases} \mathrm{Hom}(W_1, W_2) & \to & T_{W_1} \mathrm{Gr}_k(V) \\ \beta & \mapsto & \frac{d}{dt}\Big|_{t=0} \mathrm{im}(\mathrm{id}_{W_1} + t\beta) \end{cases}$$

Where $\mathrm{im}(\mathrm{id}_{W_1} + t\beta) = \{ w + t\beta(w) \,|\, w \in W_1 \} \in \mathrm{Gr}_k(V)$. Applying these isomorphisms on each fibre to the splitting $\{x\} \times_\rho V = \Sigma_+|_x \oplus \Sigma_-|_x$ and to the Grassmannians $\mathrm{Gr}_1(V) = \mathbb{P}(V)$ and $\mathrm{Gr}_{d-1}(V) = \mathbb{P}(V^*)$ yields the desired bundle isomorphisms. □

Proposition 2.6 (Dominated splittings and topological dynamics)

Consider a Γ-flow $\varphi^t : X \to X$, a representation $\rho \in \mathrm{Hom}(\Gamma, \mathrm{SL}(V))$ and a continuous section $\sigma = (\sigma_+, \sigma_-) : X \to X \times_\rho \mathbb{P}(V) \overset{\pitchfork}{\times} \mathbb{P}(V^*)$. The following are equivalent:

1. The bundle Σ_- dominates Σ_+.
2. For any neighbourhoods $\mathcal{U}^+ \subset X \times_\rho \mathbb{P}(V)$ of $\mathbb{P}(\Sigma_+)$ and $\mathcal{U}^- \subset X \times_\rho \mathbb{P}(V)$ of $\mathbb{P}(\Sigma_-)$, there is some $t_0 \geq 0$ such that
$$\forall t \geq t_0 \qquad \Phi^{-t}_{\rho, \mathbb{P}(V)}\Big(X \times_\rho \mathbb{P}(V) \setminus \mathcal{U}^- \Big) \subset \mathcal{U}^+ .$$
3. For any $\ell \in X \times_\rho \mathbb{P}(V) \setminus \mathbb{P}(\Sigma^-)$, we have $\lim_{t \to -\infty} \mathrm{dist}\big(\ell, \mathbb{P}(\Sigma_+)\big) = 0$.

Proof. Let us start with 1.⇒2. Assume that Σ^+ dominates Σ^-, consider a Euclidean metric $\|-\|$ on $X \times_\rho V$, and for $s > 0$ consider the open subsets

$$\mathcal{U}^+_s = \Big\{ [v_+ + v_-] \in X \times_\rho \mathbb{P}(V) \,\Big|\, v_+ \in \Sigma^+,\ v_- \in \Sigma^- \ \&\ \|v_+\| > e^s \|v_-\| \Big\},$$
$$\mathcal{U}^-_s = \Big\{ [v_+ + v_-] \in X \times_\rho \mathbb{P}(V) \,\Big|\, v_+ \in \Sigma^+,\ v_- \in \Sigma^- \ \&\ \|v_+\| < e^{-s} \|v_-\| \Big\}.$$

For any neighbourhoods $\mathcal{U}^\pm$ of $\mathbb{P}(\Sigma^\pm)$, compactness of X gives us some $s > 0$ such that $\mathcal{U}^+_s \subset \mathcal{U}^+$ and $\mathcal{U}^-_s \subset \mathcal{U}^-$. Using the same notations $C, a > 0$ as in the definition of a dominated splitting, we

find that $\varphi_\rho^{-t}\left(\mathsf{X}\times_\rho \mathbb{P}(\mathsf{V})\setminus \mathcal{U}_s^-\right)\subset \mathcal{U}_s^+$ as soon as $t>\frac{2s+\operatorname{Log}C}{a}$, thus proving 2.
The implication 2.⇒3. is straightforward, so let us now prove that 3.⇒1. For any transverse $\ell\in\mathbb{P}(\mathsf{V})$ and $\mathsf{H}\in\mathbb{P}(\mathsf{V}^*)$, we can consider the map

$$\exp^{\mathsf{u}}_{(\ell,\mathsf{H})}:\left\{\begin{array}{ccc} \mathsf{T}_\ell\mathbb{P}(\mathsf{V})\simeq \operatorname{Hom}(\ell,\mathsf{H}) & \to & \mathbb{P}(\mathsf{V}) \\ f & \mapsto & \operatorname{im}(\mathrm{id}_\ell+f)\end{array}\right.$$

That is $\exp^{\mathsf{u}}_{(\ell,\mathsf{H})}(f)=[v+f(v)]$ for $v\in\ell$. It establishes a diffeomorphism from $\mathsf{T}_\ell\mathbb{P}(\mathsf{V})$ to $\mathbb{P}(\mathsf{V})\setminus\mathbb{P}(\mathsf{H})$. It extends to vector bundles

$$\exp^{\mathsf{u}}_\sigma:\mathsf{X}\times_\rho \mathsf{T}_{\sigma_+}\mathbb{P}(\mathsf{V})\to \mathsf{X}\times_\rho\mathbb{P}(\mathsf{V}),$$

and conjugates the suspension flow with its vertical derivative

$$\exp^{\mathsf{u}}_\sigma\circ \mathrm{d}^{\mathsf{V}}_{\sigma_+}\Phi^t_{\rho,\mathbb{P}(\mathsf{V})}=\Phi^t_{\rho,\mathbb{P}(\mathsf{V})}\circ\exp^{\mathsf{u}}_\sigma.$$

Now let $\zeta\in\mathsf{X}\times_\rho\mathsf{T}_{\sigma_+}\mathbb{P}(\mathsf{V})$. By assumption, we have that $\Phi^t_{\rho,\mathbb{P}(\mathsf{V})}(\exp^{\mathsf{u}}_\sigma(\zeta))$ accumulates on Σ^+ as $t\to-\infty$, hence $\mathrm{d}^{\mathsf{V}}_\sigma\Phi^t_{\rho,\mathbb{P}(\mathsf{V})}(\zeta)\to 0$ as $t\to-\infty$, and by Lemma 2.3 this implies dilation, thus domination of Σ_+ over Σ_- by Proposition 2.5. □

2.4 Projective Anosov representations

2.4.1 The flow definition

In this section we introduce a slight variation on Labourie's original definition of an Anosov representation, using any coarse Gromov flow instead of the geodesic flow of a negatively curved manifold or a Gromov flow. We will see generalisations to other Lie groups in the next section. The definition will make use of the space

$$\mathbb{P}(\mathsf{V})\overset{\pitchfork}{\times}\mathbb{P}(\mathsf{V}^*)\overset{\text{def}}{=}\left\{\left([v],[\alpha]\right)\in\mathbb{P}(\mathsf{V})\times\mathbb{P}(\mathsf{V}^*)\,\middle|\,\alpha(v)\neq 0\right\}.$$

A projective Anosov representation is essentially a $\left(\mathrm{SL}(\mathsf{V}),\mathbb{P}(\mathsf{V})\overset{\pitchfork}{\times}\mathbb{P}(\mathsf{V}^*)\right)$-embedding structure on the non Hausdorff space $\Gamma\backslash\partial_\infty\Gamma^{(2)}$. Because of topological considerations, it is better to work with Γ-equivariant maps on $\partial_\infty\Gamma^{(2)}$. Consider a coarse Gromov flow $\varphi^t:\mathsf{X}\to\mathsf{X}$, a representation $\rho\in\operatorname{Hom}(\Gamma,\mathrm{SL}(\mathsf{V}))$ and let (ξ,ξ^*) be a pair of limit maps for ρ, i.e. $\xi:\partial_\infty\Gamma\to\mathbb{P}(\mathsf{V})$ and $\xi^*:\partial_\infty\Gamma\to\mathbb{P}(\mathsf{V}^*)$ are continuous ρ-equivariant maps. Such a pair induces flow-equivariant sections

$$\xi_+:\left\{\begin{array}{ccc}\mathsf{X} & \to & \mathsf{X}\times_\rho\mathbb{P}(\mathsf{V})\\ \Gamma\cdot\widehat{x} & \mapsto & \Gamma\cdot\left(\widehat{x},\xi\left(\widehat{x}^+\right)\right)\end{array}\right.\quad\text{and }\xi^*_-:\left\{\begin{array}{ccc}\mathsf{X} & \to & \mathsf{X}\times_\rho\mathbb{P}(\mathsf{V}^*)\\ \Gamma\cdot\widehat{x} & \mapsto & \Gamma\cdot\left(\widehat{x},\xi^*\left(\widehat{x}^-\right)\right)\end{array}\right.$$

If moreover the pair (ξ,ξ^*) is transverse, i.e.

$$\forall(\eta_+,\eta_-)\in\partial_\infty\Gamma^{(2)}\qquad \xi(\eta_+)\pitchfork\xi^*(\eta_-),$$

they induce a flow-equivariant section

$$(\xi_+,\xi^*_-):\left\{\begin{array}{ccc}\mathsf{X} & \to & \mathsf{X}\times_\rho\mathbb{P}(\mathsf{V})\overset{\pitchfork}{\times}\mathbb{P}(\mathsf{V}^*)\\ \Gamma\cdot\widehat{x} & \mapsto & \Gamma\cdot\left(\widehat{x},\xi\left(\widehat{x}^+\right),\xi^*\left(\widehat{x}^-\right)\right)\end{array}\right.$$

Anosov representations are defined by contraction/dilation of vertical derivatives of these sections.

Definition (Projective Anosov limit maps)

Let Γ be a Gromov hyperbolic group, $\varphi^t : X \to X$ a coarse Gromov flow, and $\rho \in \mathrm{Hom}(\Gamma, \mathrm{SL}(V))$. A pair of transverse projective limit maps (ξ, ξ^*) is called a pair of *projective Anosov limit maps* if it satisfies the following properties:

- The vertical flow $d^V_{\xi_+} \Phi^t_{\rho,\mathbb{P}(V)} : X \times_\rho T_{\xi_+} \mathbb{P}(V) \to X \times_\rho T_{\xi_+} \mathbb{P}(V)$ is dilating.
- The vertical flow $d^V_{\xi^*_-} \Phi^t_{\rho,\mathbb{P}(V^*)} : X \times_\rho T_{\xi^*_-} \mathbb{P}(V^*) \to X \times_\rho T_{\xi^*_-} \mathbb{P}(V^*)$ is contracting.

Note that thanks to Proposition 2.5, each of these two conditions implies the other.

Definition (Projective Anosov representation)

Let Γ be a hyperbolic group. A homomorphism $\rho : \Gamma \to \mathrm{SL}(V)$ is called *projective Anosov* if it admits a pair of projective Anosov limit maps for some coarse Gromov flow.

Proposition 2.7 (Contraction of Σ_+)

If $\rho \in \mathrm{Hom}(\Gamma, \mathrm{SL}(V))$ is projective Anosov, then the bundle Σ_+ is contracted.

Proof. Since the bundle $\mathrm{Hom}(\Sigma_-, \Sigma_+)$ is contracted, so is its determinant bundle

$$\det\Big(\mathrm{Hom}(\Sigma_-, \Sigma_+)\Big) \simeq \det(\Sigma_-) \otimes \Sigma_+^{\otimes(d-1)},$$

but the fact that $\det(X \times_\rho V)$ is trivialisable (because $\rho(\Gamma) < \mathrm{SL}(V)$) gives us an isomorphism

$$\det(\Sigma_-) \simeq \Sigma_+.$$

It follows that $\Sigma_+^{\otimes d}$ is contracted, and so is Σ_+. □

This proof is taken from [BCLS15, Lemma 2.4].

2.4.2 Proof of Theorem 1.12

In this section we prove the equivalence between Anosov representations and asymptotic embeddings in the sense of [KLP17]. The explanation is that any pair of transverse limit maps give sections of bundles $X \times_\rho \mathbb{P}(V)$ and $X \times_\rho \mathbb{P}(V^*)$ for any coarse Gromov flow $\varphi^t : X \to X$. The limit maps being uniformly dynamics preserving is then equivalent to the source-sink dynamics of the flow $\Phi^t_{\rho,\mathbb{P}(V)}$ described in Proposition 2.6.

Proposition 2.8 (Uniformly dynamics preserving transverse limit maps and topological dynamics of suspension bundles)

Let Γ be a hyperbolic group, $\varphi^t : \mathsf{X} \to \mathsf{X}$ a coarse Gromov flow, $\rho \in \mathrm{Hom}(\Gamma, \mathrm{SL}(V))$ a representation and (ξ, ξ^*) a pair of transverse limit maps. The following are equivalent:

1. The pair (ξ, ξ^*) is uniformly dynamics preserving.
2. For any neighbourhoods $\mathscr{U}^+ \subset \mathsf{X} \times_\rho \mathbb{P}(V)$ of $\xi_+(\mathsf{X})$ and $\mathscr{U}^- \subset \mathsf{X} \times_\rho \mathbb{P}(V)$ of $\mathbb{P}(\xi_-^*(\mathsf{X}))$, there is some $t_0 \geq 0$ such that

$$\forall t \geq t_0 \qquad \Phi^{-t}_{\rho, \mathbb{P}(V)}\Big(\mathsf{X} \times_\rho \mathbb{P}(V) \setminus \mathscr{U}^-\Big) \subset \mathscr{U}^+ .$$

Theorem 1.12 follows directly from propositions 2.5, 2.6 and 2.8. An immediate corollary is that the definition of an Anosov representation does not depend on the choice of a coarse Gromov flow. A detailed scanning of the proofs will show that the convergences involved in the definition of uniformly dynamics preserving transverse limit maps do not have to be required to be locally uniform, and that one of the two convergences is sufficient.

Proof of 1.⇒2. in Proposition 2.8. Consider some point $\ell \in \mathsf{X} \times_\rho \mathbb{P}(V) \setminus \mathbb{P}(\Sigma^-)$, so that according to Proposition 2.5 all we need to show is that

$$\lim_{t \to -\infty} \mathrm{dist}\Big(\Phi^t_{\rho, \mathbb{P}(V)}(\ell), \mathbb{P}(\Sigma^+)\Big) = 0 .$$

By compactness of $\mathsf{X} \times_\rho \mathbb{P}(V)$, it is sufficient to show that for any sequence $t_k \to -\infty$ such that $\lim \Phi^{t_k}_{\rho, \mathbb{P}(V)}(\ell)$ exists, we have $\lim \Phi^{t_k}_{\rho, \mathbb{P}(V)}(\ell) \in \mathbb{P}(\Sigma^+)$. Let us fix such a sequence, write $x = \pi_\rho(\ell) \in \mathsf{X}$, $x' = \lim_{k \to +\infty} \varphi^{t_k}(x)$ and $\ell' = \lim_{k \to +\infty} \Phi^{t_k}_{\rho, \mathbb{P}(V)}(\ell)$, so that our goal is to prove that $\ell' = \xi_+(x')$.

Let $\mathscr{D} \subset \widehat{\mathsf{X}}$ be a compact subset meeting every orbit, and write $\ell = \Gamma \cdot (\hat{x}, \ell_0)$ where $\hat{x} \in \mathscr{D}$ and $\ell_0 \pitchfork \xi^*(\hat{x}^-)$. Let $\gamma_k \in \Gamma$ be such that $\gamma_k \cdot \widehat{\varphi}^{t_k}(\hat{x}) \in \mathscr{D}$. This implies that $\gamma_k^{-1} \to \hat{x}^- \in \partial_\infty \Gamma$. Up to replacing (t_k) with a subsequence, we may assume that $\gamma_k \cdot \widehat{\varphi}^{t_k}(\hat{x})$ has a limit $\hat{x}' \in \mathscr{D}$ (which is a lift of $x' \in \mathsf{X}$), and that (γ_k) has a limit $\gamma_+ \in \partial_\infty \Gamma$. Since $\ell_0 \pitchfork \xi^*(\hat{x}^-) = \xi^*(\gamma_-)$ and the limit maps are uniformly dynamics preserving, we find that $\rho(\gamma_k) \cdot \ell_0 \to \xi(\gamma_+)$, therefore

$$\Phi^{t_k}_{\rho, \mathbb{P}(V)}(\ell) = \Gamma \cdot \Big(\widehat{\varphi}^{t_k}(\hat{x}), \ell_0\Big) = \Gamma \cdot \Big(\underbrace{\gamma_k \cdot \widehat{\varphi}^{t_k}(\hat{x})}_{\to \hat{x}'}, \underbrace{\rho(\gamma_k) \cdot \ell_0}_{\to \xi(\gamma_+)}\Big) \longrightarrow \Gamma \cdot \Big(\hat{x}', \xi(\gamma_+)\Big).$$

We now notice that for all k, $\lim_{t \to +\infty} \widehat{\varphi}^t(\gamma_k \cdot \widehat{\varphi}^{t_k}(\hat{x})) = \gamma_k \cdot \hat{x}^+$, thus $\lim_{t \to +\infty} \widehat{\varphi}^t(\hat{x}') = \xi(\gamma_+)$. This finishes the proof that $\ell' = \xi_+(x')$. ☐

Proof of 2.⇒1. in Proposition 2.8. Consider a sequence $\gamma_k \to \gamma_+$ and $\gamma_k^{-1} \to \gamma_-$, and assume that $\gamma_+ \neq \gamma_-$ (thanks to Lemma 1.9, it is sufficient to treat this case). Consider the compact subset

$$\mathscr{K} = \Big\{(\gamma_k^+, \gamma_k^-) \Big| k \in \mathbb{N}\Big\} \cup \{(\gamma_+, \gamma_-)\} \subset \partial_\infty \Gamma^{(2)},$$

and let $\mathscr{D} \subset \widehat{\mathsf{X}}$ be a compact subset that surjects onto the pre-image of $\mathscr{K}$ in $\widehat{\mathsf{X}}/\mathbb{R}$ (the existence of $\mathscr{D}$ comes from the properness of the map $\widehat{\mathsf{X}}/\mathbb{R} \to \partial_\infty \Gamma^{(2)}$). Then for all k we can choose $\hat{x}_k \in \mathscr{D}$ such that $\hat{x}_k^\pm = \gamma_k^\pm$, as well as some $t_k \in \mathbb{R}$ such that $\gamma_k \cdot \widehat{\varphi}^{t_k}(\hat{x}_k) \in \mathscr{D}$. Note that we must have $t_k \to -\infty$.

In order to prove local uniform convergence of $\rho(\gamma_k)$ to $\xi(\gamma_+)$ on $\mathbb{P}(V)\setminus\mathbb{P}(\xi^*(\gamma_-))$, by simple topological arguments involving compactness (Lemma 1.4), we only have to prove that for a converging sequence $\ell_k \to \ell \in \mathbb{P}(V)\setminus\mathbb{P}(\xi^*(\gamma^-))$, such that the limit $\ell_\infty = \lim \rho(\gamma_k)\cdot\ell_k \in \mathbb{P}(V)$ exists, we must have $\ell_\infty = \xi(\gamma_+)$. Upon extraction of a subsequence, we may assume that $\gamma_k\cdot\widehat{\varphi}^{t_k}(\widehat{x}_k)$ has a limit $\widehat{x}_\infty \in \mathscr{D}$. Set $\zeta_k = \Gamma\cdot(\widehat{x}_k,\ell_k) \in X\times_\rho \mathbb{P}(V)\setminus\mathbb{P}(\Sigma^-)$. Then

$$\lim_{k\to+\infty} \operatorname{dist}\left(\Phi^{t_k}_{\rho,\mathbb{P}(V)}(\zeta_k), \mathbb{P}(\Sigma^+)\right) = 0 .$$

But we can compute

$$\begin{aligned}\Phi^{t_k}_{\rho,\mathbb{P}(V)}(\zeta_k) &= \Gamma\cdot\left(\widehat{\varphi}^{t_k}(\widehat{x}_k),\ell_k\right)\\ &= \Gamma\cdot\Big(\underbrace{\gamma_k\cdot\widehat{\varphi}^{t_k}(\widehat{x}_k)}_{\to\widehat{x}_\infty},\underbrace{\rho(\gamma_k)\cdot\ell_k}_{\to\ell_\infty}\Big),\end{aligned}$$

thus proving that $\ell_\infty = \xi(\widehat{x}^+_\infty)$. But $\lim_{t\to+\infty}\widehat{\varphi}^t(\gamma_k\cdot\widehat{\varphi}^{t_k}(\widehat{x}_k)) = \gamma_k\cdot\widehat{x}^+_k = \gamma_k\cdot\gamma^+_k = \gamma^+_k$, therefore $\widehat{x}^+_\infty = \gamma_+$ and $\ell_\infty = \xi(\gamma_+)$. This finishes the proof for the dynamics on $\mathbb{P}(V)$. The corresponding dynamics on $\mathbb{P}(V^*)$ follow from applying this result to the dual representation $\rho^* \in \operatorname{Hom}(\Gamma, \mathrm{SL}(V^*))$, which is possible because the dynamical conditions on the suspension bundles $X\times_\rho\mathbb{P}(V)$ and $X\times_\rho\mathbb{P}(V^*)$ are equivalent by combining propositions 2.5 and 2.6. □

2.5 Proving cocompactness

This section is devoted to the proof of Theorem 2.1, and we will follow the proof of [BCLS15, Proposition 4.2]. For this purpose, let us fix a distance d_∞ on $\partial_\infty\Gamma$ defining the Gromov topology, and an adapted metric $\|\cdot\|$ on Σ_+ provided by Lemma 2.3 for the contracting flow (Proposition 2.7)

$$\Phi^t_{\rho,V} : \Sigma_+ \to \Sigma_+ ,$$

i.e. there exists a constant $b>0$ such that

$$\forall x\in X\ \forall z\in \Sigma_+|_x\ \forall t\geq 0 \quad \left\|\Phi^t_{\rho,V}\Big|_x z\right\|_{\varphi^t(x)} \geq e^{bt}\|z\|_x . \tag{1}$$

Note that applying this to reverse time, we also have

$$\forall x\in X\ \forall z\in \Sigma_+|_x\ \forall t\leq 0 \quad \left\|\Phi^t_{\rho,V}\Big|_x z\right\|_{\varphi^t(x)} \leq e^{bt}\|z\|_x .$$

Definition (The map $\widehat{F} : \widehat{X} \to \widehat{K}_\rho$)

Let $\widehat{x}\in\widehat{X}$, choose $v\in V$ such that $[v] = \xi(\widehat{x}^+)$ and $\left\|\Gamma\cdot(\widehat{x},v)\right\|_x = 1$, and $\alpha\in V^*$ such that $[\alpha] = \xi^*(\widehat{x}^-)$ and $\alpha(v) = 1$. We then define

$$\widehat{F}(\widehat{x}) \overset{\text{def}}{=} [v:\alpha] .$$

Proof that $\widehat{F} : \widehat{X}\to\widehat{K}_\rho$ is well defined. Since Σ_+ is a line bundle, the only possibility for a replacement $v'\in\xi(\widehat{x}^+)$ with $\left\|\Gamma\cdot(\widehat{x},v')\right\|_x = 1$ for v is $v' = -v$. This forces the choice of $\alpha'\in V^*$ with $[\alpha'] = \xi^*(\widehat{x}^-)$ with $\alpha'(v') = 1$ to be $\alpha' = -\alpha$, so finally

$$[v':\alpha'] = [-v:-\alpha] = [v:\alpha] .$$

□

Lemma 2.9

The restriction of the map $\widehat{\mathsf{F}} : \widehat{\mathsf{X}} \to \widehat{\mathsf{K}}_\rho$ to every flow line of $\widehat{\varphi}^t$ is a bijection onto a flow line of $\varphi^t_{\mathbb{L}}$.

Proof. Applying the definition of $\widehat{\mathsf{F}}$ to $\widehat{\varphi}^t(\widehat{x})$, let us write the corresponding elements $v_t \in \mathsf{V}$, $\alpha_t \in \mathsf{V}^*$. Since $[v_t] = [v]$ for any $t \in \mathbb{R}$ and we can freely change between v_t and $-v_t$, we can assume that $v_t = \mathrm{e}^{s(t)} v$ for some $s(t) \in \mathbb{R}$ (with $s(0) = 0$). We then have $\alpha_t = \mathrm{e}^{-s(t)}\alpha$ by virtue of $\alpha_t(v_t) = 1$, hence

$$\widehat{\mathsf{F}}(\widehat{\varphi}^t(\widehat{x})) = [v_t : \alpha_t] = \left[\mathrm{e}^{s(t)} v : \mathrm{e}^{-s(t)}\alpha\right] = \varphi_{\mathbb{L}}^{s(t)}\left(\widehat{\mathsf{F}}(\widehat{x})\right).$$

Now set $z = \Gamma \cdot (\widehat{x}, v) \in \Sigma_+|_x$ and $z_t = \Gamma \cdot (\widehat{\varphi}^t(\widehat{x}), v_t) \in \Sigma_+|_{\varphi^t(x)}$, so that we have

$$\begin{aligned} \left\| \Phi^t_{\rho,\mathsf{V}}\Big|_x z \right\|_{\varphi_t(x)} &= \left\|\Gamma \cdot (\widehat{\varphi}^t(x), v)\right\|_{\varphi_t(x)} \\ &= \mathrm{e}^{-s(t)} \left\|\Gamma \cdot (\widehat{\varphi}^t(x), v_t)\right\|_{\varphi_t(x)} \\ &= \mathrm{e}^{-s(t)} \|z_t\|_{\varphi_t(x)} \\ &= \mathrm{e}^{-s(t)}. \end{aligned}$$

Having chosen an adapted norm, we find from (1) that $s(t) \geq bt > 0$ for $t > 0$, thus $\widehat{\mathsf{F}}(\widehat{\varphi}^t(\widehat{x})) \neq \widehat{\mathsf{F}}(\widehat{x})$. Replacing $\widehat{x}$ with a different point on the same flow line, we prove that the restriction of $\widehat{\mathsf{F}}$ to a flow line is injective.
The contraction of the flow on Σ_+ also implies that $s(t) \leq bt$ for $t \leq 0$, so $s(t) \to \pm\infty$ as $t \to \pm\infty$ and the restriction of $\widehat{\mathsf{F}}$ to a flow line surjects onto the corresponding flow line of $\varphi^t_{\mathbb{L}}$. □

Lemma 2.10

The map $\widehat{\mathsf{F}} : \widehat{\mathsf{X}} \to \widehat{\mathsf{K}}_\rho$ is continuous, surjective and Γ-equivariant.

Proof. Continuity can be proved by choosing local continuous sections of the tautological bundles over $\mathbb{P}(\mathsf{V})$ and $\mathbb{P}(\mathsf{V}^*)$ in order to define v and α as continuous functions of $\xi(\widehat{x}^+)$ and $\xi^*(\widehat{x}^-)$ respectively. Surjectivity comes from the surjectivity of the map $\widehat{\mathsf{X}}/\mathbb{R} \to \partial_\infty\Gamma^{(2)}$ and the surjectivity in restriction to flow lines from Lemma 2.9. Finally, the Γ-equivariance is a straightforward consequence of equivariance of the limit maps. □

Lemma 2.11

Let $\mathcal{C} \subset \mathbb{L}$ be a compact subset and $\delta > 0$. The set

$$\mathsf{Z}_{\mathcal{C},\delta} \stackrel{\text{def}}{=} \left\{\widehat{x} \in \widehat{\mathsf{X}} \,\middle|\, \mathrm{d}_\infty\left(\widehat{x}^+, \widehat{x}^-\right) \geq \delta \;\&\; \widehat{\mathsf{F}}(\widehat{x}) \in \mathcal{C}\right\}$$

is compact.

Proof. By properness of the map $\widehat{\mathsf{X}}/\mathbb{R} \to \partial_\infty\Gamma^{(2)}$, consider a compact subset $\mathcal{K}_\delta \subset \widehat{\mathsf{X}}$ such that for any $\widehat{x} \in \widehat{\mathsf{X}}$ with $\mathrm{d}_\infty\left(\widehat{x}^+, \widehat{x}^-\right) \geq \delta$, there is some $\tau(\widehat{x}) \in \mathbb{R}$ with $\widehat{\varphi}^{\tau(\widehat{x})}(\widehat{x}) \in \mathcal{K}_\delta$. Now let $\widehat{x} \in \mathsf{Z}_{\mathcal{C},\delta}$, and using notations from the definition of $\widehat{\mathrm{F}}$ as well as from the proof of Lemma 2.9 write

$$\widehat{\mathrm{F}}(\widehat{x}) = [v : \alpha] \quad \text{and} \quad \widehat{\mathrm{F}}(\widehat{\varphi}^{\tau(\widehat{x})}(\widehat{x})) = \left[e^{s\left(\tau(\widehat{x})\right)} v : e^{-s\left(\tau(\widehat{x})\right)} \alpha\right].$$

Since $\widehat{\mathrm{F}}$ is continuous, both of these elements belong to the compact subset $\mathcal{C} \cup \widehat{\mathrm{F}}(\mathcal{K}_\delta)$ of $\mathbb{L}$. Fixing a norm $\|\cdot\|_{\mathrm{V}}$ on V, this compactness provides a constant $c > 1$ such that

$$\forall (w, \beta) \in \mathrm{V} \oplus \mathrm{V}^* \qquad [w : \beta] \in \mathcal{C} \cup \widehat{\mathrm{F}}(\mathcal{K}_\delta) \;\&\; \beta(w) = 1 \quad \Longrightarrow \quad e^{-c} \le \|w\|_{\mathrm{V}} \le e^{c}.$$

Applying this to our situation, we find

$$\forall \widehat{x} \in \mathrm{Z}_{\mathcal{C},\delta} \qquad e^{-c} \le \|v\|_{\mathrm{V}} \le e^{c} \qquad \text{and} \qquad e^{-c} \le \left\| e^{s\left(\tau(\widehat{x})\right)} v \right\|_{\mathrm{V}} \le e^{c},$$

thus $-2c \le s\left(\tau(\widehat{x})\right) \le 2c$, and we find from the contraction (1) and the expression $s(t) = -\mathrm{Log}\left\| \Phi^t_{\rho,\mathrm{V}}\Big|_x z \right\|$ that $|\tau(\widehat{x})| \le 2c/b$. It follows that $\mathrm{Z}_{\mathcal{C},\delta} \subset \bigcup_{|t| \le 2c/b} \widehat{\varphi}^t(\mathcal{K}_\delta)$ is compact. □

Corollary 2.12

The map $\widehat{\mathrm{F}} : \widehat{\mathrm{X}} \to \widehat{\mathrm{K}}_\rho$ is proper.

Proof. Denote by $\pi : \mathbb{L} \to \mathbb{P}(\mathrm{V}) \overset{\pitchfork}{\times} \mathbb{P}(\mathrm{V}^*)$ the canonical projection, and consider the continuous map

$$f : \begin{cases} \mathbb{P}(\mathrm{V}) \times \mathbb{P}(\mathrm{V}^*) & \to \quad (0, +\infty) \\ \left([v], [\alpha]\right) & \mapsto \quad \frac{|\alpha(v)|}{\|v\|_{\mathrm{V}} \|\alpha\|_{\mathrm{V}^*}} \end{cases}$$

Let $\mathcal{C} \subset \widehat{\mathrm{K}}_\rho$ be a compact subset, and fix some $\varepsilon > 0$ such that $f \circ \pi \ge \varepsilon$ on $\mathcal{C}$. We now consider a distance d_∞ on $\partial_\infty \Gamma$, and the set

$$\mathcal{C}_\infty = \left\{ (\eta_+, \eta_-) \in \partial_\infty \Gamma^{(2)} \,\middle|\, \left(\xi(\eta_+), \xi^*(\eta_-)\right) \in \pi(\mathcal{C}) \right\}.$$

Since $f \circ (\xi, \xi^*) \ge \varepsilon$ on $\mathcal{C}_\infty$ and $f(\xi(\eta), \xi^*(\eta)) = 0$ for all $\eta \in \partial_\infty \Gamma$ (Proposition 1.11), there is some $\delta > 0$ such that $\mathrm{d}_\infty(\eta_+, \eta_-) \ge \delta$ for all $(\eta_+, \eta_-) \in \mathcal{C}_\infty$. It follows that

$$\widehat{\mathrm{F}}^{-1}(\mathcal{C}) = \left\{ \widehat{x} \in \widehat{\mathrm{X}} \,\middle|\, \mathrm{d}_\infty(\widehat{x}^+, \widehat{x}^-) \ge \delta \;\&\; \widehat{\mathrm{F}}(\widehat{x}) \in \mathcal{C} \right\}$$

which is compact according to 2.11, thus proving that $\widehat{\mathrm{F}}$ is proper. □

Proof of Theorem 2.1. It follows directly from the existence of a continuous proper surjective equivariant map $\widehat{\mathrm{F}} : \widehat{\mathrm{X}} \to \widehat{\mathrm{K}}_\rho$ and the same properties for the action of Γ on $\widehat{\mathrm{X}}$. □

Remark

The proof is adapted from [BCLS15], where the authors start with a Gromov flow $\varphi^t : \mathrm{X} \to \mathrm{X}$. In this case the map $\widehat{\mathrm{F}}$ is a homeomorphism.

2.6 Tautological qualities of the refraction flow

Definition (Refraction flow)

Let Γ be a hyperbolic group, and $\rho \in \mathrm{Hom}(\Gamma, \mathrm{SL}(V))$ a projective Anosov representation. The *refraction flow* of ρ is the quotient flow $\varphi_\rho^t : \mathsf{K}_\rho \to \mathsf{K}_\rho$ induced by the restriction of $\varphi_{\mathbb{L}}^t$ to $\widehat{\mathsf{K}}_\rho$.

While the definition of an Anosov representation uses an arbitrary choice of a coarse Gromov flow, it produces a canonical flow in the form of the refraction flow $\varphi_\rho^t : \mathsf{K}_\rho \to \mathsf{K}_\rho$ on $\mathsf{K}_\rho = \Gamma \backslash \widehat{\mathsf{K}}_\rho$.

Proposition 2.13

The refraction flow $\varphi_\rho^t : \mathsf{K}_\rho \to \mathsf{K}_\rho$ is a Gromov flow for Γ. More precisely, for $[v : \alpha] \in \widehat{\mathsf{K}}_\rho$ and $(\eta_+, \eta_-) \in \partial_\infty \Gamma^{(2)}$ such that $[v] = \xi(\eta_+)$ and $[\alpha] = \xi^*(\eta_-)$, we have $\lim_{t\to+\infty} \varphi_{\mathbb{L}}^t([v : \alpha]) = \eta_+$ and $\lim_{t\to-\infty} \varphi_{\mathbb{L}}^t([v : \alpha]) = \eta_-$.

Proof. Note that the second statement implies the first. Consider a coarse Gromov flow $\varphi^t : X \to X$ for Γ and the map $\widehat{F} : \widehat{X} \to \widehat{\mathsf{K}}_\rho$ from the proof of Theorem 2.1. Let $\widehat{p} \in \widehat{\mathsf{K}}_\rho$, and choose some $\widehat{x} \in \widehat{X}$ such that $\widehat{F}(\widehat{x}) = \widehat{p}$. By Lemma 2.9, there is an increasing homeomorphism $h : \mathbb{R} \to \mathbb{R}$ such that

$$\forall t \in \mathbb{R} \qquad \varphi_{\mathbb{L}}^t(\widehat{p}) = \widehat{F}\Big(\widehat{\varphi}^{h(t)}(\widehat{x})\Big).$$

Now for sequences $t_k \in \mathbb{R}$ and $\gamma_k \in \Gamma$, and a compact subset $\mathcal{C} \subset \widehat{\mathsf{K}}_\rho$, we have

$$\rho(\gamma_k^{-1}) \cdot \varphi_{\mathbb{L}}^{t_k}(\widehat{p}) \in \mathcal{C} \iff \gamma_k^{-1} \cdot \widehat{\varphi}^{h(t_k)}(\widehat{x}) \in \widehat{F}^{-1}(\mathcal{C}).$$

Since $\widehat{F}$ is proper, it follows that for any $\eta \in \partial_\infty \Gamma$ we have

$$\gamma_k \to \eta \iff \varphi_{\mathbb{L}}^{t_k}(\widehat{p}) \to \eta \iff \widehat{\varphi}^{h(t_k)}(\widehat{x}) \to \eta.$$

Writing $\widehat{p} = [v : \alpha]$, we have by definition of $\widehat{F}$ that

$$[v] = \xi\Big(\widehat{x}^+\Big) \quad \text{and} \quad [\alpha] = \xi^*\Big(\widehat{x}^-\Big).$$

The fact that $\lim_{t\to\pm\infty} h(t) = \pm\infty$ concludes the proof. □

The sections of the bundles $\mathsf{K}_\rho \times_\rho \mathbb{P}(V)$ and $\mathsf{K}_\rho \times_\rho \mathbb{P}(V^*)$ induced by the limit maps are tautological sections defined by

$$\xi_+ : \begin{cases} \mathsf{K}_\rho & \to & \mathsf{K}_\rho \times_\rho \mathbb{P}(V) \\ \Gamma \cdot [v : \alpha] & \mapsto & \Gamma \cdot \big([v : \alpha], [v]\big) \end{cases} \quad \text{and} \quad \xi_- : \begin{cases} \mathsf{K}_\rho & \to & \mathsf{K}_\rho \times_\rho \mathbb{P}(V^*) \\ \Gamma \cdot [v : \alpha] & \mapsto & \Gamma \cdot \big([v : \alpha], [\alpha]\big) \end{cases}$$

The trivial vector bundle $\widehat{\mathsf{K}}_\rho \times V$ has a natural decomposition into tautological subbundles $\widehat{\Sigma}_+ \to \widehat{\mathsf{K}}_\rho$ and $\widehat{\Sigma}_- \to \widehat{\mathsf{K}}_\rho$ defined as

$$\widehat{\Sigma}_+\Big|_{[v:\alpha]} = \mathbb{R} \cdot v \quad \text{and} \quad \widehat{\Sigma}_-\Big|_{[v:\alpha]} = \ker \alpha.$$

By Γ-equivariance they descend to a splitting $\mathsf{K}_\rho \times_\rho V = \Sigma_+ \oplus \Sigma_-$, which is the splitting given by the sections $\xi_\pm$.

Part 3. Axiom A dynamics and consequences

So far, starting with a projective Anosov representation $\rho \in \mathrm{Hom}(\Gamma, \mathrm{SL}(V))$ of a hyperbolic group Γ, we have seen how to use the limit maps $\xi : \partial_\infty\Gamma \to \mathbb{P}(V)$, $\xi^* : \partial_\infty\Gamma \to \mathbb{P}(V^*)$ to construct a Γ-invariant open set

$$\widehat{M}_\rho = \{[v:\alpha] \in \mathbb{L} \,\big|\, \forall \eta \in \partial_\infty\Gamma \; [v] \pitchfork \xi^*(\eta) \text{ or } \xi(\eta) \pitchfork [\alpha]\}$$

as well as an invariant closed subset

$$\widehat{K}_\rho = \left\{[v:\alpha] \in \mathbb{L} \,\middle|\, \exists(\eta_+, \eta_-) \in \partial_\infty\Gamma^{(2)} \; [v] = \xi(\eta_+) \text{ and } [\alpha] = \xi^*(\eta_-)\right\}.$$

Theorem 3.1

The quotient flow $\varphi_\rho^t : M_\rho \to M_\rho$ satisfies Smale's Axiom A.

3.1 Topological dynamics

Our goal in this section is the following result.

Proposition 3.2 ([DMS25a, Theorem 2])

The non wandering set of the quotient flow $\varphi_\rho^t : X_\rho \to X_\rho$ is $\mathrm{NW}(\varphi_\rho^t) = K_\rho$, and it is equal to the closure of periodic points. Moreover, the restriction of φ_ρ^t to K_ρ is topologically transitive.

3.1.1 Basic notions of topological dynamics

Topological dynamics of a flow $\varphi^t : X \to X$ are concerned with notions of recurrence, i.e. points whose orbits will come back near to itself. This can manifest itself in different behaviours. The simplest example is that of a periodic point, i.e. an element of the set

$$\mathrm{Per}(\varphi^t) = \left\{x \in X \,\middle|\, \exists T > 0 \; \varphi^T(x) = x\right\}.$$

A radically different type of recurrence would be a dense orbit.

Definition (Topological transitivity)

A flow $\varphi^t : X \to X$ is called *topologically transitive* if it has a dense orbit, i.e. there exists $x_\circ \in X$ such that

$$\overline{\{\varphi^t(x_\circ) \,|\, t \in \mathbb{R}\}} = X.$$

Chaotic dynamics are characterized by the coexistence of very different properties of orbits, typically topological transitivity associated to the density of $\mathrm{Per}(\varphi^t)$ in X. The most general notion of recurrence that we will work with is that of a non wandering point.

Definition (Non wandering set)

The non wandering set is

$$\mathrm{NW}(\varphi^t) = \left\{x \in X \,\middle|\, \exists x_k \to x, \; t_k \to \infty \; \varphi^{t_k}(x_k) \to x\right\}.$$

Note that $\mathrm{Per}(\varphi^t) \subset \mathrm{NW}(\varphi^t)$ and that $\mathrm{NW}(\varphi^t)$ is closed. In particular, we have the inclusion

$$\overline{\mathrm{Per}(\varphi^t)} \subset \mathrm{NW}(\varphi^t).$$

These dynamical notions are purely qualitative, as they only depend on the orbits of the flow and not on their parametrisations. We will however differentiate φ^t and φ^{-t} by only considering oriented orbits, i.e. orbits of $\mathbb{R}_+$.

Definition (Topological equivalence)

Two flows $\varphi_1^t : X_1 \to X_1$ and $\varphi_2^t : X_2 \to X_2$ are called *topologically equivalent* if there is a homeomorphism $f : X_1 \to X_2$ that maps oriented orbits of φ_1^t onto oriented orbits of φ_2^t.

Periodic and non wandering points are topological notions in the sense that if $\varphi_1^t : X_1 \to X_1$ and $\varphi_2^t : X_2 \to X_2$ are topologically equivalent through a homeomorphism $f : X_1 \to X_2$, then

$$\mathrm{Per}(\varphi_2^t) = f\Big(\mathrm{Per}(\varphi_1^t)\Big) \quad \text{and} \quad \mathrm{NW}(\varphi_2^t) = f\Big(\mathrm{NW}(\varphi_1^t)\Big).$$

Assuming that φ_1^t and φ_2^t are fixed point free, there are continuous functions $\kappa_i : X_i \times \mathbb{R} \to \mathbb{R}$ for $i = 1, 2$, positive on $X_i \times (0, +\infty)$, such that

$$\begin{aligned} &\forall x_1 \in X_1\ \forall t \in \mathbb{R} & f\Big(\varphi_1^t(x_1)\Big) &= \varphi_2^{\kappa_1(x_1,t)}\Big(f(x_1)\Big), \\ &\forall x_2 \in X_2\ \forall t \in \mathbb{R} & f^{-1}\Big(\varphi_2^t(x_2)\Big) &= \varphi_1^{\kappa_2(x_2,t)}\Big(f^{-1}(x_2)\Big). \end{aligned}$$

These functions, called *reparametrisation cocycles*, satisfy the *cocycle relation*

$$\forall i = 1, 2\ \forall x_i \in X_i\ \forall t, t' \in \mathbb{R} \qquad \kappa_i(x_i, t + t') = \kappa_i(\varphi_i^t(x_i), t') + \kappa_i(x_i, t). \tag{2}$$

A consequence of the cocycle relation is the following estimate when X_1 (hence X_2) is compact: there are constants $c_1, c_2, C > 0$ such that

$$\forall i = 1, 2\ \forall x_i \in X_i\ \forall t \geq 0 \quad c_1 t - C \leq \kappa_i(x_i, t) \leq c_2 t + C. \tag{3}$$

3.1.2 Topological dynamics of the quotient flow

Lemma 3.3 ([DMS25a, Lemma 3.5])

Let $[v : \alpha] \in \widehat{M}_\rho$ and $\gamma \in \Gamma \setminus \{\mathrm{id}\}$ be such that $\rho(\gamma) \cdot [v : \alpha] = [e^{T} v : e^{-T}\alpha]$ for some $T \geq 0$. Then $[v] = \xi(\gamma^+)$, $[\alpha] = \xi^*(\gamma^-)$ and $T = \lambda_1(\rho(\gamma))$.

Proof. Consider $\beta \in V^* \setminus \{0\}$ such that $[\beta] = \xi^*(\gamma^+)$, so that $\beta \circ \rho(\gamma^{-1}) = \pm e^{-\lambda_d(\rho(\gamma))}\beta$. We find

$$\beta(v) = \Big(\beta \circ \rho(\gamma^{-1})\Big)\Big(\rho(\gamma) v\Big) = \pm e^{-\lambda_d(\rho(\gamma)) + T}\beta(v).$$

If $[v] \pitchfork \xi^*(\gamma^+)$, i.e. $\beta(v) \neq 0$, this would mean that $T = \lambda_d(\rho(\gamma)) < 0$, a contradiction with $T \geq 0$. As $[v : \alpha] \in \widehat{M}_\rho$, we must have $\xi(\gamma^+) \pitchfork [\alpha]$.
Now consider $w \in V \setminus \{0\}$ such that $[w] = \xi(\gamma^+)$, so that $\rho(\gamma) w = \pm e^{\lambda_1(\rho(\gamma))} w$. We find

$$\alpha(w) = \Big(\alpha \circ \rho(\gamma^{-1})\Big)\Big(\rho(\gamma) w\Big) = \pm e^{\lambda_1(\rho(\gamma)) - T}\alpha(w).$$

As $\alpha(w) \neq 0$, it means that $T = \lambda_1(\rho(\gamma))$, hence $[v] = \xi(\gamma^+)$ and $[\alpha] = \xi^*(\gamma^-)$. □

Corollary 3.4 ([DMS25a, Lemma 3.11])

Let $x \in \mathbf{M}_\rho$ and consider a lift $[v : \alpha] \in \widehat{\mathbf{M}}_\rho$. Then $x \in \mathrm{Per}(\varphi_\rho^t)$ if and only if there is $\gamma \in \Gamma \setminus \{\mathbf{1}_\Gamma\}$ such that $[v] = \xi(\gamma^+)$ and $[\alpha] = \xi^*(\gamma^-)$. In this case, if γ is primitive, then the period of x is $\lambda_1(\gamma)$.

Proof. The equality $\phi_\rho^T(x) = x$ means that there is $\gamma \in \Gamma$ such that $[e^T v : e^{-T}\alpha] = \rho(\gamma) \cdot [v : \alpha]$, so the result follows from Lemma 3.3. □

The relationship between the topological dynamics of the quotient flow and the limit maps is contained in the following lemma.

Lemma 3.5 ([DMS25a, Lemma 3.10])

Consider sequences $x_k \in \mathbf{M}_\rho$ and $t_k \to +\infty$ and points $x, y \in \mathbf{M}_\rho$ such that $x_k \to x \in \mathbf{M}_\rho$ and $\varphi_\rho^{t_k}(x_k) \to y \in \mathbf{M}_\rho$. Let $[v : \alpha] \in \widehat{\mathbf{M}}_\rho$ (resp. $[w : \beta] \in \widehat{\mathbf{M}}_\rho$) be a lift of x (resp. of y). Then $[v] \in \xi(\partial_\infty\Gamma)$ and $[\beta] \in \xi^*(\partial_\infty\Gamma)$.

Proof. Consider lifts $[v_k : \alpha_k] \in \widehat{\mathbf{M}}_\rho$ of x_k such that $v_k \to v$, $\alpha_k(v_k) = 1$ and $[v_k : \alpha_k] \to [v : \alpha]$ in $\widehat{\mathbf{M}}_\rho$. There is a sequence $\gamma_k \in \Gamma$ such that $\rho(\gamma_k) \cdot [e^{t_k} v_k : e^{-t_k}\alpha_k] \to [w : \beta]$, and up to a subsequence, assume that the limits $\gamma_+ = \lim \gamma_k \in \partial_\infty\Gamma$ and $\gamma_- = \lim \gamma_k^{-1} \in \partial_\infty\Gamma$ exist. Define

$$w_k \overset{\text{def}}{=} e^{t_k}\rho(\gamma_k)v_k \quad \text{and} \quad \beta_k \overset{\text{def}}{=} e^{-t_k}\rho(\gamma_k)\cdot\alpha_k.$$

Since $[w_k : \beta_k] \to [w : \beta]$ and $\beta_k(w_k) = \alpha_k(v_k) = 1$, the sequences w_k and β_k must stay in compact subsets of $V \setminus \{0\}$ and $V^* \setminus \{0\}$ respectively. Indeed, by definition there is a sequences $\lambda_k \in \mathbb{R}^\times$ such that $\lambda_k w_k \to w$ and $\lambda_k\beta_k \to \beta$, and $\lambda_k^2 = (\lambda_k\beta_k)(\lambda_k w_k) \to \beta(w) \neq 0$.
Consequently, $\rho(\gamma_k)v_k = e^{-t_k} w_k \to 0$ and $[v]$ cannot be transverse to $\xi^*(\gamma_-)$ (otherwise Proposition 1.18 would imply that $\rho(\gamma_k)v_k \to \infty$). Since $[v : \alpha] \in \widehat{\mathbf{M}}_\rho$, we must have $[\alpha] \pitchfork \xi(\gamma_-)$, therefore $\rho(\gamma_k) \cdot [\alpha_k] \to \xi^*(\gamma_+)$ (because the limit maps are uniformly dynamics preserving). This means that

$$[\beta] = \xi^*(\gamma_+) \in \xi^*(\partial_\infty\Gamma).$$

From $[w : \beta] \in \mathbb{L}$ and $[\beta] = \xi^*(\gamma_+)$, we now find that $[w] \pitchfork \xi^*(\gamma_+)$, so $\rho(\gamma_k^{-1}) \cdot [w_k] \to \xi(\gamma_-)$, hence

$$[v] = \lim_{k\to+\infty} [v_k] = \lim_{k\to+\infty} \rho(\gamma_k^{-1}) \cdot [w_k] = \xi(\gamma_-) \in \xi(\partial_\infty\Gamma).$$

□

Proof of Proposition 3.2. We find directly from Corollary 3.4, denoting by $\Gamma_\infty \subset \Gamma$ the set of infinite order elements, that

$$\mathrm{Per}(\varphi_\rho^t) = \Gamma \backslash \left\{ [v : \alpha] \in \mathbb{L} \,\middle|\, \exists \gamma \in \Gamma_\infty \ [v] = \xi(\gamma^+) \text{ and } [\alpha] = \xi^*(\gamma^-) \right\}.$$

Since the set of poles of Γ, i.e. pairs (γ^+, γ^-) for $\gamma \in \Gamma_\infty$, is dense in $\partial_\infty\Gamma^{(2)}$ [Gro87, Corollary 8.2.G], we find

$$\overline{\mathrm{Per}(\varphi_\rho^t)} = \mathbf{K}_\rho.$$

Now if $x \in \mathrm{NW}(\varphi^t)$, then by definition there exist sequences $x_k \in \mathbf{M}_\rho$ and $t_k \to +\infty$ such that

$\varphi_\rho^{t_k}(x_k) \to x$, and Lemma 3.5 implies that $x \in \mathbf{K}_\rho$. It follows that

$$\mathrm{NW}(\varphi_\rho^t) \subset \mathbf{K}_\rho = \overline{\mathrm{Per}(\varphi_\rho^t)}.$$

Since any continuous flow φ^t satisfies $\overline{\mathrm{Per}(\varphi^t)} \subset \mathrm{NW}(\varphi^t)$, we finally get

$$\mathrm{NW}(\varphi_\rho^t) = \mathbf{K}_\rho = \overline{\mathrm{Per}(\varphi_\rho^t)}.$$

For the topological transitivity, we start by considering $(\eta_+, \eta_-) \in \partial_\infty \Gamma^{(2)}$ whose Γ-orbit is dense in $\partial_\infty \Gamma^{(2)}$ [Gro87, 8.2.I]. Let $x \in \mathbf{K}_\rho$ be an element admitting a lift $[v : \alpha] \in \widehat{\mathbf{K}}_\rho$ with $[v] = \xi(\eta_+)$ and $[\alpha] = \xi^*(\eta_-)$. Let $y \in \mathbf{K}_\rho$ and consider a lift $[w : \beta] \in \widehat{\mathbf{K}}_\rho$.
Let $\mathrm{U} \subset \mathbf{K}_\rho$ be an open subset containing y, and $\widehat{\mathrm{U}} \subset \widehat{\mathbf{K}}_\rho$ its preimage. As the projection $p : \mathbb{L} \to \mathbb{P}(\mathrm{V}) \overset{\pitchfork}{\times} \mathbb{P}(\mathrm{V}^*)$ is an open map, there is $\gamma \in \Gamma$ such that $\rho(\gamma) \cdot ([v], [\alpha]) \in p(\widehat{\mathrm{U}})$, i.e. there is $t \in \mathbb{R}$ such that $\rho(\gamma) \cdot [e^t v : e^{-t} \alpha] \in \widehat{\mathrm{U}}$, hence $\varphi_\rho^t(x) \in \mathrm{U}$. This proves the density of the orbit of x. □

3.2 Hyperbolicity

We now turn to the hyperbolicity of the quotient flow.

Proposition 3.6

The subset $\mathbf{K}_\rho \subset \mathbf{M}_\rho$ is a hyperbolic set for the flow φ_ρ^t.

Before we prove this, let us describe a bit of the geometry of the homogeneous space

$$\mathbb{L} = \{[v : \alpha] \in \mathbb{P}(\mathrm{V} \oplus \mathrm{V}^*) \,|\, \alpha(v) > 0\}.$$

In order to describe the geometry of $\mathbb{L}$, it is practical to work with the affine quadric hypersurface

$$\mathbb{L}_1 \overset{\text{def}}{=} \{(v, \alpha) \in \mathrm{V} \oplus \mathrm{V}^* \,|\, \alpha(v) = 1\}$$

which is an $\mathrm{SL}(\mathrm{V})$-equivariant double cover of $\mathbb{L}$ through the restriction of the projection $\pi : (\mathrm{V} \oplus \mathrm{V}^*) \setminus \{(0,0)\} \to \mathbb{P}(\mathrm{V} \oplus \mathrm{V}^*)$. The description of the tangent space

$$\mathrm{T}_{(v,\alpha)} \mathbb{L}_1 = \{(w, \beta) \in \mathrm{V} \oplus \mathrm{V}^* \,\big|\, \alpha(w) + \beta(v) = 0\}$$

shows that there is a natural splitting $\mathrm{T}\mathbb{L}_1 = \mathrm{E}^{\mathrm{u}} \oplus \mathrm{E}^0 \oplus \mathrm{E}^{\mathrm{s}}$ where

$$\begin{aligned} \mathrm{E}^{\mathrm{u}}_{(v,\alpha)} &= \ker \alpha \times \{0\}, \\ \mathrm{E}^{\mathrm{s}}_{(v,\alpha)} &= \{0\} \times \ker \iota_v, \qquad \forall (v, \alpha) \in \mathbb{L}_1. \\ \mathrm{E}^{0}_{(v,\alpha)} &= \mathbb{R} \cdot (v, -\alpha), \end{aligned}$$

These distributions project to an $\mathrm{SL}(\mathrm{V})$-equivariant splitting of the tangent bundle $\mathrm{T}\mathbb{L}$:

$$\mathrm{T}\mathbb{L} = \mathrm{E}^{\mathrm{u}} \oplus \mathrm{E}^0 \oplus \mathrm{E}^{\mathrm{s}} \tag{4}$$

where $\mathrm{E}^{\mathrm{i}}_{[v:\alpha]} = d\pi|_{(v,\alpha)}(\mathrm{E}^{\mathrm{i}}_{(v,\alpha)})$ for $\mathrm{i} \in \{\mathrm{u}, 0, \mathrm{s}\}$ and $(v, \alpha) \in \mathbb{L}_1$. This decomposition is invariant under the differential of the flow $\varphi_{\mathbb{L}}^t$, and it is related to this flow by the formula $\mathrm{E}^0_{[v:\alpha]} = \mathbb{R} \cdot \frac{d}{dt}\big|_{t=0} \varphi_{\mathbb{L}}^t([v : \alpha])$.

The SL(V)-equivariant of the decomposition (4) means that it descends to a decomposition

$$TM_\rho = E^u \oplus E^0 \oplus E^s$$

enjoying the same relationship with φ^t_ρ as $\varphi^t_{\mathbb{L}}$ with (4).

Lemma 3.7

The restrictions of the differential of $d\varphi^t_\rho$ to $E^u|_{K_\rho}$ and $E^s|_{K_\rho}$ are respectively conjugate to the vertical flows $d^V_{\xi_+}\Phi^t_{\rho,\mathbb{P}(V)} : K_\rho \times_\rho T_{\xi_+}\mathbb{P}(V) \to K_\rho \times_\rho T_{\xi_+}\mathbb{P}(V)$ and $d^V_{\xi^*_-}\Phi^t_{\rho,\mathbb{P}(V^*)} : K_\rho \times_\rho T_{\xi^*_-}\mathbb{P}(V^*) \to K_\rho \times_\rho T_{\xi^*_-}\mathbb{P}(V^*)$.

Proof. Recall that the bundles $K_\rho \times_\rho T_{\xi_+}\mathbb{P}(V)$ and $K_\rho \times_\rho T_{\xi^*_-}\mathbb{P}(V^*)$ can be seen as the quotients

$$K_\rho \times_\rho T_{\xi_+}\mathbb{P}(V) = \Gamma\backslash\left(\widehat{\xi}_+\right)^* T\mathbb{P}(V) \quad \text{and} \quad K_\rho \times_\rho T_{\xi^*_-}\mathbb{P}(V^*) = \Gamma\backslash\left(\widehat{\xi}^*_-\right)^* T\mathbb{P}(V^*)$$

where $\widehat{\xi}_+ : \widehat{K}_\rho \to \mathbb{P}(V)$ and $\widehat{\xi}^*_- : \widehat{K}_\rho \to \mathbb{P}(V^*)$ are the tautological maps

$$\widehat{\xi}_+ : \left\{ \begin{array}{ccc} \widehat{K}_\rho & \to & \mathbb{P}(V) \\ [v:\alpha] & \mapsto & [v] \end{array} \right. \quad \text{and} \quad \widehat{\xi}^*_- : \left\{ \begin{array}{ccc} \widehat{K}_\rho & \to & \mathbb{P}(V^*) \\ [v:\alpha] & \mapsto & [\alpha] \end{array} \right.$$

Consider the projections

$$p_+ : \left\{ \begin{array}{ccc} \mathbb{L} & \to & \mathbb{P}(V) \\ [v:\alpha] & \mapsto & [v] \end{array} \right. \quad \text{and} \quad p_- : \left\{ \begin{array}{ccc} \mathbb{L} & \to & \mathbb{P}(V^*) \\ [v:\alpha] & \mapsto & [\alpha] \end{array} \right.$$

The SL(V)-equivariance of these maps induce Γ-equivariant isomorphisms

$$\left\{ \begin{array}{ccc} E^u|_{\widehat{K}_\rho} & \to & \left(\widehat{\xi}_+\right)^* T\mathbb{P}(V) \\ (\widehat{x}, w) & \mapsto & (\widehat{x}, dp_+|_{\widehat{x}}\, w) \end{array} \right. \quad \text{and} \quad \left\{ \begin{array}{ccc} E^s|_{\widehat{K}_\rho} & \to & \left(\widehat{\xi}^*_-\right)^* T\mathbb{P}(V^*) \\ (\widehat{x}, w) & \mapsto & (\widehat{x}, dp_-|_{\widehat{x}}\, w) \end{array} \right.$$

that descend to isomorphisms $E^u|_{K_\rho} \to K_\rho \times_\rho T_{\xi_+}\mathbb{P}(V)$ and $E^s|_{K_\rho} \to K_\rho \times_\rho T_{\xi^*_-}\mathbb{P}(V^*)$. The flow-invariance $p_\pm \circ \varphi^t_{\mathbb{L}} = p_\pm$ shows that these isomorphisms conjugate the flows.

□

3.3 The geometry of the stable and unstable foliations

3.3.1 Global product structure

For $[v:\alpha] \in \mathbb{L}$, the distributions E^s and E^u in the splitting

$$T\mathbb{L} = E^u \oplus E^0 \oplus E^s$$

are tangent to foliations W^s and W^u of $\mathbb{L}$ whose leaves are

$$\begin{aligned} W^u([v:\alpha]) &= \{[w:\alpha] \,|\, w \in V,\ \alpha(w) = \alpha(v)\}, \\ W^s([v:\alpha]) &= \{[v:\beta] \,\big|\, \beta \in V^*,\ \beta(v) = \alpha(v)\}. \end{aligned} \tag{5}$$

We will also consider the central unstable distribution $E^{cu} = E^u \oplus E^0$ and the central stable distribution $E^{cs} = E^s \oplus E^0$ as well as the associated foliations W^{cu}, W^{cs} of $\mathbb{L}$ whose leaves are

$$\begin{aligned} W^{cu}([v:\alpha]) &= \{[w:\alpha] \,|\, w \in V,\ \alpha(w) > 0\}, \\ W^{cs}([v:\alpha]) &= \{[v:\beta] \,\big|\, \beta \in V^*,\ \beta(v) > 0\}. \end{aligned} \tag{6}$$

Note that for $[v:\alpha], [w:\beta] \in \mathbb{L}$ such that $\alpha(w) > 0$ (resp. $\beta(v) > 0$), the intersection $W^{cu}([v:\alpha]) \cap W^s([w:\beta])$ (resp. $W^{cs}([v:\alpha]) \cap W^u([w:\beta])$) consists of exactly one point:

$$\begin{aligned} W^{cu}([v:\alpha]) \cap W^s([w:\beta]) &= \left\{\left[w : \frac{1}{\alpha(w)}\alpha\right]\right\}, \\ W^{cs}([v:\alpha]) \cap W^u([w:\beta]) &= \left\{\left[\frac{1}{\beta(v)} v : \beta\right]\right\}. \end{aligned} \tag{7}$$

3.3.2 Local product structure

The foliations W^u, W^s, W^{cu} and W^{cs} project to foliations of $\mathbf{M}_\rho$ denoted by the same names, respectively. However, the leaves of these foliations are only immersed submanifolds, so we need to be somewhat careful when discussing local properties of the leaves. From now on, we fix a complete Riemannian metric on $\mathbf{M}_\rho$, and denote by d the Riemannian distance.

Definition 3.8 (Local (central) (un)stable manifolds)

For $x \in \mathbf{M}_\rho$ and $\varepsilon > 0$, we consider

$$\begin{aligned} W^s_\varepsilon(x) &\overset{\text{def}}{=} \left\{x' \in \mathbf{M}_\rho \,\middle|\, \forall t \geq 0\ \ d(\varphi^t_\rho(x), \varphi^t_\rho(x')) \leq \varepsilon \ \text{ and } \lim_{t\to+\infty} d(\varphi^t_\rho(x), \varphi^t_\rho(x')) = 0\right\}, \\ W^u_\varepsilon(x) &\overset{\text{def}}{=} \left\{x' \in \mathbf{M}_\rho \,\middle|\, \forall t \leq 0\ \ d(\varphi^t_\rho(x), \varphi^t_\rho(x')) \leq \varepsilon \ \text{ and } \lim_{t\to-\infty} d(\varphi^t_\rho(x), \varphi^t_\rho(x')) = 0\right\}, \\ W^{cs}_\varepsilon(x) &\overset{\text{def}}{=} \bigcup_{|t|<\varepsilon} W^s_\varepsilon\left(\varphi^t_\rho(x)\right), \\ W^{cu}_\varepsilon(x) &\overset{\text{def}}{=} \bigcup_{|t|<\varepsilon} W^u_\varepsilon\left(\varphi^t_\rho(x)\right). \end{aligned}$$

The following result applies to any Axiom A flow as a consequence of the Stable Manifold Theorem (see [Sma67, §II.7 Thm 7.3] and [KH95, Thm 6.4.9] for the discrete time case and [Dya18, Thm. 5] for a detailed treatment of the flow case) and compactness of $\mathbf{K}_\rho$, but in our case it can be recovered from the explicit formulas (5), (6) and (7).

Lemma 3.9 (Local product structure and exponential decay)

There exist $\varepsilon, \delta, C, a > 0$ such that the following properties are satisfied at every $x \in \mathbf{K}_\rho$:

(1) For every $i \in \{s, u, cs, cu\}$, $W^i_\varepsilon(x)$ is an embedded submanifold of $\mathbf{M}_\rho$ with tangent space $T_{x'}W^i_\varepsilon(x) = E^i_{x'}$, at every point $x' \in W^i_\varepsilon(x)$.

(2) For any $x' \in B(x, \delta)$, the intersections $W^s_\varepsilon(x) \cap W^{cu}_\varepsilon(x')$ and $W^u_\varepsilon(x) \cap W^{cs}_\varepsilon(x')$ each consist of a single point. These points lie in $\mathbf{K}_\rho$ if, and only if, $x' \in \mathbf{K}_\rho$.

(3) $\forall x' \in W^s_\varepsilon(x)\ \forall t \geq 0\ \ d(\varphi^t_\rho(x), \varphi^t_\rho(x')) \leq Ce^{-at} d(x, x')$.

(4) $\forall x' \in W^u_\varepsilon(x)\ \forall t \leq 0\ \ d(\varphi^t_\rho(x), \varphi^t_\rho(x')) \leq Ce^{at} d(x, x')$.

Points (2), (3) and (4) were already established by Sambarino without the need of smoothness.

3.4 Consequences of having an Axiom A flow

Axiom A flows share many properties of geodesic flows of convex cocompact hyperbolic manifolds. For example, the axiom A property (added to some "non triviality") implies that the flow φ^t_ρ has positive entropy, and that periodic orbits equidistribute.

3.4.1 Equidistribution of periodic orbits

For a periodic orbit $c \subset \mathbf{K}_\rho$, let $\ell(c)$ denote its period, and λ_c the Lebesgue measure of length $\ell(c)$ supported on c. Given $U \in \mathcal{C}^0(\mathbf{K}_\rho)$, we consider

$$\ell_U(c) = \int_c U \, d\lambda_c.$$

Theorem 3.10 ([PP90, Theorem 9.4])

For any positive Hölder function $U \in \mathcal{C}^\alpha(\mathbf{K}_\rho)$, the weak-* limit

$$m_U \overset{\text{def}}{=} \lim_{T \to +\infty} \frac{\sum_{\ell(c) \leq T} e^{\ell_U(c)} \frac{\lambda_c}{\ell(c)}}{\sum_{\ell(c) \leq T} e^{\ell_U(c)}}$$

exists. Furthermore, its support is $\mathbf{K}_\rho$, it gives positive measure to any non empty open subset of $\mathbf{K}_\rho$, and it is φ^t_ρ-invariant and ergodic.

Two important cases: $U = 0$ gives the maximal entropy measure, and choosing the unstable Jacobian $U = \frac{d}{dt}\big|_{t=0} \operatorname{Log} \det d\varphi^t_\rho|_{E^u}$ we get the SRB measure.

The thermodynamical formalism also allows to prove that these Gibbs measure are mixing. For a potential $U \in C^\alpha(\mathbf{K}_\rho, \mathbb{R})$ and observables $F, G \in L^2(m_U)$, the *correlation function* is defined by

$$c^t(F, G; U) = \left| \int_{z \in \mathbf{K}_\rho} F(z) \cdot G(\varphi^t_\rho(z)) \, dm_U(z) - \int_{z \in \mathbf{K}_\rho} F(z) \, dm_U(z) \int_{z \in \mathbf{K}_\rho} G(z) \, dm_U(z) \right|. \tag{8}$$

Theorem 3.11

For any potential $U \in C^\alpha(\mathbf{K}_\rho)$ and observables $F, G \in L^2(m_U)$, we have that $\lim_{t\to\infty} c^t(F,G;U) = 0$.

3.4.2 Exponential mixing

General Axiom A flows can have arbitrarily slow mixing rates (i.e. the rate of decay of the correlation functions $c^t(F,G;U)$), but large classes (such as geodesic flows of convex cocompact groups in rank one) mix exponentially fast.

Definition

The flow φ_ρ^t is *exponentially mixing* with respect to m_U for all Hölder observables if there exists $c_\alpha(U), C_\alpha(U) > 0$ such that

$$\forall\, t \in \mathbb{R} \quad c^t(F,G;U) \le C_\alpha(U) e^{-c_\alpha(U)|t|} \|F\|_\alpha \|G\|_\alpha .$$

Theorem 3.12 ([DMS25a, Theorem C])

If ρ is irreducible, then for any Hölder potential $U \in C^\alpha(\mathbf{K}_\rho)$ the flow φ_ρ^t is exponentially mixing with respect to m_U for all Hölder observables.

The proof relies on a result of Stoyanov establishing exponential mixing for Axiom A flows under geometric assumptions on the stable and unstable laminations. As most results on exponential mixing of hyperbolic dynamical systems, Stoyanov's work is built open the *Dolgopyat method*, which is itself built on estimates on the *time separation function* called *non integrability conditions*.

For Zariski dense representations and the maximal entropy measure, exponential mixing for Hölder observables was also established by Chow and Sarkar [CS24].

Definition (Time separation and projection)

Let $x, y \in \mathbf{K}_\rho$ with $d(x,y) < \delta$ (where $\varepsilon > \delta > 0$ are given by Lemma 3.9). Define $\pi_y(x) \in \mathbf{K}_\rho$ by $\pi_y(x) \in W^s_\varepsilon(x) \cap W^{cu}_\varepsilon(y)$ and $\Delta(x,y) \in (-\varepsilon, \varepsilon)$ by $\varphi_\rho^{\Delta(x,y)}(\pi_y(x)) \in W^u_\varepsilon(y)$.

The time separation function can be understood by following a simple process (see Figure 1): any two nearby points $x, y \in \mathbf{K}_\rho$ can be joined by a path that consists in the concatenation of

1. a path in $W^s_\varepsilon(x)$,
2. flowing out for some time $\Delta(x,y)$,
3. a path in $W^u_\varepsilon(y)$.

The time separation function is the time spent flowing out in the second step. It can be computed explicitly by consider local inverses of the projection $\pi : \widehat{\mathbf{M}}_\rho \to \mathbf{M}_\rho$. Choosing nearby lifts $[v : \alpha], [w : \beta] \in \widehat{\mathbf{K}}_\rho$ of x, y, we find:

$$\pi_y(x) = \pi\left(\left[v : \frac{1}{\beta(v)}\beta\right]\right), \qquad \Delta(x,y) = -\log \beta(v). \tag{9}$$

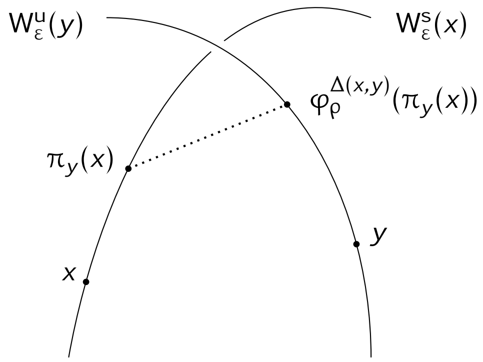


Figure 1: The time separation function and the dynamical projection.

Lower bounds on the time separation function around the diagonal are called *non integrability conditions*, as they quantify the fact that the distribution $E^s \oplus E^u$ is not integrable. In our case, the distribution $E^s \oplus E^u$ is a contact structure, so any reasonable non-integrability condition should be satisfied. However in the work of Stoyanov [Sto11] the precise non-integrability condition one needs involves the fractal geometry of $\mathbf{K}_\rho$.

Definition (Strong Local Non-Integrability Condition (SLNIC))

There exists $d_0 \in (0,1)$ and ε_0 such that, for any $\varepsilon \in (0, \varepsilon_0)$, $z \in \mathbf{K}_\rho$ and any unit vector $w \in E^u|_z$, there exist $y \in \mathbf{K}_\rho \cap W^s_\varepsilon(z)$, $\kappa > 0$ and $\varepsilon' \in (0, \varepsilon)$ such that:

$$|\Delta(\exp^u_x(\nu), \pi_y(x))| \geq \kappa \|\nu\|$$

for any $x \in \mathbf{K}_\rho \cap W^u_{\varepsilon'}(z)$ and $\nu \in E^u|_x$ with $\exp^u_x(\nu) \in \mathbf{K}_\rho$, $\|\nu\| \leq \varepsilon'$ and $\left\| \frac{\nu_z}{\|\nu_z\|} - w \right\| \leq d_0$ where $\nu_z \in E^u|_z$ is the parallel transport of ν along the geodesic in $W^u(z)$ from x to z with respect to the metric obtained by restriction of the auxiliary Riemannian metric on $\mathbf{M}_\rho$.

Definition 3.13 (Bowen's dynamical balls)

Let $x \in \mathbf{K}_\rho$, $T > 0$ and $\delta > 0$ and define

$$B^u_T(x, \delta) = \left\{ y \in W^u_{\varepsilon_0}(x) \,\middle|\, d(\varphi^t_\rho(x), \varphi^t_\rho(y)) \leq \delta, \forall 0 \leq t \leq T \right\}.$$

Definition (Uniformly regular distortion along unstable manifolds)

The flow φ^t_ρ has *uniformly regular distortion along unstable manifolds* over the basic set $\mathbf{K}_\rho$ if for some constant $\varepsilon_* > 0$ and every $\delta \in (0, \varepsilon_*)$, there exists $R_\delta > 0$ such that

$$\operatorname{diam}\left(\mathbf{K}_\rho \cap B^u_T(x, \varepsilon)\right) \leq \varepsilon R_\delta \cdot \operatorname{diam}\left(\mathbf{K}_\rho \cap B^u_T(x, \delta)\right)$$

for every $x \in \mathbf{K}_\rho$, $\varepsilon \in (0, \varepsilon_*)$ and $T > 0$.

Stoyanov proves exponential mixing when the following three hypotheses are simultaneously satisfied:

- Strong local non uniform integrability,
- Uniformly regular distortion along stable manifolds,
- Lipschitz regularity of holonomy of the stable lamination.

This last condition is automatic in our setting because of the smoothness of the distributions $E^{s/u}$. The proof in [DMS25a] consists in verifying the first two hypotheses. Strong local non uniform integrability uses the irreducibility of the representation (more precisely, the assumption that $\xi(\partial_\infty\Gamma)$ cannot be included in the projectivization of a proper vector subspace of V). The proof of uniformly regular distortion along stable manifolds follows ideas from [Sto13].

3.4.3 Counting problems

Given a projective Anosov representation $\rho \in \mathrm{Hom}(\Gamma, \mathrm{SL}(V))$, define the orbit counting function

$$N_\rho(t) = \mathrm{Card}\left\{[\gamma] \in [\Gamma]_{\mathrm{prim}} \,\middle|\, \lambda_1(\rho(\gamma)) \le t\right\}.$$

Theorem 3.14 ([DMS25a, Theorem E])

Suppose $\Gamma < \mathrm{SL}(V)$ is an irreducible projective Anosov subgroup, and denote by $h_\rho > 0$ the topological entropy of the flow φ_ρ^t on K_ρ. Then there exists $0 < c < h_\rho$ such that

$$N_\rho(t) = \frac{e^{h_\rho t}}{h_\rho t}\left(1 + O\left(e^{-ct}\right)\right).$$

The leading term $N_\rho(t) \sim \frac{e^{h_\rho t}}{h_\rho t}$ was obtained by Sambarino [Sam14]. The bridge between orbital counting and mixing rates is the Zeta function

$$\zeta_\rho(s) = \prod_{[\gamma]\in[\Gamma]_{\mathrm{prim}}} \left(1 - e^{-s\lambda_1(\rho(\gamma))}\right)^{-1}.$$

Here $[\Gamma]_{\mathrm{prim}}$ denotes the set of conjugacy classes of primitive elements in Γ (i.e. elements that are not positive powers of other elements). This product defines a holomorphic function on the half plane $\mathfrak{Re}(s) > h_\rho$.

Theorem 3.15 ([DMS25a, Theorem D])

For any projective Anosov representation, the function ζ_ρ admits a meromorphic extension to $\mathbb{C}$. If ρ is irreducible, then there is some $\varepsilon > 0$ such that ζ_ρ has no zero or pole in the vertical strip $h_\rho - \varepsilon < \mathfrak{Re}(s) \le h_\rho$ with the exception of a simple pole at $s = h_\rho$.

The meromorphic extension to $\mathbb{C}$ is due to Dyatlov-Guillarmou [DG16, DG18] for all Axiom A flows, solving a conjecture of Smale. Since our flow is real analytic and has real analytic stable and unstable distributions, the result also follows from earlier work of Fried [Fri95], building upon ideas of Rugh [Rug92, Rug96].

The zero and pole free vertical strip follows from the spectral estimates on Ruelle transfer operators achieved by Stoyanov [Sto11], combined with the work Pollicott-Sharp [PS98] (see also Dolgopyat-Pollicott [DP98]).

Part 4. Lie theory

We now leave the setting of projective Anosov representations and move to a general non compact semi-simple Lie group G (with finite centre and finitely many connected components), and denote bu $\mathfrak{g}$ its Lie algebra. The Killing form $B_{\mathfrak{g}}$ on $\mathfrak{g}$ is a non-degenerate indefinite symmetric bilinear form giving a canonical duality $\mathfrak{g} \simeq \mathfrak{g}^*$ intertwining the adjoint and coadjoint G-actions. Via the latter actions, G acts on the Grassmannians $\mathrm{Gr}_d(\mathfrak{g})$, $\mathrm{Gr}_d(\mathfrak{g}^*)$ for all $0 \le d \le \dim \mathfrak{g}$.

4.1 Flag manifolds and transverse flag spaces

This section is a summary of [DMS25b, Section 3].

4.1.1 Parabolic subgroups and flag manifolds

An element $X \in \mathfrak{g}$ is called *hyperbolic* if $\mathrm{ad}_X : \mathfrak{g} \to \mathfrak{g}$ is diagonalizable over $\mathbb{R}$. We write $\mathfrak{g}_{\mathrm{hyp}} \subset \mathfrak{g}$ for the subset of hyperbolic elements. For $X \in \mathfrak{g}_{\mathrm{hyp}}$, we consider the subsets

$$\mathfrak{p}_X \stackrel{\mathrm{def}}{=} \bigoplus_{\lambda \ge 0} \ker(\mathrm{ad}_X - \lambda \mathrm{id}) \subset \mathfrak{g} \quad \text{and} \quad P_X \stackrel{\mathrm{def}}{=} \mathrm{Stab}_G(\mathfrak{p}_X) = \{g \in G \,|\, \mathrm{Ad}(g)\mathfrak{p}_X = \mathfrak{p}_X\}.$$

Then $\mathfrak{p}_X$ is a Lie subalgebra of $\mathfrak{g}$, and P_X is a closed Lie subgroup of G with Lie algebra $\mathfrak{p}_X$ (this last fact is rather standard but not easy, see e.g. [Vog97, Ch. 7],[Kna02, VII.7], [War72, Ch. 1.2]).

Example (Running example)

In the case of $\mathfrak{g} = \mathfrak{sl}(d, \mathbb{K})$, $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, an element $X \in \mathfrak{sl}(d, \mathbb{K})$ is hyperbolic if and only if it is diagonalizable with real eigenvalues. For some integer $r \in \mathbb{N}$, consider vectors $\mathbf{d} = (d_1, ..., d_r) \in \mathbb{N}^r$ and $\mathbf{x} = (x_1, \dots, x_r) \in \mathbb{R}^r$ such that

$$d_1 + \cdots + d_r = d, \qquad d_1 x_1 + \cdots + d_r x_r = 0, \qquad \text{and } x_1 > \cdots > x_r,$$

and the diagonal matrix

$$X = \begin{pmatrix} x_1 \mathbf{1}_{d_1} & & \\ & \ddots & \\ & & x_r \mathbf{1}_{d_r} \end{pmatrix} \in \mathfrak{sl}(d, \mathbb{K}).$$

Then P_X consists of upper block-wise triangular matrices:

$$P_X = \left\{ \begin{pmatrix} * & & * \\ & \ddots & \\ 0 & & * \end{pmatrix} \right\}.$$

Definition (Parabolic subgroups and subalgebras)

A *parabolic subgroup* of G is a subgroup $P < G$ for which there is $X \in \mathfrak{g}_{\mathrm{hyp}}$ with $P = P_X$.
A *parabolic subalgebra* of $\mathfrak{g}$ is a subalgebra $\mathfrak{p} \subset \mathfrak{g}$ for which there is $X \in \mathfrak{g}_{\mathrm{hyp}}$ with $\mathfrak{p} = \mathfrak{p}_X$.

Flag manifolds are the homogeneous spaces corresponding to parabolic subgroups.

Definition (Flag manifolds)

A *flag manifold* is a G-homogeneous space $\mathcal{F}$ whose point stabilizers are parabolic subgroups.

The geometric approach to flag manifolds usually consists in interpreting them as orbits in the visual boundary of the Riemannian symmetric space of G (see e.g. [Ebe96, 2.17]), but we will use a different description by embedding them in Grassmannian manifolds of the Lie algebra $\mathfrak{g}$. For $X \in \mathfrak{g}_{\mathrm{hyp}}$, we consider

$$\mathcal{F}_X \overset{\mathrm{def}}{=} G \cdot \mathfrak{p}_X \subset \mathrm{Gr}_{\dim \mathfrak{p}_X}(\mathfrak{g}).$$

By the definition of P_X, there is an isomorphism $\mathcal{F}_X \simeq G/P_X$ as analytic G-manifolds. Since flag manifolds are compact [Kna02, VII, Prop. 7.83 (f)], $\mathcal{F}_X \subset \mathcal{G}_{\dim \mathfrak{p}_X}(\mathfrak{g})$ is closed.

Definition (Standard flag manifolds)

A *standard flag manifold* is a subset $\mathcal{F} \subset \mathcal{G}_d(\mathfrak{g})$ (for some $0 \leq d \leq \dim \mathfrak{g}$) for which there is $X \in \mathfrak{g}_{\mathrm{hyp}}$ with $\mathcal{F} = \mathcal{F}_X$.

Any flag manifold $\mathcal{F}$ is uniquely G-equivariantly diffeomorphic to a standard flag manifold, simply defined as $\{\mathfrak{stab}(x) \mid x \in \mathcal{F}\}$ where $\mathfrak{stab}(x)$ denotes the Lie algebra of the stabilizer of $x \in \mathcal{F}$.

Example (Running example)

The space $\mathcal{F}_{\mathbf{d}}$ of *flags of type* $\mathbf{d}$ is defined as

$$\mathcal{F}_{\mathbf{d}}(\mathbb{K}^d) = \left\{ V_\bullet \in \prod_{i=0}^{r} \mathrm{Gr}_{d_1+\cdots+d_i}(\mathbb{K}^d) \,\middle|\, \{0\} = V_0 \subset V_1 \subset \cdots \subset V_{r-1} \subset V_r = \mathbb{K}^d \right\}.$$

Considering a real diagonal matrix $X = \mathrm{Diag}\left(x_1 \mathbf{1}_{d_1}, \ldots, x_r \mathbf{1}_{d_r}\right) \in \mathfrak{sl}(d, \mathbb{K})$ with $x_1 > \cdots > x_r$, the map

$$\begin{cases} \mathcal{F}_{\mathbf{d}}(\mathbb{K}^d) & \to & \mathcal{F}_X \\ V_\bullet & \mapsto & \{Y \in \mathfrak{sl}(d, \mathbb{K}) \mid Y \cdot V_i \subset V_i \ \forall i\} \end{cases}$$

is an $SL(d, \mathbb{K})$-equivariant diffeomorphism.

To get a similar classification for an arbitrary semi-simple Lie group G, one considers a *Cartan subspace* $\mathfrak{a} \subset \mathfrak{g}$, i.e. a vector subspace consisting of commuting hyperbolic elements of maximal dimension. One can then write the *restricted root space decomposition*

$$\mathfrak{g} = \mathfrak{a} \oplus \mathfrak{m} \oplus_{\alpha \in \Sigma} \mathfrak{g}_\alpha.$$

Now choose a closed Weyl chamber $\mathfrak{a}^+ \subset \mathfrak{a}$ and the corresponding simple system $\Delta \subset \Sigma$, then any subset $\Theta \subset \Delta$ defines a *standard parabolic subgroup* $P_\Theta = P_X$ where $X \in \mathfrak{a}^+$ is any element such that

$\{\alpha \in \Delta \,|\, \alpha(X) \neq 0\} = \Theta$. This parabolic subgroup does not depend on the choice of such an X, neither does the standard flag manifold $\mathcal{F}_\Theta = \mathcal{F}_X$.

4.1.2 Levi subgroups and transverse flag spaces

From an element $X \in \mathfrak{g}_{hyp}$ we can also define the subsets

$$\mathfrak{l}_X \overset{\text{def}}{=} \ker(\mathrm{ad}_X) \subset \mathfrak{g} \quad \text{and} \quad L_X \overset{\text{def}}{=} \mathrm{Stab}_G(X) = \{g \in G \,|\, \mathrm{Ad}(g)X = X\} < G.$$

Then $\mathfrak{l}_X$ is a Lie subalgebra of $\mathfrak{g}$, and L_X is a closed Lie subgroup of G with Lie algebra $\mathfrak{l}_X$.

Definition (Levi subgroups)

A *Levi subgroup* of G is a subgroup $L < G$ for which there is $X \in \mathfrak{g}_{hyp}$ with $L = L_X$.

While flag manifolds will play a central role in our arguments, the most important objects will be the open G-orbits formed by the transverse pairs in products of opposite flag manifolds. Such pairs are related to Levi subgroups by the relation

$$L_X = P_X \cap P_{-X}.$$

Definition (Transversality in flag manifolds)

Consider two flag manifolds $\mathcal{F}^+$ and $\mathcal{F}^-$. A pair $(x^+, x^-) \in \mathcal{F}^+ \times \mathcal{F}^-$ is *transverse* if there is $X \in \mathfrak{g}_{hyp}$ such that $\mathrm{Stab}_G(x^+) = P_X$ and $\mathrm{Stab}_G(x^-) = P_{-X}$.
The flag manifolds $\mathcal{F}^+, \mathcal{F}^-$ are *opposite* if $\mathcal{F}^{\pitchfork} \neq \emptyset$. In this case, we call $\mathcal{F}^{\pitchfork}$ a *transverse flag space.*

Notation

We write $x^+ \pitchfork x^-$ to signify that the pair $(x^+, x^-) \in \mathcal{F}^+ \times \mathcal{F}^-$ is transverse and denote the set of all transverse pairs in $\mathcal{F}^+ \times \mathcal{F}^-$ by $\mathcal{F}^+ \overset{\pitchfork}{\times} \mathcal{F}^-$ (or $\mathcal{F}^{\pitchfork}$ for short, when the factors $\mathcal{F}^+, \mathcal{F}^-$ are clear from the context).

Example (Running example)

Considering a real diagonal matrix $X = \mathrm{Diag}\left(x_1 \mathbf{1}_{d_1}, \ldots, x_r \mathbf{1}_{d_r}\right) \in \mathfrak{sl}(d, \mathbb{K})$ with $x_1 > \cdots > x_r$, the Levi subgroup L_X consists of block-wise diagonal matrices:

$$L_X = \left\{ \begin{pmatrix} * & & 0 \\ & \ddots & \\ 0 & & * \end{pmatrix} \right\}.$$

Just as we saw how to identify $\mathcal{F}_X$ with the flag manifold $\mathcal{F}_{\mathbf{d}}(\mathbb{K}^d)$, we can identify $\mathcal{F}_{-X}$ to $\mathcal{F}_{\iota(\mathbf{d})}(\mathbb{K}^d)$ where $\iota(\mathbf{d}) = (d_r, \ldots, d_1)$, and the transverse flag space

$$\mathcal{F}_{\mathbf{d}}(\mathbb{K}^d) \overset{\pitchfork}{\times} \mathcal{F}_{\iota(\mathbf{d})}(\mathbb{K}^d) = \left\{ \left(V_\bullet, W_\bullet\right) \middle| V_i \oplus W_{r-i} = \mathbb{K}^d \ \forall i \right\}$$

can be identified with the space of *gradings of type* $\mathbf{d}$

$$\mathcal{F}^{\pitchfork}_{\mathbf{d}}(\mathbb{K}^d) \overset{\text{def}}{=} \left\{ E_\bullet \in \prod_{i=1}^{r} \mathrm{Gr}_{d_i}(\mathbb{K}^d) \middle| E_1 \oplus \cdots \oplus E_r = \mathbb{K}^d \right\}$$

through the map

$$\begin{cases} \mathcal{F}_{\mathbf{d}}(\mathbb{K}^d) \overset{\pitchfork}{\times} \mathcal{F}_{\iota(\mathbf{d})}(\mathbb{K}^d) & \to & \mathcal{F}^{\pitchfork}_{\mathbf{d}}(\mathbb{K}^d) \\ \left(V_\bullet, W_\bullet\right) & \mapsto & (V_i \cap W_{r+1-i})_{1 \le i \le r} \end{cases}$$

whose inverse maps $E_\bullet \in \mathcal{F}^{\pitchfork}_{\mathbf{d}}(\mathbb{K}^d)$ to the pair $(V_\bullet, W_\bullet)$ where $V_i = E_1 \oplus \cdots \oplus E_i$ and $W_i = E_r \oplus \cdots \oplus E_{r+1-i}$.

4.1.3 Anosov representations

Fix a semi-simple Lie group G and a pair of opposite flag manifolds $\mathcal{F}^\pm$.

Definition (Transverse limit maps)

Let Γ be a hyperbolic group, and $\rho \in \mathrm{Hom}(\Gamma, G)$. A *pair of transverse limit maps* into $\mathcal{F}^\pm$ is a pair (ξ^+, ξ^-) of continuous Γ-equivariant maps $\xi^+ : \partial_\infty \Gamma \to \mathcal{F}^+$ and $\xi^- : \partial_\infty \Gamma \to \mathcal{F}^-$ such that $\xi^+(\eta_+) \pitchfork \xi^((\eta_-)$ for every $(\eta_+, \eta_-) \in \partial_\infty \Gamma^{(2)}$.

Definition (Anosov limit maps)

Let Γ be a Gromov hyperbolic group, $\varphi^t : \mathbf{X} \to \mathbf{X}$ a coarse Gromov flow, and $\rho \in \mathrm{Hom}(\Gamma, G)$. A pair of transverse limit maps (ξ^+, ξ^-) in $\mathcal{F}^\pm$ is called a pair of *Anosov limit maps* if it satisfies the following properties:

- The vertical flow $d^V_{\xi^+_+} \Phi^t_{\rho, \mathcal{F}^+} : \mathbf{X} \times_\rho T_{\xi^+_+} \mathcal{F}^+ \to \mathbf{X} \times_\rho T_{\xi^+_+} \mathcal{F}^+$ is dilating.
- The vertical flow $d^V_{\xi^-_-} \Phi^t_{\rho, \mathcal{F}^-} : \mathbf{X} \times_\rho T_{\xi^-_-} \mathcal{F}^- \to \mathbf{X} \times_\rho T_{\xi^-_-} \mathcal{F}^-$ is contracting.

Definition (Anosov representation)

Let Γ be a hyperbolic group. A homomorphism $\rho : \Gamma \to \mathrm{SL}(V)$ is called *projective Anosov* if it admits a pair of projective Anosov limit maps for some coarse Gromov flow.

Definition (Dynamics preserving limit maps)

Let Γ be a hyperbolic group, and $\rho \in \mathrm{Hom}(\Gamma, G)$.

- A pair of transverse projective limit maps (ξ^+, ξ^-) is called *dynamics preserving* if for any infinite order element $\gamma \in \Gamma$, $\rho(\gamma)$ is proximal, $\xi^+(\gamma^+) = \mathbf{f}^+(\rho(\gamma))$ and $\xi^-(\gamma^-) = \mathbf{f}^-(\rho(\gamma))$.

- A pair of transverse projective limit maps (ξ^+, ξ^-) is called *uniformly dynamics preserving* if for any unbounded sequence $\gamma_k \in \Gamma$ with boundary limit points $\gamma_+ = \lim_{k\to+\infty} \gamma_k \in \partial_\infty\Gamma$ and $\gamma_- = \lim_{k\to+\infty} \gamma_k^{-1} \in \partial_\infty\Gamma$, the actions on $\mathcal{F}^+$ and $\mathcal{F}^-$ obey the following dynamics:

$$(1)\ \ \forall \mathbf{x}^+ \in \mathcal{F}^+ \qquad \mathbf{x}^+ \pitchfork \xi^-(\gamma_-) \implies \lim_{k\to+\infty} \rho(\gamma_k)\cdot \mathbf{x}^+ = \xi^+(\gamma_+),$$
$$(2)\ \ \forall \mathbf{x}^- \in \mathcal{F}^- \qquad \xi^+(\gamma_-) \pitchfork \mathbf{x}^- \implies \lim_{k\to+\infty} \rho(\gamma_k)\cdot \mathbf{x}^- = \xi^-(\gamma_+),$$

  and the convergences are locally uniform.

Theorem 4.1

A representation $\rho \in \mathrm{Hom}(\Gamma, G)$ is $\mathcal{F}^\pm$-Anosov if and only if it possesses a pair of uniformly dynamics preserving transverse limit maps into $\mathcal{F}^\pm$.

4.2 Constructing locally homogeneous axiom A flows

4.2.1 Flow spaces over transverse flag spaces

Following the ideas developed in the introduction, we wish to construct a flow $\varphi^t : \mathbb{L} \to \mathbb{L}$ with orbit space $\mathcal{F}^\pitchfork$. We also require the flow φ^t to have trivial dynamics, so that the dynamics of the quotient flow by a discrete subgroup only reflect the properties of this subgroup. By trivial dynamics, we usually mean that the resulting action of $\mathbb{R}$ on $\mathbb{L}$ is free and proper, i.e. that it defines an $\mathbb{R}$-principal bundle (that we also call an *affine line bundle*) $\mathbb{L} \to \mathcal{F}^\pitchfork$. We also require these affine line bundles to be G-*homogeneous*, i.e. $\mathbb{L}$ is a homogeneous G-space, the flow $\varphi^t_{\mathbb{L}}$ commutes with the action of G and the projection $\mathbb{L} \to \mathcal{F}^\pitchfork$ is G-equivariant.

Fixing a base point in $\mathcal{F}^\pitchfork$ to identify $\mathcal{F}^\pitchfork \simeq G/L$ where $L < G$ is a Levi subgroup, there is a bijective correspondence between G-homogeneous affine line bundles over G/L and non zero additive characters $\mathbf{b} \in \mathrm{Hom}(L, \mathbb{R})$. Starting with $\mathbf{b} \in \mathrm{Hom}(L, \mathbb{R}) \setminus \{0\}$, we simply define

$$\mathbb{L}_{\mathbf{b}} \overset{\text{def}}{=} G/\ker \mathbf{b},$$

where the $\mathbb{R}$-action is defined by

$$\varphi^t_{\mathbf{b}}\big(g \ker \mathbf{b}\big) \overset{\text{def}}{=} g e^{tX} \ker \mathbf{b}$$

for any $X \in \mathfrak{l}$ with $\mathrm{d}\mathbf{b}|_{1_G} X = 1$.

Reciprocally, if $\varphi^t : \mathbb{L} \to \mathbb{L}$ is a G-homogeneous affine line bundle over $\mathcal{F}^{\pitchfork}$, consider a base point $\mathbf{x}$ in the fibre over G/L, and for $g \in \mathrm{L}$ let $\mathbf{b}(g) \in \mathbb{R}$ be defined by

$$\mathbf{x} \cdot g = \varphi^{\mathbf{b}(g)}\mathbf{x}.$$

Example (Running example)

Considering a real diagonal matrix $X = \mathrm{Diag}\left(x_1 \mathbf{1}_{d_1}, \ldots, x_r \mathbf{1}_{d_r}\right) \in \mathfrak{sl}(d, \mathbb{K})$ with $x_1 > \cdots > x_r$, additive characters $\beta \in \mathrm{Hom}(\mathrm{L}_X, \mathbb{R})$ are in one-to-one correspondence with $\mathbb{R}^{r-1}$, mapping $\beta = (\beta_1 \ \ldots \ \beta_{r-1})$ to the character

$$\left\{ \begin{array}{ccc} \mathrm{L}_X & \to & \mathbb{R} \\ \begin{pmatrix} g_1 & & 0 \\ & \ddots & \\ 0 & & g_r \end{pmatrix} & \mapsto & \sum_{i=1}^{r-1} \beta_i \mathrm{Log}|\det g_i| \end{array} \right.$$

In a general semi-simple Lie group G, there is a natural identification $\mathrm{Hom}(\mathrm{L}_\Theta, \mathbb{R}) \simeq \mathfrak{a}_\Theta^*$ where $\mathfrak{a}_\Theta = \{X \in \mathfrak{a} \,|\, \forall \alpha \in \Delta \setminus \Theta \ \alpha(X) = 0\}$.

4.2.2 Benoist's limit cone

The last tool we will need is Benoist's limit cone for a representation $\rho \in \mathrm{Hom}(\Gamma, G)$. We first need to consider the *Jordan projection* $\lambda : G \to \mathfrak{a}^+$, defined through the Jordan decomposition $g = g_e g_h g_u$ and $\lambda(g) \in \mathfrak{a}^+$ is the unique element conjugate to X_h where $g_h = e^{X_h}$. For $G = \mathrm{SL}_d(\mathbb{R})$, we can set

$$\mathfrak{a}^+ = \left\{ (a_1, \ldots, a_d) \in \mathbb{R}^d \,\middle|\, a_1 + \cdots + a_d = 0 \ \& \ a_1 \geq \cdots \geq a_d \right\},$$

so that $\lambda(g) = (\lambda_1(g), \ldots, \lambda_d(g))$.

Definition 4.2 (Benoist's limit cone)

The limit cone $\mathcal{L}_\rho$ is the smallest closed cone in $\mathfrak{a}^+$ containing all elements $\lambda(\rho(\gamma))$ for $\gamma \in \Gamma$.

Back to the general case, consider a restricted root space decomposition and a subset $\Theta \subset \Delta$ of simple restricted roots, we can embed $\mathrm{Hom}(\mathrm{L}_\Theta, \mathbb{R}) \simeq \mathfrak{a}_\Theta^* \hookrightarrow \mathfrak{a}^*$ by considering the decomposition $\mathfrak{a} = \mathfrak{a}_\Theta \oplus \mathfrak{a}_{\Delta \setminus \Theta}$. So for $\mathbf{b} \in \mathrm{Hom}(\mathrm{L}_\Theta, \mathbb{R})$ and $g \in G$, it makes sense to talk about $\mathbf{b}(\lambda(g)) \in \mathbb{R}$.

Theorem 4.3 ([DMS25b])

Let $\rho \in \mathrm{Hom}(\Gamma, G)$ be Anosov with respect to the flag manifolds $\mathcal{F}_\Theta, \mathcal{F}_{\iota\Theta}$, and let $\mathbf{b} \in \mathrm{Hom}(\mathrm{L}_\Theta, \mathbb{R})$. If $\mathbf{b} > 0$ on $\mathcal{L}_r ho \setminus \{0\}$, then there is an open subset $\widehat{\mathbf{M}}_{\rho,\mathbf{b}} \subset \mathbb{L}_\mathbf{b}$ invariant under $\rho(\Gamma)$ and $\varphi_\mathbf{b}^t$, on which Γ acts properly discontinuously and such that the quotient flow

$$\varphi_{\rho,\mathbf{b}}^t : \mathbf{M}_{\rho,\mathbf{b}} \to \mathbf{M}_{\rho,\mathbf{b}} = \Gamma \backslash \widehat{\mathbf{M}}_{\rho,\mathbf{b}}$$

is an Axiom A flow.

References

[BCLS15] M. Bridgeman, R. Canary, F. Labourie, and A. Sambarino, *The pressure metric for Anosov representations*, Geom. Funct. Anal. **25** (2015), no. 4, 1089–1179.

[BPS19] J. Bochi, R. Potrie, and A. Sambarino, *Anosov representations and dominated splittings*, J. Eur. Math. Soc. (JEMS) **21** (2019), no. 11, 3343–3414.

[CK02] M. Coornaert and G. Knieper, *Growth of conjugacy classes in Gromov hyperbolic groups.*, Geom. Funct. Anal. **12** (2002), no. 3, 464–478 (English).

[CS24] M. Chow and P. Sarkar, *Exponential mixing and essential spectral gaps for Anosov subgroups*, 2024, arXiv:2408.11274.

[DG16] S. Dyatlov and C. Guillarmou, *Pollicott–Ruelle resonances for open systems*, Ann. Henri Poincaré **17** (2016), no. 11, 3089–3146.

[DG18] ______, *Afterword: Dynamical zeta functions for Axiom A flows*, Bull. Amer. Math. Soc. (2018), no. 55, 337–342.

[DMS25a] Benjamin Delarue, Daniel Monclair, and Andrew Sanders, *Locally homogeneous axiom a flows. I: projective Anosov subgroups and exponential mixing*, Geom. Funct. Anal. **35** (2025), no. 3, 673–735 (English).

[DMS25b] ______, *Locally homogeneous Axiom A flows II: geometric structures for Anosov subgroups*, Preprint, arXiv:2502.20195 [math.GT] (2025), 2025.

[DP98] D. Dolgopyat and M. Pollicott, *Addendum to: "Periodic orbits and dynamical spectra" [Ergodic Theory Dynam. Systems* **18** *(1998), no. 2, 255–292] by V. Baladi*, Ergodic Theory Dynam. Systems **18** (1998), no. 2, 293–301.

[Dya18] S. Dyatlov, *Notes on hyperbolic dynamics*, 2018, arXiv:1805.11660.

[Ebe96] P. B. Eberlein, *Geometry of nonpositively curved manifolds*, Chicago, IL: The University of Chicago Press, 1996.

[Fri95] D. Fried, *Meromorphic zeta functions for analytic flows*, Comm. Math. Phys. **174** (1995), no. 1, 161–190.

[GGKW17] F. Guéritaud, O. Guichard, F. Kassel, and A. Wienhard, *Anosov representations and proper actions*, Geom. Topol. **21** (2017), no. 1, 485–584.

[Gro87] M. Gromov, *Hyperbolic groups*, Essays in Group Theory (S. M. Gersten, ed.), Springer New York, New York, NY, 1987, pp. 75–263.

[GW12] O. Guichard and A. Wienhard, *Anosov representations: domains of discontinuity and applications.*, Invent. Math. **190** (2012), no. 2, 357–438.

[KH95] A. Katok and B. Hasselblatt, *Introduction to the modern theory of dynamical systems*, Encycl. Math. Appl., vol. 54, Cambridge: Cambridge Univ. Press, 1995.

[KLP17] M. Kapovich, B. Leeb, and J. Porti, *Anosov subgroups: dynamical and geometric characterizations*, Eur. J. Math. **3** (2017), no. 4, 808–898.

[Kna02] A. W. Knapp, *Lie groups beyond an introduction*, Progress in Mathematics, Birkhäuser Boston, 2002.

[KP22] F. Kassel and R. Potrie, *Eigenvalue gaps for hyperbolic groups and semigroups*, J. Mod. Dyn. **18** (2022), 161–208.

[Lab06] F. Labourie, *Anosov flows, surface groups and curves in projective space.*, Invent. Math. **165** (2006), no. 1, 51–114.

[PP90] William Parry and Mark Pollicott, *Zeta functions and the periodic orbit structure of hyperbolic dynamics*, Astérisque, vol. 187-188, Paris: Société Mathématique de France, 1990 (English).

[PS98] M. Pollicott and R. Sharp, *Exponential error terms for growth functions on negatively curved surfaces*, Amer. J. Math. **120** (1998), no. 5, 1019–1042.

[Rug92] Hans H. Rugh, *The correlation spectrum for hyperbolic analytic maps*, Nonlinearity **5** (1992), no. 6, 1237–1263 (English).

[Rug96] H. H. Rugh, *Generalized Fredholm determinants and Selberg zeta functions for Axiom A dynamical systems*, Ergodic Theory Dynam. Systems **16** (1996), no. 4, 805–819.

[Sam14] A. Sambarino, *Quantitative properties of convex representations*, Comment. Math. Helv. **89** (2014), no. 2, 443–488.

[Sam24] ______, *A report on an ergodic dichotomy*, Ergodic Theory Dynam. Systems **44** (2024), no. 1, 236–289.

[Sma67] S. Smale, *Differentiable dynamical systems. With an appendix to the first part of the paper: "Anosov diffeomorphisms" by John Mather*, Bull. Am. Math. Soc. **73** (1967), 747–817.

[Sto11] L. Stoyanov, *Spectra of Ruelle transfer operators for Axiom A flows*, Nonlinearity **24** (2011), no. 4, 1089–1120.

[Sto13] ______, *Pinching conditions, linearization and regularity of axiom A flows*, Discrete Contin. Dyn. Syst. **33** (2013), no. 2, 391–412.

[Vog97] D. A. Vogan, *Chapter 5 - the orbit method and unitary representations for reductive lie groups*, Algebraic and Analytic Methods in Representation Theory (B. Ørsted and H. Schlichtkrull, eds.), Perspectives in Mathematics, Academic Press, San Diego, 1997, pp. 243–339.

[War72] G. Warner, *Harmonic analysis on semi-simple Lie groups*, vol. I, Springer–Verlag, Berlin, Heidelberg, New York, 1972.