Geodesic flows in anti-de Sitter geometry

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Anti-de Sitter geometry is the study of Lorentzian manifolds of constant negative curvature. A Lorentzian metric on a manifold is a field of quadratic forms of signature (-,+,...,+), a contrast with Riemannian metrics that are positive definite. Note that while the isotropy group SO(n) of Riemannian geometry is compact, the group SO(1,n) of Lorentzian geometry is not. This non compactness often leads to interesting dynamics of groups acting on Lorentzian manifolds. The aim of this course is to describe both similarities and differences between anti-de Sitter and hyperbolic manifolds, with an emphasis on the dynamical systems that arise from their studies: isometry groups and geodesic flows. Here is the rough outline of the three lectures:

Part 1. Basics of anti-de Sitter geometry

Part 2. GHMC AdS manifolds

Part 3. Dynamics of the spacelike geodesic flow

Part 1. Basics of anti-de Sitter geometry

1.1 The anti-de Sitter space

Definition (Anti-de Sitter space)

The anti-de Sitter space is the quadric

$$\mathrm{AdS}^{d+1} = \left\{ x \in \mathbb{R}^{d+2} \,\middle|\, x_1^2 + \cdots + x_d^2 - x_{d+1}^2 - x_{d+2}^2 = -1 \right\}.$$

We can start by drawing a picture when $d = 0$: AdS^1 is the unit circle in $\mathbb{R}^2$. We then move to $d = 1$, and find that AdS^2 is a one-sheeted hyperboloid, obtained by rotating a hyperbola along one its symmetry axis that separates the two branches. This picture can be used in higher dimensions: AdS^{d+1} is obtained by rotating the hyperbolic space $\mathbb{H}^d$ around some axis, and is therefore diffeomorphic to $\mathbb{R}^d \times \mathbb{S}^1$.

More precisely, consider the hyperboloid model

$$\mathbb{H}^d = \left\{ x \in \mathbb{R}^{d+1} \,\middle|\, x_1^2 + \cdots + x_d^2 - x_{d+1}^2 = -1 \ \&\ x_{d+1} > 0 \right\}$$

and embed it into AdS^{d+1} through

$$\iota : \left\{ \begin{array}{ccc} \mathbb{H}^d & \to & \mathrm{AdS}^{d+1} \\ x & \mapsto & (x,0) \end{array} \right. .$$

We can rotate $\iota(\mathbb{H}^d)$ in the plane spanned by the last two coordinates to obtain a diffeomorphism

$$\Psi : \left\{ \begin{array}{ccc} \mathbb{H}^d \times \mathbb{S}^1 & \to & \mathrm{AdS}^{d+1} \\ (x,\theta) & \mapsto & (x_1, \dots, x_d, x_{d+1}\cos\theta, x_{d+1}\sin\theta) \end{array} \right.$$

One can find the inverse of Ψ by using polar coordinates for the last two coordinates in AdS^{d+1}.

In particular for $d = 2$, we see that AdS^3 has the topology of a solid torus.

Quadratic vocabulary

Definition (Quadratic spaces and signatures)

A *quadratic space* is a pair $(\mathrm{V}, \mathbf{b})$ where V is a finite dimensional real vector space and $\mathbf{b} : \mathrm{V} \times \mathrm{V} \to \mathbb{R}$ is a symmetric bilinear form.
The *signature* of $(\mathrm{V}, \mathbf{b})$ is the pair of integers (p, q) where:

$$p = \max\{\dim \mathrm{W} \,|\, \mathrm{W} \subset \mathrm{V}, \mathbf{b}|_{\mathrm{W}\times\mathrm{W}} \text{ is positive definite }\},$$
$$q = \max\{\dim \mathrm{W} \,|\, \mathrm{W} \subset \mathrm{V}, \mathbf{b}|_{\mathrm{W}\times\mathrm{W}} \text{ is negative definite }\}.$$

Given a vector subspace $\mathrm{W} \subset \mathrm{V}$, the *signature* of W is defined as the signature of $(\mathrm{W}, \mathbf{b}|_{\mathrm{W}\times\mathrm{W}})$.

Notation (Matrix of a quadratic space)

Given a quadratic space $(\mathrm{V}, \mathbf{b})$ and a vector basis $\mathbf{e} = (e_1, \dots, e_n)$ of V we write

$$[\mathbf{b}]_{\mathbf{e}} = \Big(\mathbf{b}(e_i, e_j)\Big)_{1 \le i,j \le n}$$

for the matrix of $\mathbf{b}$ in the basis $\mathbf{e}$.

Theorem (Sylvester's inertial law)

Let $(\mathrm{V}, \mathbf{b})$ be a quadratic space, and denote by (p, q) its signature. There exists a basis $\mathbf{e} = (e_1, \dots, e_n)$ of V for which the matrix $[\mathbf{b}]_{\mathbf{e}}$ is the block-diagonal matrix

$$[\mathbf{b}]_{\mathbf{e}} = \begin{pmatrix} \mathbf{1}_p & & \\ & -\mathbf{1}_q & \\ & & 0 \end{pmatrix}.$$

For a quadratic space $(\mathrm{V}, \mathbf{b})$ and a vector $x \in \mathrm{V}$, we will denote by $x^{\perp_{\mathbf{b}}}$ the orthogonal with respect to $\mathbf{b}$, i.e.

$$x^{\perp_{\mathbf{b}}} = \{y \in \mathrm{V} \,|\, \mathbf{b}(x, y) = 0\},$$

and also use the notation $\mathrm{W}^{\perp_{\mathbf{b}}} = \bigcap_{x\in\mathrm{W}} x^{\perp_{\mathbf{b}}}$ for a subset $\mathrm{W} \subset \mathrm{V}$.

Definition (Non-degeneracy)

A quadratic space $(\mathrm{V}, \mathbf{b})$ is called *non-degenerate* if $\mathrm{V}^{\perp_{\mathbf{b}}} = \{0\}$.

This is equivalent to $p + q = \dim \mathrm{V}$ where (p, q) is the signature of $(\mathrm{V}, \mathbf{b})$.

Proposition

Let $(\mathrm{V}, \mathbf{b})$ be a non-degenerate quadratic space. Then for any vector subspace $\mathrm{W} \subset \mathrm{V}$, we have $\dim \mathrm{W} + \dim \mathrm{W}^{\perp_{\mathbf{b}}} = \dim \mathrm{V}$. Furthermore, $\mathrm{V} = \mathrm{W} \oplus \mathrm{W}^{\perp_{\mathbf{b}}}$ if and only if W is non-degenerate. In this case, if $(\mathrm{V}, \mathbf{b})$ has signature (p, q) and W has signature (p', q'), then $\mathrm{W}^{\perp_{\mathbf{b}}}$ has signature $(p - p', q - q')$.

1.1.1 The anti-de Sitter metric

Fix a quadratic space $(\mathrm{V}, \mathbf{b})$ of dimension $d+2$ and signature $(d, 2)$. The standard choice is of course $\mathrm{V} = \mathbb{R}^{d+2}$ and

$$\mathbf{b}(x, y) = \sum_{k=1}^{d} x_k y_k - x_{d+1} y_{d+1} - x_{d+2} y_{d+2}.$$

Definition (Anti-de Sitter space)

The anti-de Sitter space is the quadric

$$\mathrm{AdS}^{d+1} = \{x \in \mathrm{V} \,|\, \mathbf{b}(x, x) = -1\}.$$

Fact

For $x \in \mathrm{AdS}^{d+1}$, the tangent space $\mathrm{T}_x \mathrm{AdS}^{d+1} = x^{\perp_{\mathbf{b}}}$ is the orthogonal subspace with respect to the bilinear form $\mathbf{b}$. The restriction of $\mathbf{b}$ to $x^{\perp_{\mathbf{b}}}$ has signature $(d, 1)$.

In general, a smooth field of symmetric bilinear forms of signature $(d, 1)$ on a $d+1$-dimensional manifold is called a *Lorentzian metric*.

Definition (Types of tangent vectors)

A tangent vector $v \in \mathrm{T}_x \mathrm{AdS}^{d+1}$ is called

- Spacelike if $\mathbf{b}(v, v) > 0$.
- Timelike if $\mathbf{b}(v, v) < 0$.
- Lightlike if $\mathbf{b}(v, v) = 0$.

A submanifold $\mathrm{M} \subset \mathrm{AdS}^{d+1}$ is called spacelike (resp. timelike) if all its non zero tangent vectors are spacelike (resp. timelike).

The diffeomorphism $\Psi : \mathbb{H}^d \times \mathbb{S}^1 \to \mathrm{AdS}^{d+1}$ is rather useful when it comes to understanding types of tangent vectors. Consider $x \in \mathrm{AdS}^{d+1}$ and a tangent vector $v \in \mathrm{T}_x \mathrm{AdS}^{d+1}$. Write $x = \Psi(y, \theta)$ for $(y, \theta) \in \mathbb{H}^d \times \mathbb{S}^1$ and $v = \mathrm{d}\Psi|_{(y,\theta)} = (w, t)$. After a calculation, one finds

$$\mathbf{b}(v, v) = g_{\mathbb{H}^d}(w, w) - y_{d+1}^2 t^2.$$

In particular, the hypersurfaces $\mathbb{H}^d \times \{*\}$ are spacelike, while the curves $\{*\} \times \mathbb{S}^1$ are timelike.

Definition (Future and past oriented causal vectors)

A tangent vector $v \in \mathrm{T}_x\mathrm{AdS}^{d+1}$ is called *causal* if $v \neq 0$ and $\mathbf{b}(v,v) \leq 0$. A causal vector is called *future directed* (resp. *past directed*) if $\mathbf{b}(v, \mathbf{T}(x)) < 0$ (resp > 0) where $\mathbf{T} : \mathrm{AdS}^{d+1} \to \mathrm{TAdS}^{d+1}$ is the vector field defined by

$$\mathbf{T}(x) = (0, \ldots, 0, -x_{d+2}, x_{d+1}).$$

A future directed vector is a causal vector that belongs to the same half of the cone $\mathbf{b} \leq 0$ as the timelike vector field $\mathbf{T}$. Note that $\Psi^*\mathbf{T}(x,\theta) = (0,1)$ on $\mathbb{H}^d \times \mathbb{S}^1$, so a timelike vector $v = \mathrm{d}\Psi|_{(y,\theta)} = (w,t)$, where $x = \Psi(y,\theta)$, is future (resp. past) directed if and only if $t > 0$ (resp. $t < 0$).

1.1.2 Isometries of AdS^{d+1}

The identity component $\mathrm{SO}_\circ(d,2) = \mathrm{SO}_\circ(\mathbf{b})$ of the group of linear transformations that preserve the bilinear form $\mathbf{b}$ acts by orientation preserving diffeomorphisms on AdS^{d+1}.

This action is isometric for the Lorentzian metric, in particuler its differential preserves the types (spacelike, timelike, lightlike) of tangent vectors. It also preserves future directed timelike vectors (this is why we restrict to the identity component).

The action of $\mathrm{SO}_\circ(d,2)$ on AdS^{d+1} is transitive, and the stabiliser of the point $x_0 = (0, \ldots, 0, 1)$ is isomorphic to $\mathrm{SO}_\circ(d,1)$ through the embedding

$$\begin{cases} \mathrm{SO}_\circ(d,1) & \to & \mathrm{SO}_\circ(d,2) \\ A & \mapsto & \begin{bmatrix} A & \\ & 1 \end{bmatrix} \end{cases}$$

so AdS^{d+1} can be seen as the homogeneous space $\mathrm{SO}_\circ(d,2)/\mathrm{SO}_\circ(d,1)$.

1.1.3 Geodesics

We will not discuss many differential geometric aspects of AdS^{d+1}, but focus on one of them: geodesics. Just as for the Euclidean sphere or the real hyperbolic space, there is no need for a general theory to define anti-de Sitter geodesics. Just as in the Riemannian world, a geodesic is prescribed by initial point and velocity.

Definition (Geodesics)

Let $x \in \mathrm{AdS}^{d+1}$ and $v \in \mathrm{T}_x\mathrm{AdS}^{d+1}$. The geodesic $c_{x,v} : \mathbb{R} \to \mathrm{AdS}^{d+1}$ is the unique curve satisfying the following properties:

1. The map $t \mapsto \mathbf{b}(\dot{c}_{x,v}(t), \dot{c}_{x,v}(t))$ is constant.
2. For all $t \in \mathbb{R}$, $c_{x,v}(t) \in \mathrm{Span}(x,v)$.

Proposition

Let $x \in \mathrm{AdS}^{d+1}$ and $v \in \mathrm{T}_x\mathrm{AdS}^{d+1}$. The geodesic $c_{x,v} : \mathbb{R} \to \mathrm{AdS}^{d+1}$ has the following parametrization:

- If $\mathbf{b}(v,v) = 1$, then $c_{x,v}(t) = \cosh tx + \sinh tv$.
- If $\mathbf{b}(v,v) = \lambda^2$ for some $\lambda > 0$, then $c_{x,v}(t) = c_{x,v/\lambda}(\lambda t)$.
- If $\mathbf{b}(v,v) = -1$, then $c_{x,v}(t) = \cos tx + \sin tv$.
- If $\mathbf{b}(v,v) = -\lambda^2$ for some $\lambda > 0$, then $c_{x,v}(t) = c_{x,v/\lambda}(\lambda t)$.
- If $\mathbf{b}(v,v) = 0$, then $c_{x,v}(t) = x + tv$.

There is a slight subtlety when looking at geodesics between two given points $x, y \in \mathrm{AdS}^{d+1}$. There are four possibilities:

- If $\mathbf{b}(x,y) < -1$, then x and y are joined by a spacelike geodesic.
- If $\mathbf{b}(x,y) = -1$, then x and y are joined by a lightlike geodesic.
- If $\mathbf{b}(x,y) \in (-1,1]$, then x are y are joint by a timelike geodesic.
- If $\mathbf{b}(x,y) > 1$, then x and y are not joined by a geodesic.

Note a subtelty in the third case: if $x, y \in \mathrm{AdS}^{d+1}$ and $\mathbf{b}(x,y) = -1$, then $y = -x$ and all timelike geodesics passing through x also pass through $-x$. In all other cases, if x and y can be joined by a geodesic line, then it is unique, and will be denoted by (xy).

1.2 Anti-de Sitter manifolds

Anti-de Sitter manifolds are usually defined as Lorentzian manifolds with constant negative sectional curvature. It will be more practical for our purpose to consider them as (G,X)-structures in the sense of Ehresmann-Thurston.

Definition (Anti-de Sitter manifold)

An *anti-de Sitter atlas* on a manifold is an atlas with values in AdS^{d+1} and transitions that are restrictions of elements of $\mathrm{SO}_\circ(d,2)$.
An *anti-de Sitter manifold* is a manifold equipped with a maximal anti-de Sitter atlas.

Remark

If one were to define a hyperbolic manifold in this fashion, it would be natural to insert completeness in the definition. This way, a hyperbolic manifold is isometric to a quotient $\Gamma\backslash\mathbb{H}^d$, where $\Gamma < \mathrm{SO}_\circ(d,1)$ is a discrete subgroup (and torsion free for the quotient to be a manifold). We will take this as a definition for hyperbolic manifolds.
We will however make no such assumption for anti-de Sitter manifolds: the interesting examples are incomplete. This should be seen as a feature of this theory rather than a bug: some of the most important results in Lorentzian geometry, known as singularity theorems, imply that "physically relevant" Lorentzian manifolds are incomplete. For instance this incompleteness is at the heart of the definition of a black hole.

Proposition (Developing map and holonomy representation)

Let (M,g) be an anti-de Sitter manifold. There exists a map $\mathrm{dev}:\widetilde{M}\to \mathrm{AdS}^{d+1}$ and a representation $\rho:\pi_1 M\to \mathrm{SO}_\circ(d,2)$ such that:

1. The map $\mathrm{dev}:\widetilde{M}\to \mathrm{AdS}^{d+1}$ is ρ-equivariant and a local diffeomorphism.
2. In anti-de Sitter charts, dev reads as an isometry.

They are well defined up to the action of $\mathrm{SO}_\circ(d,2)$ given by $g\cdot(\mathrm{dev},\rho)=(g\circ\mathrm{dev}, g\rho g^{-1})$.
Furthermore, any (equivalence class of a) pair (f,φ) where $\varphi:\pi_1 M\to \mathrm{SO}_\circ(d,2)$ is a representation and $f:\widetilde{M}\to\mathrm{AdS}^{d+1}$ is a φ-equivariant local diffeomorphism determines a unique anti-de Sitter structure on M.

Remark

The space AdS^{d+1} is not simply connected. Using the theory of (G,X)-structures, we can also see an anti-de Sitter structure on a manifold M as a pair $(\widetilde{\mathrm{dev}},\widetilde{\rho})$ where $\widetilde{\mathrm{dev}}:\widetilde{M}\to\widetilde{\mathrm{AdS}}^{d+1}$ and $\widetilde{\rho}:\pi_1 M\to\widetilde{\mathrm{SO}}_\circ(d,2)$ satisfy the same conditions as above. Here $\widetilde{\mathrm{AdS}}^{d+1}$ is the universal cover of AdS^{d+1}, diffeomorphic to $\mathbb{R}^{d+1}$, and $\widetilde{\mathrm{SO}}_\circ(d,2)$ is the group of lifts of elements of $\mathrm{SO}_\circ(d,2)$.[a]

[a] This is not quite the universal cover of $\mathrm{SO}(d,2)$, but close enough...

Note that if M is an anti-de Sitter manifold, it inherits a Lorentzian metric g, locally isometric to AdS^{d+1} (by definition). We will therefor use the notation (M,g) for an anti-de Sitter manifold.

Given an anti-de Sitter manifold (M,g), we can still define types of tangent vectors, future-directed and past-directed causal vectors, and geodesics, as all of these notions are invariant under $\mathrm{SO}_\circ(d,2)$. Note however that geodesics need not be complete.

1.2.1 A first family of examples

Of course the first example of an anti-de Sitter manifold is AdS^{d+1} itself. Other straightforward examples are open subsets of AdS^{d+1}, and its universal cover $\widetilde{\mathrm{AdS}}^{d+1}$ (recall that AdS^{d+1} is diffeomorphic to $\mathbb{R}^d\times\mathbb{S}^1$, so it is not simply connected).

We will now show that if Σ is a hyperbolic manifold, then there is an anti-de Sitter manifold diffeomorphic to $\Sigma\times\mathbb{R}$. Since we already have an explicit diffeomorphism $\Psi:\mathbb{H}^d\times\mathbb{S}^1\to\mathrm{AdS}^{d+1}$, it is tempting to use it for this purpose. There are however some shortcomings that prevent us from doing so.

One strategy would be to copy the formula for the Lorentian metric $\Psi^* g_{\mathrm{AdS}}$. This can be computed:

$$\Psi^* g_{\mathrm{AdS}} = g_{\mathbb{H}^d} - y_{d+1}^2 g_{\mathbb{S}^1} \,.$$

The coordinate function y_{d+1} is well defined on $\mathbb{H}^d$, but not on any hyperbolic manifold, so we cannot use this formula. The next naive guess consists in taking the point of view of discrete groups: write $\Sigma = \Gamma \backslash \mathbb{H}^d$ where $\Gamma < \mathrm{SO}_\circ(d,1)$ is a discrete subgroup. It is tempting to consider $\rho_\circ(\Gamma)\backslash \mathrm{AdS}^{d+1}$, where $\rho_\circ$ is the embedding

$$\rho_\circ : \begin{cases} \mathrm{SO}_\circ(d,1) & \to & \mathrm{SO}_\circ(d,2) \\ A & \mapsto & \begin{bmatrix} A & \\ & 1 \end{bmatrix} \end{cases}$$

The problem is that the quotient space is not a manifold: the action $\rho_\circ(\Gamma) \curvearrowright \mathrm{AdS}^{d+1}$ is not properly discontinuous. For instance, it has a global fixed point $(0,\dots,0,1)$. Instead, we will look for a *discontinuity domain*, i.e. an open subset $\mathcal{O}_\Gamma \subset \mathrm{AdS}^{d+1}$ invariant under $\rho_\circ(\Gamma)$, so that $\rho_\circ(\Gamma)\backslash \mathcal{O}_\Gamma$ is the manifold that we are looking for (meaning that the action of Γ on $\mathcal{O}_\Gamma$ should be properly discontinuous).

Start by considering the embedding $\iota : \mathbb{H}^d \to \mathrm{AdS}^{d+1}$ used previously, and note that it is $\rho_\circ$-equivariant. A natural way to extend this action is to try to foliate the space AdS^{d+1} with equidistant hypersurfaces to $\mathbb{H}^d$. Since there is no notion of distance in Lorentzian geometry, let us use geodesics instead. For $x \in \mathbb{H}^d$, let $\mathrm{N}(x) \in \mathrm{T}_{\iota(x)}\mathrm{AdS}^{d+1}$ be the unique future directed timelike vector that is orthogonal to $\iota(\mathbb{H}^d)$ and normalized by $\mathbf{b}(\mathrm{N}(x),\mathrm{N}(x)) = -1$. It has the simplest expression:

$$\mathrm{N}(x) = (0,\dots,0,1).$$

For $\theta \in \mathbb{S}^1$, write $\mathrm{F}(x,\theta) = c_{x,\mathrm{N}(x)}(\theta)$, that is:

$$\mathrm{F}(x,\theta) = (x_1 \cos\theta, \dots, x_{d+1}\cos\theta, \sin\theta).$$

The map $\mathrm{F} : \mathbb{H}^d \times \mathbb{S}^1 \to \mathrm{AdS}^{d+1}$ defines a diffeomorphism between $\mathbb{H}^d \times \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ and the open subset $\mathcal{O} \subset \mathrm{AdS}^{d+1}$ defined as

$$\mathcal{O} = \left\{ (x_1,\dots,x_{d+2}) \in \mathrm{AdS}^{d+1} \,\middle|\, x_{d+2} \in (-1,1) \right\}.$$

Now, note that $\mathrm{F} : \mathbb{H}^d \times \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \to \mathcal{O}$ is $\mathrm{SO}_\circ(d,1)$-equivariant, where $\mathrm{SO}_\circ(d,1)$ acts on $\mathbb{H}^d \times \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ by its isometric action on $\mathbb{H}^d$ and trivially on the second factor, and through $\rho_\circ$ on $\mathcal{O} \subset \mathrm{AdS}^{d+1}$.

Now the action of any discrete subgroup $\Gamma < \mathrm{SO}_\circ(d,1)$ on $\mathbb{H}^d \times \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ is properly discontinuous, with quotient $\left(\Gamma\backslash\mathbb{H}^d\right) \times \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. It follows that $\rho_\circ(\Gamma)\backslash\mathcal{O}$ is an anti-de Sitter manifold, diffeomorphic to $\Gamma\backslash\mathbb{H}^d \times \mathbb{R}$ as promised.

Remark

Now that we have constructed these anti-de Sitter manifolds, it is easy to compute the Lorentzian metric on $\left(\Gamma\backslash\mathbb{H}^d\right) \times \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, and find the formula $\cos^2\theta g_{\mathrm{hyp}} - d\theta^2$. We could have started with this formula and checked that it has constant sectional curvature equal to -1, but there is no fun in that...

Part 2. GHMC AdS manifolds

2.1 Global hyperbolicity and uniformisation

2.1.1 Cauchy hypersurfaces

Definition (Cauchy hypersurface)

A *Cauchy hypersurface* in a Lorentzian manifold (M, g) is a topological hypersurface $\Sigma \subset \mathrm{M}$ such that every inextensible causal curve $c : \mathbb{R} \to \mathrm{M}$ intersects Σ exactly once.

Definition (GH, GHC)

A Lorentzian manifold (M, g) is called:

- *Globally hyperbolic* (GH) if it possesses a Cauchy hypersurface.
- *Globally hyperbolic Cauchy-compact* (GHC) or *spatially compact* if it possesses a compact Cauchy hypersurface.

It is not too complicated to prove that:

- If (M, g) is globally hyperbolic, then all Cauchy hypersurfaces are homeomorphic to each other.
- If $\Sigma \subset \mathrm{M}$ is a Cauchy hypersurface, then M is homeomorphic to $\Sigma \times \mathbb{R}$. Furthermore if Σ is smooth and spacelike, then M is diffeomorphic to $\Sigma \times \mathbb{R}$.

Theorem (Geroch, Bernal-Sanchez)

If (M, g) is globally hyperbolic, there exist a smooth manifold Σ and a diffeomorphism $\Phi : \Sigma \times \mathbb{R} \to \mathrm{M}$ such that:

1. For any $x \in \Sigma$, the curve $t \mapsto \Phi(x, t)$ is timelike.
2. For any $t \in \mathbb{R}$, the submanifold $\Phi(\Sigma \times \{t\})$ is spacelike and is a Cauchy hypersurface.

Note in particular that for any $x \in \mathrm{M}$, there exists a smooth spacelike Cauchy hypersurface containing x. We will not prove this result, instead we can see it as an alternate definition of global hyperbolicity.

Definition (GHMC)

Let (M, g) be a GHC Lorentzian manifold.

- Let (N, h) be another Lorentzian manifold. A *causal embedding* $f : \mathrm{M} \to \mathrm{N}$ is an isometric embedding such that $f(\Sigma)$ is a Cauchy hypersurface in N for some Cauchy hypersurface $\Sigma \subset \mathrm{M}$.
- (M, g) is called *globally hyperbolic maximally Cauchy-compact* (GHMC) if it is GHC and any causal embedding $f : \mathrm{M} \to \mathrm{N}$ is onto.

Note that the diffeomorphism Φ is far from being unique. A hypersurface $S \subset M$ such that there is such a Φ with $S = \Phi^{-1}(\Sigma \times \{0\})$ is called a *Cauchy hypersurface*.

2.1.2 Uniformisation and causality

Definition

An anti-de Sitter manifold (M, g) is called *uniformisable* if there exist an open subset $\mathcal{O} \subset \mathrm{AdS}^{d+1}$ and a discrete subgroup $\Gamma \subset \mathrm{SO}_\circ(d, 2)$ such that:

1. The group Γ preserves $\mathcal{O}$.
2. The action of Γ on $\mathcal{O}$ is properly discontinuous.
3. The quotient space $\Gamma \backslash \mathcal{O}$ is isometric to (M, g).

Theorem (Mess $d = 2$, Barbot $d \geq 3$)

Every spatially compact anti-de Sitter manifold is uniformisable. Furthermore, the corresponding open subset $\mathcal{O} \subset \mathrm{AdS}^{d+1}$ is simply connected.

In order to prove this result, we will need a few notions of causality theory.

Definition (Chronological future and past)

Let $X \subset M$ be a subset. The *chronological future* (resp. *chronological past*) of X is the set $I^+(X)$ (resp. $I^-(X)$) of points $x \in M$ such that there exists a future directed timelike curve $c : [0, 1] \to M$ with $c(0) \in X$ and $c(1) = x$ (resp. $c(0) = x$ and $c(1) \in X$).

Definition (Causal future and past)

- A *causal curve* $c : I \to M$ (where $I \subset \mathbb{R}$ is an interval) is a $\mathcal{C}^1$ curve such that $g(\dot{c}, \dot{c}) \leq 0$.
- Let $X \subset M$ be a subset. The *causal future* (resp. *causal past*) of X is the set $J^+(X)$ (resp. $J^-(X)$) of points $x \in M$ such that there exists a future directed causal curve $c : [0, 1] \to M$ with $c(0) \in X$ and $c(1) = x$ (resp. $c(0) = x$ and $c(1) \in X$).

Example (AdS^{d+1})

These notions are not so interesting in AdS^{d+1}, since for any $x \in \mathrm{AdS}^{d+1}$ we have $I^+(x) = I^-(x) = J^+(x) = J^-(x) = \mathrm{AdS}^{d+1}$ (and the same goes for any non empty subset $X \subset \mathrm{AdS}^{d+1}$).

Example ($\widetilde{\mathrm{AdS}}^{d+1}$)

There is more to say about the universal cover $\widetilde{\mathrm{AdS}}^{d+1}$. Recall the diffeomorphism $\Psi : \mathbb{H}^d \times \mathbb{S}^1 \to \mathrm{AdS}^{d+1}$, and consider its lift $\widetilde{\Psi} : \mathbb{H}^d \times \mathbb{R} \to \widetilde{\mathrm{AdS}}^{d+1}$. We then have $\widetilde{\Psi}^* g_{\widetilde{\mathrm{AdS}}} = g_{\mathbb{H}^d} - y_{d+1}^2 dt^2$. Now consider the Riemannian metric $h = y_{d+1}^{-2} g_{\mathbb{H}^d}$ on $\mathbb{H}^d$. This happens to be a metric of constant positive curvature, isometric to a hemisphere. To be more precise, consider the hemisphere

$$\mathbb{S}^d_+ = \left\{ y \in \mathbb{S}^d \subset \mathbb{R}^{d+1} \,\middle|\, y_{d+1} > 0 \right\},$$

and the following diffeomorphism is an isometry:

$$\Phi : \begin{cases} \mathbb{S}^d_+ & \to & \mathbb{H}^d \\ y & \mapsto & \left(\frac{y_1}{y_{d+1}}, \dots, \frac{y_d}{y_{d+1}}, \frac{1}{y_{d+1}} \right) \end{cases}$$

In particular, $(\mathbb{H}^d, h)$ has diameter π.

Now consider a curve $c : [0,1] \to \widetilde{\mathrm{AdS}}^{d+1}$, and write $c(s) = \widetilde{\Psi}(y(s), t(s))$. Any future directed causal curve must satisfy $\frac{dt}{ds} > 0$, so it may be reparametrised as $c(f(t)) = \widetilde{\Psi}(z(t), t)$. Such a curve is causal (resp. timelike) if and only if $\left\| \frac{dz}{dt} \right\|_h \leq 1$ (resp. $\left\| \frac{dz}{dt} \right\|_h < 1$).

For $x = \widetilde{\Psi}(y, t)$, $y \in \mathbb{H}^d$ and $t \in \mathbb{R}$, we find

$$\begin{aligned} \mathrm{J}^+(x) &= \left\{ \widetilde{\Psi}(y', t') \,\middle|\, t' \geq t + d_h(y, y') \right\} \\ \mathrm{J}^-(x) &= \left\{ \widetilde{\Psi}(y', t') \,\middle|\, t' \leq t - d_h(y, y') \right\} \\ \mathrm{I}^+(x) &= \left\{ \widetilde{\Psi}(y', t') \,\middle|\, t' > t + d_h(y, y') \right\} \\ \mathrm{I}^-(x) &= \left\{ \widetilde{\Psi}(y', t') \,\middle|\, t' < t - d_h(y, y') \right\} \end{aligned}$$

Proof of uniformisation of GHMC AdS manifolds. Consider a smooth spacelike Cauchy hypersurface $\Sigma \subset M$, and consider its lift $\widetilde{\Sigma} \subset \widetilde{M}$. and let $f : \widetilde{\Sigma} \to \mathbb{H}^d$ be the composition of the restriction of dev to $\widetilde{\Sigma}$, the diffeomorphism $\Psi^{-1} : \mathrm{AdS}^{d+1} \to \mathbb{H}^d \times \mathbb{S}^1$ and the projection on the first factor.

Since $\widetilde{\Sigma}$ is spacelike, the restriction of the lift $\widetilde{g}$ is a Riemannian metric. The key observation is that $f : \widetilde{\Sigma} \to \mathbb{H}^d$ cannot decrease the length of a tangent vector. This is a consequence of the formula $\Psi^* g_{\mathrm{AdS}} = g_{\mathbb{H}^d} - y_{d+1} d\theta^2$.

By a standard completeness argument (see e.g. the proof of the Cartan-Hadamard Theorem), this implies that f is a covering map, hence a diffeomorphism. The rest of the proof relies on *causality*, and is much easier if we consider the lifts $\widetilde{\mathrm{dev}} : \widetilde{M} \to \widetilde{\mathrm{AdS}}^{d+1}$ and $\widetilde{\rho} : \pi_1 M \to \widetilde{\mathrm{SO}}_\circ(d, 2)$. Here, start by noticing that $\widetilde{\Sigma}$ is a Cauchy hypersurface in $\widetilde{M}$, and consequently it separates $\widetilde{M}$ into two connected components: the past $\mathrm{I}^-(\widetilde{\Sigma})$ and the future $\mathrm{I}^+(\widetilde{\Sigma})$. A significant difference between AdS^{d+1} and its universal cover is that $\widetilde{\mathrm{AdS}}^{d+1}$ is *causal*: there are no causal loops. So $\widetilde{\mathrm{dev}} \circ c$ is injective whenever $c : [0,1] \to \widetilde{M}$ is timelike. This means that the three subsets $\widetilde{\mathrm{dev}}(\widetilde{\Sigma})$, $\widetilde{\mathrm{dev}}(\mathrm{J}^-(\widetilde{\Sigma}))$, and $\widetilde{\mathrm{dev}}(\mathrm{J}^+(\widetilde{\Sigma}))$ are pairwise disjoint. We conclude that $\widetilde{\mathrm{dev}}$ is injective by using the fact that there is a Cauchy hypersurface through any point.

We now know that $\widetilde{\mathrm{dev}}$ is a diffeomorphism from $\widetilde{M}$ to some open subset $\widetilde{\mathcal{O}} \subset \widetilde{\mathrm{AdS}}^{d+1}$, it remains to prove that the projection to AdS^{d+1} is injective when restricted to $\mathcal{O}$. This is related to another important (almost defining) properties of globally hyperbolic spacetimes: for any $x, y \in \widetilde{M}$, the *causal interval* $\mathrm{J}^+(x) \cap \mathrm{J}^-(y)$ is compact. This forces the desired injectivity: if $x, y \in \widetilde{\mathcal{O}}$ have the same projection in AdS^{d+1}, then y is the image of x under some deck transformation, which implies that $\mathrm{J}^+(x) \cap \mathrm{J}^-(y)$ (or $\mathrm{J}^+(y) \cap \mathrm{J}^-(x)$ depending on their relative position) is not compact. □

This means that a spatially compact anti-de Sitter manifold (M, g) can be seen as a quotient space $\Gamma \backslash \mathcal{O}$, where $\Gamma < \mathrm{SO}_\circ(d,2)$ is a discrete subgroup. Since $\mathcal{O}$ is simply connected, the group Γ is isomorphic to $\pi_1(\mathrm{M}) = \pi_1(\Sigma)$ where $\Sigma \subset \mathrm{M}$ is any Cauchy hypersurface. The representation $\rho : \pi_1(\mathrm{M}) \to \Gamma < \mathrm{SO}_\circ(d,2)$ is the holonomy representation of the anti-de Sitter structure.

2.2 Holonomy representations of GHMC AdS manifolds

Proposition

A maximal spatially compact anti-de Sitter manifold is caracterized up to isometry by its holonomy representation.

This means we have an embedding

$$\{\text{ isometry classes of spatially compact anti-de Sitter structures on M}\} \to \mathrm{Hom}(\pi_1(\mathrm{M}), \mathrm{SO}_\circ(d,2))/\mathrm{SO}_\circ(d,2).$$

Incompleteness translates as $\mathcal{O} \neq \mathrm{AdS}^{d+1}$.

From the previous section, describing GMHC AdS structures on a given manifold M is the same as understanding the subset $\mathrm{Hom}_{\mathrm{GHMC}}(\pi_1(\mathrm{M}), \mathrm{SO}_\circ(d,2)) \subset \mathrm{Hom}(\pi_1(\mathrm{M}), \mathrm{SO}_\circ(d,2))$ of holonomy representations of such structures.

Theorem (Mess, Barbot-Mérigot, Barbot)

$\mathrm{Hom}_{\mathrm{GHMC}}(\pi_1(\mathrm{M}), \mathrm{SO}_\circ(d,2))$ is open and closed in $\mathrm{Hom}(\pi_1(\mathrm{M}), \mathrm{SO}_\circ(d,2))$

Theorem (Mess)

For $d = 2$, if M has genus $g \geq 2$, then $\mathrm{Hom}_{\mathrm{GHMC}}(\pi_1(\mathrm{M}), \mathrm{SO}_\circ(d,2))/\pi_1(\mathrm{M})$ is homeomorphic to $\mathbb{R}^{12g-12}$.

This can be understood by the exceptional isomorphism $\mathrm{SO}_\circ(2,2) \simeq \mathrm{PSL}_2(\mathbb{R}) \times \mathrm{PSL}_2(\mathbb{R})$.

Theorem (Lee-Marquis, M.-Schlenker-Tholozan)

For any $d \geq 4$, there exist GHMC AdS^{d+1}-manifolds with Cauchy hypersurface admitting negatively curved metrics, but no hyperbolic metric.

Part 3. Dynamics of GHMC AdS manifolds

3.1 The limit set

3.1.1 The boundary of AdS

Just as we can compactify the hyperbolic space $\mathbb{H}^d$ by adding a sphere at infinity $\partial_\infty \mathbb{H}^d \simeq \mathbb{S}^{d-1}$, we can compactify $\mathrm{AdS}^{d+1} \simeq \mathbb{H}^d \times \mathbb{S}^1$ by adding a boundary $\partial_\infty \mathrm{AdS}^{d+1} \simeq \partial_\infty \mathbb{H}^d \times \mathbb{S}^1$. If we are to do this simply through the diffeomorphism Ψ encountered previously, this is a rather vacuous statement. What makes it interesting is that this compactification is geometric in three ways:

- The action $\mathrm{SO}_\circ(d,2) \curvearrowright \mathrm{AdS}^{d+1}$ extends to $\partial_\infty \mathrm{AdS}^{d+1}$.
- Geodesics in AdS^{d+1} can have endpoints on $\partial_\infty \mathrm{AdS}^{d+1}$.
- The space $\partial_\infty \mathrm{AdS}^{d+1}$ has a geometry of its own, invariant under $\mathrm{SO}_\circ(d,2)$.

Since AdS^{d+1} is closed in $\mathbb{R}^{d+2}$, we cannot define a boundary directly there. The trick is to projectivise. Instead of projecting to $\mathbb{RP}^{d+1}$, we will consider the spherical projective space $\mathbb{S}(\mathbb{R}^{d+2})$ defined as the quotient of $\mathbb{R}^{d+2} \setminus \{0\}$ under the relation identifying x and λx for $\lambda > 0$. Of course restricting the projection $\mathbb{S} : \mathbb{R}^{d+2} \setminus \{0\} \to \mathbb{S}(\mathbb{R}^{d+2})$ to the unit sphere yields a diffeomorphism $\mathbb{S}(\mathbb{R}^{d+2}) \simeq \mathbb{S}^{d+1}$, but this point of view is more convenient to consider the action of the group $\mathrm{SL}_{d+2}(\mathbb{R})$ on $\mathbb{S}(\mathbb{R}^{d+2})$.

The restriction of the projection map to AdS^{d+1} is a diffeomorphism onto an open subset, and we define

$$\partial_\infty \mathrm{AdS}^{d+1} \overset{\text{def}}{=} \partial \mathbb{S}(\mathrm{AdS}^{d+1}) \subset \mathbb{S}(\mathbb{R}^{d+2}).$$

We obtain a topology on $\mathrm{AdS}^{d+1} \sqcup \partial_\infty \mathrm{AdS}^{d+1}$ by identifying AdS^{d+1} with its image in $\mathbb{S}(\mathbb{R}^{d+2})$. The group $\mathrm{SO}_\circ(d,2)$ acts continuously on this compactification.

A spacelike geodesic $c_{x,v}$ in AdS^{d+1} (assume that $\mathbf{b}(v,v) = 1$) has two distinct endpoints

$$\lim_{t\to\pm\infty} c_{x,v}(t) = [x \pm v] \in \partial_\infty \mathrm{AdS}^{d+1}.$$

A lightlike geodesic $c_{x,v}$ has a unique endpoint $\lim_{t\to\pm\infty} c_{x,v}(t) = [v]$. A timelike geodesic in AdS^{d+1} is a compact subset, is has no endpoint in $\partial_\infty \mathrm{AdS}^{d+1}$.

We will discuss the intrinsic geometry of $\partial_\infty \mathrm{AdS}^{d+1}$ later.

3.1.2 The limit set and the convex core

There are several ways of defining the limit set of a GHMC AdS manifold $\mathrm{M} = \Gamma \backslash \mathcal{O}$.

Definition

The *limit set* of a GHMC AdS manifold (M, g) is $\Lambda = \overline{\mathcal{O}} \cap \partial_\infty \mathrm{AdS}^{d+1}$, where $\mathrm{M} = \Gamma \backslash \mathcal{O}$.

This is also the same as accumulation points of the Γ-orbit of a point $x \in \mathcal{O}$, however this is not true if we pick a point $x \in \mathrm{AdS}^{d+1} \setminus \mathcal{O}$ (we will not prove either of these facts).

Definition

The *convex core* of a GHMC AdS manifold (M, g) is the quotient $\mathrm{C}(\mathrm{M})$ of the convex hull $\mathrm{C}(\Lambda)$ by $\pi_1 \mathrm{M}$.

3.1.3 The boundary of a Cauchy hypersurface

Let (M, g) be a GHMC AdS manifold, and consider a smooth spacelike Cauchy hypersurface $\Sigma \subset \mathrm{M}$, as well as a lift $\widetilde{\Sigma} \subset \mathcal{O} \subset \mathrm{AdS}^{d+1}$.

Fact

There is a smooth map $f_\Sigma : \mathbb{H}^d \to \mathbb{S}^1$ such that $\|df\|_h < 1$ and $\widetilde{\Sigma} = \Psi(\mathrm{Gr} f_\Sigma)$.

Proof. We have already seen that the projection on the first factor $\widetilde{\Sigma} \to \mathbb{H}^d$ is a diffeomorphism, so there is a smooth map $f_\Sigma : \mathbb{H}^d \to \mathbb{S}^1$ such that $\widetilde{\Sigma} = \Psi(\mathrm{Gr} f_\Sigma)$. Now the fact that $\mathrm{G}r(f_\Sigma)$ is spacelike is equivalent to $\|df\|_h < 1$. □

Write $\mathbb{S}^d_+$ for the open half-hemisphere isometric to $(\mathbb{H}^d, h)$. Since $f_\Sigma : \mathbb{S}^d_+ \to \mathbb{S}^1$ is 1-Lipschitz, it extends uniquely as a 1-Lipschitz map to the closure $\overline{\mathbb{S}^d_+} \subset \mathbb{S}^d$. Since the boundary of $\mathbb{S}^d_+$ is $\mathbb{S}^{d-1}$, we have constructed a 1-Lipschitz map $\partial f_\Sigma : \mathbb{S}^{d-1} \to \mathbb{S}^1$, and we have

$$\partial\widetilde{\Sigma} = \Psi(\mathrm{Gr}\,\partial f_\Sigma) \subset \partial_\infty \mathrm{AdS}^{d+1},$$

where $\partial\widetilde{\Sigma} = \overline{\widetilde{\Sigma}} \cap \partial_\infty \Sigma$.

Fact

For any Cauchy hypersurface $\Sigma \subset \mathrm{M}$, $\partial\widetilde{\Sigma} = \Lambda$.

Proof. It really follows from causality considerations: the domain $\mathcal{O} \subset \mathrm{AdS}^{d+1}$ is included in the *domain of dependence* $\mathrm{D}(\widetilde{\Sigma})$ of the lift of Σ (still denoted by $\widetilde{\Sigma}$). The domain of dependence (also called the Cauchy development) is the set of points $x \in \widetilde{\mathrm{AdS}}^{d+1}$ such that any inextendible causal curve going through x must meet $\widetilde{\Sigma}$. With a good understanding of causal curves in AdS^{d+1}, one can show that $\partial \mathrm{D}(\widetilde{\Sigma}) = \partial\widetilde{\Sigma}$. □

This has a major consequence: the limit set Λ is the graph of a Lipschitz map $\mathbb{S}^{d-1} \to \mathbb{S}^1$. In particular it is a Lipschitz submanifold of dimension $d-1$.

3.1.4 Acausality of the limit set and the convex core

From now on, we will make an extra assumption on our GHCM AdS manifolds: we assume that the set $\Lambda = \overline{\mathcal{O}} \cap \partial_\infty \mathrm{AdS}^{d+1}$ is *acausal*, meaning that the map ∂f_Σ decreases distances.[1]

Theorem (Many people)

For a GHMC AdS manifold (M, g), the following are equivalent:

1. The limit set Λ is acausal.
2. The convex core $\mathrm{C}(\mathrm{M})$ is compact.
3. The fundamental group $\pi_1 \mathrm{M}$ is Gromov hyperbolic.
4. There exists a "convex" Cauchy hypersurface.

The limit is always a Lipschitz submanifold, but never $\mathcal{C}^1$ except in the Fuchsian case (Glorieux-M.).

3.2 The spacelike geodesic flow

On any Lorentzian manifold (M, g), we can define the unit spacelike tangent bundle

$$\mathrm{T}^1\mathrm{M} = \{v \in \mathrm{TM} \,|\, g(v,v) = 1\},$$

[1] Meaning for any distinct $x, y \in \mathbb{S}^{d-1}$, we have $d_{\mathbb{S}^1}(\partial f_\Sigma(x), \partial f_\Sigma(y)) < d_{\mathbb{S}^{d-1}}(x,y)$.

and the geodesic flow $\varphi^t : \mathrm{T}^1\mathrm{M} \to \mathrm{T}^1\mathrm{M}$. In general, it is incomplete.

Definition (The subset $\mathrm{K} \subset \mathrm{T}^1\mathrm{M}$)

Let (M, g) be a GHMC AdS manifold with acausal limit set. Denote by $\mathrm{K} \subset \mathrm{T}^1\mathrm{M}$ the subset of initial data of geodesics whose lifts to AdS^{d+1} have both endpoints in Λ.

Proposition

Let (M, g) be a GHMC AdS manifold with acausal limit set.

1. The set $\mathrm{K} \subset \mathrm{T}^1\mathrm{M}$ is compact.
2. A geodesic in M is complete if and only if it lifts to a spacelike geodesic in AdS^{d+1} with both endpoints in Λ.

Proposition

We can embed $\mathrm{T}^1\mathrm{M}$ into a manifold N on which the geodesic flow φ^t extends to a complete flow, such that the non wandering set is K.

Idea of proof. Consider the open subset

$$\mathrm{U} = \left\{(x, v) \in \mathrm{T}^1\mathrm{AdS}^{d+1} \,\middle|\, \exists t \in \mathbb{R} \;\; \varphi^t(x, v) \in \mathcal{O}\right\}.$$

Then Γ acts properly discontinuously and freely on U, and we can take $\mathrm{U} = \Gamma\backslash\mathrm{U}$. □

3.3 Smale's axiom A

3.3.1 Definition

Definition (Basic axiom A flow)

A smooth flow $\varphi^t : M \to M$ on a smooth manifold M is called a *basic axiom A flow* if the following properties are satisfied:

1. The *non wandering* set $NW(\varphi)$ is compact, where

$$NW(\varphi) = \left\{ x \in M \,\middle|\, \exists x_k \to x \; \exists t_k \to \infty \; \varphi^{t_k}(x_k) \to x \right\}.$$

2. The non wandering set is equal to the closure of the set of periodic points:

$$NW(\varphi) = \overline{Per(\varphi)} \quad \text{where} \quad Per(\varphi) = \left\{ x \in M \,\middle|\, \exists T > 0 \; \varphi^T(x) = x \right\}.$$

3. The restriction $\varphi^t|_{NW(\varphi)} : NW(\varphi) \to NW(\varphi)$ has a dense orbit.

4. The restricted tangent bundle $TM|_{NW(\varphi)}$ admits a $d\varphi^t$-invariant splitting $TM|_{NW(\varphi)} = E^s \oplus E^0 \oplus E^u$ where:

   (a) $\dim E^0 = 1$ and $E^0 = \mathbb{R} \cdot \frac{d}{dt}\big|_{t=0} \varphi^t$.

   (b) E^s is contracted by $d\varphi^t$ and E^u is dilated by $d\varphi^t$.

Proposition

If (M, g) is a GHMC AdS manifold with acausal limit set, the extended geodesic flow $\varphi^t : N \to N$ is axiom A, with non wandering set equal to K.

3.3.2 Consequences of having an axiom A flow

A good example of the information that we gain by having an axiom A flow is orbit counting. Consider the orbit counting function.

$$\pi_\varphi(T) = \sharp\{ \text{ periodic orbits of least period } \leq T\}.$$

Theorem (Parry-Pollicott (83))

If $\varphi^t : M \to M$ is a basic axiom A flow, there is some $h_\varphi > 0$ such that

$$\pi_\varphi(T) \sim \frac{e^{h_\varphi T}}{h_\varphi T}$$

as $T \to +\infty$.

Here, h_φ is the topological entropy. But the spacelike geodesic flow has nicer properties than the generic axiom A flow, which can be reflected in more precise orbit counting.

Theorem (Delarue-M.-Sanders)

Let (M, g) be a GHMC AdS manifold, and denote by $h > 0$ the topological entropy of the extended geodesic flow φ^t. Then there exists $0 < c < h$ such that

$$\pi_\varphi(t) = \mathrm{Li}(e^{ht})\left(1 + O\left(e^{-ct}\right)\right).$$

The logarithmic integral function Li is defined by

$$\mathrm{Li}(x) = \int_2^x \frac{du}{\mathrm{Log} u} \sim \frac{x}{\mathrm{Log} x}.$$

3.3.3 Sketch of proof of the axiom A property

Let us start by recalling the formula for the spacelike geodesic flow on $T^1\mathrm{AdS}^{d+1}$:

$$\varphi^t(x, v) = (\cosh tx + \sinh tv, \sinh tx + \cosh tv).$$

Now consider the tangent space of the unit spacelike tangent bundle:

$$T_{(x,v)}T^1\mathrm{AdS}^{d+1} = \{(w, z) \,|\, \mathbf{b}(x, w) = \mathbf{b}(v, z) = \mathbf{b}(x, z) + \mathbf{b}(w, v) = 0\}.$$

We find a decomposition $T_{(x,v)}T^1\mathrm{AdS}^{d+1} = E^u_{(x,v)} \oplus E^s_{(x,v)} \oplus E^0_{(x,v)}$ where

$$E^u_{(x,v)} = \{(w, w) \,|\, \mathbf{b}(x, w) = \mathbf{b}(v, w) = 0\},$$
$$E^u_{(x,v)} = \{(w, -w) \,|\, \mathbf{b}(x, w) = \mathbf{b}(v, w) = 0\},$$
$$E^0_{(x,v)} = \mathbb{R} \cdot (v, x) = \mathbb{R} \cdot \left.\frac{d}{dt}\right|_{t=0} \varphi^t(x, v).$$

One immediately finds

$$d\varphi^t|_{(x,v)}(w, w) = e^t(w, w),$$
$$d\varphi^t|_{(x,v)}(w, -w) = e^{-t}(w, w).$$

From there, it is very tempting to believe that E^s and E^u will give the distributions we are looking for. It is true, but there is still a non trivial computation involved. The problem is that we are not considering a fixed norm on $\mathbb{R}^{d+2} \times \mathbb{R}^{d+2}$, but a Riemannian metric on $U \subset T^1\mathrm{AdS}^{d+1}$ which is Γ-invariant (a fixed norm will never be Γ-invariant). So the previous computations gives:

$$\left\| d\varphi^t|_{(x,v)}(w, w) \right\|_{\varphi^t(x,v)} = e^t \|(w, w)\|_{\varphi^t(x,v)},$$
$$\left\| d\varphi^t|_{(x,v)}(w, -w) \right\|_{\varphi^t(x,v)} = e^{-t} \|(w, w)\|_{\varphi^t(x,v)}.$$

However, for the axiom A property, we need to compare $\left\| d\varphi^t|_{(x,v)}(w, w) \right\|_{\varphi^t(x,v)}$ and $\|(w, w)\|_{(x,v)}$. There is however no direct way to relate $\|(w, w)\|_{\varphi^t(x,v)}$ and $\|(w, w)\|_{(x,v)}$.

A good way of understanding the axiom A mechanism is to look at periodic orbits. A periodic point $p \in N$ of φ^t lifts to $(x, v) \in U \subset T^1\mathrm{AdS}^{d+1}$ such that $\varphi^T(x, v) = (\gamma x, \gamma v)$ for some $\gamma \in \Gamma$ and $T > 0$. Then the differential

$$d\varphi^T|_p : T_pN \to T_pN$$

is conjugate to the composition

$$\gamma^{-1} d\varphi^{\mathsf{T}}|_{(x,v)} : \mathrm{T}_{(x,v)}\mathrm{T}^1\mathrm{AdS}^{d+1} \to \mathrm{T}_{(x,v)}\mathrm{T}^1\mathrm{AdS}^{d+1}.$$

In particular, the actions on E^u and E^s are

$$\begin{cases} \mathrm{E}^u_{(x,v)} & \to & \mathrm{E}^u_{(x,v)} \\ (w,w) & \mapsto & e^{\mathsf{T}}(\gamma^{-1}w, \gamma^{-1}w) \end{cases} \qquad \begin{cases} \mathrm{E}^s_{(x,v)} & \to & \mathrm{E}^s_{(x,v)} \\ (w,-w) & \mapsto & e^{-\mathsf{T}}(\gamma^{-1}w, -\gamma^{-1}w) \end{cases}$$

So the matter is to check that $e^{\mathsf{T}}\gamma^{-1}$ (resp. $e^{-\mathsf{T}}\gamma^{-1}$) is dilating (resp. contracting) vectors in $x^{\perp_\flat} \cap v^{\perp_\flat}$. A simple computation yields:

$$\gamma \cdot (x+v) = e^{\mathsf{T}}(x+v),$$
$$\gamma \cdot (x-v) = e^{-\mathsf{T}}(x-v).$$

Now the fact that $(x,v) \in \mathrm{U}$ implies that this eigenvalue e^{T} (resp. $e^{-\mathsf{T}}$) is the highest (resp. lowest) eigenvalue of γ. So the spectral radius on $x^{\perp_\flat} \cap v^{\perp_\flat}$ is the second highest eigenvalue of γ, and controlling the contraction of $d\varphi^{\mathsf{T}}$ on E^s becomes a question of controlling the gap between the two highest eigenvalues of γ, which is one of the aspects of the theory of *Anosov representations*.