

Introduction to mixing for Anosov representations

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The goal of this course is to understand how to use tools from dynamical systems (mixing) in a geometric context (Anosov representations). The first part aside, these notes are a shortened version of [Mon25]. The plan for the three lectures is the following:

Part 1. Counting closed geodesics

Part 2. Anosov representations and their flows

Part 3. Mixing properties of Anosov representations

Part 1. Counting closed geodesics

1.1 Closed hyperbolic manifolds

Let (N, g_0) be a closed oriented hyperbolic manifold of dimension $d \geq 2$.

Remark (The group point of view)

We can consider $(N, g_0) \simeq \Gamma \backslash \mathbb{H}^d$, where $\Gamma < \text{Isom}(\mathbb{H}^d)_o \simeq \text{SO}(d, 1)_o$ is a discrete subgroup, acting on \mathbb{H}^d with a compact quotient (such a subgroup is called a *uniform lattice*). We then have $\Gamma \simeq \pi_1 N$. While the content of this first section can be described in terms of Riemannian geometry, stating everything in terms of groups will allow for a natural generalisation to the case of Anosov representations in the next sections.

From Riemannian geometry, we know that every free homotopy class on N contains a unique closed geodesic. For any $T > 0$, we consider

$$\pi_{g_0}(T) = \#\{\text{non trivial primitive closed geodesics of length } \leq T\}.$$

By primitive, we mean that we do not consider closed geodesic obtained by running through the same geodesic several times. The negative curvature assumption shows that $\pi_{g_0}(T)$ is finite.

Remark (The group point of view)

Free homotopy classes correspond to conjugacy classes in the fundamental group $\pi_1 N$, or equivalently in $\Gamma \simeq \pi_1 N$. We will use the notation $[\Gamma]$ for the set of conjugacy classes of non trivial elements of Γ , and $[\Gamma]_{\text{prim}} \subset [\Gamma]$ for the subset of conjugacy classes of *primitive elements*, i.e. that are not equal to γ^k for some $k \geq 2$. Any $[\gamma] \in [\Gamma]_{\text{prim}}$ then corresponds to a primitive closed geodesic on $\Gamma \backslash \mathbb{H}^d$ whose length will be denoted by $\ell(\gamma)$. We then have $\pi_{g_0}(T) = \pi_\Gamma(T)$, where

$$\pi_\Gamma(T) = \#\{[\gamma] \in [\Gamma]_{\text{prim}} \mid \ell(\gamma) \leq T\}.$$

The length $\ell(\gamma)$ can be understood algebraically: it is the logarithm of the spectral radius of $\gamma \in \text{SO}(d, 1)_\circ < \text{SL}_{d+1}(\mathbb{R})$.

Theorem 1.1 (The prime geodesic theorem)

If (N, g_0) is a closed hyperbolic manifold, then

$$\pi_{g_0}(T) \sim \frac{e^{(d-1)T}}{(d-1)T}$$

as $T \rightarrow +\infty$.

Let us have an very short and outrageously incomplete discussion of the proofs of this theorem:

- The first proof is attributed to Huber (59) for $d = 2$. This proof is based on Selberg's Trace Formula, meaning that it has some "representation theoretic content", so it cannot adapt to variable curvature (unless locally symmetric). Hejhal (76) gave a proof based on Selberg's Trace Formula for $d \geq 3$.
- In his thesis (69), Margulis gave a very different proof based on the dynamics of the geodesic flow. This approach has the advantage of applying to variable negative curvature.

Recall that the geodesic flow $\varphi_{g_0}^t : T^1 N \rightarrow T^1 N$ is defined on the unit tangent bundle

$$T^1 N = \{(x, v) \in TN \mid \|v\|_x = 1\}$$

by $\varphi_{g_0}^t(x, v) = (c_{x,v}(t), \dot{c}_{x,v}(t))$ where $c_{x,v} : \mathbb{R} \rightarrow N$ is the geodesic with initial position $c_{x,v}(0) = x$ and velocity $\dot{c}_{x,v}(0) = v$.

Closed geodesics on N correspond to periodic orbits of the geodesic flow $\varphi_{g_0}^t$. Margulis developed a strategy that allows to count periodic orbits of a certain class of flows, called *topologically mixing Anosov flows*.

1.2 Anosov flows

Definition (Anosov flow)

A smooth flow $\varphi^t : M \rightarrow M$ on a closed manifold M is called *Anosov* if the tangent bundle TM has a $d\varphi^t$ -invariant splitting $TM = E^s \oplus E^0 \oplus E^u$ where

1. $\dim E^0 = 1$ and $E^0 = \mathbb{R} \cdot \frac{d}{dt} \Big|_{t=0} \varphi^t$.
2. E^s is *contracted* by $d\varphi^t$, i.e. there exist $C, \lambda > 0$ such that:

$$\forall x \in M \forall v \in E^s|_x \forall t \geq 0 \quad \|d\varphi^t|_x\|_{\varphi^t(x)} \leq Ce^{-\lambda t} \|v\|_x.$$

3. E^u is *dilated* by $d\varphi^t$, i.e. E^u is contracted by $d\varphi^{-t}$.

Theorem 1.2 (Anosov? Hadamard?)

The geodesic flow of a closed negatively curved Riemannian manifold is Anosov.

In the case of geodesic flows, there is a nice description of the stable and unstable distributions in terms of the visual boundary $\partial_\infty \widetilde{N}$ of the universal cover and horospheres. Constant curvature geodesic flows stand out amongst other Anosov geodesic flows by the smoothness of these (un)stable distributions.

Theorem 1.3 (Ghys, Benoist-Foulon-Labourie, Besson-Courtois-Gallot, Hamendstadt)

If (N, g) is a closed negatively curved Riemannian manifold and the distributions E^s and E^u of its geodesic flow are smooth, then g is locally symmetric of rank one.

Definition (Topological mixing)

A flow $\varphi^t : M \rightarrow M$ is *topologically mixing* if for any non empty open subsets $U, V \subset M$, there is some $T > 0$ such that:

$$\forall t \geq T \quad \varphi^{-t}(U) \cap V \neq \emptyset.$$

For Anosov flows, topological mixing should be considered as an "irrational slope" condition on the distributions E^s and E^u .

Proposition (Anosov)

The geodesic flow of a closed negatively curved Riemannian manifold is topologically mixing.

For a flow $\varphi^t : M \rightarrow M$, the appropriate counting function is

$$\pi_\varphi(T) = \#\{\text{periodic orbits of least period} \leq T\}.$$

Theorem 1.4 (Margulis (69))

If $\varphi^t : M \rightarrow M$ is a topologically mixing Anosov flow, there is some $h_\varphi > 0$ such that

$$\pi_\varphi(T) \sim \frac{e^{h_\varphi T}}{h_\varphi T}$$

as $T \rightarrow +\infty$.

For the geodesic flow $\varphi_g : T^1N \rightarrow T^1N$ of a closed negatively curved Riemannian manifold (N, g) , Margulis notices that h_{φ_g} is equal to the *volume entropy* h_g , i.e.

$$h_g = \lim_{r \rightarrow +\infty} \frac{\log \text{Vol}_{\tilde{g}} B_{\tilde{g}}(o, r)}{r}$$

where $o \in \tilde{N}$ is any point and \tilde{g} is the lift of g to the universal cover \tilde{N} . In particular, for a hyperbolic manifold, we have $h_g = d - 1$.

For general topologically mixing Anosov flows, Bowen proved that the number h_φ has a dynamical meaning: it is the *topological entropy* of the flow φ^t .

Part 2. Anosov representations

2.1 The action of isometries on $\partial_\infty \mathbb{H}^d$

All the computations will be done by using the hyperboloid model

$$\mathbb{H}^d = \{x \in \mathbb{R}^{d+1} \mid \langle x, x \rangle = -1 \text{ \& } x_{d+1} > 0\}$$

where $\langle x, y \rangle = x_1 y_1 + \dots + x_d y_d - x_{d+1} y_{d+1}$. In this model, the boundary is presented as

$$\partial_\infty \mathbb{H}^d = \{[x] \in \mathbb{R}P^d \mid \langle x, x \rangle = 0\}.$$

It is homeomorphic to \mathbb{S}^{d-1} by the map

$$\begin{cases} \mathbb{S}^{d-1} & \rightarrow & \partial_\infty \mathbb{H}^d \\ (x_1, \dots, x_d) & \mapsto & [x_1 : \dots : x_d : 1] \end{cases}.$$

The topology on $\mathbb{H}^d \cup \partial_\infty \mathbb{H}^d$ is obtained by identifying \mathbb{H}^d with its image in $\mathbb{R}P^d$.

Lemma (Boundary limits of diverging sequences of isometries)

Given a sequence $(g_k)_{k \geq 0}$ in $\text{Isom}(\mathbb{H}^d)$, points $x, y \in \mathbb{H}^d$ and $\eta \in \partial_\infty \mathbb{H}^d$, we have the following equivalence:

$$\lim_{k \rightarrow +\infty} g_k \cdot x = \eta \iff \lim_{k \rightarrow +\infty} g_k \cdot y = \eta.$$

Definition (Notation)

When such limits exist, we write $\eta = \lim g_k$.

Example (Hyperbolic isometries)

An element $g \in \text{Isom}(\mathbb{H}^d)$ is called *hyperbolic* if it acts by a non trivial translation along a geodesic ℓ of \mathbb{H}^d . This geodesic is then unique, and its endpoints on $\partial_\infty \mathbb{H}^d$ are $g^+ = \lim_{k \rightarrow +\infty} g^k \in \partial_\infty \mathbb{H}^d$ and $g^- = \lim_{k \rightarrow +\infty} g^{-k} \in \partial_\infty \mathbb{H}^d$.

The typical example of a hyperbolic isometry is the element $a^t \in \text{Isom}(\mathbb{H}^d)_o \simeq \text{SO}(d, 1)_o$ (for $t \neq 0$) given by

$$a^t = \begin{pmatrix} \cosh t & 0 & \sinh t \\ 0 & \mathbf{1}_{d-1} & 0 \\ \sinh t & 0 & \cosh t \end{pmatrix}.$$

It stabilises the geodesic with initial data

$$x_0 = (0, \dots, 0, 1); \quad v_0 = (1, 0, \dots, 0).$$

We can compute the limits $a^+ = \lim_{t \rightarrow +\infty} a^t = [1 : 0 : \dots : 0 : 1]$ and $a^- = \lim_{t \rightarrow -\infty} a^t = [-1 : 0 : \dots : 0 : 1]$. Note that for any $\eta = [\eta_1 : \dots : \eta_{d+1}] \in \partial_\infty \mathbb{H}^d$, if $\eta \neq [-1 : 0 : \dots : 0 : 1]$, i.e. $\eta_1 + \eta_{d+1} \neq 0$, we have $\lim_{t \rightarrow +\infty} a^t \cdot \eta = a^+$. Any hyperbolic isometry is conjugate to a^t for some $t > 0$, so they all present the same dynamics on $\partial_\infty \mathbb{H}^d$.

Proposition (Convergence property)

Let $(g_k)_{k \geq 0} \in \text{Isom}(\mathbb{H}^d)_o^{\mathbb{N}}$ be a diverging sequence (i.e. $g_k \rightarrow \infty$ in $\text{Isom}(\mathbb{H}^d)$) admitting boundary limits $g_+ = \lim_{k \rightarrow +\infty} g_k \in \partial_\infty \mathbb{H}^d$ and $g_- = \lim_{k \rightarrow +\infty} g_k^{-1} \in \partial_\infty \mathbb{H}^d$. Then for any $\eta \in \partial_\infty \mathbb{H}^d \setminus \{g_-\}$, we have $g_k \cdot \eta \rightarrow g_+$, and the convergence is uniform on compact subsets.

Proof. We can use Cartan's KAK decomposition of $\text{Isom}(\mathbb{H}^d)_o \simeq \text{SO}(n, 1)_o$. For this, we fix two distinct points $x^\pm \in \partial_\infty \mathbb{H}^d$ and a point $x \in \mathbb{H}^d$ on the geodesic with endpoints x^\pm . Let $K \subset \text{Isom}(\mathbb{H}^d)$ be the stabiliser of x (it is a compact subgroup, $K \simeq \text{SO}(n)$), and $A = \{a_t \mid t \in \mathbb{R}\}$ be the one-parameter group of translations along the geodesic $(x^- x^+)$. Then every element $g \in \text{Isom}(\mathbb{H}^d)_o$ can be written as $g = r a_t s$ where $r, s \in K$ and $t \geq 0$. Applying this decomposition to every $g_k = r_k a_{t_k} s_k$, the divergence $g_k \rightarrow \infty$ means that $t_k \rightarrow +\infty$, and the convergences $\lim g_k^{\pm 1} = g_\pm$ mean that $r_k \cdot x^+ \rightarrow g_+$ and $s_k^{-1} \cdot x^- \rightarrow g_-$. For $\eta \neq g_-$, we decompose $g_k \cdot \eta = r_k a_{t_k} s_k \cdot \eta$ and follow three steps:

- The fact that $\eta \neq g_-$ implies that $s_k \cdot \eta$ stays bounded away from $\lim s_k \cdot g_- = x^-$ (from the equicontinuity of the action of K on $\partial_\infty \mathbb{H}^d$),
- The fact that $s_k \cdot \eta$ stays bounded away from $\lim s_k \cdot g_- = x^-$ implies that $a_{t_k} \cdot (s_k \cdot \eta) \rightarrow x^+$,
- The fact that $a_{t_k} \cdot (s_k \cdot \eta) \rightarrow x^+$ implies that $r_k \cdot (a_{t_k} s_k \cdot \eta) \rightarrow g_+$ (again from the equicontinuity of the action of K on $\partial_\infty \mathbb{H}^d$).

□

Definition

The *limit set* $\Lambda_\Gamma \subset \partial_\infty \mathbb{H}^d$ of a discrete subgroup $\Gamma < \text{Isom}(\mathbb{H}^d)$ is

$$\Lambda_\Gamma = \left\{ \lim \gamma_k \mid (\gamma_k)_{k \geq 0} \in \Gamma^{\mathbb{N}} \gamma_k \rightarrow \infty \right\} \subset \partial_\infty \mathbb{H}^d.$$

Remark

Using a diagonal argument, we can see that Λ_Γ is always closed in $\partial_\infty \mathbb{H}^d$.

Example

Let (N, g) be a closed hyperbolic manifold, and consider $\Gamma < \text{Isom}(\mathbb{H}^d)_o$ such that $(N, g) \simeq \Gamma \backslash \mathbb{H}^d$. Then $\Lambda_\Gamma = \partial_\infty \mathbb{H}^d$.

2.2 Convex cocompactness and hyperbolic groups

Definition (Convex cocompact subgroup)

A discrete subgroup $\Gamma < \text{Isom}(\mathbb{H}^d)$ is called *convex cocompact* if there is a non empty convex subset $C \subset \mathbb{H}^d$ preserved by Γ such that the quotient $\Gamma \backslash C$ is compact.

If C is such a convex subset, then $\Lambda_\Gamma = \overline{C} \cap \partial_\infty \mathbb{H}^d$.

Proposition

Let $\Gamma < \text{Isom}(\mathbb{H}^d)_o$ be a discrete subgroup. The following are equivalent:

1. Γ is convex cocompact.
2. For any pair of distinct points $\eta_+, \eta_- \in \Lambda_\Gamma$, there is a diverging sequence $(\gamma_k)_{k \geq 0} \in \Gamma^{\mathbb{N}}$ such that $\eta_+ = \lim \gamma_k$ and $\eta_- = \lim \gamma_k^{-1}$.

We can consider a topology on $\Gamma \cup \Lambda_\Gamma$ by embedding Γ in \mathbb{H}^d through the orbit map $\gamma \mapsto \gamma \cdot x$ (for a generic $x \in \mathbb{H}^d$).

Theorem 2.1 (Bowditch)

A countable group Γ is a hyperbolic group if and only if it admits a metrisable compactification $\overline{\Gamma} = \Gamma \cup \partial_\infty \Gamma$ on which the left action of Γ extends continuously such that:

1. If $\gamma \in \Gamma$ has infinite order, then the limits

$$\gamma^+ = \lim_{k \rightarrow +\infty} \gamma^k \quad \text{and} \quad \gamma^- = \lim_{k \rightarrow -\infty} \gamma^k$$

exist, are distinct elements of $\partial_\infty \Gamma$ and are the only fixed points of γ on $\Gamma \sqcup \partial_\infty \Gamma$. Moreover, the set of pairs (γ^+, γ^-) for $\gamma \in \Gamma$ of infinite order is dense in $\partial_\infty \Gamma^{(2)}$.

2. If a sequence $\gamma_k \in \Gamma$ satisfies $\gamma_k \rightarrow \gamma_+ \in \partial_\infty \Gamma$ and $\gamma_k^{-1} \rightarrow \gamma_- \in \partial_\infty \Gamma$, then $\gamma_k \cdot x \rightarrow \gamma_+$ for all $x \in \overline{\Gamma} \setminus \{\gamma_-\}$, and the convergence is locally uniform.
3. For any pair of distinct points $\eta_+, \eta_- \in \partial_\infty \Gamma$, there is a diverging sequence $(\gamma_k)_{k \geq 0} \in \Gamma^{\mathbb{N}}$ such that $\eta_+ = \lim \gamma_k$ and $\eta_- = \lim \gamma_k^{-1}$.

2.3 Proximity

Notation

Let $g \in \text{GL}(V)$ for some vector space V of dimension $d \in \mathbb{N}$. We denote by

$$\lambda_1(g) \geq \dots \geq \lambda_d(g)$$

the logarithms of the moduli of the complex eigenvalues of g .

Example

If $g \in \text{SO}(d, 1)$, then $\lambda_2(g) = \dots = \lambda_d(g) = 0$, and $\lambda_{d+1}(g) = -\lambda_d(g)$. An element $g \in \text{SO}(d, 1)_o \simeq \text{Isom}(\mathbb{H}^d)_o$ is a hyperbolic isometry if and only if $\lambda_1(g) > 0$.

Proposition 2.2 (Definition of proximality)

For any $g \in \text{GL}(V)$, the following are equivalent:

- (1) $\lambda_1(g) > \lambda_2(g)$.
- (2) The action of g on $\mathbb{P}(V)$ has an attracting fixed point (i.e. a fixed point $\ell \in \mathbb{P}(V)$ such that $\lambda_1(dg|_\ell) < 0$).
- (3) There exists a line $\ell^+ \in \mathbb{P}(V)$ and a hyperplane $H^- \in \mathbb{P}(V^*)$ such that $\ell^+ \notin \mathbb{P}(H^-)$ and $\lim_{k \rightarrow +\infty} g^k = \ell^+$ for all $\ell \in \mathbb{P}(V) \setminus \mathbb{P}(H^-)$.

An element $g \in \text{GL}(V)$ satisfying these conditions is called *proximal*. If g is proximal, then:

- There is a unique pair $(\ell^+(g), H^-(g)) \in \mathbb{P}(V) \times \mathbb{P}(V^*)$ satisfying (3), and $\ell^+(g) \pitchfork H^-(g)$.
- The convergence in (3) is locally uniform.
- $\ell^+(g) \in \mathbb{P}(V)$ is the only attractive fixed point of g .

2.4 Projective Anosov representations

We state everything in terms of hyperbolic groups, but we can also think of a closed hyperbolic manifold $(N, g) \simeq \Gamma \backslash \mathbb{H}^d$, then $\partial_\infty \Gamma \simeq \partial_\infty \mathbb{H}^d$.

Definition (Transverse limit maps)

Let $\rho \in \text{Hom}(\Gamma, \text{SL}(V))$. A *pair of transverse projective limit maps* is a pair (ξ, ξ^*) of continuous Γ -equivariant maps $\xi : \partial_\infty \Gamma \rightarrow \mathbb{P}(V)$ and $\xi^* : \partial_\infty \Gamma \rightarrow \mathbb{P}(V^*)$ such that $\xi(\eta_+) \pitchfork \xi^*(\eta_-)$ for every distinct elements $\eta_+, \eta_- \in \partial_\infty \Gamma$.

There are many equivalent characterisations of (projective) Anosov representations that are stated in terms of the existence of a pair of transverse (projective) limit maps with additional dynamical properties.

Definition (Uniformly dynamics preserving limit maps)

Let $\rho \in \text{Hom}(\Gamma, \text{SL}(V))$. A pair of transverse projective limit maps (ξ, ξ^*) is called *uniformly dynamics preserving* if for any diverging sequence $\gamma_k \in \Gamma$ with boundary limit points $\gamma_+ = \lim_{k \rightarrow +\infty} \gamma_k \in \partial_\infty \Gamma$ and $\gamma_- = \lim_{k \rightarrow +\infty} \gamma_k^{-1} \in \partial_\infty \Gamma$, the actions on $\mathbb{P}(V)$ and $\mathbb{P}(V^*)$ obey the following dynamics:

- (1) $\forall \ell \in \mathbb{P}(V) \quad \ell \pitchfork \xi^*(\gamma_-) \implies \lim_{k \rightarrow +\infty} \rho(\gamma_k) \cdot \ell = \xi(\gamma_+),$
- (2) $\forall H \in \mathbb{P}(V^*) \quad \xi(\gamma_-) \pitchfork H \implies \lim_{k \rightarrow +\infty} \rho(\gamma_k) \cdot H = \xi^*(\gamma_+),$

and the convergences are locally uniform.

Theorem 2.3 (Kapovich-Leeb-Porti [KLP17])

A representation $\rho \in \text{Hom}(\Gamma, \text{SL}(V))$ is projective Anosov if and only if it admits a uniformly dynamics preserving pair of transverse limit maps.

Remark

A representation possessing a uniformly dynamics preserving pair of transverse limit maps is called an *asymptotic embedding* in [KLP17].

This definition makes sense in the more general setting of a Gromov hyperbolic group Γ , but it was originally introduced with the goal of studying surface groups.

Example (Hitchin representations)

For any $n \geq 2$, there is an irreducible representation $\rho_{\text{irr}} : \text{Isom}(\mathbb{H}^2)_\circ \simeq \text{PSL}_2(\mathbb{R}) \rightarrow \text{PSL}_n(\mathbb{R})$, unique up to conjugacy. A *Hitchin representation* of a uniform lattice $\Gamma < \text{Isom}(\mathbb{H}^2)$ is a representation $\rho \in \text{Hom}(\Gamma, \text{PSL}_n(\mathbb{R}))$ such that there exists a continuous family $(\rho_t)_{t \in [0,1]}$ with $\rho_1 = \rho$ and $\rho_0 = \rho_{\text{irr}}|_\Gamma$.

Labourie proved in [Lab06] that Hitchin representations are projective Anosov.

2.5 Convex cocompact representations

We are still in the context of a closed hyperbolic manifold $(N, g) \simeq \Gamma \backslash \mathbb{H}^d$, and we now study representations $\rho : \Gamma \rightarrow \text{Isom}(\mathbb{H}^n)_\circ$ for some $n \geq 2$. We can see them as representations in $\text{SO}(n, 1)_\circ < \text{SL}_{n+1}(\mathbb{R})$.

Proposition

A faithful representation $\rho : \Gamma \rightarrow \text{Isom}(\mathbb{H}^n)$ is projective Anosov if and only if $\rho(\Gamma)$ is convex cocompact. If it is the case, the limit maps $\xi : \partial_\infty \Gamma \rightarrow \mathbb{P}(V)$ and $\xi^* : \partial_\infty \Gamma \rightarrow \mathbb{P}(V^*)$ are related by

$$\forall \eta \in \partial_\infty \Gamma \quad \xi^*(\eta) = \langle \xi(\eta), \cdot \rangle.$$

Furthermore, ξ is a homeomorphism from $\partial_\infty \Gamma$ to $\Lambda_{\rho(\Gamma)}$.

Idea of proof. If ρ is projective Anosov, then Γ acts cocompactly on the convex hull of $\xi(\partial_\infty \Gamma)$.

If ρ is convex cocompact, fix some $x \in \mathbb{C}$. Cocompactness implies that $\gamma \mapsto \rho(\gamma) \cdot x$ is a quasi-isometry, so it extends to a continuous map $\xi : \partial_\infty \Gamma \rightarrow \partial_\infty \mathbb{H}^n$, which is the Anosov limit map. \square

A convex cocompact representation $\rho \in \text{Hom}(\Gamma, \text{Isom}(\mathbb{H}^n)_o)$ produces a hyperbolic manifold $\mathbf{N}_\rho = \rho(\Gamma) \backslash \mathbb{H}^n$, which cannot be compact if $n > d$ (note that we necessarily have $n \geq d$). The typical examples are *quasi-Fuchsian representations*, i.e. $d = 2$ and $n = 3$.

One can still study the geodesic flow $\varphi_\rho^t : \mathbf{M}_\rho \rightarrow \mathbf{M}_\rho$, where $\mathbf{M}_\rho = T^1 \mathbf{N}_\rho$, of such hyperbolic manifolds. They are no longer Anosov, but still have nice dynamical properties, allowing for counting results. Introduce the counting function

$$\pi_\rho(T) = \#\{[\gamma] \in [\Gamma]_{\text{prim}} \mid \lambda_1(\rho(\gamma)) \leq T\}.$$

Theorem 2.4 (Parry-Pollicott (83))

If $\rho \in \text{Hom}(\Gamma, \text{Isom}(\mathbb{H}^d)_o)$ is convex cocompact, there is some $h_\rho > 0$ such that

$$\pi_\rho(T) \sim \frac{e^{h_\rho T}}{h_\rho T}$$

as $T \rightarrow +\infty$.

Part 3. Mixing

3.1 Axiom A flows

Definition (Basic axiom A flow)

A smooth flow $\varphi^t : M \rightarrow M$ on a smooth manifold M is called a *basic axiom A flow* if the following properties are satisfied:

1. The *non wandering* set $NW(\varphi)$ is compact, where

$$NW(\varphi) = \left\{ x \in M \mid \exists x_k \rightarrow x \exists t_k \rightarrow \infty \varphi^{t_k}(x_k) \rightarrow x \right\}.$$

2. The non wandering set is equal to the closure of the set of periodic points:

$$NW(\varphi) = \overline{\text{Per}(\varphi)} \text{ where } \text{Per}(\varphi) = \left\{ x \in M \mid \exists T > 0 \varphi^T(x) = x \right\}.$$

3. The restriction $\varphi^t|_{NW(\varphi)} : NW(\varphi) \rightarrow NW(\varphi)$ is topologically mixing.

4. The restricted tangent bundle $TM|_{NW(\varphi)}$ admits a $d\varphi^t$ -invariant splitting $TM|_{NW(\varphi)} = E^s \oplus E^0 \oplus E^u$ where:

- (a) $\dim E^0 = 1$ and $E^0 = \mathbb{R} \cdot \frac{d}{dt} \Big|_{t=0} \varphi^t$.

- (b) E^s is contracted by $d\varphi^t$ and E^u is dilated by $d\varphi^t$.

Topologically mixing Anosov flows are basic axiom A flows (this is a consequence of Anosov's closing lemma). Axiom A flows were introduced by Smale in the context of structural stability, and studied in depth by Ruelle and Bowen.

Theorem 3.1 (Parry-Pollicott (83))

If $\varphi^t : M \rightarrow M$ is a basic axiom A flow, there is some $h_\varphi > 0$ such that

$$\pi_\varphi(T) \sim \frac{e^{h_\varphi T}}{h_\varphi T}$$

as $T \rightarrow +\infty$.

Here again, h_φ is the topological entropy (the fact that $h_\varphi > 0$ was proved by Bowen). Part of the proof follows the same strategy as Margulis: constructing appropriate invariant measures. There are two differences: the use of symbolic codings (through what is known as *Markov partitions*), and Ruelle's Zeta function

$$\zeta_\varphi(s) = \prod_c \left(1 - e^{-s\lambda(c)} \right)^{-1}$$

where the product runs over periodic orbits, and $\lambda(c)$ denotes the least period.

Proposition

A discrete subgroup $\Gamma < \text{Isom}(\mathbb{H}^d)_o$ is convex cocompact if and only if the geodesic flow on $\Gamma \backslash T^1\mathbb{H}^d$ is a basic axiom A flow.

The most important part to understand is the non wandering set.

Proposition

Let $\Gamma < \text{Isom}(\mathbb{H}^d)_o$ be a discrete subgroup. The non wandering set $\text{NW}(\varphi_\Gamma^t)$ of the geodesic flow on $\Gamma \backslash T^1\mathbb{H}^d$ is the projection of

$$\left\{ (x, v) \in T^1\mathbb{H}^d \mid \lim_{t \rightarrow \pm\infty} \varphi^t(x, v) \in \Lambda_\Gamma \right\}.$$

Let us state a deep result connecting the dynamics of the geodesic flow and the fractal geometry of the limit set.

Theorem 3.2 (Patterson, Sullivan)

If $\rho : \Gamma \rightarrow \text{Isom}(\mathbb{H}^n)$ is convex cocompact, then the topological entropy h_{φ_ρ} is equal to the Hausdorff dimension of the limit set $\Lambda_\rho \subset \partial_\infty \mathbb{H}^n$.

3.2 A flow for projective Anosov representations

Given a projective Anosov representation $\rho \in \text{Hom}(\Gamma, \text{SL}(V))$, we can still define the counting function

$$\pi_\rho(T) = \#\{[\gamma] \in [\Gamma]_{\text{prim}} \mid \lambda_1(\rho(\gamma)) \leq T\}.$$

In order to estimate the growth of this counting function, we wish to introduce a flow with nice properties. This was done by A. Sambarino.

Theorem 3.3 (Sambarino (16))

If $\rho \in \text{Hom}(\Gamma, \text{SL}(V))$ is projective Anosov, there is some $h_\rho > 0$ such that

$$\pi_\rho(T) \sim \frac{e^{h_\rho T}}{h_\rho T}$$

as $T \rightarrow +\infty$.

We will now present a construction of the flow introduced in [DMS25], different from Sambarino's approach. As a replacement for the geodesic flow of \mathbb{H}^n , we consider the $\text{SL}(V)$ -homogeneous space

$$\mathbb{L} = \{[v : \alpha] \in \mathbb{P}(V \oplus V^*) \mid \alpha(v) > 0\}$$

equipped with the homogeneous flow

$$\varphi_{\mathbb{L}}^t([v : \alpha]) = [e^t v : e^{-t} \alpha].$$

Note that the action of $SL(V)$ on $\mathbb{P}(V \oplus V^*)$ is given by $g \cdot [v : \alpha] = [g \cdot v : \alpha \circ g^{-1}]$.

For a projective Anosov representation $\rho \in \text{Hom}(\Gamma, SL(V))$, we denote by (ξ, ξ^*) its limit maps.

Definition

For a projective Anosov representation $\rho \in \text{Hom}(\Gamma, SL(V))$, we introduce the following subsets:

$$\widehat{\mathbf{M}}_\rho \stackrel{\text{def}}{=} \{[v : \alpha] \in \mathbb{L} \mid \forall \eta \in \partial_\infty \Gamma, [v] \pitchfork \xi^*(\eta) \text{ or } \xi(\eta) \pitchfork [\alpha]\},$$

$$\widehat{\mathbf{K}}_\rho \stackrel{\text{def}}{=} \{[v : \alpha] \in \mathbb{L} \mid \exists \eta_+, \eta_- \in \partial_\infty \mathbb{H}^d, \eta_+ \neq \eta_-, [v] = \xi(\eta_+) \text{ and } [\alpha] = \xi^*(\eta_-)\}.$$

Theorem 3.4 ([DMS25, Theorem A.(1)])

Let $\rho \in \text{Hom}(\Gamma, SL(V))$ be a projective Anosov representation.

1. The subset $\widehat{\mathbf{M}}_\rho \subset \mathbb{L}$ is open, non empty, invariant under $\rho(\Gamma)$ and the flow $\varphi_\mathbb{L}^t$.
2. The action of Γ on $\widehat{\mathbf{M}}_\rho$ is properly discontinuous.
3. If Γ is torsion-free, then the action of Γ on $\widehat{\mathbf{M}}_\rho$ is free.
4. The flow φ_ρ^t induced by $\varphi_\mathbb{L}^t$ on the quotient manifold $\mathbf{M}_\rho = \rho(\Gamma) \backslash \widehat{\mathbf{M}}_\rho$ is a basic axiom A flow with non wandering set $\mathbf{K}_\rho = \rho(\Gamma) \backslash \widehat{\mathbf{K}}_\rho$.

Items 1 and 3 are quite straightforward and are left as exercises (or see [Mon25]).

The proof of item 2 relies on the study the dynamics of a projective Anosov representation $\rho : \Gamma \rightarrow SL(V)$ acting on V . With this in mind, it is important to notice that contraction for the action on the projective space translates to expansion for the linear action, as a contracting fixed point in $\mathbb{P}(V)$ is the eigendirection for the largest eigenvalue.

Lemma ([DMS25, Lemma 3.2])

Let $\rho \in \text{Hom}(\Gamma, SL(V))$ be a projective Anosov representation, and let $\gamma_k \in \Gamma$ be a sequence admitting limits $\gamma_+ = \lim \gamma_k \in \partial_\infty \Gamma$ and $\gamma_- = \lim \gamma_k^{-1} \in \partial_\infty \Gamma$. For any sequence $v_k \rightarrow v \in V \setminus \{0\}$ such that $[v] \pitchfork \xi^*(\gamma_-)$, one has $\rho(\gamma_k)v_k \rightarrow \infty$ as $k \rightarrow \infty$.

In order to understand the axiom A property, we can look at periodic orbits. A periodic point $p \in \mathbf{M}_\rho$ of φ_ρ^t lifts to $[v : \alpha] \in \widehat{\mathbf{M}}_\rho$ such that $[e^T v : e^{-T} \alpha] = \rho(\gamma) \cdot [v : \alpha]$. This means that v (resp. α) is an eigenvector (resp. eigenform) for $\rho(\gamma)$ with eigenvalue e^T (resp. e^{-T}).

The first step is to understand that the condition $[v : \alpha] \in \widehat{\mathbf{M}}_\rho$ implies that $T = \lambda_1(\rho(\gamma))$, so there is a correspondence between periodic orbits of the flow and conjugacy classes in Γ . Furthermore, $[v] = \xi(\gamma^+)$ and $[\alpha] = \xi^*(\gamma^-)$, so $p \in \mathbf{K}_\rho$. To understand the derivative of the flow $d\varphi_\rho^T|_p$, we see that it is conjugate to $\rho(\gamma)^{-1} d\varphi_\mathbb{L}^T|_{[v:\alpha]}$.

The description of the tangent space

$$T_{[v:\alpha]} \mathbb{L} \simeq \{(w, \beta) \in V \oplus V^* \mid \alpha(w) + \beta(v) = 0\}$$

shows that there is a natural splitting $T\mathbb{L} = E^u \oplus E^0 \oplus E^s$ where

$$\begin{aligned} E_{[v:\alpha]}^u &= \ker \alpha \times \{0\}, \\ E_{[v:\alpha]}^s &= \{0\} \times \ker \iota_v, \quad \forall [v:\alpha] \in \mathbb{L}. \\ E_{[v:\alpha]}^0 &= \mathbb{R} \cdot (v, -\alpha), \end{aligned}$$

The differential of $\varphi_{\mathbb{L}}^t$ restricted to these subbundles is just multiplication by e^t (resp. e^{-t}) on E^u (resp. E^s). These distributions descend to \mathbf{M}_ρ , and find that:

$$\lambda_1 \left(d\varphi_\rho^T|_{E^u|_p} \right) = \lambda_1(\rho(\gamma)) - \lambda_2(\rho(\gamma)).$$

So the contraction/dilation of φ_ρ^t is related to the uniformity of proximality of $\rho(\gamma)$ for $\gamma \in \Gamma$, which is exactly the Anosov condition.

3.3 Smooth ergodic theory

Axiom A flows share many properties of geodesic flows of convex cocompact hyperbolic manifolds. For example, the axiom A property (added to some "non triviality") implies that the flow φ_ρ^t has positive entropy, and that periodic orbits equidistribute.

Ergodic theory studies the statistical distribution of orbits. For a flow $\varphi^t : \mathbf{M} \rightarrow \mathbf{M}$, we consider a φ^t -invariant probability measure μ on \mathbf{M} , and ask if this measure is:

- Ergodic, i.e. there any φ^t -invariant subset has zero or full measure.
- Mixing, i.e. the correlations $c^t(F, G) = \int F(G \circ \varphi^t) d\mu - \left(\int F d\mu \right) \left(\int G d\mu \right)$ go to 0 as $t \rightarrow \infty$ for any $F, G \in L^2(\mu)$.

3.3.1 Equidistribution of periodic orbits

For a periodic orbit $c \subset \mathbf{K}_\rho$, let $\ell(c)$ denote its period, and λ_c the Lebesgue measure of length 1 supported on c . For $T > 0$, consider the measure

$$\mu_T = \frac{1}{\pi_\varphi(T)} \sum_{\ell(c) \leq T} \lambda_c.$$

Theorem 3.5 (Bowen)

As $T \rightarrow +\infty$, the family μ_T weak- * converges to a probability measure μ with the following properties:

1. $\text{Supp}(\mu) = \mathbf{K}_\rho$, and μ gives positive measure to any non empty open subset of \mathbf{K}_ρ .
2. μ is φ_ρ^t -invariant and mixing.

This measure is known as the Bowen-Margulis, Bowen-Margulis-Sullivan measure. The construction was extended to general Gibbs measures (by considering weighted measures along periodic orbits) by Parry-Pollicott [PP90].

3.3.2 Ruelle-Pollicott resonances

Denote by $X_\rho : M_\rho \rightarrow TM_\rho$ the generator of the flow φ_ρ^t , i.e. $X_\rho(x) = \frac{d}{dt}\big|_{t=0} \varphi_\rho^t(x)$. One way of interpreting a vector field is as a derivation, i.e. a linear differential operator of order one

$$X_\rho : \begin{cases} \mathcal{C}^\infty(M_\rho) & \rightarrow \mathcal{C}^\infty(M_\rho) \\ f & \mapsto df(X_\rho) \end{cases} .$$

It generates a one-parameter group $e^{tX_\rho} \cdot f = f \circ \varphi_\rho^t$, called the *transfer operator*. If one wants to define a spectrum for X_ρ , the space $\mathcal{C}^\infty(M_\rho)$ is a very bad choice. A solution consists in using complex analysis. For $s \in \mathbb{C}$ with $\Re(s) > 0$, consider

$$R_s^X \cdot f(x) \stackrel{\text{def}}{=} \int_0^{+\infty} e^{-st} f(\varphi_\rho^t(x)) dt.$$

This can be interpreted as $(X_\rho + s)^{-1}$.

Theorem 3.6 (Dyatlov-Guillarmou)

The family R_s^X extends meromorphically to $R_s^X : \mathcal{C}^\infty(M_\rho) \rightarrow \mathcal{D}'(M_\rho)$ for $s \in \mathbb{C}$.

This extension was previously proved for Anosov flows by Butterley-Liverani, Faure-Sjöstrand, Dyatlov-Zworski.

Definition 3.7

A pole of this extension is called a *Ruelle-Pollicott resonance*.

3.3.3 Zeta function

The bridge between orbital counting and mixing rates is the Zeta function

$$\zeta_\rho(s) = \prod_{[\gamma] \in [\Gamma]_{\text{prim}}} \left(1 - e^{-s\lambda_1(\rho(\gamma))}\right)^{-1}.$$

Here $[\Gamma]_{\text{prim}}$ denotes the set of conjugacy classes of primitive elements in Γ (i.e. elements that are not positive powers of other elements). This product defines a holomorphic function on the half plane $\Re(s) > h_\rho$.

Theorem 3.8 ([DMS25, Theorem D])

For any projective Anosov representation, the function ζ_ρ admits a meromorphic extension to \mathbb{C} . If ρ is irreducible, then there is some $\varepsilon > 0$ such that ζ_ρ has no zero or pole in the vertical strip $h_\rho - \varepsilon < \Re(s) \leq h_\rho$ with the exception of a simple pole at $s = h_\rho$.

The meromorphic extension to \mathbb{C} is due to Dyatlov-Guillarmou [DG16, DG18] for all Axiom A flows, solving a conjecture of Smale. Since our flow is real analytic and has real analytic stable and unstable distributions, the result also follows from earlier work of Fried [Fri95], building upon ideas of Rugh [Rug92, Rug96].

The zero and pole free vertical strip follows from the spectral estimates on Ruelle transfer operators achieved by Stoyanov [Sto11], combined with the work Pollicott-Sharp [PS98] (see also Dolgopyat-Pollicott [DP98]).

3.3.4 Exponential mixing

General Axiom A flows can have arbitrarily slow mixing rates (i.e. the rate of decay of the correlation functions $c^t(F, G)$), but large classes (such as geodesic flows of convex cocompact groups in rank one) mix exponentially fast.

Definition

The flow φ_ρ^t is *exponentially mixing* with respect to μ for all Hölder observables if there exists $c_\alpha, C_\alpha > 0$ such that

$$\forall t \in \mathbb{R} \quad |c^t(F, G)| \leq C_\alpha e^{-c_\alpha |t|} \|F\|_\alpha \|G\|_\alpha.$$

Theorem 3.9 ([DMS25, Theorem C])

If ρ is irreducible, the flow φ_ρ^t is exponentially mixing with respect to μ for all Hölder observables.

The proof relies on a result of Stoyanov establishing exponential mixing for Axiom A flows under geometric assumptions on the stable and unstable laminations. As most results on exponential mixing of hyperbolic dynamical systems, Stoyanov's work is built upon the *Dolgopyat method*, which is itself built on estimates on the *time separation function* called *non integrability conditions*.

For Zariski dense representations and the maximal entropy measure, exponential mixing for Hölder observables was also established by Chow and Sarkar [CS24].

3.3.5 Counting problems

Given a projective Anosov representation $\rho \in \text{Hom}(\Gamma, \text{SL}(V))$, define the orbit counting function

$$\pi_\rho(t) = \text{Card}\left\{[\gamma] \in [\Gamma]_{\text{prim}} \mid \lambda_1(\rho(\gamma)) \leq t\right\}.$$

Theorem 3.10 ([DMS25, Theorem E])

Suppose $\rho : \Gamma \rightarrow \text{SL}(V)$ is an irreducible projective Anosov representation, and denote by $h_\rho > 0$ the topological entropy of the flow φ_ρ^t on \mathbf{K}_ρ . Then there exists $0 < c < h_\rho$ such that

$$\pi_\rho(t) = \text{Li}(e^{h_\rho t}) \left(1 + O(e^{-ct})\right).$$

The logarithmic integral function Li is defined by

$$\text{Li}(x) = \int_2^x \frac{du}{\text{Log} u} \sim \frac{x}{\text{Log} x}.$$

The leading term $N_\rho(t) \sim \frac{e^{h_\rho t}}{h_\rho t}$ was obtained by Sambarino [Sam14].

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