Groups and geometry

Final exam

Exercise 1

A few results on left-invariant Riemannian metrics

1. Prove that a left-invariant Riemannian metric on a Lie group is complete.

   Solution: Let $T > 0$ be such that any geodesic starting at e is defined up to time T. If $c : I \to G$ is a maximal geodesic, for any $t_0 \in I$ then $L_{c(t_0)^{-1}} \circ c$ is a geodesic going through e, hence $]t_0 - T, t_0 + T[\subset I$, and $I = \mathbb{R}$.

2. Prove that if ∇ is the Levi-Civita connection of a left-invariant Riemannian metric on a Lie group G, and $X, Y \in {}^G\mathcal{X}(G)$, then $\nabla_X Y \in {}^G\mathcal{X}(G)$.

   Solution: If $X, Y, Z \in \mathcal{X}(G)$ are left-invariant, then the products $\langle X, Y\rangle$, $\langle Y, Z\rangle$ and $\langle Z, X\rangle$ are constant so the Koszul formula is simplified:

   $$2\langle \nabla_X Y, Z\rangle = \langle [X, Y], Z\rangle - \langle [X, Z], Y\rangle - \langle [Y, Z], X\rangle$$

   This shows that $\langle \nabla_X Y, Z\rangle$ is constant.
   Now let $V \in \mathcal{X}(G)$ be such that $\langle V, Z\rangle$ is constant for any $Z \in {}^G\mathcal{X}(G)$. We have $\left\langle L_g^* V, Z\right\rangle = \langle V, Z\rangle \circ L_g = \langle V, Z\rangle$, so we find that $L_g^* V = V$. Applying this to $V = \nabla_X Y$ answers the question.

3. Let G and H be connected Lie groups, and consider a Lie group morphism $f : G \to H$. Let us assume that $d_e f$ is invertible. Given a left-invariant Riemannian metric $\langle\cdot,\cdot\rangle$ on H, prove that the pull-back $f^*\langle\cdot,\cdot\rangle$ is a left-invariant Riemannian metric on G. Infer that f is a covering map.

   Solution: Since $d - ef$ is invertible, the map f is an immersion and $f^*\langle\cdot,\cdot\rangle$ is a Riemannian metric on G. For $g \in G$, we find:

   $$\begin{aligned} L_g^* f^*\langle\cdot,\cdot\rangle &= (f \circ L_g)^*\langle\cdot,\cdot\rangle \\ &= (L_{f(g)} \circ f)^*\langle\cdot,\cdot\rangle \\ &= f^* L_{f(g)}^*\langle\cdot,\cdot\rangle \\ &= f^*\langle\cdot,\cdot\rangle \end{aligned}$$

   It follows from the first question that $(G, f^*\langle\cdot,\cdot\rangle)$ is complete, since this metric makes f a local isometry it is a covering map.

4. Let G be a connected Lie group whose Lie algebra $\mathfrak{g}$ is abelian. Prove that G is abelian and that $\exp_G$ is surjective.

   Solution: If $\mathfrak{g}$ is abelian, then $\exp_G$ is a Lie group morphism from $(\mathfrak{g}, +)$ to G, and $d_0 \exp_G = \mathrm{Id}_{\mathfrak{g}}$ is invertible, so the previous question shows that $\exp_G$ is a covering map, hence onto. Because $\exp_G(X)\exp_G(Y) = \exp_G(Y)\exp_G(X)$, it also follows that G is abelian.

Exercise 2

Construction of a Lie algebra

Let V be a finite dimensional real vector space, and consider a linear form $\ell : V \to \mathbb{R}$.

1. Prove that the map $[\cdot,\cdot] : \left\{ \begin{array}{ccc} V \times V & \to & V \\ (x,y) & \mapsto & \ell(x)y - \ell(y)x \end{array} \right.$ is a Lie bracket.

   Solution: It is bilinear because ℓ is linear. Skew-symmetry is straightforward. Note that $\ell([x,y]) = 0$, which will make the the Jacobi identity easier to establish:

$$\begin{aligned}
[[x,y],z] + [[y,z],x] + [[z,x],y] &= -\ell(z)[x,y] - \ell(x)[y,z] - \ell(y)[z,x] \\
&= -\ell(z)\ell(x)y + \ell(z)\ell(y)x - \ell(x)\ell(y)z + \ell(x)\ell(z)y - \ell(y)\ell(z)x + \ell(y)\ell(x)z \\
&= 0
\end{aligned}$$

We now denote by $\mathfrak{g}$ the Lie algebra that we obtain.

2. Is the Lie algebra $\mathfrak{g}$ solvable? Nilpotent? Semi-simple?

   Solution: If $\ell = 0$ then $\mathfrak{g}$ is abelian, so we now assume that $\ell \neq 0$. As discussed, we have $[\mathfrak{g},\mathfrak{g}] \subset \ker \ell$. If $\dim V = 1$, this shows that $\mathfrak{g}$ is abelian, and we now assume $\dim V \geq 2$.

   Let $z \in \mathfrak{g}$ be such that $\ell(z) = 1$. If $x \in \ker \ell$ then $[z,x] = x$, so $[\mathfrak{g},\mathfrak{g}] = \ker \ell$.
   We also have $[z,[x,y]] = [x,y]$ for all $x,y \in \mathfrak{g}$, and this shows that $C_2(\mathfrak{g}) = C_1(\mathfrak{g})$, so $\mathfrak{g}$ is not nilpotent. However $[x,y] = 0$ whenever $x,y \in \ker \ell = [\mathfrak{g},\mathfrak{g}]$, so $D_2(\mathfrak{g}) = \{0\}$ and $\mathfrak{g}$ is solvable (hence not semi-simple).

The Lie subalgebras $\mathfrak{n}$ and $\mathfrak{a}$

Consider an inner product $\langle \cdot,\cdot \rangle$ on $\mathfrak{g}$. Let $\mathfrak{n} = \ker \ell \subset \mathfrak{g}$ and $\mathfrak{a} \subset \mathfrak{g}$ its orthogonal. Consider the vector $X_\ell \in \mathfrak{a}$ such that $\ell(X) = \langle X_\ell, X \rangle_e$ for all $X \in \mathfrak{g}$.

3. Prove that $\mathfrak{n}$ is an abelian Lie subalgebra of $\mathfrak{g}$. Is it an ideal?

   Solution: We have already seen that $\mathfrak{n}$ is an abelian subalgebra in the previous answer. If $x \in \mathfrak{g}$ and $y \in \mathfrak{n}$, then $\ell([x,y]) = \ell(x)\ell(y) = 0$ so $\mathfrak{n}$ is an ideal of $\mathfrak{g}$.

4. Prove that $\mathfrak{a}$ is an abelian Lie subalgebra of $\mathfrak{g}$. Is it an ideal?

   Solution: If $\ell = 0$, then $\mathfrak{a} = \{0\}$ is an abelian ideal. If $\ell \neq 0$, then $\mathfrak{a}$ is a vector subspace of dimension 1 of $\mathfrak{g}$, therefore an abelian ideal. If $X \in \mathfrak{n} \setminus \{0\}$, then $[X_\ell, X] = \ell(X_\ell)X = \|X_\ell\|^2 X \notin \mathfrak{a}$, so $\mathfrak{a}$ is not an ideal unless $\dim V = 1$.

5. Given $X \in \mathfrak{n}$, describe the matrix of $\mathrm{ad}(X)$ in a basis adapted to the decomposition $\mathfrak{g} = \mathfrak{n} \oplus \mathfrak{a}$. Do the same for $\mathrm{ad}(X_\ell)$.

Solution: Consider a basis $(X_1,\dots,X_d)$ of $\mathfrak{n}$. For $X\in\mathfrak{n}$, we have $\mathrm{ad}(X)X_i = 0$ for $1\le i\le d$. We also have $\mathrm{ad}(X)X_\ell = -\|\ell\|^2 X$, so the matrix of $\mathrm{ad}(X)$ in the basis $(X_1,\dots,X_d,X_\ell)$ is

$$\mathrm{ad}(X) = \begin{pmatrix} 0 & -\|\ell\|^2 X \\ 0 & 0 \end{pmatrix}$$

Here we identified X with the column of its coordinates in $(X_1,\dots,X_d)$.
Since $[X_\ell, X_i] = \|\ell\|^2 X_i$, we find:

$$\mathrm{ad}(X_\ell) = \begin{pmatrix} \|\ell\|^2 1_d & 0 \\ 0 & 0 \end{pmatrix}$$

The Lie group G

Let G be a connected Lie group whose Lie algebra is $\mathfrak{g}$. Consider the connected immersed Lie subgroup N (resp. A) whose Lie algebra is $\mathfrak{n}$ (resp. $\mathfrak{a}$).

6. Prove that for $X\in\mathfrak{n}$ or $X\in\mathfrak{a}$, if $\exp_G(X) = e$, then $X = 0$.
   Hint: describe the matrix of $\mathrm{Ad}(\exp_G(X))$ *in a basis adapted to* $\mathfrak{g} = \mathfrak{n}\oplus\mathfrak{a}$.

   Solution: Since $\mathrm{Ad}(\exp_G(X)) = \exp(\mathrm{ad}(X))$, in the basis previously described we find for $X\in\mathfrak{n}$

   $$\mathrm{Ad}(\exp_G(X)) = \begin{pmatrix} 1_d & -\|\ell\|^2 X \\ 0 & 1 \end{pmatrix}$$

   If $\exp_G(X) = e$, then $\mathrm{Ad}(\exp_G(X)) = \mathrm{Id}$ and we get $X = 0$.

   Similarly, for $X = \lambda X_\ell \in \mathfrak{a}$, we find

   $$\mathrm{Ad}(\exp_G(X)) = \begin{pmatrix} e^{\lambda\|\ell\|^2} 1_d & 0 \\ 0 & 1 \end{pmatrix}$$

   If $\exp_G(X) = e$, then $e^{\lambda\|\ell\|^2} = 1$ and $X = 0$.

7. Prove that $\exp_G$ is a diffeomorphism from $\mathfrak{n}$ to N, and also from $\mathfrak{a}$ to A.

   Solution: This result is false if $\ell = 0$, so we now assume for the rest of the exercise that $\ell\neq 0$.
   The result of Question 4. of Exercise 1 shows that N is a abelian and that $\exp_N$ is onto. The previous question shows that $\exp_N = \exp_G|_{\mathfrak{n}}$ is injective (because it is a Lie group morphism), so it is a Lie group isomorphism (the injectivity of $\exp_G|_{\mathfrak{n}}$ is the part that uses $\ell\neq 0$).
   This only shows that $\exp_N$ is a diffeomorphism when N is considered with its intrinsic manifold structure, it is not clear that it is embedded. It is however the case because $\exp_G|_{\mathfrak{n}}$ is proper: if $X_k\to\infty$ in N, then $\mathrm{Ad}(\exp_G(X_k))\to\infty$ in $\mathrm{GL}(\mathfrak{g})$ thanks to the previous computation, therefore $\exp_G(X_n)\to\infty$ in G. This shows that N is an embedded Lie subgroup, and $\exp_G$ is a diffeomorphism from $\mathfrak{n}$ to N.
   The same reasoning applies to A because the formula for $\mathrm{Ad}(\exp_G(X_\ell))$ also implies the properness of $\exp_G|_{\mathfrak{a}}$.

8. Prove that the restriction to A of the projection $G \to G/N$ is a covering map.

Solution: We have seen in the previous answer that N is an embedded Lie subgroup. It is normal in G because it is connected and its Lie algebra $\mathfrak{n}$ is an ideal of $\mathfrak{g}$. So G/N is a Lie group and the projection $G \to G/N$ is a Lie group morphism, and so is its restriction to A. The differential at e is the restriction to $\mathfrak{a}$ of the projection $\mathfrak{g} \to \mathfrak{g}/\mathfrak{n}$, hence an isomorphism. The result of Question 3. of Exercise 1 states that it is a covering map.

9. Given $g \in G$, prove that there is a unique pair $(n,a) \in N \times A$ such that $g = na$.

Solution: Let $\pi : G \to G/N$ be the projection. First, let us show that its restriction to A is an isomorphism, which amounts to proving that $A \cap N = \{e\}$. If $g \in A \cap N$, there is $X \in \mathfrak{n}$ such that $g = \exp_G(X)$, and following the answer to question 6. we find

$$\mathrm{Ad}(g) = \begin{pmatrix} 1_d & -\|\ell\|^2 X \\ 0 & 1 \end{pmatrix}$$

There is also $\lambda \in \mathbb{R}$ such that $g = \exp_G(\lambda X_\ell)$, and the answer to question 6. know yields

$$\mathrm{Ad}(g) = \begin{pmatrix} e^{\lambda\|\ell\|^2} 1_d & 0 \\ 0 & 1 \end{pmatrix}$$

The equality between these matrices implies that $X = 0$, hence $g = e$.

Now let $g \in G$. There is a unique $b \in A$ such that $\pi(g^{-1}) = \pi(b)$, i.e. $g = na$ where $a = b^{-1} \in A$ and $n = gb \in N$.

10. Prove that the map $G \to N \times A$ thus defined is a diffeomorphism.

Solution: Using the inverse of the isomorphism $\pi|_A : A \to G/N$, we see that $g \mapsto (n,a)$ is smooth. It is inverse is the multiplication map of G, so it is also smooth, hence a diffeomorphism.

11. Prove that G is simply connected.

Solution: We have seen that $\exp_G$ is a diffeomorphism from $\mathfrak{n}$ to N (resp. from $\mathfrak{a}$ to A) so N (resp. A) is simply connected. Thanks to the previous question, G is diffeomorphic to $N \times A$, so it is also simply connected.

12. Can the Lie group G possess a bi-invariant Riemannian metric?

Solution: No. If such a metric $\langle \cdot, \cdot \rangle$ were to exist, we would then have:

$$\forall g \in G \; \forall X, Y \in \mathfrak{g} \quad \langle \mathrm{Ad}(g)X, Y \rangle_e + \langle X, \mathrm{Ad}(g)Y \rangle_e = 0$$

Applying this with $g = \exp_G(X_\ell)$ and $X = Y = X_\ell$, we find that $\mathrm{Ad}(g)X = X$ because of question 6., so $\langle X, X \rangle = 0$, which is a contradiction since $X_\ell \neq 0$.

Curvature of G

Let $\langle\cdot,\cdot\rangle$ be a left-invariant Riemannian metric on G.

13. Given $X, Y \in {}^G\mathcal{X}(G)$ such that $X(e) \in \mathfrak{n}$ and $Y(e) \in \mathfrak{n}$, prove that $\nabla_X Y(e) = \langle X, Y\rangle X_\ell$.

    Solution: As seen in Question 2. of Exercise 1., for $Z \in {}^G\mathcal{X}(G)$ the Koszul formula becomes

    $$2\langle \nabla_X Y, Z\rangle = \langle [X,Y], Z\rangle + \langle [Z,X], Y\rangle + \langle [Z,Y], X\rangle$$

    Evaluating these constant functions at e, we find

    $$\begin{aligned} 2\langle \nabla_X Y, Z\rangle &= 0 + \ell(Z(e))\langle X, Y\rangle + \ell(Z(e))\langle Y, X\rangle \\ &= 2\langle X, Y\rangle\langle X_\ell, Z(e)\rangle \end{aligned}$$

    This shows that $\nabla_X Y(e) = \langle X, Y\rangle X_\ell$.

14. Given $Y \in {}^G\mathcal{X}(G)$, prove that $\nabla_e Y(X_\ell) = 0$.

    Solution: Let $Z \in \mathfrak{g}$.

    $$\begin{aligned} 2\langle \nabla_e Y(X_\ell), Z\rangle &= \langle [X_\ell, Y(e)], Z\rangle_e + \langle [Z, X_\ell], Y(e)\rangle_e + \langle [Z, Y(e)], X_\ell\rangle_e \\ &= \ell(X_\ell)\langle Y(e), Z\rangle_e - \ell(Y(e))\langle X_\ell, Z\rangle_e + \ell(Z)\langle X_\ell, Y(e)\rangle_e - \ell(X_\ell)\langle Z, Y(e)\rangle_e \\ &\qquad + \ell([Z, Y(e)]) \\ &= \ell(X_\ell)\langle Y(e), Z\rangle_e - \ell(Y(e))\ell(Z) + \ell(Z)\ell(Y(e)) - \ell(X_\ell)\langle Z, Y(e)\rangle_e \\ &\qquad + 0 \\ &= 0 \end{aligned}$$

15. Calculate $\nabla_X Y(e)$ when $X, Y \in {}^G\mathcal{X}(G)$ satisfy $X(e) \in \mathfrak{n}$ and $Y(e) \in \mathfrak{a}$.

    Solution: Let $Z \in \mathfrak{g}$. We also identify X, Y with $X(e), Y(e) \in \mathfrak{g}$.

    $$2\langle \nabla_X Y, Z\rangle = \langle [X,Y], Z\rangle + \langle [Z,X], Y\rangle + \langle [Z,Y], X\rangle$$

    Let us treat the three terms separately.

    - $[X, Y] = -\ell(Y)X$ so $\langle [X,Y], Z\rangle = -\ell(Y)\langle X, Z\rangle$.
    - Since $[Z, X] \in \mathfrak{n}$ and $Y \in \mathfrak{a} = \mathfrak{n}^\perp$, we have $\langle [Z,X], Y\rangle = 0$.
    - $[Z, Y] = \ell(Z)Y - \ell(Y)Z$ so $\langle [Z,Y], X\rangle = -\ell(Y)\langle Z, X\rangle = 0$ because $\langle X, Y\rangle = 0$.

    It follows that

    $$2\langle \nabla_X Y, Z\rangle = 2\langle -\ell(Y)X, Z\rangle$$

    So $\nabla_X Y = -\ell(Y)X$.

16. For which $X \in \mathfrak{g}$ is the curve $t \mapsto \exp_G(tX)$ a geodesic?

Solution: First, note that this is equivalent to $\nabla_X X = 0$. Indeed, if $\nabla_X X = 0$, then all integral curves of the left-invariant vector field X are geodesics, and $t \mapsto \exp_G(tX)$ is one of them. Reciprocally, if $t \mapsto \exp_G(tX)$ is a geodesic, then the geodesic equation at $t = 0$ yields $\nabla_X X(e) = 0$, but $\nabla_X X$ is left-invariant so $\nabla_X X = 0$.
If $X \in \mathfrak{a}$, then question 14. shows that $\nabla_X X = 0$, so $t \mapsto \exp_G(tX)$ is a geodesic.
In general, write $X = X_\mathfrak{n} + \lambda X_\ell$ where $X_\mathfrak{n} \in \mathfrak{n}$ and $\lambda \in \mathbb{R}$. The results of the previous questions lead to

$$\nabla_X X = \|X_\mathfrak{n}\|^2 X_\ell - \frac{\lambda}{2}\|X_\ell\|^2 X_\mathfrak{n}$$

This shows that $\nabla_X X = 0 \iff X \in \mathfrak{a}$.

17. Give a simple expression of $R(X,Y)Z$ for $X, Y, Z \in {}^G\mathcal{X}(G)$.

Solution: If $X, Y \in \mathfrak{a}$, then $R(X,Y) = 0$ because of skew-symmetry and $\dim \mathfrak{a} = 1$.

Now assume that $X, Y \in \mathfrak{n}$. If $Z \in \mathfrak{n}$, then according to Question 13. we find $\nabla_Y Z = \langle Y, Z\rangle X_\ell$, and following Question 15. we find $\nabla_X \nabla_Y Z = -\ell(X_\ell)\langle Y, Z\rangle X = -\|\ell\|^2 \langle Y, Z\rangle X$. Since $[X,Y] = 0$, we get the following expression:

$$R(X,Y)Z = \|\ell\|^2 \Big(\langle X, Z\rangle Y - \langle Y, Z\rangle X\Big) \tag{1}$$

If $Z \in \mathfrak{a}$, then

$$\begin{aligned}\nabla_X \nabla_Y Z &= \nabla_X \Big(-\ell(Z)Y\Big)\\ &= -\ell(Z)\langle X, Y\rangle X_\ell\end{aligned}$$

This leads to

$$\begin{aligned}R(X,Y)Z &= \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z\\ &= -\ell(Z)\langle X, Y\rangle + \ell(Z)\langle Y, X\rangle - 0\\ &= 0\end{aligned}$$

Note that (1) is still valid for $Z \in \mathfrak{a}$, because $\langle X, Z\rangle = \langle Y, Z\rangle = 0$. This means that (1) is valid for $X, Y \in \mathfrak{a}$ and any $Z \in \mathfrak{g}$.
It is also valid for $X, Y \in \mathfrak{n}$ and any $Z \in \mathfrak{g}$. Indeed, the right-hand side of (1) is skew-symmetric in X, Y and $\dim \mathfrak{a} = 1$, so it must vanish.

Now assume that $X \in \mathfrak{n}$ and $Y \in \mathfrak{a}$. Because of Question 14. we know that $\nabla_Y Z = 0$, hence $\nabla_X \nabla_Y Z = 0$ and also $\nabla_Y \nabla_X Z = 0$. This simplifies the expression of the curvature tensor:

$$R(X,Y)Z = -\nabla_{[X,Y]} Z = \ell(Y)\nabla_X Z$$

If $Z \in \mathfrak{n}$, then $\nabla_X Z = \langle X, Z\rangle X_\ell$, but $\ell(Y) X_\ell = \langle X_\ell, Y\rangle X_\ell = \|\ell\|^2 Y$ (since $Y \in \mathfrak{a} = \mathbb{R}X_\ell$), so we find

$$R(X,Y)Z = \|\ell\|^2 \langle X, Z\rangle Y$$

Since $\langle Y, Z\rangle = 0$, we find that (1) also holds in this case.
If $Z \in \mathfrak{a}$, then $\nabla_X Z = -\ell(Z)X$ and in this case $\ell(Y)\ell(Z) = \|\ell\|^2 \langle Y, Z\rangle$, so $R(X,Y)Z = -\|\ell\|^2 \langle Y, Z\rangle X$ and once again (1) is valid because $\langle X, Z\rangle = 0$.

In conclusion, we have shown that (1) is valid for any $Z \in \mathfrak{g}$ in the three following cases: $(X,Y) \in \mathfrak{n}\times\mathfrak{n}$, $(X,Y) \in \mathfrak{a}\times\mathfrak{a}$ and $(X,Y) \in \mathfrak{n}\times\mathfrak{a}$. Since both sides of the equality are multi-linear and skew-symmetric in X, Y, we find that it always holds, i.e.

$$\forall X,Y,Z \in \mathfrak{g} \quad R(X,Y)Z = \|\ell\|^2 \Big(\langle X, Z\rangle Y - \langle Y, Z\rangle X \Big)$$

18. Prove that the sectional curvature of G is constant, and give its value.

Solution: Let $P \subset \mathfrak{g} = T_e G$ be a plane, and (X,Y) an orthonormal basis of P. The previous question shows $R(X,Y)Y = -\|\ell\|^2$, and the sectional curvature is $\kappa(P) = \langle R(X,Y)Y, X\rangle = -norm\ell^2$.
If $P \subset T_g G$ is a plane, then $\kappa(P) = \kappa(d_g L_{g^{-1}}(P)) = -\|\ell\|^2$ because the metric is left-invariant.

19. Infer that $(G, \langle\cdot,\cdot\rangle)$ is isometric to a Riemannian manifold seen in the lectures, and that it is a symmetric space.

Solution: The Riemannian manifold $(G, \langle\cdot,\cdot\rangle)$ is complete (question 1. of exercise 1.), simply connected (question 11. of exercise 2.) and has constant negative sectional curvature (question 18. of exercise 2.), so it is homothetic to the hyperbolic space $\mathbb{H}^{\dim G}$, which is a symmetric space.

The isometry group of G

20. Given a linear isometry $\varphi : \mathfrak{g} \to \mathfrak{g}$ (for the inner product $\langle\cdot,\cdot\rangle_e$), prove that there is a unique isometry $\Phi \in \mathrm{Isom}(G)$ such that $\Phi(e) = e$ and $d_e\Phi = \varphi$.

Solution: This is a consequence of the result of question 19. and the fact that the statement is true for the real hyperbolic space $\mathbb{H}^d$ (in a general symmetric space, in order to use the Cartan-Ambrose-Hicks Theorem one would need to add the condition that φ preserves the Riemann tensor, but the expression of R in question 17. shows that it is true for any linear isometry). Note that $\Phi = \exp_e \circ \varphi \circ \exp_e^{-1}$ where $\exp_e$ is the Riemannian exponential map (which is different from the Lie exponential as seen in question 16.).

21. Let $K \subset \mathrm{Isom}(G)$ be the stabilizer of e. Prove that the map

$$\begin{cases} K \times A \times N & \to & \mathrm{Isom}(G) \\ (k,a,n) & \mapsto & k \circ L_a \circ L_n \end{cases}$$

is a diffeomorphism.

Solution: Denote by $\Psi : K \times A \times N \to \mathrm{Isom}(G)$ this map, and $\Phi : G \to N \times A$ the diffeomorphism obtained in question 9. Then Ψ is a diffeomorphism with inverse

$$\Psi^{-1}(f) = \Big(f \circ L_{f^{-1}(e)}, \Phi(f^{-1}(e)) \Big) \in K \times A \times N$$

22. When $\dim \mathfrak{g} = 2$, give an explicit representation of G in $\mathrm{PSL}(2,\mathbb{R})$, and describe the images of A and N.

Solution: $G = \left\{ \begin{bmatrix} a & b \\ 0 & a^{-1} \end{bmatrix} \middle| a > 0, b \in \mathbb{R} \right\}$, $A = \left\{ \begin{bmatrix} a & 0 \\ 0 & a^{-1} \end{bmatrix} \middle| a > 0 \right\}$, $N = \left\{ \begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix} \middle| b \in \mathbb{R} \right\}$.

The linear form ℓ on the Lie algebra $\mathfrak{g} = \left\{ \begin{pmatrix} x & y \\ 0 & -x \end{pmatrix} \middle| x, y \in \mathbb{R} \right\}$ is given by $\ell \begin{pmatrix} x & y \\ 0 & -x \end{pmatrix} = x$

Exercise 3

Curvature of a graph

Let $U \subset \mathbb{R}^2$ be an open set, and $f : U \to \mathbb{R}$ a smooth map. Consider its graph $S = \{(x, y, f(x,y)) \,|\, (x,y) \in U\} \subset \mathbb{R}^3$. At a point $(x_0, y_0, f(x_0, y_0)) \in S$ such that $d_{(x_0,y_0)} f = 0$, calculate the second fundamental form of S and its Gauß curvature.

Solution: We first need to find a unit normal vector field. Consider the parametrization $\varphi(x,y) = (x, y, f(x,y))$, which gives us a way to compute the normal direction:

$$\frac{\partial \varphi}{\partial x} \wedge \frac{\partial \varphi}{\partial y} = \begin{pmatrix} 1 \\ 0 \\ \frac{\partial f}{\partial x} \end{pmatrix} \wedge \begin{pmatrix} 0 \\ 1 \\ \frac{\partial f}{\partial y} \end{pmatrix} = \begin{pmatrix} -\frac{\partial f}{\partial x} \\ -\frac{\partial f}{\partial y} \\ 1 \end{pmatrix}$$

We get a unit normal vector field by normalizing.

$$\vec{n} = \frac{\frac{\partial \varphi}{\partial x} \wedge \frac{\partial \varphi}{\partial y}}{\left\| \frac{\partial \varphi}{\partial x} \wedge \frac{\partial \varphi}{\partial y} \right\|} = \frac{1}{\sqrt{1 + \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2}} \begin{pmatrix} -\frac{\partial f}{\partial x} \\ -\frac{\partial f}{\partial y} \\ 1 \end{pmatrix}$$

Note that $\vec{n}(\varphi(x_0, y_0)) = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$. In order to differentiate $\vec{n}$, we need to compute some partial derivatives.

$$\frac{\partial}{\partial x}\left(\frac{1}{\sqrt{1 + \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2}} \right) = \frac{\frac{\partial f}{\partial x}\frac{\partial^2 f}{\partial x^2} + \frac{\partial f}{\partial y}\frac{\partial^2 f}{\partial x \partial y}}{\left(1 + \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2\right)^{\frac{3}{2}}}$$

$$\frac{\partial}{\partial y}\left(\frac{1}{\sqrt{1 + \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2}} \right) = \frac{\frac{\partial f}{\partial y}\frac{\partial^2 f}{\partial y^2} + \frac{\partial f}{\partial x}\frac{\partial^2 f}{\partial x \partial y}}{\left(1 + \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2\right)^{\frac{3}{2}}}$$

Both of these expression vanish at (x_0, y_0). The tangent space of S is $T_{\varphi(x_0,y_0)} S = \left\{ \begin{pmatrix} u \\ v \\ 0 \end{pmatrix} \middle| u, v \in \mathbb{R} \right\}$ and we find

$$d_{\varphi(x_0,y_0)}\vec{n}\begin{pmatrix} u \\ v \\ 0 \end{pmatrix} = \begin{pmatrix} -u\frac{\partial^2 f}{\partial x^2}(x_0,y_0) - v\frac{\partial^2 f}{\partial x \partial y}(x_0,y_0) \\ -u\frac{\partial^2 f}{\partial x \partial y}(x_0,y_0) - v\frac{\partial^2 f}{\partial y^2}(x_0,y_0) \\ 0 \end{pmatrix}$$

The second fundamental form is:

$$\begin{aligned} \vec{\mathrm{II}}_{\varphi(x_0,y_0)}\left(\begin{pmatrix} u_1 \\ v_1 \\ 0 \end{pmatrix}, \begin{pmatrix} u_2 \\ v_2 \\ 0 \end{pmatrix}\right) &= \begin{pmatrix} u_1 \\ v_1 \\ 0 \end{pmatrix} . d_{\varphi(x_0,y_0)}\vec{n}\begin{pmatrix} u_2 \\ v_2 \\ 0 \end{pmatrix} \\ &= -u_1 u_2 \frac{\partial^2 f}{\partial x^2}(x_0,y_0) - (u_1 v_2 + u_2 v_1)\frac{\partial^2 f}{\partial x \partial y}(x_0,y_0) - u_2 v_2 \frac{\partial^2 f}{\partial y^2}(x_0,y_0) \\ &= -\mathrm{Hess}(f)_{\varphi(x_0,y_0)}\left(\begin{pmatrix} u_1 \\ v_1 \\ 0 \end{pmatrix}, \begin{pmatrix} u_2 \\ v_2 \\ 0 \end{pmatrix}\right) \end{aligned}$$

The Gauß curvature is the determinant

$$K(\varphi(x_0,y_0)) = \frac{\partial^2 f}{\partial x^2}(x_0,y_0)\frac{\partial^2 f}{\partial y^2}(x_0,y_0) - \left(\frac{\partial^2 f}{\partial x \partial y}(x_0,y_0)\right)^2$$