Groups and geometry

Final exam

Exercise 1

Let $\mathbb{X}$ be a Riemannian symmetric space.

1. For $x \in \mathbb{X}$, let $\varphi_x \in \mathrm{End}(T_x\mathbb{X})$ be the self-adjoint operator such that :

$$\forall u, v \in T_x\mathbb{X} \quad \langle \varphi_x(u), v\rangle_x = \mathrm{Ric}_x(u,v)$$

   Show that for $f \in \mathrm{Isom}(\mathbb{X})$, we have $d_x f \circ \varphi_x = \varphi_{f(x)} \circ d_x f$.

   Solution: It follows from the invariance formula for the Ricci curvature $\mathrm{Ric}_{f(x)}(d_x f(u), d_x f(v)) = \mathrm{Ric}_x(u,v)$.

2. Let $o \in \mathbb{X}$. If $\mathbb{X}$ is irreducible, prove that there is $\lambda \in \mathbb{R}$ such that $\varphi_o = \lambda \mathrm{Id}$.
   Hint: consider the action of $K = \mathrm{Stab}_G(o)$ on the eigenspaces of φ_o.

   Solution: The group K acts on $T_x o\mathbb{X}$ by $g.u = d_o g(u)$. It follows from the previous question that this action commutes with φ_o, so it preserves the eigenspaces of φ_o. Since φ_o is diagonalisable (it is self-adjoint) and the action of K is irreducible, we find that φ_o has only one eigenspace, i.e. there is $\lambda \in \mathbb{R}$ such that $\varphi_o = \lambda \mathrm{Id}$.

3. Prove that any irreducible symmetric space is an Einstein manifold.

   Solution: From the previous question we find $\mathrm{Ric}_o(u,v) = \lambda \langle u, v\rangle_o$ for all $u, v \in T_o\mathbb{X}$. Using the homogeneity of $\mathbb{X}$, we also have $\mathrm{Ric}_x(u,v) = \lambda \langle u, v\rangle_x$ for all $x \in \mathbb{X}$ and $u, v \in T_x\mathbb{X}$.

Exercise 2

Consider two smooth manifolds M and N.

1. Let $f : M \to N$ be a smooth map. Prove that $\ker df$ is a vector sub-bundle of TM if and only if f has constant rank.

   Solution: If $\ker df$ is a vector sub-bundle of codimension r of TM, then the rank of $d_x f$ is r for any $x \in M$, so f has constant rank.
   If f has constant rank r, then for any $x \in M$ we can find coordinates $(x^1, \ldots, x^d)$ on M around x and coordinates $(y^1, \ldots, y^n)$ on N around $f(x)$ such that $f(x^1, \ldots, x^d) = (x^1, \ldots, x^r, 0, \ldots, 0)$. So $\ker df$ is spanned by $\partial_{r+1}, \ldots, \partial_d$, and it is a vector sub-bundle of TM of codimension r.

2. Under which additional condition(s) is $\mathrm{Im}\, df$ a vector sub-bundle of TN ?

   Solution: If $\mathrm{Im}\, df$ is a vector sub-bundle of TN, then f must be surjective so that it defines a vector subspace of every $T_x N$ for $x \in N$. Also $\mathrm{Im}\, d_x f$ should have the same dimension for every $x \in M$, so f has constant rank. If f is surjective and has constant rank, then f is a submersion and $\mathrm{Im}\, df = TN$ is a vector sub-bundle of TN.

Exercise 3

Let (M, g) be a Riemannian manifold, and let X be the total space of the unit tangent bundle T^1M, i.e.

$$X = \{(x,v) \,|\, x \in M,\ v \in T_xM,\ g_x(v,v) = 1\}$$

Let $\pi : X \to M$ be the projection (given by $\pi(x,v) = x$).

The tangent bundle of X

1. For $(x,v) \in X$, let $V_{(x,v)} = \ker d_{(x,v)}\pi \subset T_{(x,v)}X$. What is the dimension of $V_{(x,v)}$? Prove that this defines a vector sub-bundle V of TX.

   Solution: The map π is a submersion, so $\dim V_{(x,v)} = \dim X - \dim M = d - 1$ where $d = \dim M$. Question 1. of exercise 1 shows that V is a vector sub-bundle of TX.

2. Let $(x,v) \in X$ and $z \in T_{(x,v)}X$. Consider a path $\gamma(t) = (c(t), X(t)) \in X$ such that $\gamma(0) = (x,v)$ and $\dot\gamma(0) = z$. Show that the map

$$\varphi_{(x,v)} : \begin{cases} T_{(x,v)}X & \to & T_xM \\ z & \mapsto & \frac{D}{dt}X(0) \end{cases}$$

   is well defined (i.e. $\varphi_{(x,v)}(z)$ only depends on z and not on the path γ) and defines a linear isomorphism from $V_{(x,v)} = \ker d_{(x,v)}\pi$ to $v^\perp \subset T_xM$.

   Hint: using local coordinates, we can assume that $M \subset \mathbb{R}^d$ is an open set and that $X \subset \mathbb{R}^{2d}$ is a submanifold whose equation we will give. This leads to equations for the subspaces $T_{(x,v)}X$ and $V_{(x,v)}$ of $\mathbb{R}^{2d}$, and an explicit formula for $\varphi_{(x,v)}$.

   Solution: As suggested, we assume that $M \subset \mathbb{R}^d$ is an open set, and write g_{ij} the components of the Riemannian metric g. So X is defined as:

$$X = \left\{(x,v) \in \mathbb{R}^d \times \mathbb{R}^d \,\middle|\, g_{ij}(x)v^iv^j = 1\right\}$$

   Note that we use Einstein's summation convention. Write $f(x,v) = g_{ij}(x)v^iv^j = 1$. Its differential is:

$$d_{(x,v)}f(y,w) = \partial_k g_{ij}(x)y^kv^iv^j + 2g_{ij}(x)v^iw^j$$

   In particular, if $(x,v) \in X$ then $d_{(x,v)}f(0,v) = 2 \neq 0$ so f is a submersion around (x,v), and the tangent space $T_{(x,v)}X \subset \mathbb{R}^d \times \mathbb{R}^d$ is equal to $\ker d_{(x,v)}f$.

$$T_{(x,v)}X = \left\{(y,w) \in \mathbb{R}^d \times \mathbb{R}^d \,\middle|\, \partial_k g_{ij}(x)y^kv^iv^j + 2g_{ij}(x)v^iw^j = 0\right\}$$

   Since the projection π is given by $\pi(x,v) = x$, we have $d_{(x,v)}\pi(y,w) = y$ and we find

$$\begin{aligned} V_{(x,v)} &= \left\{(0,w) \in \mathbb{R}^d \times \mathbb{R}^d \,\middle|\, g_{ij}(x)v^iw^j = 0\right\} \\ &= \left\{(0,w) \in \mathbb{R}^d \times \mathbb{R}^d \,\middle|\, g_x(v,w) = 0\right\} \\ &= \left\{(0,w) \in \mathbb{R}^d \times \mathbb{R}^d \,\middle|\, w \in v^\perp\right\} \end{aligned}$$

Now let $z = (0, w) \in V_{(x,v)}$ and $\gamma(t) = (c(t), X(t))$ a curve in X such that $\gamma(0) = (x, v)$ and $\dot\gamma(0) = z$. Hence $\dot c(0) = 0$ and $\dot X(0) = w$. Now

$$\begin{aligned}\left(\frac{D}{dt}X(0)\right)^k &= \dot X^k(0) + \Gamma^k_{ij}(x)\dot c^i(0)X^j(0)\\ &= \dot X^k(0) + 0\\ &= w^k\end{aligned}$$

This means that $\frac{D}{dt}X(0) = w$, i.e. $\varphi_{(x,v)}(0, w) = w$. This shows that $\varphi_{(x,v)}(z)$ does not depend on the path γ, and that it defines a linear isomorphism from $V_{(x,v)}$ to $v^\perp$.

3. For $(x, v) \in X$ and $w \in T_xM$, we set

$$\psi_{(x,v)}(w) = \left.\frac{d}{dt}\right|_{t=0}(c_w(t), V(t))$$

where c_w is the geodesic with initial velocity $\dot c_w(0) = w$, and V is the parallel vector field along c_w such that $V(0) = v$. Prove that the map $\psi_{(x,v)} : T_xM \to T_{(x,v)}X$ is linear and injective.

Solution: We use the same local setting $M \subset \mathbb{R}^d$ as in the previous question. Now we find $\psi_{(x,v)}(w) = (\dot c_w(0), \dot V(0))) = (w, \dot V(0))$. Since V is parallel and $V(0) = v$, we find:

$$\begin{aligned}\dot V^k(0) &= \left(\frac{D}{dt}V(0)\right)^k - \Gamma^k_{ij}(x)w^iV^j(0)\\ &= 0 - \Gamma^k_{ij}(x)w^iv^j\end{aligned}$$

Therefore $\psi_{(x,v)}(w) = (w, -\Gamma^k_{ij}(x)w^iv^j)$. This shows that $\psi_{(x,v)}$ is linear and injective.

4. Let $H_{(x,v)} = \psi_{(x,v)}(T_xM) \subset T_{(x,v)}X$. Prove that this defines a vector sub-bundle H of TX, and that $TX = H \oplus V$.

Solution: Locally, for $M \subset \mathbb{R}^d$, we find that $H_{(x,v)}$ is spanned by the linearly independent vector fields $\psi_{(x,v)}(\partial_1), \dots, \psi_{(x,v)}(\partial_d)$, so it defines a vector sub-bundle of rank d of TX. The descriptions of $H_{(x,v)}$ and $V_{(x,v)}$ in coordinates show that $H_{(x,v)} \cap V_{(x,v)} = \{0\}$. But we also have $\dim H_{(x,v)} = d$, $\dim V_{(x,v)} = d-1$ and $\dim T_{(x,v)}X = 2d-1$, hence $\dim T_{(x,v)}X = H_{(x,v)} \oplus V_{(x,v)}$.

5. Show that there is a unique Riemannian metric $\widetilde g$ on X with the following three properties:

- $\forall (x,v) \in X\ \forall z_1, z_2 \in V_{(x,v)} \quad \widetilde g_{(x,v)}(z_1, z_2) = g_x\left(\varphi_{(x,v)}(z_1), \varphi_{(x,v)}(z_2)\right)$
- $\forall (x,v) \in X\ \forall w_1, w_2 \in T_xM \quad \widetilde g_{(x,v)}\left(\psi_{(x,v)}(w_1), \psi_{(x,v)}(w_2)\right) = g_x(w_1, w_2)$
- $\forall (x,v) \in X\ \forall (z_V, z_H) \in V_{(x,v)} \times H_{(x,v)} \quad \widetilde g_{(x,v)}(z_V, z_H) = 0$

Solution: It follows from the decomposition $TX = H \oplus V$ and the fact that φ and ψ are isomorphisms. The (useless) local formula is

$$\widetilde g_{(x,v)}((y_1, w_1), (y_2, w_2)) = g_{ij}(x)y_1^iy_2^j + g_{ij}(x)\left(w_1^i + \Gamma^i_{jk}(x)y_1^jv^k\right)\left(w_2^i + \Gamma^i_{jk}(x)y_2^jv^k\right)$$

6. Let $\mathcal{Z} \in \mathcal{X}(X)$ be the geodesic spray (i.e. the vector field whose flow $\varphi_{\mathcal{Z}}$ is the geodesic flow $\varphi^t_{\mathcal{Z}}(x,v) = (c_v(t), \dot{c}_v(t))$). What is the decomposition of $\mathcal{Z}$ along $TX = H \oplus V$? Compute $\widetilde{g}(\mathcal{Z},\mathcal{Z})$.

   Solution: Since $\dot{c}_v$ is a parallel vector fiend along c_v (because c_v is a geodesic), we find that $\mathcal{Z}(x,v) = \frac{d}{dt}\big|_{t=0}(c_v(t), \dot{c}_v(t)) = \psi_{(x,v)}(v)$. This shows that $\mathcal{Z}(x,v) \in H_{(x,v)}$ and $\widetilde{g}(\mathcal{Z},\mathcal{Z}) = 1$.

The geodesic flow of $\mathbb{H}^d$

In question 7. to 14. we consider that (M,g) is the real hyperbolic space $\mathbb{H}^d$, and we work with the hyperboloid model $\mathbb{H}^d \subset \mathbb{R}^{d,1}$. Set $\langle x,y\rangle = x_1y_1 + \cdots + x_dy_d - x_{d+1}y_{d+1}$ for $x = (x_1,\ldots,x_{d+1})$ and $y = (y_1,\ldots,y_{d+1})$, thus $\mathbb{H}^d = \left\{x \in \mathbb{R}^{d+1} \,\middle|\, \langle x,x\rangle = -1,\ x_{d+1} > 0\right\}$.

7. Describe $X \subset \mathbb{R}^{d+1} \times \mathbb{R}^{d+1}$.

   Solution: For $x \in \mathbb{H}^d$, we have $T_x\mathbb{H}^d = x^\perp = \left\{x \in \mathbb{R}^{d+1} \,\middle|\, \langle x,v\rangle = 0\right\}$, so

$$X = \left\{(x,v) \in \mathbb{R}^{d+1} \times \mathbb{R}^{d+1} \,\middle|\, \langle x,x\rangle = -1, x_{d+1} > 0, \langle x,v\rangle = 0, \langle v,v\rangle = 1\right\}$$

8. For $(x,v) \in X$, describe $T_{(x,v)}X \subset \mathbb{R}^{d+1} \times \mathbb{R}^{d+1}$.

   Solution: We have that X is an open subset of $F^{-1}(\{0\})$ where

$$F : \begin{cases} \mathbb{R}^{d+1} \times \mathbb{R}^{d+1} & \to & \mathbb{R}^3 \\ (x,v) & \mapsto & (\langle x,x\rangle, \langle x,v\rangle, \langle v,v\rangle) \end{cases}$$

   The map F is smooth, and its differential is

$$d_{(x,v)}F(y,w) = (2\langle x,y\rangle, \langle y,v\rangle + \langle x,w\rangle, 2\langle v,w\rangle)$$

   If $(x,v) \in X$ then

$$\begin{aligned} d_{(x,v)}F(x,0) &= (-2,0,0) \\ d_{(x,v)}F(0,v) &= (0,0,2) \\ d_{(x,v)}F(v,-x) &= (0,2,0) \end{aligned}$$

   This shows that F is a submersion at (x,v), so $T_{(x,v)}X = \ker d_{(x,v)}F$, i.e.

$$T_{(x,v)}X = \left\{(y,w) \in \mathbb{R}^{d+1} \times \mathbb{R}^{d+1} \,\middle|\, \langle x,y\rangle = \langle v,w\rangle = \langle y,v\rangle + \langle x,w\rangle = 0\right\}$$

9. Let $(x,v) \in X$. Describe $V_{(x,v)}$, and give an explicit formula for $\varphi_{(x,v)}$.

   Solution: Since $d_{(x,v)}\pi(y,w) = y$, we find

$$\begin{aligned} V_{(x,v)} = \ker d_{(x,v)}\pi &= \left\{(0,w) \in \mathbb{R}^{d+1} \times \mathbb{R}^{d+1} \,\middle|\, \langle x,w\rangle = \langle v,w\rangle = 0\right\} \\ &= \left\{(0,w) \in \mathbb{R}^{d+1} \times \mathbb{R}^{d+1} \,\middle|\, w \in x^\perp \cap v^\perp\right\} \\ &= \left\{(0,w) \in \mathbb{R}^{d+1} \times \mathbb{R}^{d+1} \,\middle|\, w \in T_x\mathbb{H}^d, w \in v^\perp\right\} \end{aligned}$$

Let $z = (0, w) \in V_{(x,v)}$ and consider a path $\gamma(t) = (c(t), X(t)) \in X$ such $\gamma(0) = (x, v)$ and $\dot\gamma(0) = (0, w)$. Now $\frac{D}{dt}X(0)$ is the orthogonal projection of $\dot X(0)$ on $T_x\mathbb{H}^d$, but $\dot X(0) = w \in T_x\mathbb{H}^d$ so $\frac{D}{dt}X(0) = w$, and $\varphi_{(x,v)}(z) = w$.

10. Let $(x, v) \in X$. Describe $H_{(x,v)}$, and give an explicit formula for $\psi_{(x,v)}$.
Hint: first compute $\psi_{(x,v)}(w)$ for $w \in v^\perp$, then compute $\psi_{(x,v)}(v)$.

Solution: First start with $w \in v^\perp$ such that $\|w\| = 1$. Then we know that $c_w(t) = \cosh tx + \sinh tw$. The formula $V(t) = v$ is a vector field along c_w which is parallel (because $\dot V(t) = 0$ implies that the projection $\frac{D}{dt}V(0)$ is also zero). This leads to the formula

$$\psi_{(x,v)}(w) = (w, 0)$$

The definition of $\psi_{(x,v)}$ leads to $\psi_{(x,v)}(v) = (\dot c_v(0), \ddot c_v(0)) = (v, x)$.
An arbitrary vector $w \in T_xM$ decomposes as $w = \langle w, v\rangle v + (w - \langle w, v\rangle v) \in \mathbb{R}.v \oplus v^\perp$, so the general formula is:

$$\psi_{(x,v)}(w) = (w, \langle w, v\rangle x)$$

By definition we have $H_{(x,v)} = \psi_{(x,v)}(T_x\mathbb{H}^d)$, so $H_{(x,v)} = (x^\perp \cap v^\perp) \times \{0\} \oplus \mathbb{R}.(v, x)$.

11. Let $(x, v) \in X$ and $t \in \mathbb{R}$. Give an explicit formula for $\varphi^t_{\mathcal{Z}}(x, v)$ and $d_{(x,v)}\varphi^t_{\mathcal{Z}}$.

Solution: Recall the formula for unit speed geodesics in $\mathbb{H}^d$:

$$c_v(t) = \cosh tx + \sinh tv$$

This leads to

$$\begin{aligned}\varphi^t_{\mathcal{Z}}(x, v) &= (c_v(t), \dot c_v(t)) \\ &= (\cosh tx + \sinh tv, \sinh tx + \cosh tv)\end{aligned}$$

It is the restriction to $X \subset \mathbb{R}^{d+1} \times \mathbb{R}^{d+1}$ of a linear map, so its differential is

$$d_{(x,v)}\varphi^t_{\mathcal{Z}}(y, w) = (\cosh ty + \sinh tw, \sinh ty + \cosh tw)$$

12. Let $(x, v) \in X$. We set:

$$\begin{aligned}E^s_{(x,v)} &= \left\{(y, -y) \in \mathbb{R}^{d+1} \times \mathbb{R}^{d+1} \,\middle|\, \langle x, y\rangle = \langle v, y\rangle = 0\right\} \\ E^u_{(x,v)} &= \left\{(y, y) \in \mathbb{R}^{d+1} \times \mathbb{R}^{d+1} \,\middle|\, \langle x, y\rangle = \langle v, y\rangle = 0\right\}\end{aligned}$$

Check that $T_{x,v)}X = E^s_{(x,v)} \oplus E^u_{(x,v)} \oplus \mathbb{R}.\mathcal{Z}(x, v)$.

Solution: Decomposing $(y, w) = \frac{1}{2}(y+w, y+w) + \frac{1}{2}(y-w, w-y)$ we find that $E^s_{(x,v)} \oplus E^u_{(x,v)} = (x^\perp \cap v^\perp) \times (x^\perp \cap v^\perp)$, so the description of $T_{(x,v)}X$ in question 8. associated to the formula $\mathcal{Z}(x, v) = \psi_{(x,v)}(v) = (v, x)$ gives the desired decomposition.

13. For $(x,v) \in X$ and $t \in \mathbb{R}$, show that $d_{(x,v)}\varphi_{\mathcal{Z}}^t\left(E^s_{(x,v)}\right) = E^s_{\varphi_{\mathcal{Z}}^t(x,v)}$ and $d_{(x,v)}\varphi_{\mathcal{Z}}^t\left(E^u_{(x,v)}\right) = E^u_{\varphi_{\mathcal{Z}}^t(x,v)}$.

Solution: If $(y,-y) \in E^s_{(x,v)}$, then

$$\begin{aligned} d_{(x,v)}\varphi_{\mathcal{Z}}^t(y,-y) &= (\cosh ty - \sinh ty, \sinh ty - \cosh ty) \\ &= e^{-t}(y,-y) \end{aligned}$$

Since $\langle y,x\rangle = \langle y,v\rangle = 0$, we also have $\langle y, c_v(t)\rangle = \langle y, \dot{c}_v(t)\rangle$, therefore $(y,-y) \in E^s_{\varphi_{\mathcal{Z}}^t(x,v)}$. The same works for E^u.

14. For $z \in E^s_{(x,v)}$ and $t \in \mathbb{R}$, prove that:

$$\widetilde{g}_{\varphi_{\mathcal{Z}}^t(x,v)}\left(d_{(x,v)}\varphi_{\mathcal{Z}}^t(z), d_{(x,v)}\varphi_{\mathcal{Z}}^t(z)\right) = e^{-t}\widetilde{g}_{(x,v)}(z,z)$$

Solution: Write $z = (y,-y)$. From questions 9. and 10. we see that $(0,y) \in V_{(x,v)}$ and $(y,0) = \psi_{(x,v)}(y) \in H_{(x,v)}$, so $\widetilde{g}_{(x,v)}(z,z) = 2\langle y,y\rangle$. The previous questions shows that $\widetilde{g}_{\varphi_{\mathcal{Z}}^t(x,v)}\left(d_{(x,v)}\varphi_{\mathcal{Z}}^t(z), d_{(x,v)}\varphi_{\mathcal{Z}}^t(z)\right) = 2e^{-t}\langle y,y\rangle$, hence the result.

Jacobi fields and the geodesic flow

We are back to an arbitrary Riemannian manifold (M,g) (except for question 18.).

15. Let $(x,v) \in X$, and $J : I_v \to TM$ a Jacobi field along c_v. Prove that the following propositions are equivalent:

    - $g_x\left(v, \frac{D}{dt}J(0)\right) = 0$
    - $\forall t \in I_v \;\; g_{c_v(t)}\left(\dot{c}_v(t), \frac{D}{dt}J(t)\right) = 0$
    - $\forall t \in I_v \;\; g_{c_v(t)}(\dot{c}_v(t), J(t)) = g_x(v, J(0))$

    Solution: This is a consequence of the fact that $g\left(\dot{c}_v, \frac{D}{dt}J\right)$ is constant, which itself follows from

$$\frac{d}{dt}g\left(\dot{c}_v, \frac{D}{dt}J\right) = g\left(\dot{c}_v, \frac{D}{dt}\frac{D}{dt}J\right) = R(J, \dot{c}_v, \dot{c}_v, \dot{c}_v) = 0$$

16. For $(x,v) \in X$, let $E_{(x,v)}$ be the set of Jacobi fields J along c_v such that $g_x\left(v, \frac{D}{dt}J(0)\right) = 0$. Show that there is a unique linear isomorphism $J_{(x,v)} : T_{(x,v)}X \to E_{(x,v)}$ satisfying the following two properties:

    - If $w \in T_xM$ and $z = \psi_{(x,v)}(w)$ then $J_{(x,v)}(z)(0) = w$ and $\frac{D}{dt}J_{(x,v)}(z)(0) = 0$.
    - If $z \in V_{(x,v)}$ then $J_{(x,v)}(z)(0) = 0$ and $\frac{D}{dt}J_{(x,v)}(z)(0) = \varphi_{(x,v)}(z)$.

    Solution: This is a consequence of the facts that a Jacobi field is uniquely determined by $J(0)$ and $\frac{D}{dt}J(0)$, and that $\varphi_{(x,v)}$ and $\psi_{(x,v)}$ are isomorphisms.

17. Let $(x,v) \in X$. Consider $z \in T_{(x,v)}X$, and write $J = J_{(x,v)}(z)$. For $t \in \mathbb{R}$, we set $J_t = J_{\varphi^t_Z(x,v)}\left(d_{(x,v)}\varphi^t_Z(z)\right)$. Show that for all $s \in I_{\dot{c}_v(t)}$, we have $J_t(s) = J(t+s)$.

Solution: This is by far the most complicated question in this exam! First we need to understand the relationship between Jacobi field and the tangent bundle of X more deeply, and this goes through variations of geodesics. We have seen Jacobi fields are variation fields of geodesic variations. Because we are working with the unit tangent bundle T^1M and not the whole TM, we need to make sure that we can choose a variation by unit speed geodesics.

Fact 1: If J is a Jacobi field along c_v such that $g_x\left(v, \frac{D}{dt}J(0)\right) = 0$, and $T \in I_v$, then there is a geodesic variation $f : U \to M$, where $U \subset \mathbb{R}^2$ is open and contains $[0,T] \times \{0\}$ such that $f(t,0) = c_v(t)$, $\frac{\partial f}{\partial s}(t,0) = J(t)$ for all $t \in [0,T]$ and $\left\|\frac{\partial f}{\partial t}(t,s)\right\| = 1$ for all $(t,s) \in U$.

Proof. We are looking for a smooth curve $\gamma :]-\varepsilon,\varepsilon[\to M$ and a vector field Z along γ such that we can define $f(t,s) = \exp_{\gamma(s)}(tZ(s))$.
First, note that since the domain of the exponential map is open, it will be defined on some appropriate open set $U \subset \mathbb{R}^2$ up to shrinking ε.
The property $f(t,0) = c_v(t)$ simply means that $\gamma(0) = 0$. Also $\frac{\partial f}{\partial s}(0) = J(0)$ translates as $\dot\gamma(0) = 0$. We will see that any curve γ with these properties will work (e.g. $\gamma = c_{J(0)}$).
Now in order to make sure that $\frac{\partial f}{\partial s}(t,0) = J(t)$ for all t, we just need to show that $\frac{D}{\partial t}\frac{\partial f}{\partial s}(0,0) = \frac{D}{dt}J(0)$. Since the Levi-Civita connection is torsion-free, this is the same as $\frac{D}{\partial s}\frac{\partial f}{\partial t}(0,0) = \frac{D}{dt}J(0)$, i.e. $\frac{D}{ds}Z(0) = \frac{D}{dt}J(0)$.
The condition that $\left\|\frac{\partial f}{\partial t}(t,s)\right\| = 1$ for all $(t,s) \in U$ is also a condition on the vector field Z since geodesic have constant speed, so it is equivalent to $\|Z(s)\| = 1$ for all $s \in]-\varepsilon,\varepsilon[$.
Set $w = \frac{D}{dt}J(0)$, so we are looking for a vector field Z along γ such that $\|Z\| \equiv 1$ and $\frac{D}{ds}Z(0) = w$.
If $w = 0$, then the parallel vector field along γ such that $Z(0) = v$ works.
If $w \neq 0$, let V (resp. W) be the parallel vector field along γ such that $V(0) = v$ (resp. $W(0) = 0$). The vector field $Z(s) = \cos(\|w\|s)V(s) + \frac{\sin(\|w\|s)}{\|w\|}W(s)$ works. □

We can now have an other description of the map $J_{(x,v)}$ involving geodesic variations.

Fact 2: If $J = J_{(x,v)}(z)$ for $z \in T_xM$ and $f : U \to M$ is a geodesic variation given by Fact 1, then $z = \frac{d}{ds}\big|_{s=0}\left(f(s,0), \frac{\partial f}{\partial t}(s,0)\right)$.

Remark. *We need a geodesic variation by unit speed geodesics in order to have $\left(f(s,0), \frac{\partial f}{\partial t}(s,0)\right) \in X$ for all s.*

Proof. Write $y = J(0)$ and $w = \frac{D}{dt}J(0)$. By definition of the map $J_{(x,v)}$, the decomposition $z = z_V + z_H \in V_{(x,v)} \oplus H_{(x,v)}$ satisfies $\varphi_{(x,v)}(z_V) = w$ and $\psi_{(x,v)}(y) = z_H$.
Now consider $\widetilde{z} = \frac{d}{ds}\big|_{s=0}\left(f(s,0), \frac{\partial f}{\partial t}(s,0)\right)$ Our goal is to show that $\widetilde{z} = z$.
Since this is a local consideration, we can assume that $M \subset \mathbb{R}^d$ is an open set and write

$$\widetilde{z} = \left(\frac{\partial f}{\partial s}(0,0), \frac{\partial^2 f}{\partial s \partial t}(0,0)\right) = (J(0), \dot{J}(0))$$

Now in coordinates we have $\dot{J}(0)^k = \left(\frac{D}{dt}J(0)\right)^k - \Gamma^k_{ij}v^iJ(0)^j$, i.e. $\dot{J}(0)^k = w^k - \Gamma^k_{ij}v^iy^j$. Using the coordinate formula for $\psi_{(x,v)}$ found in question 3., we get

$$\widetilde{z} = (y, \dot{J}(0)) = \psi_{(x,v)}(y) + (0,w) = z_H + (0,w)$$

In question 2. we saw that $(0,w) \in V_{(x,v)}$ and $\varphi_{(x,v)}(0,w) = w$, hence $(0,w) = z_V$ and $\widetilde{z} = z_H + z_V = z$. $\square$

Now to answer the question, we consider a geodesic variation $f : U \to M$ given by Fact 1. Because it is a variation by unit speed geodesics, we have that $\varphi_{\mathcal{Z}}^t\left(f(0,s), \frac{\partial f}{\partial t}(0,s)\right) = \left(f(t,s), \frac{\partial f}{\partial t}(t,s)\right)$ whenever defined. Now according to Fact 2, differentiation at $s = 0$ shows that J_t is the variation field of $f(\cdot + t, \cdot)$, hence the formula.

18. For $(M,g) = \mathbb{H}^d$, give an explicit formula for $J_{(x,v)}$, then describe $J_{(x,v)}\left(E^s_{(x,v)}\right)$ and $J_{(x,v)}\left(E^u_{(x,v)}\right)$.

Solution: Let $z \in T_{(x,v)}X$. We start by decomposing $z = \lambda(v,x) + (y,w)$ where $\lambda \in \mathbb{R}$ and $y, w \in x^\perp \cap v^\perp$. Note that $J_{(x,v)}(v,x)(0) = J_{(x,v)}(\mathcal{Z}(x,v))(0) = v$ and $\frac{D}{dt}J_{(x,v)}(v,x)(0) = 0$, so $J_{(x,v)}(v,x)(t) = v$ for all $t \in \mathbb{R}$. $(y,w) = (0,w) + (y,0) \in V_{(x,v)} \oplus H_{(x,v)}$ and the computations of $\varphi_{(x,v)}$ and $\psi_{(x,v)}$ yield

$$J_{(x,v)}(y,w)(t) = \cosh tw + \sinh ty$$

In particular, for $(y,-y) \in E^s_{(x,v)}$ we get $J_{(x,v)}(y,-y)(t) = -e^{-t}y$ and for $(y,-y) \in E^u_{(x,v)}$ we get $J_{(x,v)}(y,y)(t) = e^t y$. This shows that Jacobi fields $J \in J_{(x,v)}\left(E^s_{(x,v)}\right)$ (resp. $J \in J_{(x,v)}\left(E^u_{(x,v)}\right)$) are characterized by $J(0) = -\frac{D}{dt}J(0) \in v^\perp$ (resp. $J(0) = \frac{D}{dt}J(0) \in v^\perp$).