Contents

Chapter 1

Basics of Lie groups and Lie algebras

1.1 Lie groups

A Lie group is just on group on which one can do calculus. They were first considered as models for the symmetry groups of several physical systems, but they quickly gained a purely mathematical interest as the symmetry groups of many different geometries.

Definition 1.1.1. *A **Lie group** is a smooth manifold G endowed with a group structure such that the group operation*

$$\begin{cases} G \times G & \to & G \\ (x,y) & \mapsto & xy \end{cases}$$

is smooth.

Remarks.

- *A manifold is not necessarily connected.*
- *This is the definition of a real Lie group. One can also define a complex Lie group using complex manifolds and requiring holomorphicity of the group operation. This class only presents real Lie groups, but it is a good exercise to check which properties and which proofs also hold in the complex setting.*
- *Here smooth means C^∞. Requiring everything to be real analytic leads to the same theory.*
- *Quite often, smoothness of the inverse map $x \mapsto x^{-1}$ is also required. However, we will see that it is a consequence of this definition.*

1.1.1 Examples of Lie groups

1. Countable groups are exactly 0-dimensional Lie groups.

2. $(\mathbb{R},+)$, $(\mathbb{R}^*,\times)$ and $(\mathbb{U},\times)$ where $\mathbb{U}=\{z\in\mathbb{C}:|z|=1\}$ are 1-dimensional Lie groups. Similarly, $(\mathbb{C},+)$ and $(\mathbb{C}^*,\times)$ are 1-dimensional complex Lie groups (and 2-dimensional real Lie groups).

3. The additive group of a real (resp. complex) vector space is a real (resp. complex) Lie group.

4. A complex Lie group can be seen as a real Lie group.

5. The product of Lie groups is a Lie group.

6. Given a finite dimensional real vector space V, the general linear group $\mathrm{GL}(V)$ is a Lie group. Indeed, it is a open subset of $\mathrm{End}(V)$, hence a manifold. The group operation is smooth because it is the restriction of a bilinear map between finite dimensional vector spaces.

7. For the same reasons, $\mathrm{GL}(n,\mathbb{R})$ and $\mathrm{GL}(n,\mathbb{C})$ are Lie groups ($\mathrm{GL}(n,\mathbb{C})$ being a complex Lie group).

8. Classical Lie groups: let n, p and q be positive integers. Let $I_n\in\mathcal{M}_n(\mathbb{R})$ be the identity matrix, and

$$I_{p,q}=\begin{pmatrix} I_p & 0 \\ 0 & -I_q \end{pmatrix}\in\mathcal{M}_{p+q}(\mathbb{R}),\quad J_n=\begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}\in\mathcal{M}_{2n}(\mathbb{R}).$$

$$\begin{array}{lcl}
\mathrm{SL}(n,\mathbb{R}) & = & \{x\in \mathrm{GL}(n,\mathbb{R})\,|\,\det x = 1\}\\
\mathrm{SL}(n,\mathbb{C}) & = & \{x\in \mathrm{GL}(n,\mathbb{C})\,|\,\det x = 1\}\\
\mathrm{SL}(n,\mathbb{H}) & = & \{x\in \mathcal{M}_n(\mathbb{H})\,|\,\det x = 1\}\\
\mathrm{O}(n,\mathbb{R}) & = & \left\{x\in \mathrm{GL}(n,\mathbb{R})\,\middle|\,{}^t x\, x = I_n\right\}\\
\mathrm{SO}(n,\mathbb{R}) & = & \mathrm{O}(n,\mathbb{R})\cap \mathrm{SL}(n,\mathbb{R})\\
\mathrm{O}(p,q) & = & \left\{x\in \mathrm{GL}(p+q,\mathbb{R})\,\middle|\,{}^t x\, I_{p,q}\, x = I_{p,q}\right\}\\
\mathrm{SO}(p,q) & = & \mathrm{O}(p,q)\cap \mathrm{SL}(p+q,\mathbb{R})\\
\mathrm{O}(n,\mathbb{C}) & = & \left\{x\in \mathrm{GL}(n,\mathbb{C})\,\middle|\,{}^t x\, x = I_n\right\}\\
\mathrm{SO}(n,\mathbb{C}) & = & \mathrm{O}(n,\mathbb{C})\cap \mathrm{SL}(n,\mathbb{C})\\
\mathrm{SO}(n,\mathbb{H}) & = & \left\{x\in \mathrm{SL}(n,\mathbb{H})\,\middle|\,{}^t \overline{x}\, x = I_n\right\}\\
\mathrm{SO}^*(2n) & = & \left\{x\in \mathrm{SO}(2n,\mathbb{C})\,\middle|\,{}^t \overline{x}\, J_n\, x = J_n\right\}\\
\mathrm{U}(n) & = & \left\{x\in \mathrm{GL}(n,\mathbb{C})\,\middle|\,{}^t \overline{x}\, x = I_n\right\}\\
\mathrm{SU}(n) & = & \mathrm{U}(n)\cap \mathrm{SL}(n,\mathbb{C})\\
\mathrm{U}(p,q) & = & \left\{x\in \mathrm{GL}(p+q,\mathbb{C})\,\middle|\,{}^t \overline{x}\, I_{p,q}\, x = I_{p,q}\right\}\\
\mathrm{SU}(p,q) & = & \mathrm{U}(p,q)\cap \mathrm{SL}(p+q,\mathbb{C})\\
\mathrm{SU}^*(2n) & = & \{x\in \mathrm{SL}(2n,\mathbb{C})\,|\,J_n\, x = \overline{x} J_n\}\\
\mathrm{Sp}(2n,\mathbb{R}) & = & \left\{x\in \mathrm{GL}(2n,\mathbb{R})\,\middle|\,{}^t x\, J_n\, x = J_n\right\}\\
\mathrm{Sp}(2n,\mathbb{C}) & = & \left\{x\in \mathrm{GL}(2n,\mathbb{C})\,\middle|\,{}^t x\, J_n\, x = J_n\right\}\\
\mathrm{Sp}(p,q) & = & \left\{x\in \mathcal{M}_{p+q}(\mathbb{H})\,\middle|\,{}^t \overline{x}\, I_{p,q}\, x = I_{p,q}\right\}\\
\mathrm{Sp}(n) & = & \left\{x\in \mathcal{M}_n(\mathbb{H})\,\middle|\,{}^t \overline{x}\, x = I_n\right\}\\
\mathrm{Sp}_*(n) & = & \mathrm{Sp}(2n,\mathbb{C})\cap \mathrm{U}(2n)
\end{array}.$$

Here $\mathbb{H}$ is the quaternion algebra: the associative 4-dimensional $\mathbb{R}$-algebra possessing a basis $(1,i,j,k)$ such that 1 is a unit, $i^2 = j^2 = k^2 = -1$ and $ij = -ji = k$.

1.1.2 Multiplication and inverse maps

Given a Lie group G and an element $g \in G$, we let L_g (resp. R_g) denote the left (resp. right) multiplication by g, i.e. the maps:

$$L_g : \left\{ \begin{array}{ccc} G & \to & G \\ x & \mapsto & gx \end{array} \right. \text{ and } R_g : \left\{ \begin{array}{ccc} G & \to & G \\ x & \mapsto & xg \end{array} \right.$$

Note that L_g and R_g are diffeomorphisms, their inverses being $(L_g)^{-1} = L_{g^{-1}}$ and $(R_g)^{-1} = R_{g^{-1}}$.

For any $g, h \in G$, we have that $L_g \circ R_h = R_h \circ L_g$ (this is a rewriting of the associativity of G).

Consider the maps:

$$m : \left\{ \begin{array}{ccc} G \times G & \to & G \\ (x,y) & \mapsto & xy \end{array} \right. \text{ and inv} : \left\{ \begin{array}{ccc} G & \to & G \\ x & \mapsto & x^{-1} \end{array} \right.$$

Smoothness of m is part of the definition of a Lie group. The map inv is also smooth.

Proposition 1.1.2. *Let G be a Lie group. The inverse map* inv : $G \to G$ *is smooth.*

Proof. Let $x_0 \in G$. The partial differential of m with respect to its second variable at the point (x_0, x_0^{-1}) is $d_{x_0^{-1}} L_{x_0}$, therefore invertible. According to the Implicit Function Theorem, there are an open neighbourhood $U \subset G$ of x_0 and a smooth map $\varphi : U \to G$ such that $m(x, \varphi(x)) = e$ for any $x \in U$. Since $x\varphi(x) = e$, we find that $\varphi(x) = x^{-1}$, and inv is smooth on U. □

This implies that a Lie group is a topological group.

Definition 1.1.3. *A **topological group** is a topological space G endowed with a group structure such that the maps* $\left\{ \begin{array}{ccc} G \times G & \to & G \\ (x,y) & \mapsto & xy \end{array} \right.$ *and* $\left\{ \begin{array}{ccc} G & \to & G \\ x & \mapsto & x^{-1} \end{array} \right.$ *are continuous.*

Note that in the topological setting, continuity of the inverse map is not a consequence of the continuity of the multiplication map.

Corollary 1.1.4. *Every Lie group is a topological group.*

Remark. *The fact that the smoothness of* inv *is usually included in the definition of a Lie group is a habit coming from topological groups.*

It should also be a part of the definitions of some infinite dimensional generalizations of Lie groups (especially if we want a group such as Diff(M) *to be a Lie group, where M is a manifold), but we will not discuss this further.*

We will now see that up to order one, there is no distinction between $\mathbb{R}^n$ and any other Lie group.

Proposition 1.1.5. *Let G be a Lie group. For $X, Y \in T_eG$, we have that:*

$$d_{(e,e)}m(X,Y) = X + Y \text{ and } d_e\mathrm{inv}(X) = -X$$

Proof. Differentiating $m(x,e) = x$ (resp. $m(e,y) = y$), we find that $d_{(e,e)}m(X,0) = X$ (resp. $d_{(e,e)}m(0,Y) = Y$), hence:

$$d_{(e,e)}m(X,Y) = d_{(e,e)}m(X,0) + d_{(e,e)}m(0,Y) = X + Y$$

Differentiating $m(x, \mathrm{inv}(x)) = e$ at e now yields $X + d_e\mathrm{inv}(X) = 0$. □

This tells us that in order to tell Lie groups apart using infinitesimal quantities, we will need to work with second order differentials, which is why the study of Lie groups has a geometric nature: as we will see in the other chapters of this course, differential geometry treats second order differentials as curvature.

Before going further, we should also have in mind that only few manifolds can carry a Lie group structure, as there are some strong topological obstructions.

Proposition 1.1.6. *If G is a Lie group, then the tangent bundle TG is trivialisable (i.e. G is parallelisable).*

Proof. The map

$$\begin{cases} G \times T_e G & \to & TG \\ (g,v) & \mapsto & (g, d_e L_g(v)) \end{cases}$$

is a trivialisation of TG. It is smooth because $d_e L_g(v) = d_{(g,e)} m(0,v)$, and its inverse is $(g,w) \mapsto (g, d_{(g^{-1},g)} m(0,w))$ which is also smooth because m and inv are. □

As a consequence, there is no Lie group structure on $\mathbb{S}^2$.

1.1.3 Lie group morphisms

Since we will talk a lot about morphisms between Lie groups, we need to make the definitions clear.

Definition 1.1.7. *Let G and H be Lie groups. A **Lie group morphism** from G to H is a smooth map $f : G \to H$ which is a group homomorphism.*

*A **linear representation** of G is a Lie group morphism $f : G \to \mathrm{GL}(V)$ where V is a finite dimensional real vector space.*

*A **matrix representation** of G is a Lie group morphism $f : G \to \mathrm{GL}(n,\mathbb{R})$ or $f : G \to \mathrm{GL}(n,\mathbb{C})$ for some $n \in \mathbb{N}$.*

*A **representation** of G is either a linear representation or a matrix representation.*

*A **Lie group isomorphism** between G and H is a smooth diffeomorphism $f : G \to H$ which is a group isomorphism. When $G = H$, we call f a **Lie group automorphism**.*

Remark. *The fact that f is a group homomorphism translates as $f \circ L_g = L_{f(g)} \circ f$ for all $g \in G$. Note that we use the same notation $L_g : G \to G$ and $L_{f(g)} : H \to H$ as there is very little risk for confusion.*

Recall that a smooth map $f : M \to N$ has **constant rank** if the rank of $d_x f : T_x M \to T_{f(x)} N$ does not depend on $x \in M$. Recall that a constant rank map is **linearisable** (i.e. for any $x_0 \in M$, there are diffeomorphisms p, ψ such that $f = \psi \circ d_{x_0} f \circ \varphi$ near x_0).
A constant rank map is a **submersion** if its differential is always onto, an **immersion** if its differential is always injective.

Exercise.

1. *If $f : M \to N$ has constant rank and is injective, then it is an immersion.*
2. *If $f : M \to N$ has constant rank and is surjective, then it is a submersion.*

Hint: show that the image of a constant rank map which is not a submersion has empty interior using Baire's theorem.

Proposition 1.1.8.

1. *A Lie group morphism has constant rank.*
2. *A bijective Lie group morphism is a Lie group isomorphism.*

Proof. 1. Given $g \in G$, differentiating the expression $f \circ L_g = L_{f(g)} \circ f$ at e yields:

$$d_g f \circ d_e L_g = d_{f(g)} L_{f(g)} \circ d_e f$$

Since L_g and $L_{f(g)}$ are diffeomorphisms, it follows that $d_g f$ and $d_e f$ have the same rank.

2. As a consequence of the previous statement and of the result of the exercise, a bijective Lie group morphism is a local diffeomorphism, therefore a diffeomorphism.

□

1.1.4 The adjoint representation of a Lie group

Let G be a Lie group. Given $g \in G$, we denote by $i_g = L_g \circ R_{g^{-1}}$ the conjugacy by g (i.e. $i_g(x) = gxg^{-1}$). It is a Lie group automorphism of G.

Definition 1.1.9. *Let G be a Lie group. Given $g \in G$, we set $\mathrm{Ad}(g) = d_e i_g : T_e G \to T_e G$. The map*

$$\mathrm{Ad} : G \to \mathrm{GL}(T_e G)$$

*is called the **adjoint representation** G.*

Note that Ad is indeed a representation.

Definition 1.1.10. *The **centre** of G is:*

$$Z(G) = \{g \in G | \forall x \in G\, gx = xg\}$$

Note that we always have $Z(G) \subset \ker \mathrm{Ad}$. However it is not always an equality.

1.2 Lie algebras

Definition 1.2.1. *Let $\mathbb{K}$ be a field. A **Lie algebra** $\mathfrak{g}$ over $\mathbb{K}$ is a pair $(V,[\cdot,\cdot])$ where V is a vector space over $\mathbb{K}$ and $[\cdot,\cdot]:\mathfrak{g}\times\mathfrak{g}\to\mathfrak{g}$ is a bilinear map satisfying:*

1. $\forall X,Y \in V\,[Y,X] = -[X,Y]$
2. $\forall X,Y,Z \in V\,[X,[Y,Z]]+[Y,[Z,X]]+[Z,[X,Y]] = 0$

We will only consider the cases $\mathbb{K}=\mathbb{R}$ and $\mathbb{K}=\mathbb{C}$. A bilinear map $[\cdot,\cdot]$ satisfying *1.* and *2.* is called a **Lie bracket**. Condition *2.* is called the **Jacobi identity**.

Remarks.

- *The vector space V need not be finite dimensional. We will actually consider one infinite dimensional example.*
- *We will use the same notation for $\mathfrak{g}$ and V.*

Note that properties of morphisms between Lie algebras are more straightforward.

Definition 1.2.2. *Let $\mathfrak{g},\mathfrak{h}$ be Lie algebras over a field $\mathbb{K}$. A **Lie algebra morphism** from $\mathfrak{g}$ to $\mathfrak{h}$ is a $\mathbb{K}$-linear map $\varphi:\mathfrak{g}\to\mathfrak{h}$ such that $\forall X,Y\in\mathfrak{g}\,[\varphi(X),\varphi(Y)] = [X,Y]$.*
*A **Lie algebra isomorphism** is a bijective Lie algebra morphism (its inverse is also a Lie algebra morphism). When $\mathfrak{g}=\mathfrak{h}$, a Lie algebra isomorphism is called a **Lie algebra automorphism**.*
*A **Lie subalgebra** of $\mathfrak{g}$ is a vector subspace $V\subset\mathfrak{g}$ such that $[X,Y]\in V$ for all $X,Y\in V$.*

1.2.1 First examples of Lie algebras

1. Any vector space V can be endowed with the Lie bracket defined by $[X,Y]=0$ for all $X,Y\in V$. Such a Lie algebra $(V,[\cdot,\cdot])$ is called **abelian**.

2. Let $\mathcal{A}$ be an associative algebra over $\mathbb{K}$. For $X,Y\in\mathcal{A}$, we set $[X,Y]=XY-YX$. Tedious yet easy computations show that it is a Lie bracket. This applies to the associative algebras $\mathrm{End}(V)$ and $\mathcal{M}_n(\mathbb{K})$. These Lie algebras will be denoted respectively by $\mathfrak{gl}(V)$ ans $\mathfrak{gl}(n,\mathbb{K})$.
   A **representation** of a Lie algebra $\mathfrak{g}$ is a Lie algebra morphism $\varphi:\mathfrak{g}\to\mathrm{End}(V)$ where V is a vector space (or $\varphi:\mathfrak{g}\to\mathfrak{gl}(n,\mathbb{K})$).

   Remark. *Note that we do not ask representations of Lie algebras to be finite dimensional, while we do for Lie groups (so that a Lie group representation is a Lie group morphism).*

3. If $\mathfrak{g}$ is a complex Lie algebra, we denote by $\mathfrak{g}_{\mathbb{R}}$ the underlying Lie algebra (obtained by considering $\mathfrak{g}$ as a real vector space). If $\mathfrak{g}$ is a real Lie algebra, then $\mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C}$ is a complex Lie algebra when endowed with the only Lie bracket satisfying $[X \otimes z, X' \otimes z'] = [X, X'] \otimes zz'$ for all $X, X' \in \mathfrak{g}$ and $z, z' \in \mathbb{C}$.

4. The product $\mathfrak{g}_1 \times \cdots \times \mathfrak{g}_k$ of Lie algebras is a Lie algebra with bracket:

$$[(X_1, \dots, X_k), (Y_1, \dots, Y_k)] = ([X_1, Y_1], \dots, [X_k, Y_k])$$

5. The image and the kernel of a Lie algebra morphism are Lie subalgebras.

1.2.2 The Lie algebra of a Lie group

Let G be a Lie group. We have already defined the adjoint representation $\mathrm{Ad} : G \to \mathrm{GL}(T_e G)$ of G. We set $\mathrm{ad} = d_e \mathrm{Ad} : T_e G \to \mathrm{End}(T_e G) = T_{\mathrm{Id}} \mathrm{GL}(T_e G)$.

Proposition 1.2.3. *Let G be a Lie group. The map*

$$[\cdot, \cdot] : \begin{cases} T_e G \times T_e G & \to & T_e G \\ (X, Y) & \mapsto & \mathrm{ad}(X) Y \end{cases}$$

is a Lie bracket.

Let us start with two elementary examples.

- If G is abelian, then $i_g = \mathrm{Id}$ for any $g \in G$, which yields $\mathrm{Ad}(g) = \mathrm{Id}$, hence $\mathrm{ad} = 0$. Therefore $[X, Y] = \mathrm{ad}(X) Y$ is a Lie bracket, and $(T_e G, [\cdot, \cdot])$ is an abelian Lie algebra.
- If $G = \mathrm{GL}(V)$, then $T_{\mathrm{Id}} G = \mathrm{End}(V)$ (since $\mathrm{GL}(V)$ is an open subset of $\mathrm{End}(V)$). Given $g \in G$, the map $i_g : G \to G$ is the restriction to G of a linear map defined on $\mathrm{End}(V)$, its differential is $\mathrm{Ad}(g) Y = g Y g^{-1}$. Differentiating with respect to g at Id, yields $\mathrm{ad}(X) Y = XY - YX$, the usual Lie bracket on $\mathrm{End}(V)$.

Proof. The bracket is bilinear because differentials are linear maps. To prove anti-symmetry, consider elements $X, Y \in T_e G$ and paths $x, y : \mathbb{R} \to G$ such that $x(0) = y(0) = e$, $\dot{x}(0) = X$ and $\dot{y}(0) = Y$.

Consider the function:

$$c : \begin{cases} \mathbb{R}^2 & \to & G \\ (s, t) & \mapsto & x(s) y(t) x(s)^{-1} y(t)^{-1} \end{cases}$$

The element $c(s, t)$ is the commutator of $x(s)$ and $y(t)$ (usually denoted by $[x(s), y(t)]$).

Notice that for all $s, t \in \mathbb{R}$ we have $c(s,0) = c(0,t)$. It follows that the first derivatives $\frac{\partial c}{\partial s}(0,t)$ and $\frac{\partial c}{\partial t}(s,0)$ are elements of T_eG, and so are the crossed second order derivatives $\frac{\partial^2 c}{\partial s \partial t}(0,0)$ and $\frac{\partial^2 c}{\partial t \partial s}(0,0)$. The Schwarz Lemma ensures that they are equal.

Let us compute $\frac{\partial c}{\partial t}(s,0)$. Since $c(s,t) = i_{x(s)}(y(t))y(t)^{-1} = m(i_{x(s)}(y(t)), \mathrm{inv}(y(t)))$, we find:

$$\frac{\partial c}{\partial t}(s,0) = \mathrm{Ad}(x(s))Y - Y$$

It follows that $\frac{\partial^2 c}{\partial s \partial t}(0,0) = \mathrm{ad}(X)Y$.

Let us now compute the same second order derivative in the opposite order. We now use that $c(s,t) = x(s) i_{y(t)}(x(s)^{-1}) = m(x(s), i_{y(t)}(\mathrm{inv}(x(s))))$.

$$\frac{\partial c}{\partial s}(0,t) = X - \mathrm{Ad}(y(t))X$$

It follows that $\frac{\partial^2 c}{\partial s \partial t}(0,0) = -\mathrm{ad}(Y)X$, and the equality between crossed derivatives yields $\mathrm{ad}(X)Y = -\mathrm{ad}(Y)X$, i.e. $[X,Y] = -[Y,X]$.

In order to prove the Jacobi identity, we will start by considering the behaviour of the bracket under Lie group morphisms.

Lemma 1.2.4. *If G, H are Lie groups and $f : G \to H$ is a Lie group morphism, then:*

- $\forall g \in G \ \mathrm{Ad}(f(g)) \circ d_e f = d_e f \circ \mathrm{Ad}(g)$.
- $\forall X \in T_eG \ \mathrm{ad}(d_e f(X)) \circ d_e f = d_e f \circ \mathrm{ad}(X)$.

  i.e. $\forall Y \in T_eG \ [d_e f(X), d_e f(Y)] = d_e f([X,Y])$.

Proof of Lemma 1.2.4. Since f is a Lie group morphism, we have that $f \circ i_g = i_{f(g)} \circ f$. Differentiating these two maps at e yields the first point. Differentiating with respect to g at e yields the second. □

Given $g \in G$, we can apply Lemma 1.2.4 in the case where $H = G$ and $f = i_g$, the second point leads to:

$$\forall X \in T_eG \ \mathrm{ad}(\mathrm{Ad}(g)X) \circ \mathrm{Ad}(g) = \mathrm{Ad}(g) \circ \mathrm{ad}(X)$$

Which translates as:

$$\forall X, Y \in T_eG \, [\mathrm{Ad}(g)X, \mathrm{Ad}(g)Y] = \mathrm{Ad}(g)[X,Y]$$

Differentiating this identity with respect to g at e, along the tangent vector $Z \in T_eG$, we find:

$$[\mathrm{ad}(Z)X, Y] + [X, \mathrm{ad}(Z)Y] = \mathrm{ad}(Z)[X,Y]$$

Reordering the terms leads to the Jacobi identity. □

1.2.3 Structural constants

Let $\mathfrak{g}$ be a finite dimensional Lie algebra, and $e = (e_1, \dots, e_d)$ be a basis of $\mathfrak{g}$. We can decompose brackets between elements of e in the basis e:

$$[e_i, e_j] = \sum_{k=1}^{d} C_{i,j}^k e_k$$

The numbers $(C_{i,j}^k)_{1 \le i,j,k \le d}$ are called the **structural constants** of $\mathfrak{g}$ (in the basis $(e_1, \dots, e_d)$).

Knowing the structural constants of a Lie algebra makes for easy computations (especially if there is a basis in which they have very simple expressions, e.g. most of them vanish).

One can use a basis to show that a map $\mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ is a Lie bracket. Anti-symmetry has a simple expression:

$$\forall i, j, k \in \{1, \dots, d\} \quad C_{i,j}^k + C_{j,i}^k = 0,$$

The Jacobi identity is more difficult to handle in this form:

$$\forall i_1, i_2, i_3, j \in \{1, \dots, d\} \quad \sum_{k=1}^{d} \left(C_{i_1,i_2}^k C_{i_3,k}^j + C_{i_2,i_3}^k C_{i_1,k}^j + C_{i_3,i_1}^k C_{i_2,k}^j \right) = 0.$$

However, small dimensions allow for simpler proofs. In dimension 2, anti-symmetry implies the Jacobi identity. In dimension 3, once anti-symmetry is proven, it is sufficient to prove the Jacobi identity for one linearly independent family (X, Y, Z). In terms of structural constants, we only have to check three equations:

$$\forall j \in \{1, 2, 3\} \quad \sum_{k=1}^{d} \left(C_{1,2}^k C_{3,k}^j + C_{2,3}^k C_{1,k}^j + C_{3,1}^k C_{2,k}^j \right) = 0.$$

1.2.4 The adjoint representation of a Lie algebra

Definition 1.2.5. *Let $\mathfrak{g}$ be a Lie algebra. For $X \in \mathfrak{g}$, we let $\mathrm{ad}(X) : \mathfrak{g} \to \mathfrak{g}$ be the map defined by $\mathrm{ad}(X)Y = [X, Y]$. We call $\mathrm{ad} : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$ the* ***adjoint representation*** *of $\mathfrak{g}$.*

A simple rewriting of the Jacobi identity shows that ad is a Lie algebra morphism.

In the case of the Lie algebra of a Lie group, both definitions coincide.

Definition 1.2.6. *The* ***centre*** *of $\mathfrak{g}$ is*

$$\mathfrak{z}(\mathfrak{g}) = \{X \in \mathfrak{g} | \forall Y \in \mathfrak{g} \; [X, Y] = 0\}$$

By definition of ad, we have that $\mathfrak{z}(\mathfrak{g}) = \ker \mathrm{ad}$.

1.2.5 Derivations of a Lie algebra

In order to understand the nature of the map $\mathrm{ad}(X)$ for a given $X \in \mathfrak{g}$, we have to look at derivations of Lie algebras.

Definition 1.2.7. *Let $\mathfrak{g}$ be a Lie algebra. A* ***derivation*** *of $\mathfrak{g}$ is a linear map $\delta : \mathfrak{g} \to \mathfrak{g}$ satisfying*

$$\forall X, Y \in \mathfrak{g} \; \delta[X,Y] = [\delta X, Y] + [X, \delta Y]$$

We denote by $\mathrm{Der}(\mathfrak{g}) \subset \mathfrak{gl}(\mathfrak{g})$ the set of derivations of $\mathfrak{g}$.

Proposition 1.2.8.

- $\mathrm{Der}(\mathfrak{g})$ *is a Lie subalgebra of* $\mathfrak{gl}(\mathfrak{g})$.
- *For any $X \in \mathfrak{g}$,* $\mathrm{ad}(X) \in \mathrm{Der}(\mathfrak{g})$.

Proof. If $\delta, \delta' \in \mathrm{Der}(\mathfrak{g})$ ans $X, Y \in \mathfrak{g}$, we find:

$$\delta \circ \delta'[X,Y] = [\delta \circ \delta' X, Y] + [\delta X, \delta' Y] + [\delta' X, \delta Y] + [X, \delta \circ \delta' Y]$$

This leads to $[\delta, \delta'] \in \mathrm{Der}(\mathfrak{g})$.
The fact that $\mathrm{ad}(X)$ is a derivation is a rewriting of the Jacobi identity. □

1.2.6 The Killing form of a Lie algebra

A very remarkable property of finite dimensional Lie algebras is that they come with a symmetric bilinear form, which will be an important tool in the structure theory of Lie algebras.

Definition 1.2.9. *Let $\mathfrak{g}$ be a finite dimensional Lie algebra. The Killing form of $\mathfrak{g}$ is the bilinear form B on $\mathfrak{g}$ defined by*

$$\forall X, Y \in \mathfrak{g} \; B(X,Y) = \mathrm{Tr}(\mathrm{ad}(X) \circ \mathrm{ad}(Y)).$$

Proposition 1.2.10. *Let $\mathfrak{g}$ be a finite dimensional Lie algebra, and B its Killing form. Then B is symmetric and* ad*-invariant, i.e.*

$$\forall X, Y, Z \in \mathfrak{g} \; B(\mathrm{ad}(X)Y, Z) + B(Y, \mathrm{ad}(X)Z) = 0.$$

Proof. The symmetry of B is a consequence of the symmetry of the bilinear map $A, B \mapsto \mathrm{Tr}(AB)$ on $\mathfrak{gl}(\mathfrak{g})$. The *ad*-invariance of B is a consequence of the fact that ad is a Lie algebra morphism:

$$\begin{aligned} B(\mathrm{ad}(X)Y, Z) &= \mathrm{Tr}(\mathrm{ad}([X,Y]) \circ \mathrm{ad}(Z)) \\ &= \mathrm{Tr}((\mathrm{ad}(X) \circ \mathrm{ad}(Y) - \mathrm{ad}(Y) \circ \mathrm{ad}(X)) \circ \mathrm{ad}(Z)) \\ &= \mathrm{Tr}(\mathrm{ad}(X) \circ \mathrm{ad}(Y) \circ \mathrm{ad}(Z)) - \mathrm{Tr}(\mathrm{ad}(Y) \circ \mathrm{ad}(X) \circ \mathrm{ad}(Z)) \\ &= \mathrm{Tr}(\mathrm{ad}(Y) \circ \mathrm{ad}(Z) \circ \mathrm{ad}(X)) - \mathrm{Tr}(\mathrm{ad}(Y) \circ \mathrm{ad}(X) \circ \mathrm{ad}(Z)) \\ &= \mathrm{Tr}(\mathrm{ad}(Y) \circ (\mathrm{ad}(Z) \circ \mathrm{ad}(X) - \mathrm{ad}(X) \circ \mathrm{ad}(Z))) \\ &= \mathrm{Tr}(\mathrm{ad}(Y) \circ \mathrm{ad}([Z,X])) \\ &= -B(Y, \mathrm{ad}(X)Z) \end{aligned}$$

□

Remark. *Here* ad*-invariance means that* $\mathrm{ad}(\mathfrak{g})$ *is included in the Lie algebra* $\mathfrak{o}(B)$ *of the Lie group* $\mathrm{O}(B)$ *of linear isomorphisms of* $\mathfrak{g}$ *preserving* B.

1.3 The Lie algebra of a Lie group

Definition 1.3.1. *Let* G *be a Lie group. The Lie algebra* $(T_eG, [\cdot,\cdot])$ *defined by* $[X,Y] = \mathrm{ad}(X)Y$ *is called the* ***Lie algebra of*** G*, it will be denoted by* $\mathrm{Lie}(G)$ *or* $\mathfrak{g}$.

We now wish to understand how a Lie group and its Lie algebra work together. The main goal is see how complicated calculations in a Lie group can boil down to some simple linear algebra in its Lie algebra.

1.3.1 Locally isomorphic Lie groups

A Lie group morphism induces a Lie algebra morphism.

Proposition 1.3.2. *Let* G, H *be Lie groups with respective Lie algebras* $\mathfrak{g}, \mathfrak{h}$*, and* $f : G \to H$ *a Lie group morphism. The map* $d_e f : \mathfrak{g} \to \mathfrak{h}$ *is a Lie algebra morphism.*
Moreover, if f *is a Lie groups isomorphism, then* $d_e f$ *is a Lie algebra isomorphism.*

Proof. The fact that $d_e f$ is a Lie algebra morphism is exactly the second point of lemma 1.2.4. If f is a Lie grop isomorphism, then $d_e(f^{-1}) = (d_e f)^{-1}$ is also a Lie algebra morphism, so $d_e f$ is a Lie algebra isomorphism. □

This tells us that isomorphic Lie groups have isomorphic Lie algebras. The converse is false: $\mathbb{R}$ and $\mathbb{R}/\mathbb{Z}$ are not isomorphic, but their Lie algebras are.

Definition 1.3.3. *Two Lie groups are called* ***locally isomorphic*** *if their Lie algebras are isomorphic.*

It is also possible to check that $\mathrm{SL}(2,\mathbb{R})$ and $\mathrm{PSL}(2,\mathbb{R})$ are locally isomorphic, but not isomorphic. We will see that it is always the case for Lie groups that are coverings of Lie groups.

Proposition 1.3.4. *Let* G, H *be Lie groups with respective Lie algebras* $\mathfrak{g}, \mathfrak{h}$*, and* $f : G \to H$ *a Lie group morphism. If* f *is a local diffeomorphism, then* $d_e f : \mathfrak{g} \to \mathfrak{h}$ *is a Lie algebra isomorphism.*

Proof. The map $d_e f$ is a Lie algebra morphism and a linear isomorphism, hence a Lie algebra isomorphism. □

1.3.2 The identity component of a Lie group

Given a Lie group G and its neutral element e, we denote by $G_\circ$ the connected component of e in G, called the **neutral component** or **identity component**.

Proposition 1.3.5. *Let G be a Lie group. The neutral component $G_\circ$ is a normal subgroup of G. It is both open and closed in G.*

Remarks.

- *Since it is open, it is also a manifold of the same dimension as G, and a Lie group.*
- *The quotient group $G/G_\circ$ can be identified with the set of connected components of G. It is usually denoted by $\pi_0(G)$. The quotient topology is the discrete topology.*

(because $G_\circ$ is open).

Proof. The image of $G_\circ \times G_\circ$ under the continuous map $(x,y) \mapsto xy^{-1}$ is connected, hence a subset of $G_\circ$, which shows that $G_\circ$ is a subgroup.
Given $g \in G$, the set $gG_\circ g^{-1} = L_g \circ R_{g^{-1}}(G_\circ)$ is connected, hence a subset of $G_\circ$, which shows that $G_\circ$ is a normal subgroup.
Closedness and openness of $G_\circ$ are consequences of the more general facts that connected components of a topological space are always closed, and connected components of a manifold are always open. □

By studying the Lie algebra, we cannot see the difference between a Lie group and its identity component.

Proposition 1.3.6. *Let G be a Lie group. The identity component $G_\circ$ is locally isomorphic to G.*

Proof. One can apply Proposition 1.3.4 to the inclusion $G_\circ \hookrightarrow G$. □

Studying the Lie algebra of a Lie group will, at first glance, only allow us to understand the group near the identity element. However, for connected groups, this will be enough to recover the whole algebraic structure.

Proposition 1.3.7. *If G is a connected Lie group, then any neighbourhood of e generates G as a group.*

The proof will use a well known fact about open subgroups of topological groups.

Lemma 1.3.8. *If G is a topological group and $H \subset G$ is an open subgroup, then G is closed.*

Proof. Simply notice that $G \setminus H = \bigcup_{g \notin H} gH$ is also open. □

Proof of Proposition 1.3.7. Let $V \subset G$ be a neighbourhood of e, and H be the subgroup of G generated by V. It is an open subgroup of G, hence closed because of lemma 1.3.8. Since G is connected, we find that $H = G$. □

1.3.3 The action of a Lie group on its Lie algebra

The adjoint representation of a Lie group is an action on its Lie algebra. The first important point is that it preserves the Lie bracket.

Proposition 1.3.9. *Let G be a Lie group with Lie algebra $\mathfrak{g}$. For all $g \in G$, the map $\mathrm{Ad}(g) \in \mathrm{GL}(\mathfrak{g})$ is a Lie algebra automorphism of $\mathfrak{g}$.*

Proof. Since $\mathrm{Ad} : G \to \mathrm{GL}(\mathfrak{g})$ is a representation, it is enough to check that $\mathrm{Ad}(g)$ is a Lie algebra morphism. This was already proved in Lemma 1.2.4 (it is the second point applied to $f = i_g$ for some $g \in G$, as discussed in the proof of proposition 1.2.3).

□

Note that the Lie algebra of a Lie group is finite dimensional, so it is possible to define its Killing form (Definition 1.2.9).

Proposition 1.3.10. *Let G be a Lie group with Lie algebra $\mathfrak{g}$. For all $g \in G$, the map $\mathrm{Ad}(g)$ preserves the Killing form of $\mathfrak{g}$, i.e.*

$$\forall X, Y \in \mathfrak{g}\ B(\mathrm{Ad}(g)X, \mathrm{Ad}(g)Y) = B(X, Y).$$

Proof. The result of proposition 1.3.9 translates as

$$\mathrm{ad}(\mathrm{Ad}(g)X) = \mathrm{Ad}(g) \circ \mathrm{ad}(X) \circ \mathrm{Ad}(g)^{-1}$$

for $g \in G$ and $X \in \mathfrak{g}$. Invariance of the Killing form by $\mathrm{Ad}(g)$ is a consequence of the invariance of the trace by conjugation in $\mathrm{GL}(\mathfrak{g})$. □

Remark. *This means that the adjoint representation of a Lie group G lands in the subgroup $\mathrm{O}(B)$ of $\mathrm{GL}(\mathfrak{g})$.*

1.3.4 Reminders on vector fields

Let M be a manifold and $X \in \mathcal{X}(M) = \Gamma(TM)$ a vector field M. The flow φ_X^t of X is the maximal solution to the ordinary differential equation:

$$\begin{cases} \frac{d}{ds}\Big|_{s=t} \varphi_X^s(x) & = & X(\varphi_X^t(x)) \\ \varphi_X^0(x) & = & x \end{cases}$$

A vector field is **complete** it its flow is defined for all times.
If $f : M \to N$ is a local diffeomorphism and $X \in \mathcal{X}(N)$, the **pull-back** $f^*X \in$

$\mathcal{X}(M)$ is defined by $f^*X(x) = (d_x f)^{-1}(X(f(x)))$.

The flow of f^*X is related to the flow of X by the formula:

$$\varphi^t_{f^*X} = f^{-1} \circ \varphi^t_X \circ f$$

The **Lie bracket of vector fields** $X, Y \in \mathcal{X}(M)$ is:

$$[X,Y] = \left.\frac{d}{dt}\right|_{t=0} \left(\varphi^t_X\right)^* Y$$

The right hand term is also the Lie derivative $\mathcal{L}_X Y$ (which can be defined for any type of tensor replacing Y).

In local coordinates, the Lie bracket of vector fields can be computed through the formula:

$$[X,Y](x) = d_x Y(X(x)) - d_x X(Y(x))$$

A simple computation shows that for any $f \in \mathcal{C}^\infty(M)$, we have that:

$$[X,Y]\cdot f = X\cdot(Y\cdot f) - Y\cdot(X\cdot f)$$

This formula shows that $\mathcal{X}(M)$ endowed with the Lie bracket of vector fields is a Lie algebra (more precisely, a Lie subalgebra of $\mathrm{End}(\mathcal{C}^\infty(M))$).

1.3.5 Left invariant vector fields

If G is a Lie group, then G acts on the space of vector fields $\mathcal{X}(G)$ by $g.X = L_g^* X$ for $g \in G$ and $X \in \mathcal{X}(G)$. Vector fields which are invariant under this action are called **left-invariant**. We will denote by ${}^G\mathcal{X}(G)$ the space of left-invariant vector fields on G.

Proposition 1.3.11. *The G be a lie group. The space of left-invariant vector fields ${}^G\mathcal{X}(G)$ is a Lie subalgebra of $\mathcal{X}(G)$.*

Proof. Simply notice that $L_g^*[X,Y] = [L_g^*X, L_g^*Y]$ (more generally, $f^*[X,Y] = [f^*X, f^*Y]$ for any diffeomorphism f). □

Left-invariant vector fields provide a second way of associating a Lie algebra to a Lie group. We will see that they are isomorphic, but first let us check that they have the same dimension:

Proposition 1.3.12. *Let G be a Lie group. The map*

$$\varphi : \begin{cases} {}^G\mathcal{X}(G) & \to & T_eG \\ X & \mapsto & X(e) \end{cases}$$

is a linear isomorphism of vector spaces.

Proof. It is linear by definition of the vector space structure on the space of vector fields. Given $g \in G$, the identity $L_g^* X = X$ evaluated at e yields $X(g) = d_e L_g(X(e))$. The injectivity of φ follows.
Given $X \in T_e G$, we denote by $\overline{X} \in \mathcal{X}(G)$ the vector field $\overline{X}(g) = d_e L_g(X)$. For $g, x \in G$, we compute:

$$\begin{aligned}(L_g^*\overline{X})(x) &= (d_x L_g)^{-1}(\overline{X}(gx)) \\ &= d_{gx} L_{g^{-1}}(d_e L_{gx}(X)) \\ &= d_e(L_{g^{-1}} \circ L_{gx})(X) \\ &= d_e L_x(X) \\ &= \overline{X}(x)\end{aligned}$$

Therefore $\overline{X} \in {}^G\mathcal{X}(G)$ and $\varphi(\overline{X}) = X$, so φ is also onto. □

We will use the notation $\overline{X} = \varphi^{-1}(X)$ for $X \in \mathfrak{g}$ repeatedly in this section.

We will see later on that φ is also a Lie algebra isomorphism. This will be done through the study of the most important tool relating a Lie group and its Lie algebra: the exponential map.

The exponential map encodes the flow of left-invariant Lie groups, and we will first need to check that it is defined for all times.

Lemma 1.3.13. *A left-invariant vector field on a Lie group is complete.*

Proof. Let $X \in {}^G\mathcal{X}(G)$ and $x \in G$. If $y = \varphi_X^t(x)$ is defined for some $t \in \mathbb{R}$, we set $g = yx^{-1}$, and notice that $L_g^* X = X$ yields:

$$\varphi_X^s(x) = g^{-1}\varphi_X^s(y)$$

This formula shows that $\varphi^s(x)$ is defined if and only if $\varphi_X^s(y) = \varphi_X^{t+s}(x)$ is defined. □

1.3.6 The exponential map of a Lie group

Definition 1.3.14. *Let G be a Lie group with Lie algebra $\mathfrak{g}$. The* ***exponential map of*** *G is defined as follows:*

$$\exp_G : \left\{ \begin{array}{ccc} \mathfrak{g} & \to & G \\ X & \mapsto & \varphi_{\overline{X}}^1(e) \end{array} \right.$$

where $\overline{X} \in {}^G\mathcal{X}(G)$ satisfies $\overline{X}(e) = X(e)$.

Remark. *It is well defined because of Lemma 1.3.13, and smooth because the flow of a smooth vector field is.*

Examples

1. For $G = \mathbb{U} = \{z \in \mathbb{C} | \, |z| = 1\}$, the Lie algebra is $\mathfrak{g} = i\mathbb{R}$. For $i\theta \in \mathfrak{g}$, we find that $\exp_{\mathbb{U}}(i\theta) = e^{i\theta}$.
2. For $G = \mathrm{GL}(n, \mathbb{K})$ (where $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$), the Lie exponential is equal to the matrix exponential $\exp(A) = \sum_{n=0}^{+\infty} \frac{A^n}{n!}$.

The exponential map is sufficient to retrieve to whole flow of a left-invariant vector field.

Proposition 1.3.15. *Let G be a Lie group with Lie algebra $\mathfrak{g}$, and let $X \in \mathfrak{g}$. The flow of $\overline{X} \in {}^G\mathcal{X}(G)$ is given by $\varphi^t_{\overline{X}} = R_{\exp_G(tX)}$.*

Proof.

$$\begin{aligned} \varphi^t_{\overline{X}}(g) &= \varphi^t_{L^*_{g^{-1}}\overline{X}}(g) \\ &= L_g(\varphi^t_{\overline{X}}(g^{-1}g)) \\ &= g\varphi^t_{\overline{X}}(e) \\ &= g\varphi^1_{t\overline{X}}(e) \end{aligned}$$

□

Most of the properties of the matrix exponential also hold for the exponential map of a Lie group.

Proposition 1.3.16. *Let G be a Lie group with Lie algebra $\mathfrak{g}$. Then $d_0 \exp_G = \mathrm{Id}_{\mathfrak{g}}$.*

Proof. Proposition 1.3.15 yields $\exp_G(tX) = \varphi^t_{\overline{X}}(e)$. Differentiating at $t = 0$ gives $d_0 \exp_G(X) = \overline{X}(e) = X$. □

The Local Inverse Function Theorem guarantees that $\exp_G$ is a local diffeomorphism between a neighbourhood of 0 in $\mathfrak{g}$ and a neighbourhood of e in G. Because of Proposition 1.3.7, we get the following:

Proposition 1.3.17. *Let G be a Lie group with Lie algebra $\mathfrak{g}$. The subgroup generated by $\exp_G(\mathfrak{g})$ is $G_\circ$.*

However, the exponential map of a Lie group is not always onto, even for a connected Lie group (such as $\mathrm{SL}(2, \mathbb{R})$).

Proposition 1.3.18. *Let G be a Lie group with Lie algebra $\mathfrak{g}$. For any $X \in \mathfrak{g}$, the map:*

$$\begin{cases} \mathbb{R} & \to & G \\ t & \mapsto & \exp_G(tX) \end{cases}.$$

is a Lie group morphism. Every Lie group morphism from $\mathbb{R}$ to G has this form for a unique $X \in \mathfrak{g}$.

Such a Lie group morphism is called a **one-parameter subgroup.**

Proof. Since $\exp_G(tX) = \varphi^t_{\overline{X}}(e)$, the additive property of the flow of a vector field shows that $t \mapsto \exp_G(tX)$ is a group morphism. It is smooth because $\exp_G$ is smooth.
Let $\varphi : \mathbb{R} \to G$ be a Lie group morphism. Set $X = \varphi'(0) \in \mathfrak{g}$. Differentiating the expression $\varphi(t+s) = \varphi(t)\varphi(s)$ with respect to s at $s = 0$, we find:

$$\varphi'(t) = d_e L_{\varphi(t)}(X) = \overline{X}(\varphi(t))$$

Since $\varphi(0) = e$, uniqueness of the solution of ordinary differential equations (Cauchy-Lipschitz Theorem) yields $\varphi(t) = \exp_G(tX)$ for any $t \in \mathbb{R}$. As $Y = \frac{d}{dt}\Big|_{t=0} \exp_G(tY)$ for all $Y \in \mathfrak{g}$, the uniqueness of X follows. □

Proposition 1.3.19. *Let G, H be Lie groups with Lie algebras $\mathfrak{g}, \mathfrak{h}$. If $f : G \to H$ is a Lie group morphism, then:*

$$\forall X \in \mathfrak{g} \ f(\exp_G(X)) = \exp_H(d_e f(X))$$

Proof. The map $t \mapsto f(\exp_G(tX))$ is a one parameter subgroup of H. □

Corollary 1.3.20. *Let G be a Lie group with Lie algebra $\mathfrak{g}$. For $g \in G$ and $X \in \mathfrak{g}$, we have that*

$$\exp_G(\mathrm{Ad}(g)(X)) = i_g(\exp_G(X))$$

and

$$\mathrm{Ad}(\exp_G(X)) = \exp(\mathrm{ad}(X))$$

(the right-hand term being the matrix exponential in $\mathfrak{gl}(\mathfrak{g})$).

Proof. Apply Proposition 1.3.19 first to $f = i_g$, then to $f = \mathrm{Ad}$. □

The exponential map of a Lie group is in general not a group morphism. There is no explicit formula for the exponential of the sum of two elements of the Lie algebra. There is however an asymptotic formula.

Proposition 1.3.21. *Let G be a Lie group with Lie algebra $\mathfrak{g}$. For $X, Y \in \mathfrak{g}$, we have that:*

$$\exp_G(X+Y) = \lim_{n \to +\infty} \left(\exp_G\left(\frac{X}{n}\right) \exp_G\left(\frac{Y}{n}\right)\right)^n$$

Proof. Following Proposition 1.3.16 and the Local Inverse Function Theorem, we can consider a neighbourhood U of 0 in $\mathfrak{g}$ and a neighbourhood V of e in G such that $\exp_G$ restricts to a diffeomorphism from U to V. Let $\mathrm{Log}_G : V \to U$ be its inverse.

Consider a neighbourhood $U' \subset U$ of 0 in $\mathfrak{g}$ such that $\exp_G(X)\exp_G(Y) \in V$ for all $X, Y \in U'$. Consider the map:

$$f : \begin{cases} U' \times U' & \to & U \\ (X,Y) & \mapsto & \mathrm{Log}_G(\exp_G(X)\exp_G(Y)) \end{cases}$$

Let $X, Y \in \mathfrak{g}$. For large enough n, we have that $\frac{X}{n} \in U'$ and $\frac{Y}{n} \in U'$, so we can consider $f\left(\frac{X}{n}, \frac{Y}{n}\right)$.

Since $f(0,0) = 0$ and $d_{(0,0)}f(X,Y) = X + Y$, we find:

$$f\left(\frac{X}{n}, \frac{Y}{n}\right) = \frac{X}{n} + \frac{Y}{n} + \underset{n\to+\infty}{o}\left(\frac{1}{n}\right)$$

Therefore:

$$\lim_{n\to+\infty} nf\left(\frac{X}{n}, \frac{Y}{n}\right) = X + Y$$

The continuity of $\exp_G$ leads to:

$$\lim_{n\to+\infty} \exp_G\left(nf\left(\frac{X}{n}, \frac{Y}{n}\right)\right) = \exp_G(X+Y)$$

Using Proposition 1.3.18, we get:

$$\exp_G\left(nf\left(\frac{X}{n}, \frac{Y}{n}\right)\right) = \left(\exp_G\left(f\left(\frac{X}{n}, \frac{Y}{n}\right)\right)\right)^n$$

The definition of f simplifies the right-hand term:

$$\exp_G\left(f\left(\frac{X}{n}, \frac{Y}{n}\right)\right) = \exp_G\left(\frac{X}{n}\right)\exp_G\left(\frac{Y}{n}\right)$$

Finally:

$$\exp_G\left(nf\left(\frac{X}{n}, \frac{Y}{n}\right)\right) = \left(\exp_G\left(\frac{X}{n}\right)\exp_G\left(\frac{Y}{n}\right)\right)^n.$$

□

Remark. *There is an explicit formula (as a formal series) for the function f that we used, called the Baker-Campbell-Hausdorff formula.*

Proposition 1.3.22. *Let G be a Lie group with Lie algebra* $\mathfrak{g}$*. If* $X, Y \in \mathfrak{g}$ *and* $[X, Y] = 0$*, then* $\exp_G(X+Y) = \exp_G(X)\exp_G(Y)$.

Proof. Since $\mathrm{ad}(X)Y = 0$, we have that $\exp(\mathrm{ad}(X))Y = Y$, and Corollary 1.3.20 yields $\mathrm{Ad}(\exp_G(X))Y = Y$.
We also have that $\exp_G(\mathrm{Ad}(\exp_G(X))Y) = i_{\exp_G(X)}(\exp_G(Y))$ according to the same corollary, therefore $i_{\exp_G(X)}(\exp_G(Y)) = \exp_G(Y)$, i.e. $\exp_G(X)$ and $\exp_G(Y)$ commute.
The same applies to $\frac{X}{n}$ and $\frac{Y}{n}$ for $n > 0$, and we find:

$$\begin{aligned}\left(\exp_G\left(\frac{X}{n}\right)\exp_G\left(\frac{Y}{n}\right)\right)^n &= \left(\exp_G\left(\frac{X}{n}\right)\right)^n\left(\exp_G\left(\frac{Y}{n}\right)\right)^n \\ &= \exp_G(X)\exp_G(Y)\end{aligned}$$

Proposition 1.3.21 shows that $\exp_G(X+Y) = \exp_G(X)\exp_G(Y)$. □

While we are still on the subject of the exponential map, let us mention that there is an explicit formula for the differential of the exponential map of a Lie group at any point of the Lie algebra. We will not use nor prove this formula.

Proposition 1.3.23. *Let* $\Theta : \mathrm{End}(\mathfrak{g}) \to \mathrm{End}(\mathfrak{g})$ *be defined by* $\Theta(f) = \sum_{n=1}^{+\infty} \frac{f^{n-1}}{n!}$ *(i.e.* $\Theta(z) = \frac{e^z - 1}{z}$*). For* $X \in \mathfrak{g}$*, the differential* $d_X \exp_G$ *is given by:*

$$d_X \exp_G = d_e L_{\exp_G(X)} \circ \Theta(-\mathrm{ad}(X))$$

We now have everything needed to show the equivalence between the two Lie algebras associated to a Lie group.

Theorem 1.3.24. *Let* G *be a Lie group with Lie algebra* $\mathfrak{g}$*. The map* $X \mapsto \overline{X}$ *is a Lie algebra isomorphism between* $\mathfrak{g}$ *and* ${}^G\mathcal{X}(G)$*.*

Proof. It only remains to show that it is a Lie algebra morphism. For $X, Y \in \mathfrak{g}$, we can compute:

$$\begin{aligned}
[\overline{X}, \overline{Y}](e) &= \left.\frac{d}{dt}\right|_{t=0} \left(\varphi_{\overline{X}}^{t}\right)^{*} \overline{Y}(e) \\
&= \left.\frac{d}{dt}\right|_{t=0} \left(d_e \varphi_{\overline{X}}^{t}\right)^{-1} \left(\overline{Y}(\varphi_{\overline{X}}^{t}(e))\right) \\
&= \left.\frac{d}{dt}\right|_{t=0} \left(d_e R_{\exp_G(tX)}\right)^{-1} \left(\overline{Y}(\exp_G(tX))\right) \\
&= \left.\frac{d}{dt}\right|_{t=0} \left(d_e R_{\exp_G(tX)}\right)^{-1} \left(d_e L_{\exp_G(tX)}(Y)\right) \\
&= \left.\frac{d}{dt}\right|_{t=0} d_e i_{\exp_G(tX)}(Y) \\
&= \left.\frac{d}{dt}\right|_{t=0} \mathrm{Ad}(\exp_G(tX)) Y \\
&= \mathrm{ad}(X) Y
\end{aligned}$$

□

Chapter 2

Correspondence between Lie groups and Lie algebras

Our goal is now to understand up to which extent the classification of Lie groups reduces to the classification of Lie algebras. We will work on two distinct aspects: the relationship between subgroups of a given Lie group and subalgebras of its Lie algebra, and the relationship between Lie groups whose Lie algebras are isomorphic.

2.1 Correspondence between Lie subgroups and subalgebras

The natural definition for a Lie subgroup would be a subgroup which is a submanifold. However we will see that it is necessary to work with immersed submanifolds.

2.1.1 Immersed submanifolds and foliations

Definition 2.1.1. *Let M be a manifold. An* ***immersed submanifold*** *is the data of a subset $N \subset M$ and a manifold structure on N for which the inclusion $i : N \to M$ is a smooth immersion.*

Remark. *The manifold structure may not be unique, i.e. there can be two manifold structures on $N \subset M$ such that the inclusion $i : N \to M$ is an immersion for both structures on N, yet the identity map Id_N seen from N with one structure to N with the other is not smooth.*
In order to avoid this confusion, it is better to work with an abstract manifold S and an injective immersion $f : S \to M$, so the immersed manifold is the set $f(S)$ with the manifold structure that turns f into a diffeomorphism.
One can actually define an immersed submanifold in this way, as an equivalence

class of pairs (S,f) where $f: S \to M$ is an injective immersion, identifying (S,f) with (S',f') if there is a diffeomorphism $\varphi: S \to S'$ such that $f = f' \circ \varphi$.

An immersed submanifold $N \subset M$ has two topologies: the topology induced from M (just as any subset of M), and the manifold topology (sometimes called the intrinsic topology). Unless otherwise specified, any topological condition on N (e.g. connectedness) is considered for the manifold topology.
For $x \in N$, we will identify the tangent space T_xN (for the manifold structure on N) with its image $d_x i(T_xN) \subset T_xM$.
In order to avoid confusion, we will refer to a submanifold (i.e. with the usual definition) as an embedded submanifold. Note that an immersed submanifold is embedded if and only if the subset topology is equal to the manifold topology.

Examples: Consider $M = \mathbb{R}^2/\mathbb{Z}^2$ (the 2-dimensional torus), and $\pi: \mathbb{R}^2 \to M$ the canonical projection. Given $\alpha \in \mathbb{R}^*$, we consider

$$D_\alpha = \{\pi(x, \alpha x) | x \in \mathbb{R}\}$$

If $\alpha \notin \mathbb{Q}$, then D_α is an immersed submanifold which is not embedded. Indeed, the map $x \mapsto \pi(x, \alpha x)$ is an injective immersion, and D_α is dense in M, therefore not embedded.

Another example is Bernoulli's lemniscate: consider the maps $f: \left\{ \begin{array}{ccc} \mathbb{R} & \to & \mathbb{R}^2 \\ t & \mapsto & \left(\frac{t+t^3}{1+t^4}, \frac{t-t^3}{1+t^4}\right) \end{array} \right.$ and $g: \left\{ \begin{array}{ccc} \mathbb{R} & \to & \mathbb{R}^2 \\ t & \mapsto & \left(\frac{t+t^3}{1+t^4}, \frac{t^3-t}{1+t^4}\right) \end{array} \right.$. They are injective immersions with the same image, but the composition $\varphi = f \circ g^{-1}$ is given by $\varphi(t) = \frac{1}{t}$ if $t \neq 0$, and $\varphi(0) = 0$, so it is not continuous. This means that the set $f(\mathbb{R}) = g(\mathbb{R})$ has two distinct differentiable structures for which it is an immersed submanifold.

Definition 2.1.2. *Let M be a smooth d-dimensional manifold, and let $p \in \{1, \ldots, d\}$. A **distribution** Δ of rank p on M is a collection of p-dimensional vector subspaces $\Delta_x \subset T_xM$ for each $x \in M$ with the following regularity property: M can be covered by open sets U on which there are vector fields $X_1, \ldots, X_p$ such that:*

$$\forall x \in U \ \Delta_x = \mathrm{Vect}(X_1(x), \ldots, X_p(X))$$

*We say that Δ is **integrable** if we can choose $X_i = \partial_i$ for a coordinate system $(x^1, \ldots, x^d)$ on U.*

In all this course, we consider that **piecewise smooth curves** $\gamma: [a,b] \to M$ are continuous (note that this is not the same convention as you usually encounter in a course on Fourier series). For $t \in [a,b]$ we can consider $\dot\gamma(t^-)$

(resp. $\dot{\gamma}(t^+)$) the left (resp. right) derivative of γ at t. When we impose a condition on the derivative of a piecewise smooth curve, we mean that both $\dot{\gamma}(t^-)$ and $\dot{\gamma}(t^+)$ satisfy it.

Definition 2.1.3. *Given a distribution Δ on a manifold M, we denote by $\sim_\Delta$ the equivalence relation on M defined by $x \sim_\Delta y$ if there is a piecewise smooth curve $\gamma : [0,1] \to M$ such that $\gamma(0) = x$, $\gamma(1) = y$ and $\dot{\gamma}(t) \in \Delta_{\gamma(t)}$ for all $t \in [0,1]$.*
A p-dimensional ***foliation*** *of M is an equivalence class $\mathcal{F}$ on M for which there exists an integrable distribution Δ of rank p on M with $\mathcal{F} = \sim_\Delta$.*
The ***leaf*** *$\mathcal{F}(x)$ through $x \in M$ is the equivalence class of x.*

Proposition 2.1.4. *Let M be a manifold, Δ an integrable distribution of rank p on M, and $\mathcal{F} = \sim_\Delta$ the associated foliation of M. Then each leaf of $\mathcal{F}$ carries a unique manifold structure that makes it an immersed submanifold such that $\Delta_x = T_x\mathcal{F}(x)$ for all $x \in M$.*

Moreover, any connected immersed submanifold $N \subset M$ such that $T_xN = \Delta_x$ for all $x \in N$ is an open subset of a leaf.

This is far from true if Δ is not integrable: there are examples for which $\sim_\Delta$ only has one equivalence class.

The leaves of a foliation have an additional property compared to general immersed submanifolds, which allows us to check the smoothness of some functions without working out the differentiable structure of the leaves.

Definition 2.1.5. *A smooth map $\varphi : N \to M$ is called a* ***weak embedding*** *if it is an injective immersion, and if for any manifold P and map $f : P \to N$, the smoothness of $\varphi \circ f$ implies the smoothness of f.*

Definition 2.1.6. *Let M be a manifold and $N \subset M$ an immersed submanifold. We say that N is* ***weakly embedded*** *if the inclusion $i : N \to M$ is a weak embedding, i.e. any smooth map $f : P \to M$ whose range lies in N is also smooth seen as a map from P to N.*

Remark. *Note that when $f : P \to N$ is smooth, the differentials $df : TP \to TN \subset TM$ and $TP \to TN$ coincide.*

Proposition 2.1.7. *The leaves of a foliation are weakly embedded.*

Note that the restriction of a smooth map $f : M \to P$ to an immersed submanifold $N \subset M$ is always smooth for the differentiable structure on N (it is the composition of f with the inclusion map).

Remark. *In this course we only consider smooth foliations, i.e. the coordinate system appearing in the definition is smooth. In other contexts (especially in the study of hyperbolic dynamical systems), we treat separately the regularity of the leaves (they usually are smooth) and the transverse regularity (that of the coordinate system, which is usually less regular).*

Definition 2.1.8. *Let M be a manifold. A distribution Δ of M is called **involutive** if it is stable under the Lie bracket of vector fields, i.e. the space $\Gamma(\Delta) = \{X \in \mathcal{X}(M) | \forall x \in M \ X(x) \in \Delta_x\}$ is a Lie subalgebra of $\mathcal{X}(M)$.*

Theorem 2.1.9 (Frobenius Theorem)**.** *Let M be a manifold, and Δ a distribution on M. Then Δ is integrable if and only if it is involutive.*

2.1.2 Lie subgroups

Definition 2.1.10. *Let G be a Lie group. A **Lie subgroup** of G is the data of a subset $H \subset G$ and a Lie group structure on H for which the inclusion in G is a Lie group morphism.*
*An **embedded Lie subgroup** is a subset $H \subset G$ which is a subgroup and an embedded submanifold.*

Remarks.

- *A Lie subgroup is an immersed submanifold (because an injective Lie group morphism is an immersion).*
- *Whenever we consider an immersed Lie subgroup $H \subset G$, we have implicitly fixed a Lie group structure on H for which the inclusion is a Lie group morphism and an immersion.*

Examples:

- The example $D_\alpha \subset \mathbb{R}^2/\mathbb{Z}^2$ previously described (for $\alpha \notin \mathbb{Q}$) is an immersed Lie subgroup which is not embedded.
- The kernel of a Lie group morphism is an embedded Lie subgroup (this is a consequence of Proposition 1.1.8).

Let us see how the exponential map of a Lie group restricts to a subgroup. If $H \subset G$ is an immersed Lie subgroup, then $\mathrm{Lie}(H) = T_eH \subset T_eG = \mathrm{Lie}(G)$. Since the inclusion of H into G is a Lie group morphism, we find:

Proposition 2.1.11. *Let G be a Lie group, and $H \subset G$ an immersed Lie subgroup. For $X \in T_eH \subset T_eG$, we have $\exp_G(X) = \exp_H(X)$ (hence $\exp_G(X) \in H$).*

We will see more implications of this later, once we know how to recover a Lie subgroup from a Lie subalgebra.

2.1.3 From Lie subalgebras to Lie subgroups

We will now see that there is a one on one correspondence between connected Lie subgroups (i.e. connected for the intrinsic topology) and Lie subalgebras.

Theorem 2.1.12. *Let G be a Lie group, with Lie algebra $\mathfrak{g}$. The map $H \mapsto \mathrm{Lie}(H)$ is a bijection from the set of connected immersed Lie subgroups $H \subset G$ to the set of Lie subalgebras of $\mathfrak{g}$.*

The main ingredient of the proof is that any Lie subalgebra is associated to an integrable distribution.

Lemma 2.1.13. *Let G be a Lie group with Lie algebra $\mathfrak{g}$, and let $\mathfrak{h} \subset \mathfrak{g}$ be a Lie subalgebra. Then $\Delta_g = d_e L_g(\mathfrak{h})$ for $g \in G$ defines an involutive distribution Δ on g.*

Proof. Given a vector basis $(X_1, \dots, X_p)$ of $\mathfrak{h}$, we consider the associated left-invariant vector fields $\overline{X}_1, \dots, \overline{X}_p$. Notice that $(\overline{X}_1(x), \dots, \overline{X}_p(x))$ is a vector basis of Δ_x for all $x \in G$, therefore Δ is a distribution of rank $p = \dim \mathfrak{h}$.
If $X, Y \in \mathcal{X}(G)$ take values in Δ, we can consider functions $f_1, \dots, f_p, g_1, \dots, g_p \in \mathcal{C}^\infty(G)$ such that $X = \sum_{i=1}^p f_i \overline{X}_i$ and $Y = \sum_{i=1}^p g_i \overline{X}_i$. By developing the expression of $[X, Y]$, we see that it is a linear combination of $[\overline{X}_i, \overline{X}_j]$, $\overline{X}_i$ and $\overline{X}_j$, which all take values in Δ because $\mathfrak{h}$ is a Lie subalgebra of $\mathfrak{h}$. □

Proof of Theorem 2.1.12. Let us start with the surjectivity. Let $\mathfrak{h} \subset \mathfrak{g}$ be a Lie subalgebra. Define the distribution Δ on G by:

$$\forall x \in G \ \Delta_x = d_e L_x(\mathfrak{h})$$

It is involutive by Lemma 2.1.13, therefore integrable because of Frobenius' Theorem. We let $\mathcal{F} = \sim_\Delta$ be the associated foliation, and let $H = \mathcal{F}(e)$. It is a connected immersed submanifold of G.

We need to show that it is a subgroup. Consider $x, y \in H$, and a piecewise smooth path $\gamma : [0,1] \to G$ such that $\dot{\gamma}(t) \in \Delta_{\gamma(t)}$ for all $t \in [0,1]$, $\gamma(0) = e$ and $\gamma(1) = x$. Let $\delta(t) = y\gamma(t)$. We have that:

$$\dot{\delta}(t) = d_{\gamma(t)} L_y(\dot{\gamma}(t)) \in d_{\gamma(t)} L_y \circ d_e L_{\gamma(t)}(\mathfrak{h}) = d_e L_{\delta(t)}(\mathfrak{h}) = \Delta_{\delta(t)}$$

Since $\delta(0) = y$ and $\delta(1) = yx$, we find that $y \sim_\Delta yx$, hence $yx \sim_\Delta e$, i.e. $yx \in H$.

Similarly, by considering the path $t \mapsto x^{-1}\gamma(t)$, we find that $x^{-1} \in H$, therefore $H \subset G$ is a subgroup.

The fact that the group operation on H is smooth is a consequence of Proposition 2.1.7, so H is a Lie group.

We now tackle the injectivity. Consider a Lie subalgebra $\mathfrak{h} \subset \mathfrak{g}$, and a connected immersed Lie subgroup $H \subset G$ whose Lie algebra is $\mathfrak{h}$. We wish to show that H is the leaf $\mathcal{F}(e)$ of the foliation $\mathcal{F}$ above.

Since H is a connected Lie group, according to Proposition 1.3.7 it is the group generated by $\exp_H(\mathfrak{h})$. Because of Proposition 2.1.11, we find that $\exp_H(\mathfrak{h}) = \exp_G(\mathfrak{h})$. The same being true for $\mathcal{F}(e)$, we find that H and

$\mathcal{F}(e)$ are equal as sets.

It remains to prove that the differentiable structures are the same. This is a consequence of the uniqueness of the differentiable structure on leaves (Proposition 2.1.4), since $T_gH = d_eL_g(H) = \Delta_g$ for all $g \in H$. □

One of the consequences of this proof is that a connected Lie subgroup is a leaf of a foliation. Associated to Proposition 2.1.7, we get that they are weakly embedded.

Corollary 2.1.14. *Connected Lie subgroups are weakly embedded.*

This allows for a description of the Lie algebra of an immersed Lie subgroup that does not involve any differentiation.

Proposition 2.1.15. *Let G be a Lie group, and H ⊂ an immersed Lie subgroup. Then*

$$\mathrm{Lie}(H) = \left\{X \in \mathrm{Lie}(G) \,\middle|\, \forall t \in \mathbb{R}\ \exp_G(tX) \in H\right\}.$$

Proof. We have seen in Proposition 2.1.11 that $\exp_G(X) = \exp_H(X)$ for all $X \in \mathrm{Lie}(H)$, hence $\exp_G(tX) \in H$ for all $(t, X) \in \mathbb{R} \times \mathrm{Lie}(H)$.
Let $X \in \mathrm{Lie}(G)$ be such that: $\forall t \in \mathbb{R}\ \exp_G(tX) \in H$. Naively, we want to say that $X = \frac{d}{dt}\big|_{t=0} \exp_G(tX) \in T_eH = \mathrm{Lie}(H)$. This is valid because $t \mapsto \exp_G(tX)$ is smooth for the manifold structure on H and has the same derivative as in G, a consequence of Proposition 2.1.7 and of the proof of Theorem 2.1.12. □

2.1.4 Closed subgroups of Lie groups

The difference between immersed and embedded Lie subgroups is purely topological. Just as Lie groups have restrictions on their topology, so do Lie subgroups.

Proposition 2.1.16. *An embedded Lie subgroup is closed.*

Proof. Let G be a Lie group, and $H \subset G$ a Lie subgroup. Since H is embedded, it is locally closed (i.e. open in its closure $\overline{H}$). But $\overline{H}$ is a subgroup of G, and an open subgroup of $\overline{H}$ is also closed in $\overline{H}$ (see Lemma 1.3.8) hence closed in G. □

One of the most remarkable facts in Lie theory is that the converse holds: any closed subgroup is a Lie subgroup. This may seem extremely useful, as closed subgroups appear all the time, e.g. as stabilizers for some group action, but most of the time they are clearly level set of constant rank maps.

Theorem 2.1.17 (Cartan-Von Neumann Theorem)**.** *Let G be a Lie group. If $H \subset G$ is a closed subgroup, then H is an embedded Lie subgroup.*

Proof. With Proposition 2.1.15 in mind, we set:

$$V = \{X \in \mathfrak{g} | \forall t \in \mathbb{R} \ \exp_G(tX) \in H\}.$$

First step: Show that V is a vector subspace of $\mathfrak{g}$.

We have $0 \in V$. It is also straightforward that if $X \in V$ and $\lambda \in \mathbb{R}$ then $\lambda X \in V$.
Consider $X, Y \in V$, and $t \in \mathbb{R}$. According to Proposition 1.3.21, we have:

$$\exp_G(t(X+Y)) = \lim_{n \to +\infty} \left(\exp_G\left(\frac{tX}{n}\right) \exp_G\left(\frac{tY}{n}\right)\right)^n$$

Since $X \in V$ and $Y \in V$, we find:

$$\exp_G\left(\frac{tX}{n}\right) \exp_G\left(\frac{tY}{n}\right) \in H$$

Since H is a subgroup of G, we get:

$$\left(\exp_G\left(\frac{tX}{n}\right) \exp_G\left(\frac{tY}{n}\right)\right)^n \in H$$

Finally, because H is closed, we find:

$$\lim_{n \to +\infty} \left(\exp_G\left(\frac{tX}{n}\right) \exp_G\left(\frac{tY}{n}\right)\right)^n \in H$$

It follows that $X + Y \in V$, and V is a vector subspace of $\mathfrak{g}$.

Second step: Let $W \subset \mathfrak{g}$ be a supplementary subspace of V (i.e. $V \oplus W = \mathfrak{g}$). Prove the existence of a neighbourhood U of 0 in W such that $\exp_G(X) \notin H$ for all $X \in U \setminus \{0\}$.

Let us prove this by contradiction: if it were not the case, we could find $X_n \in W \setminus \{0\}$ such that $X_n \to 0$ and $\exp_G(X_n) \in H$.
Consider a norm $\| \cdot \|$ on W, and set $\alpha_n = \frac{1}{\|X_n\|}$. Using the compactness of spheres in finite dimension normed spaces, up to considering subsequences we can assume that $\alpha_n X_n \to X \in W \setminus \{0\}$. Let us show that $X \in V$ (which is a contradiction with $X \in W \setminus \{0\}$).
Let $t \in \mathbb{R}$. Set $k_n = \lfloor t\alpha_n \rfloor \in \mathbb{Z}$ and $r_n = \{t\alpha_n\} \in [0, 1[$, so that $t\alpha_n = k_n + r_n$. Since $tX = \lim_{n \to +\infty} t\alpha_n X_n$, we find:

$$\begin{aligned} \exp_G(tX) &= \lim_{n \to +\infty} \exp_G(t\alpha_n X) \\ &= \lim_{n \to +\infty} \left(\exp_G(X_n) \right)^{k_n} \exp_G(r_n X_n) \end{aligned}$$

Since (r_n) is bounded and $X_n \to 0$, we have that $r_n X_n \to 0$, hence $\exp_G(r_n X_n) \to e$. We get:

$$\exp_G(tX) = \lim_{n\to+\infty} \left(\exp_G(X_n)\right)^{k_n}$$

As $\exp_G(X_n) \in H$, and H is a closed subgroup of G, we know that $\exp_G(tX) \in H$, hence $X \in V$, which is the aforementioned contradiction.

Third step: Build a trivialising chart for H on a neighbourhood of e.

Consider the map:

$$\varphi : \begin{cases} \mathfrak{g} = V \oplus W & \to & G \\ X + Y & \mapsto & \exp_G(X)\exp_G(Y) \end{cases}$$

Since $d_0\varphi = \mathrm{Id}$, the Local Inverse Function Theorem provides us with an open subset $U_V \subset V$ (resp. $U_W \subset W$, $U_G \subset G$) containing 0 (resp. 0, e) such that φ restricts to a diffeomorphism from $U_V + U_W$ onto U_G. According to the previous step, we can assume that $\exp_G(X) \notin H$ for all $X \in U_W \setminus \{0\}$. It follows that $\varphi(U_V) = H \cap U_G$, therefore φ is a trivialising chart for H on a neighbourhood of e.

Fourth step: For all $g \in H$, the map $L_g \circ \varphi$ is a trivialising chart for H around g, which shows that H is an embedded submanifold of G. □

2.2 Lie groups with a given Lie algebra

We will now study Lie groups that have the same Lie algebra (up to isomorphism). Note that according to Proposition 1.3.6, a Lie group and its neutral component share the same Lie algebra, so the problem reduces to connected Lie groups.

The aim of this section is to partially prove the following result.

Theorem 2.2.1. *Let $\mathfrak{g}$ be a finite dimensional Lie algebra over $\mathbb{R}$. There is a simply connected Lie group $\widetilde{G}$ whose Lie algebra is isomorphic to $\mathfrak{g}$.*
Moreover, if G is a connected Lie group whose Lie algebra is isomorphic to $\mathfrak{g}$, then G is isomorphic to a quotient $\widetilde{G}/\Gamma$ where Γ is a discrete subgroup of $Z(\widetilde{G})$.

2.2.1 Reminders on covering maps

Covering maps have a definition close to that of a local diffeomorphism, except that the locality should be considered on the target manifold rather than on the source.

Definition 2.2.2. *Let M, N be smooth manifolds. A smooth map $p : M \to N$ is called a **covering map** if every $y \in N$ has a open neighbourhood V such that,* *for every connected component U of $p^{-1}(V)$, the restriction $p|_U : U \to V$ is a diffeomorphism.*

Remarks.

- *For $y \in N$, the set $p^{-1}(\{y\})$ is called the **fibre over** y.*
- *If N is connected, then all fibres have the same cardinality. This number (eventually infinite) is called the **order** of p.*

Definition 2.2.3. *Let $p : M \to N$ and $\varphi' : M' \to N$ be covering maps. A **covering isomorphism** is a diffeomorphism $\varphi : M \to M'$ such that $p' \circ \varphi = p$. If $p = p'$, we call it a **deck transformation**.*

Definition 2.2.4. *A **Galois covering** is a covering map $p : M \to N$ such that the group of deck transformations acts transitively on each fibre.*

If $p : M \to N$ is a Galois covering and Γ is the group of deck transformations, then N identifies with $\Gamma\backslash M$ as a set. We will discuss this further later on.

For now let us recall the notion of universal covering.

Definition 2.2.5. *A manifold M is **simply connected** M if it is connected and any continuous map $f : \mathbb{S}^1 \to M$ extends to a continuous map $\overline{\mathbb{D}} \to M$.*

Definition 2.2.6. *A universal cover of a manifold N is a covering map $f : M \to N$ where M is simply connected.*

Theorem 2.2.7. *Let N be a connected manifold. Then N admits a universal cover $\pi = \widetilde{N} \to N$. If $p : M \to N$ is another universal cover, then for all $(x, x', y) \in \widetilde{N} \times M \times N$ satisfying $\pi(x) = p(x') = y$, there is a unique covering isomorphism $\varphi : \widetilde{N} \to M$ such that $\varphi(x) = x'$.*

In particular, the universal cover is Galois.

Definition 2.2.8. *Let N be a connected manifold. Its **fundamental group** $\pi_1(N)$ is the group of deck transformations of its universal cover.*

Remark. *The fundamental group is only well defined up to isomorphism.*

This is a geometric definition of the fundamental group: we define it through one of its actions (compare with the topological definition through homotopy classes).

Covering maps are nice to work with because of the lifting property.

Theorem 2.2.9. *Let M, N, P be smooth manifolds, $p : M \to N$ a covering map and $f : P \to N$ a smooth map. Assume that P is simply connected. Then for all $(x, y) \in P \times M$ satisfying $p(y) = f(x)$, there is a unique smooth map $\widetilde{f} : P \to M$ such that $p \circ \widetilde{f} = f$ and $\widetilde{f}(x) = y$ (such a map $\widetilde{f}$ is a called a **lift** of f).*

The covering maps that we will work with are quotients by actions of discrete groups.

Definition 2.2.10. *An action of a group Γ on a manifold M is called **properly discontinuous** if for every compact subset $K \subset M$, the set*

$$\{\gamma \in \Gamma | \gamma K \cap K \neq \emptyset\}$$

is finite.
*The action is called **free** if*

$$\forall x \in M \forall \gamma \in \Gamma \ \gamma.x = x \Rightarrow \gamma = e.$$

Proposition 2.2.11. *Let M be a manifold and Γ a group that acts on M by diffeomorphisms. If the action is free and properly discontinuous, then there is a unique manifold structure on the quotient $\Gamma \backslash M$ for which the quotient map $\pi : M \to \Gamma \backslash M$ is a covering map.*

Remark. *The covering maps that we obtain this way are exactly Galois coverings.*

2.2.2 Coverings of Lie groups

We will now study coverings of Lie groups.

Definition 2.2.12. *Let G and $\widehat{G}$ be Lie groups. A map $f : \widehat{G} \to G$ is a **Lie group covering** if it is both a Lie group morphism and a covering map.*

We will see that a Lie group covering $\widehat{G} \to G$ is always a quotient by a subgroup $\Gamma \subset \widehat{G}$. Such a subgroup Γ must be normal (so that the quotient $\Gamma \backslash \widehat{G}$ inherits a group structure) and discrete (for the quotient to have the same dimension as that of G, the subgroup Γ must have dimension 0). We first notice that such a group must be central, i.e. included in the centre $Z(\widehat{G})$.

Proposition 2.2.13. *Let G be a connected Lie group, and $\Gamma \subset G$ a discrete subgroup. Then Γ is normal if and only if $\Gamma \subset Z(G)$.*

Proof. Assume that Γ is normal. Let $\gamma \in \Gamma$. The image of G by the continuous map $g \mapsto g\gamma g^{-1}$ is connected and included in the discrete set Γ, hence reduced to γ. This shows that $\gamma \in Z(G)$.
If $\Gamma \subset Z(G)$, we find $g\Gamma g^{-1} = \Gamma$ for all $g \in G$, so Γ is normal. □

The quotient by such a subgroup is always a Lie group covering.

Proposition 2.2.14. *Let G be a connected Lie group, and let $\Gamma \subset G$ be a discrete normal subgroup. There is a unique Lie group structure on $\Gamma \backslash G$ for which the projection $\pi : G \to \Gamma \backslash G$ is a Lie group covering. The Lie algebras of G and $\Gamma \backslash G$ are isomorphic to each other.*

Proof. Let us first show that the left action of Γ on G is free and properly discontinuous.
It is free because $gx = x \Rightarrow g = xx^{-1} = e$.
For any subset $K \subset G$, and $g \in G$, we have that:

$$gK \cap K \neq \emptyset \iff g \in KK^{-1}$$

where $KK^{-1} = \{xy^{-1} | x, y \in K\}$. If K is compact, then so is KK^{-1}. Since Γ is discrete, the intersection $\Gamma \cap KK^{-1}$ is finite, and it follows that the action on G is properly discontinuous.

Following Proposition 2.2.11, there is a unique manifold structure on $\Gamma \backslash G$ for which π is a smooth covering map.

Since the group operation in $\Gamma \backslash G$ can be expressed through local inverses of π, it is smooth, i.e. $\Gamma \backslash G$ is a Lie group, and π a Lie group covering. □

It happens that every Lie group covering can be obtained in this way.

Proposition 2.2.15. *Let $\widehat{G}$ and G be connected Lie groups, and $p : \widehat{G} \to G$ a Lie group covering. There exists a discrete subgroup $\Gamma \subset Z(\widehat{G})$ and a Lie group isomorphism $\varphi : \Gamma \backslash \widehat{G} \to G$ such that $\varphi \circ \pi = p$ where $\pi : \widehat{G} \to \Gamma \backslash \widehat{G}$ is the projection.*

Proof. Set $\Gamma = p^{-1}(\{e\})$. It is a discrete normal subgroup of $\widehat{G}$, and $p(x) = p(y) \iff xy^{-1} \in \Gamma$, which allows us to construct φ using the universal property of the quotient. □

We will see that any smooth covering of a Lie group is a Lie group covering. Let us start with the universal cover.

Lemma 2.2.16. *Let G be a Lie group, and let $p : \widetilde{G} \to G$ be its universal cover. There is a Lie group structure on $\widetilde{G}$ for which p is a Lie group morphism. Moreover, the Lie algebra of $\widetilde{G}$ is isomorphic to G.*

Proof. We first need to choose a point in the fibre of e, which we will set to be the neutral element of $\widetilde{G}$. Let us still call it $e \in \widetilde{G}$.

Consider the map $f : \widetilde{G} \times \widetilde{G} \to G$ defined by $f(x, x') = p(x)p(x')$. Since $\widetilde{G} \times \widetilde{G}$ is simply connected, according the the Lifting Theorem (Theorem **??**), there is a unique smooth map $\widetilde{f} : \widetilde{G} \times \widetilde{G} \to \widetilde{G}$ such that $p \circ \widetilde{f} = f$ and $f(e, e) = e$. We now want to check that $\widetilde{f}$ is a group operation that meets the requirements.

Since $f(e, \cdot) = p$, and $\widetilde{f}(e, e) = e$, it follows from the uniqueness of the lift to the universal cover that $\widetilde{f}(e, \cdot) = \mathrm{Id}$, i.e. e is neutral for multiplication on the left: $\forall x \in \widetilde{G}\ ex = x$. A similar argument shows that e is neutral for multiplication on the right.

We now have to prove associativity, i.e. $\forall x, y, z \in \widetilde{G}\ \widetilde{f}(\widetilde{f}(x, y), z) = \widetilde{f}(x, \widetilde{f}(y, z))$.

For this, consider both expressions as functions in z, with x and y fixed. They both coincide at $z = e$, and they are both lifts of $z \mapsto p(x)p(y)p(z)$ (using the associativity of G). It follows from the uniqueness of the lift that they are equal, i.e. $\widetilde{G}$ is associative.

We now have to prove the existence of an inverse. First notice that if $\varphi : \widetilde{G} \to \widetilde{G}$ is a deck transformation, then: $\forall x \in \widetilde{G}\ \varphi(x) = \widetilde{f}(x, \varphi(e)) = \widetilde{f}(\varphi(e), x)$. Indeed, the maps $x \mapsto \varphi(x)$, $x \mapsto \widetilde{f}(x, \varphi(e))$ and $x \mapsto \widetilde{f}(\varphi(e), x)$ are lifts of p and coincide at e. Let $x \in \widetilde{G}$, and consider $x' \in \widetilde{G}$ such that $p(x') = p(x)^{-1}$. Since $p(xx') = e$ and p is a Galois covering, there is a deck transformation φ such that $\varphi(xx') = e$. It follows that $x'\varphi(e)$ is a right inverse for x, and that $\varphi(e)x' = x'\varphi(e)$ is a left inverse for x.

The fact that p is a group morphism is exactly the lifting property $p \circ \widetilde{f} = f$.

Finally, since p is a Lie group morphism, its differential $d_e p$ is a Lie algebra morphism from $\mathrm{Lie}(\widetilde{G})$ to $\mathrm{Lie}(G)$. Since p is a local diffeomorphism, it is a Lie algebra isomorphism. □

Corollary 2.2.17. *Let G be a connected Lie group. Its fundamental group $\pi_1(G)$ is abelian.*

Proof. According to Lemma 2.2.16 and Proposition 2.2.15, the fundamental group $\pi_1(G)$ is isomorphic to a discrete normal subgroup of $\widetilde{G}$, hence abelian because of Proposition 2.2.13. □

Proposition 2.2.18. *Let G be a connected Lie group, and let $p : M \to G$ be a smooth covering. There is a Lie group structure on M for which p is a Lie group morphism. Moreover, the Lie algebra of M is isomorphic to the Lie algebra of G.*

Proof. Every covering of G is isomorphic to $\widetilde{G}/\Gamma'$ where Γ' is a subgroup of $\pi_1(G)$. The result is a consequence of Proposition 2.2.14. □

2.2.3 Lie's third theorem

In order to finish the proof of Theorem 2.2.1, we need to prove that any finite dimensional Lie algebra is the Lie algebra of a Lie group. We will use the following result without proof.

Theorem 2.2.19 (Ado's Theorem)**.** *Let $\mathfrak{g}$ be a real finite dimensional Lie algebra. Then $\mathfrak{g}$ possesses a finite dimensional faithful linear representation.*

Remark. *The equivalent statement for Lie groups is false. The group* $\mathrm{SL}(2,\mathbb{R})$ *is diffeomorphic to $\mathbb{R}^2 \times \mathbb{S}^1$ (which can be shown using Iwasawa decomposition), so its fundamental group is isomorphic to $\mathbb{Z}$. One can show that any finite dimensional linear representation of the universal cover of* $\mathrm{SL}(2,\mathbb{R})$ *factorizes through* $\mathrm{SL}(2,\mathbb{R})$*, so it cannot be faithful.*

Since a Lie subalgebra of $\mathfrak{gl}(V)$ is the Lie algebra of a subgroup of $\mathrm{GL}(V)$ according to Theorem 2.1.12, Ado's Theorem has the following consequence:

Theorem 2.2.20. *Let $\mathfrak{g}$ be a real finite dimensional Lie algebra. There is a Lie group G whose Lie algebra is isomorphic to $\mathfrak{g}$.*

Combining Theorem 2.2.20, Proposition 2.2.18 and Proposition 2.2.15, we get a proof of Theorem 2.2.1.

Chapter 3

Classifying Lie algebras

Classifying Lie groups seems to be a daunting task, since they are non linear objects. However classifying Lie algebras seems like a more reasonable goal.

3.1 Ideals of a Lie algebra

3.1.1 Operations on ideals

We can define an ideal of a Lie algebra just as we would for a ring.

Definition 3.1.1. *Let $\mathfrak{g}$ be a Lie algebra. An* ***ideal*** *of $\mathfrak{g}$ is a vector subspace $\mathfrak{I} \subset \mathfrak{g}$ such that:*

$$\forall X \in \mathfrak{g} \; \forall Y \in \mathfrak{I} \; [X, Y] \in \mathfrak{I}$$

Note that an ideal is a Lie subalgebra (but the converse is false). Ideals are related to normal subgroups.

Proposition 3.1.2. *Let G be a Lie group with Lie algebra $\mathfrak{g}$. If $H \subset G$ is a normal immersed Lie subgroup, then its Lie algebra $\mathfrak{h}$ is an ideal of $\mathfrak{g}$.*

Proof. Consider $g \in G$ and $Y \in \mathfrak{h}$. According to Corollary 1.3.20, we have $\exp_G(s\,\mathrm{Ad}(g)Y) = i_g(\exp_G(sY)) \in H$ since H is normal. Derivating at $s = 0$ yields $\mathrm{Ad}(g)Y \in T_eH$.
Applying this formula to $g = \exp_G(tX)$ for $X \in \mathfrak{g}$, derivating at $t = 0$ gives $[X, Y] \in T_eH$. □

Quotients of Lie algebras by ideals are Lie algebras.

Proposition 3.1.3. *Let $\mathfrak{g}$ be a Lie algebra and $\mathfrak{I} \subset \mathfrak{g}$ an ideal. There is a unique Lie bracket on the vector space $\mathfrak{g}/\mathfrak{I}$ for which the projection is a Lie algebra morphism.*
If $f : \mathfrak{g} \to \mathfrak{h}$ is a Lie algebra morphism, then $\ker f$ is an ideal and $\mathrm{im} f$ is a Lie algebra isomorphic to $\mathfrak{g}/\ker f$.
The intersection and the sum of ideals are ideals.

This last property allows us to define the ideal generated by a subset $A \subset \mathfrak{g}$ as the smallest ideal of $\mathfrak{g}$ containing A.

Proposition 3.1.4. *Let $\mathfrak{g}$ be a Lie algebra, and $\mathfrak{I}, \mathfrak{J} \subset \mathfrak{g}$ be ideals. The vector space $[\mathfrak{I}, \mathfrak{J}]$ spanned by all $[X, Y]$ for $X \in \mathfrak{I}$ and $Y \in \mathfrak{J}$ is an ideal of $\mathfrak{g}$.*

Proof. Let $(X, Y, Z) \in \mathfrak{g} \times \mathfrak{I} \times \mathfrak{J}$. The Jacobi identity yields:

$$[X,[Y,Z]] = [Y,\underbrace{[X,Z]}_{\in \mathfrak{J}}] + [\underbrace{[X,Y]}_{\in \mathfrak{I}},Z] \in [\mathfrak{I},\mathfrak{J}]$$

□

3.1.2 Lie algebras with many or few ideals

Definition 3.1.5. *Let $\mathfrak{g}$ be a Lie algebra. The **derived algebra** of $\mathfrak{g}$ is the ideal $[\mathfrak{g},\mathfrak{g}]$.*
The lower central series $C_i(\mathfrak{g})$ is defined by $C_0(\mathfrak{g}) = \mathfrak{g}$ and $C_{i+1}(\mathfrak{g}) = [\mathfrak{g}, C_i(g)]$.
The derived series $D_i(\mathfrak{g})$ is defined by $D_0(\mathfrak{g}) = \mathfrak{g}$ and $D_{i+1}(\mathfrak{g}) = [D_i(\mathfrak{g}), D_i(\mathfrak{g})]$.
*A Lie algebra $\mathfrak{g}$ is called **nilpotent** if there is some n such that $C_n(\mathfrak{g}) = \{0\}$.*
*A Lie algebra $\mathfrak{g}$ is called **solvable** if there is some n such that $D_n(\mathfrak{g}) = \{0\}$.*

By definition, solvable Lie algebras have many ideals. We will mostly focus on Lie algebras that have few ideals.

Definition 3.1.6. *A Lie algebra $\mathfrak{g}$ is called **perfect** if $[\mathfrak{g},\mathfrak{g}] = \mathfrak{g}$.*
*It is called **simple** if $\dim \mathfrak{g} \geq 2$ and the only ideals of $\mathfrak{g}$ are $\{0\}$ and $\mathfrak{g}$.*
*It is called **semi-simple** if the only abelian ideal is $\{0\}$.*

Let us start with a few elementary properties of solvable Lie algebras.

Proposition 3.1.7. *Given a short exact sequence $0 \to \mathfrak{a} \to \mathfrak{b} \to \mathfrak{c} \to 0$ of Lie algebras, $\mathfrak{b}$ if solvable if and only if $\mathfrak{a}$ and $\mathfrak{c}$ le are solvable.*
If $\mathfrak{I}, \mathfrak{J} \subset \mathfrak{g}$ are solvable ideals, then $\mathfrak{I} + \mathfrak{J}$ is solvable.

This allows us to define the **solvable radical** $\mathrm{Rad}(\mathfrak{g})$ of a finite dimensional Lie algebra $\mathfrak{g}$ as its largest solvable ideal.

3.2 Semi-simple Lie algebras

3.2.1 Cartan's semi-simplicity criterion

We will admit the following result (and most results in this section).

Theorem 3.2.1 (Cartan's semi-simplicity criterion)**.** *Let $\mathfrak{g}$ be a finite dimensional Lie algebra. The following are equivalent:*

1. *$\mathfrak{g}$ is semi-simple.*

2. *The solvable radical of* $\mathfrak{g}$ *is trivial.*

3. *The Killing form of* $\mathfrak{g}$ *is non-degenerate.*

4. $\mathfrak{g}$ *is the direct sum of simple ideals.*

Remark. *In particular, for any finite dimensional Lie algebra* $\mathfrak{g}$*, the quotient* $\mathfrak{g}/\mathrm{Rad}(\mathfrak{g})$ *is semi-simple.*

The third property is by far the one that we will use most often, as it turns a semi-simple Lie algebra into a geometric object. Because of this, it would be useful to know the Killing form of the classical Lie algebras. We will admit the following result.

Lemma 3.2.2. *Let* $\mathfrak{g} \subset \mathfrak{sl}(n,\mathbb{C})$ *be an irreducible Lie subalgebra (i.e. the only vector subspaces* $V \subset \mathbb{C}^n$ *invariant under* $\mathfrak{g}$ *are* $\{0\}$ *and* $\mathbb{C}^n$*). Then* $\mathfrak{g}$ *is semi-simple, and there is* $\lambda \in \mathbb{C}^*$ *such that:*

$$\forall X, Y \in \mathfrak{g} \; B(X,Y) = \lambda \operatorname{Tr}(XY).$$

In general, the Killing form of a subalgebra has little to do with the Killing form of the ambient Lie algebra. However, they coincide for ideals.

Proposition 3.2.3. *Let* $\mathfrak{g}$ *be a finite dimensional Lie algebra, and* $\mathfrak{I} \subset \mathfrak{g}$ *an ideal. The Killing form of* $\mathfrak{I}$ *is the restriction to* $\mathfrak{I} \times \mathfrak{I}$ *of the Killing form of* $\mathfrak{g}$*.*

Proof. Consider a vector basis of $\mathfrak{g}$ adapted to a decomposition $\mathfrak{g} = \mathfrak{I} \oplus V$. The matrices of operators $\mathrm{ad}(X)$ for $X \in \mathfrak{I}$ in this basis are bloc diagonal with a vanishing bloc, so the trace of the whole matrix is equal to the trace of the (potentially) non-vanishing bloc. □

Even though we will not prove Cartan's semi-simplicity criterion, let us have a look at the link between the Killing form and ideals.

Proposition 3.2.4. *Let* $\mathfrak{g}$ *be a finite dimensional Lie algebra. The kernel of the Killing form*

$$\ker B = \{X \in \mathfrak{g} \mid \forall Y \in \mathfrak{g} \; B(X,Y) = 0\}$$

is an ideal of $\mathfrak{g}$*.*

Proof. Let $X \in \ker B$, and $Y \in \mathfrak{g}$. For $Z \in \mathfrak{g}$, we find:

$$\begin{aligned} B([X,Y],Z) &= -B(\mathrm{ad}(Y)X, Z) \\ &= B(X, \mathrm{ad}(Y)Z) \\ &= 0 \end{aligned}$$

This yields $[X,Y] \in \ker B$. Since $\ker B$ is also a vector subspace of $\mathfrak{g}$, it is an ideal. □

Remark. *Following Proposition 3.2.3, the Killing form of* $\ker(B)$ *is trivial. According to a theorem (of Cartan, obviously), this implies that* $\ker(B)$ *is solvable.*

3.2.2 Cartan subalgebras

Definition 3.2.5. *Let $\mathfrak{g}$ be a semi-simple Lie algebra. A* ***Cartan subalgebra*** *of $\mathfrak{g}$ is a Lie subalgebra $\mathfrak{h} \subset \mathfrak{g}$ such that:*

1. *$\mathfrak{h}$ is maximal abelian.*
2. *For every $X \in \mathfrak{h}$, $\mathrm{ad}(X)$ is semi-simple.*

Recall that a linear map is semi-simple if and only if it is diagonalisable in the algebraic closure of the base field (in zero characteristic).

Example: For $\mathfrak{g} = \mathfrak{sl}(n, \mathbb{K})$, we set:

$$\mathfrak{h} = \left\{ \left. \begin{pmatrix} h_1 & & 0 \\ & \ddots & \\ 0 & & h_n \end{pmatrix} \right| h_1 + \cdots + h_n = 0 \right\} \subset \mathfrak{sl}(n, \mathbb{K})$$

Let us show that it is a Cartan subalgebra of $\mathfrak{sl}(n, \mathbb{K})$. It is an abelian subalgebra. It is maximal because if $\mathfrak{h}' \supset \mathfrak{h}$ is abelian, then any element of $\mathfrak{h}'$ must commute with the matrix:

$$\begin{pmatrix} 1 & & & & \\ & 2 & & & \\ & & \ddots & & \\ & & & n-1 & \\ & & & & -\frac{n(n-1)}{2} \end{pmatrix},$$

and therefore be diagonal, i.e. be an element of $\mathfrak{h}$.

Consider the canonical basis $(E_{i,j})_{1 \leq i,j \leq n}$ of $\mathcal{M}_n(\mathbb{K})$. For $H = \begin{pmatrix} h_1 & & 0 \\ & \ddots & \\ 0 & & h_n \end{pmatrix}$, we get $\mathrm{ad}(H)E_{i,j} = (h_i - h_j)E_{i,j}$. Therefore $\mathrm{ad}(H)$ is diagonalisable for every $H \in \mathfrak{h}$, and $\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{sl}(n, \mathbb{K})$.

Proposition 3.2.6. *Let $\mathfrak{g}$ be a finite dimensional Lie algebra over a field $_b bK$. If $\mathfrak{g}$ is semi-simple, then it has a Cartan subalgebra.*
Moreover, if $\mathbb{K}$ is algebraically closed, then $\mathrm{Aut}(\mathfrak{g})$ acts transitively on the set of Cartan subalgebras.

Consider a complex finite dimensional semi-simple Lie algebra $\mathfrak{g}$. Let $\mathfrak{h} \subset \mathfrak{g}$ be a Cartan subalgebra. For every $H \in \mathfrak{h}$, the linear map $\mathrm{ad}(H) \in \mathfrak{gl}(\mathfrak{g})$ is diagonalisable, and they all commute with each other. This implies the existence of a vector basis $(X_1, \ldots, X_n)$ of $\mathfrak{g}$ in which the matrices of all the $\mathrm{ad}(H)$ for $H \in \mathfrak{h}$ are diagonal, i.e. $\mathrm{ad}(H)X_i = \alpha_i(H)X_i$.

Note that the eigenvalues $\alpha_i(H)$ are linear forms on $\mathfrak{h}$, which leads us to the concept of roots.

Definition 3.2.7. *Let $\mathfrak{g}$ be a complex finite dimensional semi-simple Lie algebra, and $\mathfrak{h} \subset \mathfrak{g}$ a Cartan subalgebra.*
For $\alpha \in \mathfrak{h}^*$, we set

$$\mathfrak{g}_\alpha = \{X \in \mathfrak{g} | \forall H \in \mathfrak{h} \ \mathrm{ad}(H)X = \alpha(H)X\}$$

*A **root** is a linear form $\alpha \in \mathfrak{h}^* \setminus \{0\}$ such that $\mathfrak{g}_\alpha \neq \{0\}$. We call $\mathfrak{g}_\alpha$ the **root space** associated to α.*

Proposition 3.2.8. *Let $\mathfrak{g}$ be a complex finite dimensional semi-simple Lie algebra, $\mathfrak{h} \subset \mathfrak{g}$ a Cartan subalgebra, and $\Phi \subset \mathfrak{h}^*$ the set of roots. We have the following decomposition of $\mathfrak{g}$:*

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Phi} \mathfrak{g}_\alpha.$$

Proof. It is a consequence of the previous discussion, and of the fact that the eigenspace $\mathfrak{g}_0$ (i.e. the **centralizer** of $\mathfrak{h}$) is equal to $\mathfrak{h}$ (because $\mathfrak{h}$ is maximal abelian). □

3.3 Abstract root systems

3.3.1 Manipulations of root systems

Let $(V, \langle\cdot|\cdot\rangle)$ be a (finite dimensional) Euclidean vector space. For $x \in V \setminus \{0\}$, we let $x^\vee = \frac{2x}{\langle x|x\rangle}$ and denote by $s_x : V \to V$ the orthogonal reflection with respect to $x^\perp$, which writes as:

$$s_x(y) = y - \langle y|x^\vee\rangle x.$$

Note that $(x^\vee)^\vee = x$, $s_{x^\vee} = s_x$ and $s_x(y^\vee) = (s_x(y))^\vee$ for all $x, y \in V \setminus \{0\}$.

Definition 3.3.1. *Let $(V, \langle\cdot|\cdot\rangle)$ be a Euclidean vector space. A **root system** of V is a subset $\Phi \subset V$ such that:*

1. *Φ is finite, $0 \notin \Phi$ and $\mathrm{Vect}(\Phi) = V$.*
2. *$\forall \alpha \in \Phi$, $s_\alpha(\Phi) = \Phi$.*
3. *$\forall \alpha, \beta \in \Phi$, $\langle \alpha|\beta^\vee\rangle \in \mathbb{Z}$.*

*We call Φ **reduced** if it also satisfies:*

4. *$\forall \alpha \in \Phi \forall t \in \mathbb{R}$, $t\alpha \in \Phi \Rightarrow t = \pm 1$.*

*The elements of Φ are called **roots**. The **rank** of Φ is $\dim V$.*
*The group $W \subset \mathrm{O}(V)$ generated by the s_α for $\alpha \in \Phi$ is called the **Weyl group** of Φ.*
For $\alpha, \beta \in \Phi$ we note $n(\alpha, \beta) = \langle \alpha|\beta^\vee\rangle$.

The notion of isomorphism between root systems may seem unnecessarily complicated at first, but it happens to be the right one.

Definition 3.3.2. *Let V, V' be Euclidean vector spaces, and let $\Phi \subset V$, $\Phi' \subset V'$ be root systems. An* ***isomorphism*** *from Φ to Φ' is a linear isomorphism $f : V \to V'$ such that $f(\Phi) = \Phi'$ et $f \circ s_\alpha \circ f = s_{f(\alpha)}$ pour tout $\alpha \in \Phi$.*

Proposition 3.3.3. *Let V be a Euclidean vector space, and $\Phi \subset V$ a root system.*

1. *The Weyl group W is finite.*
2. *If $\alpha \in \Phi$ then $-\alpha \in \Phi$.*
3. *If $\alpha \in \Phi$ and $t \in \mathbb{R}$ satisfies $t\alpha \in \Phi$ then $t \in \{\pm\frac{1}{2}, \pm 1, \pm 2\}$. We cannot have $\frac{1}{2}\alpha \in \Phi$ and $2\alpha \in \Phi$ at the same time.*

Proof.
1. The group W acts on the finite set Φ, and this action is faithful because $\mathrm{Vect}\Phi = V$.
2. We have $s_\alpha(\alpha) = -\alpha$.
3. We have $n(\alpha, t\alpha) = \frac{2}{t} \in \mathbb{Z}$ and $n(t\alpha, \alpha) = 2t \in \mathbb{Z}$, which yields $t \in \{\pm\frac{1}{2}, \pm 1, \pm 2\}$. Since $n(\frac{1}{2}\alpha, 2\alpha) = \frac{1}{2}$, these two elements cannot be simultaneously in Φ.

□

Exercise: Classify rank one root systems.

Let $\Phi \subset V$ be a root system. Given two roots $\alpha, \beta \in \Phi$, let φ be the angle between α and β. We have:

$$n(\alpha, \beta) = 2\frac{\|\alpha\|}{\|\beta\|}\cos\varphi.$$

It follows that $n(\alpha,\beta)n(\beta,\alpha) = 4\cos^2(\varphi) \in \mathbb{Z}$, hence $n(\alpha,\beta)n(\beta,\alpha) \in \{0,1,2,3,4\}$, and $n(\alpha,\beta) \in \{0, \pm 1, \pm 2, \pm 3, \pm 4\}$.

Notice that the case $n(\alpha,\beta) = \pm 4$ corresponds exactly to proportional roots ($\varphi = \pm\pi$). Let us now assume that it is not the case.

Replacing α by $-\alpha \in \Phi$ if necessary, we can assume that $n(\alpha,\beta) \le 0$ (geometrically, this means that $\varphi \ge \frac{\pi}{2}$, i.e. the angle between α and β is obtuse). Switching α and β if necessary, we can also assume that $\|\alpha\| \le \|\beta\|$. We now have that $n(\alpha,\beta) \in \{0, -1\}$, and only the four following cases are possible:

$$\begin{array}{llll}
n(\alpha,\beta) = 0 & n(\beta,\alpha) = 0 & \varphi = \frac{\pi}{2} & \\
n(\alpha,\beta) = -1 & n(\beta,\alpha) = -1 & \varphi = \frac{2\pi}{3} & \|\beta\| = \|\alpha\| \\
n(\alpha,\beta) = -1 & n(\beta,\alpha) = -2 & \varphi = \frac{3\pi}{4} & \|\beta\| = \sqrt{2}\|\alpha\| \\
n(\alpha,\beta) = -1 & n(\beta,\alpha) = -3 & \varphi = \frac{5\pi}{6} & \|\beta\| = \sqrt{3}\|\alpha\|
\end{array}$$

Corollary 3.3.4. *Let V be a Euclidean vector space and $\Phi \subset V$ a root system. If two roots $\alpha, \beta \in \Phi$ form and obtuse (resp. acute) angle, then $\alpha + \beta$ (resp. $\alpha - \beta$) is a root.*

Proof. If the angle is obtuse, then $n(\alpha, \beta) \leq 0$. According to the previous comments, up to switching α and β, we have that $n(\alpha, \beta) = -1$, hence $\beta + \alpha = \beta - n(\alpha, \beta)\alpha = s_\alpha(\beta) \in \Phi$.
If the angle is acute, then substituting $-\beta$ for β leads to the previous case. □

In particular, if the roots α, β are not proportional nor orthogonal, then either $\alpha + \beta$ or $\alpha - \beta$ is a root.

Exercise: Classify rank two root systems.

3.3.2 Classification of root systems

Definition 3.3.5. *Let V be a Euclidean vector space, and $\Phi \subset V$ a root system. A **basis** of Φ is a subset $\Pi \subset \Phi$ such that:*

- *Π is a vector basis of V.*
- *Any root $\alpha \in \Phi$ decomposes as $\alpha = \sum_{\pi \in \Pi} \alpha_\pi \pi$ where all the coefficients α_π are integer and have the same sign.*

*Elements of Π are called **simple roots**. A root $\alpha = \sum_{\pi \in \Pi} \alpha_\pi \pi$ is called **positive** (resp. **negative**) if all α_π are non negative (resp. non positive).*
*The **Cartan matrix** is the matrix $(n(\alpha, \beta))_{\alpha, \beta \in \Pi}$.*

We will admit the following fact.

Proposition 3.3.6. *Any root system has a basis.*

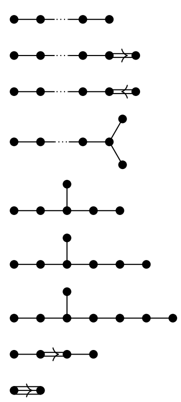
Definition 3.3.7. *Let V be a Euclidean vector space, $\Phi \subset V$ a root system, Π a basis of Φ, and $(n(\alpha, \beta))_{\alpha, \beta \in \Pi}$ the Cartan matrix. The **Dynkin diagram** of Φ (relatively to Π) is the oriented multi-edged graph Π, and two vertices $s, t \in \Pi$ are linked by:*

- *A single edge $\bullet\!-\!\bullet$ if $n(s,t) = n(t,s) = -1$.*
- *A double edge $\bullet\!=\!\bullet$ if $n(s,t) = -1$ and $n(t,s) = -2$.*
- *A triple edge $\bullet\!\equiv\!\bullet$ if $n(s,t) = -1$ et $n(t,s) = -3$.*

The edges are oriented from s towards t if $\|s\| > \|t\|$ ($\bullet\!\Rightarrow\!\bullet$ or $\bullet\!\Rrightarrow\!\bullet$).

Theorem 3.3.8 (Classification of root systems)**.**

1. *Any non empty connected Dynkin diagram is isomorphic to exactly one of the following diagrams:*



A_n	$n \geq 1$
B_n	$n \geq 2$
C_n	$n \geq 3$
D_n	$n \geq 4$
E_6	
E_7	
E_8	
F_4	
G_2	

2. *For every diagram in this list, there is, up to isomorphism, a unique irreducible reduced root system of which it is the Dynkin diagram.*

3. *For every $n \geq 1$, there is, up to isomorphism, a unique irreducible non reduced root system of rank n, denote by BC_n.*

3.3.3 Classification of complex semi-simple Lie algebras

3.3.4 Root systems of complex semi-simple Lie algebras

Let $\mathfrak{g}$ be a complex finite dimensional semi-simple Lie algebra, and $\mathfrak{h} \subset \mathfrak{g}$ a Cartan subalgebra. Recall that for $\alpha \in \mathfrak{h}^*$, we set:

$$\mathfrak{g}_\alpha = \{X \in \mathfrak{g} | \forall H \in \mathfrak{h}\ [H, X] = \alpha(H)X\}.$$

We write $\Phi = \{\alpha \in \mathfrak{h}^* \setminus \{0\} | \mathfrak{g}_\alpha \neq \{0\}\}$ the set of roots of $\mathfrak{g}$ relatively to $\mathfrak{h}$, and $\mathfrak{h}^*_{\mathbb{R}} \subset \mathfrak{h}^*$ the $\mathbb{R}$-vector subspace spanned by Φ. We now wish to endow $\mathfrak{h}^*_{\mathbb{R}}$ with an inner product, so that Φ is a root system. It will be constructed by using the Killing form of $\mathfrak{g}$. There is actually a strong relationship between the decomposition of $\mathfrak{g}$ into root spaces and the Killing form (it is almost an orthogonal decomposition).

Proposition 3.3.9. *Let $\mathfrak{g}$ be a complex finite dimensional semi-simple Lie algebra, $\mathfrak{h} \subset \mathfrak{g}$ a Cartan subalgebra, $\Phi \subset \mathfrak{h}^*$ the set of roots and $\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Phi} \mathfrak{g}_\alpha$ the root space decomposition.*

1. $\mathrm{Vect}_{\mathbb{C}} \Phi = \mathfrak{h}^*$.

2. $\forall \alpha, \beta \in \mathfrak{h}^*\ [\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subset \mathfrak{g}_{\alpha+\beta}$.

3. $\forall \alpha \in \Phi$, $\mathfrak{g}_\alpha \perp \mathfrak{h}$ *for the Killing form B of $\mathfrak{h}$.*

4. *$B|_{\mathfrak{h}\times\mathfrak{h}}$ is non degenerate.*

Proof.

1. We have that $\cap_{\alpha\in\Phi}\ker\alpha \subset \mathfrak{z}(\mathfrak{g}) = \{0\}$. Considering duals, we find that $\mathrm{Vect}_{\mathbb{C}}\Phi = \mathfrak{h}^*$.

2. It is a consequence of the Jacobi identity. Consider $X_\alpha \in \mathfrak{g}_\alpha$, $X_\beta \in \mathfrak{g}_\beta$, and $H \in \mathfrak{h}$. We calculate:

$$\begin{aligned}[H,[X_\alpha,X_\beta]] &= -[X_\alpha,[X_\beta,H]] - [X_\beta,[H,X_\alpha]] \\ &= -[X_\alpha,-\beta(H)X_\beta] - [X_\beta,\alpha(H)X_\alpha] \\ &= (\alpha(H)+\beta(H))[X_\alpha,X_\beta]\end{aligned}$$

3. If $H \in \mathfrak{h}$ and $X_\alpha \in \mathfrak{g}_\alpha$, the matrix of $\mathrm{ad}(H)\circ\mathrm{ad}(X_\alpha)$ in a basis adapted to the decomposition $\mathfrak{g} = \mathfrak{h}\oplus\bigoplus_{\beta\in\Phi}\mathfrak{g}_\beta$ has vanishing diagonal coefficients, since $\mathrm{ad}(H)\circ\mathrm{ad}(X_\alpha)(\mathfrak{g}_\beta) \subset \mathfrak{g}_{\alpha+\beta}$. Hence $\mathrm{Tr}(\mathrm{ad}(H)\circ\mathrm{ad}(X_\alpha)) = 0$, i.e. $B(H,X_\alpha) = 0$.

4. Let $H \in \ker(B|_{\mathfrak{h}\times\mathfrak{h}})$. Since H is orthogonal to all the root spaces, we find that $H \in \ker B$, hence $H = 0$ because $\mathfrak{g}$ is semi-simple.

□

The Killing form induces a non degenerate bilinear form $\langle\cdot|\cdot\rangle$ on $\mathfrak{h}^*$ by duality.

Theorem 3.3.10. *Let $\mathfrak{g}$ be a complex finite dimensional semi-simple Lie algebra, $\mathfrak{h} \subset \mathfrak{g}$ a Cartan subalgebra, $\Phi \subset \mathfrak{h}^*$ the set of roots, $\mathfrak{g} = \mathfrak{h}\oplus\bigoplus_{\alpha\in\Phi}\mathfrak{g}_\alpha$ the root space decomposition and $\langle\cdot|\cdot\rangle$ the bilinear form induced on $\mathfrak{h}^*$ by the Killing form B of $\mathfrak{g}$.*

1. *The restriction of $\langle\cdot|\cdot\rangle$ to $\mathfrak{h}^*_{\mathbb{R}}$ is a scalar product.*

2. *The set Φ is a reduced root system of $(\mathfrak{h}^*_{\mathbb{R}},\langle\cdot|\cdot\rangle)$.*

3. *Up to isomorphism, this root system only depends on $\mathfrak{g}$.*

So we can talk about the root system of a complex finite dimensional semi-simple Lie algebra, and its Dynkin diagram.

3.3.5 The classification theorem

Theorem 3.3.11 (Classification of complex semi-simple Lie algebras)**.**

1. *Two complex finite dimensional semi-simple Lie algebras are isomorphic if and only if their root systems are isomorphic.*

2. *A complex finite dimensional semi-simple Lie algebra is simple if and only if its Dynkin diagram is connected.*

3. *Every Dynkin diagram can be obtained by a complex finite dimensional semi-simple Lie algebra.*

Let us pick up the case of $\mathfrak{sl}(n,\mathbb{C})$ where we left it. We saw that the subalgebra $\mathfrak{h}$ of diagonal traceless matrices is a Cartan subalgebra. We also saw that the roots are the forms $\alpha_{i,j}$ defined by $\alpha_{i,j}(H) = h_i - h_j$ for $H = \mathrm{diag}(h_1,\dots,h_n)$ (with $i \neq j$), and that the root spaces are $\mathfrak{sl}(n,\mathbb{C})_{\alpha_{i,j}} = \mathbb{C}E_{i,j}$.

Let us prove that $\alpha_{1,2},\dots,\alpha_{n-1,n}$ is a basis of Φ. It is a family of $n-1$ linearly independent elements of $\mathfrak{h}^*$, hence a vector basis. For $i < j$, we find:

$$\alpha_{i,j} = \alpha_{i,i+1} + \cdots + \alpha_{j-1,j}$$

This shows that $\alpha_{i,j}$ and $\alpha_{j,i} = -\alpha_{i,j}$ have integer coefficients all sharing the same sign in the basis Π.

In order to compute scalar products and establish the Dynkin diagram, we need to find the vectors $H_1,\dots,H_{n-1} \in \mathfrak{h}$ such that $B(H_i,\bullet) = \alpha_{i,i+1}$. Recall that $B(X,Y) = 2n\,\mathrm{Tr}(XY)$. It follows that

$$H_i = \frac{1}{2n}\mathrm{diag}(0,\dots,1,-1,0,\dots)$$

where the 1 is in the i^{th} position.

We have that $\left\langle \alpha_{i,i+1} \middle| \alpha_{j,j+1} \right\rangle = B(H_i,H_j)$, which yields:

$$\langle \alpha_{i,i+1} | \alpha_{j,j+1} \rangle = \begin{cases} \frac{1}{n} & \text{if } i = j \\ \frac{-1}{2n} & \text{if } |i-j| = 1 \\ 0 & \text{if } |i-j| > 1 \end{cases}$$

We now can find the coefficients $n(\alpha,\beta) = \frac{2\langle \alpha|\beta\rangle}{\langle \beta|\beta\rangle}$, which are:

$$n(\alpha_{i,i+1},\alpha_{j,j+1}) = \begin{cases} 2 & \text{if } i = j \\ -1 & \text{if } |i-j| = 1 \\ 0 & \text{if } |i-j| > 1 \end{cases}$$

This shows that the Dynkin diagram of $\mathfrak{sl}(n,\mathbb{C})$ is of type A_{n-1}. Similar calculations allows us to find the Dynkin diagrams of all classical semi-simple Lie algebras.

Proposition 3.3.12 (Dynkin diagrams of classical Lie algebras)**.**
The Dynkin diagram of the Lie algebra $\mathfrak{sl}(n+1,\mathbb{C})$ is of type A_n.
The Dynkin diagram of the Lie algebra $\mathfrak{so}(2n+1,\mathbb{C})$ is of type B_n.
The Dynkin diagram of the Lie algebra $\mathfrak{sp}(2n,\mathbb{C})$ is of type C_n.
The Dynkin diagram of the Lie algebra $\mathfrak{so}(2n,\mathbb{C})$ is of type D_n.

Types E_6, E_7, E_8, F_4 and G_2, called **exceptional** correspond to Lie algebras that are much more complicated to describe.

Chapter 4

Homogeneous spaces

4.1 Smooth actions of Lie groups

We will now start to relate Lie groups and the geometry of manifolds, through actions of Lie groups.

The two reference examples are obtained from a group G and a subgroup $H \subset G$. The group H acts on G by left multiplication ($h.g = hg$). One can consider the quotient G/H of G by this action, it consists of cosets gH for $g \in G$. The group G acts on G/H by $g.g'H = gg'H$.

4.1.1 Some vocabulary

Definition 4.1.1. *Let G be a Lie group, and X a smooth manifold. An action $G \curvearrowright X$ is called **smooth** if the map*

$$\begin{cases} G \times X & \rightarrow & X \\ (g, x) & \mapsto & gx \end{cases}$$

is smooth.

Definition 4.1.2. *Let G be a Lie group, X a smooth manifold, and consider a smooth action $G \curvearrowright X$.*
For $g \in G$, we denote by $m_g : X \rightarrow X$ the diffeomorphism defined by $m_g(x) = gx$.
*For $x \in X$, the **orbit map** is $\varphi_x : G \rightarrow X$ defined by $\varphi_x(g) = gx$.*
*The **orbit** $G.x$ of x is the range of φ_x.*
*The **stabiliser** of x (also called its **isotropy subgroup**) is $G_x = \{g \in G | gx = x\}$.*
We denote by $\Theta_x : G/G_x \rightarrow X$ the map induced by the orbit map φ_x.

Definition 4.1.3. *Let G be a Lie group, X a smooth manifold, and consider a smooth action $G \curvearrowright X$.*
*We say that $G \curvearrowright X$ is **transitive** if there is $x \in X$ such that $G.x = X$.*
*We say that $G \curvearrowright X$ is **free** if for all $x \in X$, we have $G_x = \{e\}$.*
*We say that $G \curvearrowright X$ is **proper** if for all compact subset $K \subset X$, the set $\{g \in G | gK \cap k \neq \emptyset\}$ is compact.*

Definition 4.1.4. *A homogeneous space is a manifold X that possesses a smooth transitive action of a Lie group.*

Lemma 4.1.5. *Let G be a Lie group, X a smooth manifold, and consider a smooth action $G \curvearrowright X$. Let $x \in X$.*

1. *The orbit map $\varphi_x : G \to X$ has constant rank.*
2. *The stabiliser G_x is an embedded Lie subgroup of G, and $T_e G_x = \ker d_e\varphi_x$.*

Remark. *The rank of φ_x may depend on $x \in X$.*

Proof.

1. Differentiating $\varphi_x \circ L_g = m_g \circ \varphi_x$ (i.e. the equivariance of $\varphi_x : G \to X$) at e yields $d_g\varphi_x \circ d_e L_g = d_x m_g \circ d_e\varphi_x$, which implies that $d_g\varphi_x$ and $d_e\varphi_x$ have the same rank since L_g and m_g are diffeomorphisms.
2. It is a level set of a constant rank map, hence an embedded submanifold with tangent space the kernel of $d_e\varphi_x$.

□

4.1.2 Topology of the quotient by a smooth action

We will now discuss manifolds that are obtained as quotients by a smooth action of a Lie group. Before discussing manifold structures on quotients, we need to discuss the topology, which will always be the quotient topology. Given an action $G \curvearrowright X$, if $\pi : X \to X/G$ is the canonical projection, then we know that π is surjective and continuous (recall that $V \subset X/G$ is open if and only if $\pi^{-1}(V)$ is open), which is the case for all quotient topologies. In the case of the quotient by a smooth group action, we have an additional property (which only uses the continuity of the action).

Proposition 4.1.6. *Let G be a Lie group, X a manifold, and consider a smooth action $G \curvearrowright X$. The quotient map $\pi : X \to X/G$ is open, i.e. the image of an open set is open.*

Proof. Let $U \subset X$ be open. Then $\pi^{-1}(\pi(U)) = \bigcup_{g \in G} m_g(U)$ is open because m_g is a homeomorphism, therefore $\pi(U)$ is open. □

Lemma 4.1.7. *Let G be a Lie group, X a manifold, and consider a smooth proper action $G \curvearrowright X$. For all compact subset $K \subset X$, the set $GK = \{gx \mid (g,x) \in G \times K\} \subset X$ is closed. In particular, orbits are closed.*

Proof. Recall that the topology of a manifold is metrisable, so we can use sequences to show that a subset is closed. Si $(g_n) \in G^{\mathbb{N}}$, $(x_n) \in K^{\mathbb{N}}$ and $g_n x_n \to y \in X$, we wish to show that $y \in GK$. Since K is compact, we can assume that $x_n \to x \in K$. Since X is locally compact, there is a compact set $L \subset X$ such that $y \in \mathring{L}$ and $x \in \mathring{L}$. For large enough n, we find that $g_n x_n \in g_n L \cap L$, therefore (g_n) stays in a compact set of G (because the action is proper), and we can assume that $g_n \to g \in G$. We get $y = gx \in GK$. □

Lemma 4.1.8. *Let G be a Lie group, and $H \subset G$ a Lie subgroup. The left action $H \curvearrowright G$ is proper if and only if G is closed.*

Proof. If $H \curvearrowright G$ is proper, then the orbit $H.e$ is closed according to Lemma 4.1.7, i.e. H is closed.

Reciprocally, if H is closed and $K \subset G$ is compact, then $\{h \in H \,|\, h.K \cap K \neq \emptyset\} = H \cap (K \cap K^{-1})$ is the intersection of a closed set and a compact set, hence closed, i.e. the action is proper. □

Lemma 4.1.9. *Let G be a Lie group, X a manifold, and consider a smooth proper action $G \curvearrowright X$. The quotient topology on X/G is Hausdorff and second countable.*

Proof. First notice that the image of a second countable space under a continuous and open map is also second countable, so it is the case for X/G.

Let $x, y \in X$ be such that $\pi(x) \neq \pi(y)$. Since $x \notin Gy$ and Gy is closed according to Lemma 4.1.7, we can consider an open set $U \subset X$ containing x, such that $U \cap Gy = \emptyset$ (because the topology of X is metrisable). Up to shrinking U, we can assume that $\overline{U}$ is compact, and that $\overline{U} \cap Gy = \emptyset$. This directly implies that $G\overline{U} \cap Gy = \emptyset$.

According to Lemma 4.1.7, the set $G\overline{U}$ is closed. Since $G\overline{U} \cap Gy = \emptyset$, we can consider open sets $V, W \subset X$ such that $G\overline{U} \subset V$, $Gy \subset W$ and $V \cap W = \emptyset$ (once again, because the topology of X is metrisable).

Since π is an open map, the sets $\pi(U)$ and $\pi(W)$ are open in G/X. We have that $\pi(x) \in \pi(U)$ and $\pi(y) \in \pi(W)$. Moreover, the inclusion $\pi^{-1}(\pi(U)) \subset V$ shows that $\pi(U) \cap \pi(W) = \emptyset$, therefore X/G is Hausdorff. □

4.2 Quotients of Lie groups

We will now study quotients of Lie groups by Lie subgroups. Note that the quotient by an immersed subgroup need not be a manifold (think of an irrational line in the torus). In order to guarantee the uniqueness of the manifold structure on the quotient, we will need to understand submersions a little better.

Proposition 4.2.1. *Let M, N, P be manifolds, $p : M \to N$ a surjective submersion, and $f : N \to P$ a map. Then f is smooth if and only if $f \circ p$ is smooth.*

Proof. The Constant Rank Theorem associated to the surjectivity of p implies that every point in N has a neighbourhood U on which we can find a smooth function $\varphi : U \to M$ satisfying $p \circ \varphi = \mathrm{Id}_U$. It follows that $f = f \circ p \circ \varphi$ on U. □

Theorem 4.2.2. *Let G be a Lie group, $H \subset G$ an embedded Lie subgroup, and $\pi : G \to G/H$ the projection. There is a unique manifold structure on G/H for which π is a submersion. Moreover, the action of G on G/H is smooth.*
If H is a normal subgroup of G, then G/H is a Lie group, π is a Lie group morphism, $\mathrm{Lie}(H)$ *is an ideal of* $\mathrm{Lie}(G)$ *and* $Lie(G/H)$ *is isomorphic to* $\mathrm{Lie}(G)/\mathrm{Lie}(H)$.

Remarks.

- $\dim G/H = \dim G - \dim H$.
- $T_{\pi(e)}G/H \approx T_eG/T_eH$, *more generally* $T_{\pi(g)}G/H \approx T_gG/T_g(gH)$.
- *If M is a manifold, a map $f : G/H \to M$ is smooth if and only if $f \circ \pi$ is smooth.*
- *If $f : G \to G'$ is a Lie group morphism, then $f(G)$ is an immersed Lie subgroup of G'. Indeed, the quotient $G/\ker f$ is a Lie group, and $f(G)$ is the image of $\overline{f} : G/\ker f \to G'$ which is an injective Lie group morphism, hence an immersion.*

Proof. The uniqueness of the manifold structure follows from applying Proposition 4.2.1 to the identity map. For the existence, we already know that the quotient topology is Hausdorff and second countable (according to Lemma 4.1.9, because $H \curvearrowright G$ is proper, se Lemma 4.1.8). We only need to find an atlas.

Let $n = \dim G$ and $p = \dim H$.

First step: We look for a $n-p$-dimensional submanifold $\mathcal{W} \subset G$ such that $e \in \mathcal{W}$, $T_eH \oplus T_e\mathcal{W} = T_eG$ and $\pi|_{\mathcal{W}}$ is injective.

For this, first consider a $n-p$-dimensional submanifold $\mathcal{W} \subset G$ such that $e \in \mathcal{W}$ and $T_e\mathcal{W} \oplus T_eH = T_eG$. Define $\varphi : H \times \mathcal{W} \to G$ by $\varphi(h,x) = hx$. We find $d_{(e,e)}\varphi(X,Y) = X+Y$, so up to shrinking $\mathcal{W}$, the Local Inverse Function Theorem provides us with open sets $U \subset H$ and $V \subset G$ both containing e such that φ restricts to a diffeomorphism from $U \times \mathcal{W}$ to V.

Let us now show that we can shrink $\mathcal{W}$ to make $\pi|_{\mathcal{W}}$ injective. Were it not the case, we could find sequences $(h_k) \in H^{\mathbb{N}}$ and $(w_k) \in \mathcal{W}^{\mathbb{N}}$ such that $w_k \to e$, $h_k w_k \in \mathcal{W}$, $h_k w_k \to e$ and $h_k w_k \neq w_k$. We then would have $h_k = h_k w_k w_k^{-1} \to e$, hence $h_k \in U$ for large enough k. But $\varphi(h_k, w_k) = \varphi(e, h_k w_k)$ is a contradiction with the fact that $\varphi|_{U\times\mathcal{W}}$ is a diffeomorphism.

Second step: Build a chart around $\varphi(e)$.

Consider the map $\varphi : H \times \mathcal{W} \to G$ from the first step, and open sets $U \subset H$, $V \subset G$ containing e such that φ restricts to a diffeomorphism from $U \times H$ to V. Let $\psi : V \to U \times \mathcal{W}$ be its inverse.

Notice that $\pi(V) = \pi(\mathcal{W})$, therefore $\pi(\mathcal{W})$ is an open subset of G/H (since π is an open map) containing $\pi(e)$. Let $\Phi : \pi(\mathcal{W}) \to \mathcal{W}$ be defined by $\Phi(\pi(w)) = w$ for $w \in \mathcal{W}$ (it is well defined because $\pi|_{\mathcal{W}}$ is injective). Let us show that Φ is a homeomorphism.

Its construction makes it a bijection, and its inverse $\pi|_{\mathcal{W}}$ is continuous. Continuity of Φ is a consequence of the fact that π is an open map: if $\mathcal{O} \subset \pi(\mathcal{W})$ is open, then $\Phi^{-1}(\mathcal{O}) = \pi^{-1}(\mathcal{O}) \cap \mathcal{W}$ is open.

Third step: Build an atlas on G/H.

Given $g \in G$, we let $U_g = m_g(\pi(\mathcal{W}))$ and $\Phi_g = \Phi \circ m_{g^{-1}}$. This defines a homeomorphism from an open subset of G/H containing $\pi(g)$ to a manifold $\mathcal{W}$ (which we can assume to be diffeomorphic to $\mathbb{R}^{n-p}$).

Given $g, g' \in G$, the transition map $\Phi_g(U_g \cap U_{g'}) \to \Phi_{g'}(U_g \cap U_{g'})$ is the restriction to $\mathcal{W} \subset G$ of $L_{g'}^{-1} \circ L_g$, hence smooth.

Fourth step: Smoothness of the action.

In the charts defined above, the action of G reads as the restrictions to some submanifolds of left multiplication in G, which guarantees the smoothness.

Fifth step: The normal subgroup case.

If H is a normal subgroup, then G/H is a group, and by applying Proposition 4.2.1 to the projection $G \times G \to H \times H$ and the map $(g, h) \mapsto \pi(gh) = g.\pi(h)$, we find that G/H is a Lie group and that π is a Lie group morphism. Then $d_e\pi$ gives a short exact sequence of Lie algebras $0 \to \mathrm{Lie}(H) \to \mathrm{Lie}(G) \to \mathrm{Lie}(G/H) \to 0$.

□

4.3 Transitive actions of Lie groups

Proposition 4.3.1. *Let G be a Lie group, X a manifold, and consider a smooth action $G \curvearrowright X$. For all $x \in X$, the canonical map $\Theta_x : G/G_x \to X$ is an injective immersion.*

Remark. *In particular, the orbit $G.x$ is an immersed submanifold, and $T_xG.x = d_e\varphi_x(\mathfrak{g})$.*

Proof. Let $\pi : G \to G/G_x$ be the projection. The map Θ_x is smooth because $\Theta_x \circ \pi = \varphi_x$ is smooth.

It is also injective, and its range is that of φ_x, i.e. $G.x$.

We are left to show that $d_{\pi(g)}\Theta_x$ is injective for all $g \in G$. Using the equivariance, we only need to prove it for $g = e$.

We know that $T_eG_x = \ker d_e\varphi_x$, and $\varphi_x = \Theta_x \circ \pi$ where π is a submersion satisfying $\ker d_e\pi = T_eG_x$, so the formula $d_e\varphi_x = d_{\pi(e)}\Theta_x \circ d_e\pi$ yields $\ker d_{\pi(e)}\Theta_x = \{0\}$. □

Theorem 4.3.2. *Let G be a Lie group and X a manifold. If a smooth action $G \curvearrowright X$ is transitive, then for every $x \in X$, the canonical map $\Theta_x : G/G_x \to X$ is a diffeomorphism.*

Proof. According to Proposition 4.3.1, Θ_x is an injective immersion. Since the action is transitive, it is also surjective. A bijective immersion is a diffeomorphism. □

4.4 Embedded orbits

Proposition 4.4.1. *Let G be a Lie group, and X a manifold. If a smooth action $G \curvearrowright X$ is proper, then orbits are embedded submanifolds.*

Remarks.

- *According to Proposition 4.3.1 and Lemma 4.1.7, we know that the orbits are closed immersed submanifolds. But that is not enough to guarantee that they are embedded submanifolds.*
- *If G is compact, then all smooth actions are proper, so orbits are always embedded submanifolds.*

Proof. Since $\Theta_x : G/G_x \to X$ is an injective immersion with range $G.x$, we only need to show that it is a homeomorphism onto $G.x$.

Let $\Phi : Gx \to G/G_x$ be its inverse. Consider a sequence $(y_k) \in (Gx)^{\mathbb{N}}$ and $y \in Gx$ such that $y_k \to y$. We can write $y_k = g_kx$ and $y = gx$ where $g_k, g \in G$. It follows that $\Phi(y_k) = \pi(g_k)$ and $\Phi(y) = \pi(g)$ where $\pi : G \to G_x$ is the projection.

Since $g_kx \to y$, properness of the action implies that (g_k) lies in a compact subset of G. Up to considering a subsequence, we can assume that $g_k \to h \in G$.

We find that $hx = y$, hence

$$\Phi(y) = \pi(h) = \lim_{k\to+\infty} \pi(g_k) = \lim_{k\to+\infty} \Phi(y_k)$$

and Φ is continuous. □

Theorem 4.4.2. *Let $G \curvearrowright X$ be a smooth action of a Lie group G on a manifold X, and let $x \in X$.*

1. *The orbit Gx is an embedded submanifold if and only if it is locally closed.*

2. *If Gx is an embedded submanifold, then Θ_x is a diffeomorphism from G/G_x to Gx.*

Remark. *In particular, every closed orbit is an embedded submanifold.*

Proof.

1. Any embedded submanifold is locally closed. Reciprocally, assume that $G.x$ is locally closed. Then $G.x$ is locally compact, and satisfy Baire's property. Since Θ_x is an injective immersion, we only need to show that it is a homeomorphism onto its image. It is enough for this to show that φ_x is an open map. Using the equivariance, we are left to show that for every open neighbourhood U of e, the image $\varphi_x(U) = U.x$ is an open subset of $G.x$. Let V be a compact neighbourhood of e such that $V^{-1}V \subset U$. Consider a dense sequence $(g_i)_{i\in\mathbb{N}}$ in G. We have that $G = \bigcup_{i\in\mathbb{N}} g_i V$, and $G.x = \bigcup_{i\in\mathbb{N}} g_i(V.x)$. Since V is compact, so is $V.x = \varphi_x(V)$, and it must be closed in $G.x$. According to Baire's property, there is some $i \in \mathbb{N}$ such that $g_i(V.x)$ has non empty interior. If $g \in V$ is such that gx lies in the interior of $g_i(V.x)$, then $g^{-1}Vx$ is a neighbourhood of x in $G.x$ that lies in $\varphi_x(U)$.

2. A proper injective immersion is a diffeomorphism onto its image.

□

4.5 Examples of homogeneous spaces

1. Spheres.
$$\mathbb{S}^n \approx \mathrm{O}(n+1)/\mathrm{O}(n) \approx \mathrm{SO}(n+1)/\mathrm{SO}(n)$$

2. Projective spaces.
$$\mathbb{RP}^n \approx \mathrm{PO}(n+1)/\mathrm{O}(n) \approx \mathrm{PSL}(n,\mathbb{R})/P$$

3. Grassmannians.
$$\mathcal{G}_k(\mathbb{R}^n) \approx \mathrm{O}(n)/\mathrm{O}(k)\times\mathrm{O}(n-k) \approx \mathrm{GL}(n,\mathbb{R})/P_{n,k}$$

4. The upper half plane.
$\mathrm{SL}(2,\mathbb{R})$ acts on $\mathcal{H} = \{z \in \mathbb{C} | \mathrm{Im}(z) > 0\}$ via $\begin{pmatrix} a & b \\ c & d \end{pmatrix}.z = \frac{az+b}{cz+d}$.
$$\mathcal{H} \approx \mathrm{SL}(2,\mathbb{R})/\mathrm{SO}(2,\mathbb{R}) \approx \mathrm{PSL}(2,\mathbb{R})/\mathrm{PSO}(2,\mathbb{R})$$