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When first learning about functions, your attention was drawn to the importance of figuring out its domain, formally this means the set E on which a map $f : E \to F$ is defined. For the purpose of calculus, you first learned that E should be an interval in $\mathbb{R}$, then an open subset of $\mathbb{R}^d$, later on that E should be a manifold.

But the target space F received little attention so far, with the exception of the study of local inverses, where you learned that you can always replace F with the range of f in order to make it artificially surjective.

A strange situation tends to happen when working on the geometry of manifolds (or in physics): we may define maps f, for which the source is a manifold M, but the point $f(x)$ belongs to a set F_x which depends on x. We will mostly focus on the case where each F_x is a vector space.

From a set theoretic point of view, this is not a huge problem, as you can always consider such maps to have image in the disjoint union $\sqcup_{x\in M} F_x$. From a differential point of view, it is not so straightforward, as we naturally want to define a differential as $d_x f(v) = \lim_{t\to 0} \frac{1}{t}(f(\gamma(t)) - f(x))$ where $\gamma : \mathbb{R} \to M$ is a smooth curve satisfying $\gamma(0) = x \in M$ and $\dot{\gamma}(0) = v \in T_x M$. The problem is that the difference between $f(\gamma(t))$ and $f(\gamma(0))$ is not defined as they live in different vector spaces. To make some sense of this formula, we need to find a way of connecting $F_{\gamma(0)}$ and $F_{\gamma(t)}$, which is possible, but not canonical.

This situation happens naturally when considering second order derivatives of maps defined on manifolds. Start with a smooth function $f : M \to \mathbb{R}$. Then for $x \in M$, we have a linear map $d_x f : T_x M \to \mathbb{R}$, i.e. $d_x f \in (T_x M)^* = T_x^* M$. So df can be seen as a map of M, but the image $d_x f$ lies in a set $T_x^* M$ that depends on x.

Chapter 1

Fibre bundles

1.1 General fibre bundles

1.1.1 Submersions and trivialisations

Consider two manifolds M and F. A fibre bundle over M with fibre F is the assignment to each $x \in M$ of a manifold ξ_x which is diffeomorphic to F in a way that "depends smoothly on x". The whole point of the following definitions is to make some sense of this smooth dependence.

Let us first focus on the task of assigning a manifold ξ_x to each $x \in M$. Since constructing a manifold structure on a set is a tedious task, we want to define ξ_x as a submanifold of a given manifold E. The simplest setting is to consider a submersion $p : E \to M$, and to set $\xi_x = p^{-1}(x)$.

For this to actually produce a manifold for every $x \in M$, we require p to be surjective. In this case, all manifolds ξ_x have the same dimension, but they need not be diffeomorphic to each other.

Even if they are all diffeomorphic to the same manifold F, just knowing it abstractly is not enough, we really need the diffeomorphism to "depend smoothly on x". If we consider a diffeomorphism $\theta_x : F \to \xi_x$ for each $x \in M$, then one can simply express the smooth dependence of θ_x in x by requiring the smoothness of the map $(x,y) \mapsto \theta_x(y)$. This is what we will call a trivialisation.

Definition 1.1.1. *Consider a surjective submersion $p : E \to M$ and a manifold F. A **trivialisation of** p **with respect to** F is a collection of diffeomorphisms $(\theta_x : F \to p^{-1}(x))_{x \in M}$ such that the map*

$$\Theta : \begin{cases} M \times F & \to & E \\ (x,z) & \mapsto & \theta_x(z) \end{cases}$$

*is smooth. We say that p is **trivialisable with respect to** F if it possesses a trivialisation with respect to F.*

The simplest example of a submersion onto M which is trivialisable with respect to F is the projection $\pi_1 : M \times F \to M$ onto the first factor. Here we assign to each $x \in M$ the manifold $\xi_x = \pi_1^{-1}(\{x\}) = \{x\} \times F$ which is diffeomorphic to F, and by setting $\theta_x(y) = (x, y)$ we find a diffeomorphism $\theta_x : F \to \xi_x$ which is a smooth function on $M \times F$.

Up to a diffeomorphism, it is the only example.

Proposition 1.1.2. *Consider manifolds E, M, F, a surjective submersion $p : E \to M$, and $\theta = (\theta_x)_{x \in M}$ a trivialisation of p with respect to F. Then the map*

$$\Theta : \begin{cases} M \times F & \to & E \\ (x, z) & \mapsto & \theta_x(z) \end{cases}$$

is a diffeomorphism.

Remark. *The important property of Θ is that $p \circ \Theta = \pi_1$ where $\pi_1(x, z) = x$.*

Proof. The map Θ is smooth by definition of a trivialisation of p with respect to F.
Note that Θ is bijective, and that $\Theta^{-1}(y) = (p(x), \theta_{p(x)}^{-1}(y))$ for all $y \in E$.

The fact that $p \circ \Theta(x, z) = x$ for all $(x, z) \in M \times F$ differentiates to $d_{(x,z)}(p \circ \Theta)(\dot{x}, \dot{z}) = \dot{x}$ for $\dot{x} \in T_x M$ and $\dot{z} \in T_z F$.

If $d_{(x,z)}\Theta(\dot{x}, \dot{z}) = 0$ for some $x \in M$, $\dot{x} \in T_x M$, $z \in F$, $\dot{z} \in T_z F$, it follows from the differentiation of $p \circ \Theta$ that $\dot{x} = 0$. Now $d_{(x,z)}\Theta(0, \dot{z}) = d_z \theta_x(\dot{z}) = 0$, therefore $\dot{z} = 0$ since θ_x is a diffeomorphism.

We have seen that Θ is a bijective immersion, hence a diffeomorphism. □

For now it may seem that we are stuck in a dead end, since the assignment to each $x \in M$ of a manifold ξ_x which is diffeomorphic to F in a way that depends smoothly on x is the same as considering the product $M \times F$.

For this to lead to a rich theory, we only need to ask for the trivialisations to be defined locally.

To make sense of this, we need to consider restrictions of surjective submersions. It is important to notice that the notion of surjective submersions behaves well under restrictions to open sets of the target space: if $p : E \to M$ is a surjective submersion, and $U \subset M$ is open, then the restriction $p|_{p^{-1}(U)} : p^{-1}(U) \to U$ is also a surjective submersion.

Definition 1.1.3. *A **fibre bundle** $\xi = (E, p, M, F)$ is the data of manifolds E, M, F and a map $p : E \to M$ such that:*

- *The map $p : E \to M$ is a surjective submersion.*
- *For every $x \in M$, there is an open neighbourhood $U \subset M$ such that $p|_{p^{-1}(U)} : p^{-1}(U) \to U$ is trivialisable with respect to F.*

*The **total space** of ξ is the manifold E, the **projection** of ξ is the map $p : E \to M$, the **base** of ξ is the manifold M and the **fibre** of ξ is the manifold F. For $x \in M$, the **fibre over** x is the manifold $\xi_x = p^{-1}(x)$.*

*We say that ξ is **trivialisable** if p is trivialisable with respect to F.*

Remarks.

- *As mentionned above, given any open set $U \subset M$, the map $p|_{p^{-1}(U)} : p^{-1}(U) \to U$ is a surjective submersion, which is important for this definition to make sense.*
- *For every $x \in M$, the fibre $\xi_x = p^{-1}(x)$ is a submanifold of E diffeomorphic to F.*
- *A map $p : E \to M$ is called a **locally trivial fibration** if there is a manifold F such that (E, p, M, F) is a fibre bundle.*

The standard example is the **trivial bundle**, namely $\xi = (M \times F, \pi_1, M, F)$. We will have many examples of fibre bundles in this course that will not be equivalent to the trivial bundle in any relevant way.

Just as mentioned for surjective submersions, the notion of restriction of fibre bundles behave well when restricting to open subsets of the base.

Definition 1.1.4. *Let $\xi = (E, p, M, F)$ be a fibre bundle, and $U \subset M$ an open set. The **restriction** of ξ to U is $\xi|_U = (p^{-1}(U), p|_{p^{-1}(U)}, U, F)$.*

Remark. *It is also a fibre bundle.*

Definition 1.1.5. *Let $\xi = (E, p, M, F)$ be a fibre bundle.*
*An open set $U \subset M$ is a **trivialising domain** if $\xi|_U$ is trivialisable.*
*A **trivialising chart** is a pair (U, θ) where $U \subset M$ is a trivialisation domain and θ is a trivialisation of $\xi|_U$.*
*A **trivialising atlas** $\mathcal{A}$ is a set of trivialising charts such that $\bigcup_{(U,\theta)\in\mathcal{A}} U = M$.*

Remark. *For all the set theory nerds out there, a trivialising chart can be seen as a subset of $M \times F \times E$, so an atlas is a subset of $\mathcal{P}(M \times F \times E)$, and all set theoretic manipulations are valid.*

Fibre bundles are not all trivialisable, however by definition they possess a trivialising atlas.

1.1.2 Constructing fibre bundles

Constructing fibre bundles, even the most basic examples such as the tangent bundle of a manifold, can be quite tricky and rather annoying because of one aspect: the manifold structure on the total space. It turns out that is usually the most technical and time consuming part of the proof that something is a fibre bundle. As a result, this manifold structure is usually quite poorly understood. Moreover, this differentiable structure ends up getting more attention than it deserves, as the study of fibre bundles basically boils down to techniques that allow us to avoid manipulating this differentiable structure in an abstract way. For this reason, we will now see how this manifold structure can systematically be derived from local trivialisations.

Since we usually consider that the base M and the fibre F are familiar manifolds, it is much more convenient to only work with the issue of differentiabily for maps defined on and with values in open subsets of the familiar M, F, or $M\times F$ rather than the very abstract total space E.

Theorem 1.1.6. *Let M, F be manifolds. Consider a collection of manifolds $(\xi_x)_{x\in M}$, an open cover $\mathcal{U}$ of M, and for each $U \in \mathcal{U}$ a collection of diffeomorphisms $(\theta_x^U : F \to \xi_x)_{x\in U}$.*

Assume that for every $U, V \in \mathcal{U}$, the map

$$\begin{cases} (U\cap V)\times F & \to & F \\ (x,z) & \mapsto & \left(\theta_x^U\right)^{-1}\circ\theta_x^V(z) \end{cases}$$

is smooth. Then there is a unique manifold structure on the disjoint union $E = \sqcup_{x\in M}\xi_x$ satisfying the following properties:

- *The quadruple (E, p, M, F) is a fibre bundle, where $p : E \to M$ is defined by $p(z) = x$ when $z \in \xi_x$.*

- *For each $U \in \mathcal{U}$, the map $\Theta^U : \begin{cases} U\times F & \to & E \\ (x,y) & \mapsto & \theta_x^U(y) \end{cases}$ is smooth.*

Proof. We start with the existence. We wish to show that the maps θ^U for $U \in \mathcal{U}$ provide an atlas for E. Note that they are injective, and that $\Theta^U(U\times F) = p^{-1}(U) = \sqcup_{x\in U}\xi_x$.

First step: define a topology on E.

Declare that a set $O \subset E$ is open if for every $y \in O$ there are:

- An element $U \in \mathcal{U}$, and an open subset $V \subset U$ containing $p(y)$.

- An open subset $W \subset F$ containing $\left(\theta_{p(y)}^U\right)^{-1}(y)$.

Such that $\Theta^U(V \times W) \subset O$. One easily checks that it defines a topology on E: $\emptyset$ is open because the condition is then empty, E is open because $\bigcup_{U\in\mathcal{U}} U = M$, stability by union is a tautology, and stability by finite intersections is a consequence of the same properties for the topologies of M

and F.

Second step: show that E has the right topological properties.

First notice that p is continuous: given $z \in E$, we pick $U \in \mathcal{U}$ such that $p(z) \in \mathcal{U}$, so that for any open subset $V \subset M$ containing $p(z)$, we have that $p^{-1}(U \cap V)$ is open, so $p^{-1}(V)$ is a neighbourhood of z.

We now wish to show that E is Hausdorff. Let $z, z' \in E$ be distinct. Set $x = p(z)$ and $x' = p(z')$. If $x \neq x'$, then the continuity of p and the fact that M is Hausdorff provide disjoint open sets containing z and z'. If $x = x'$, we consider $U \in \mathcal{U}$ that contains x, and set $y = \left(\theta_x^U\right)^{-1}(z), y' = \left(\theta_x^U\right)^{-1}(z')$. Since $z \neq z'$, we also have $y \neq y'$, and we can consider disjoint open sets $W, W' \subset F$ such that $y \in W$ and $y' \in W'$. We now have that $\Theta^U(U \times W)$ and $\Theta^U(U \times W')$ are disjoint open subsets of E containing y and y' respectively.

In order to show that E is secound countable, we first use the fact that M is Lindelöf (i.e. every open cover has a countable subcover) to consider a sequence $(U_i)_{i \in \mathbb{N}}$ in $\mathcal{U}$ such that $\bigcup_{i \in \mathbb{N}} U_i = M$. For every $i \in \mathbb{N}$, we consider a countable base $(U_{i,j})_{j \in \mathbb{N}}$ of the topology of U_i. We also consider a countable base $(W_k)_{k \in \mathbb{N}}$ of the topology of F. For $i, j, k \in \mathbb{N}$, set:

$$O_{i,j,k} = \Theta^{U_i}(U_{i,j} \times W_k)$$

One easily checks that $(O_{i,j,k})_{(i,j,k) \in \mathbb{N}^3}$ is a countable base of the topology of E.

Third step: Find an atlas.

Given $U \in \mathcal{U}$, the map Θ^U is injective and its image is $p^{-1}(U)$. Let $\Phi_U : p^{-1}(U) \to M \times F$ be its inverse. Note that its domain $p^{-1}(U)$ is an open subset of E, and that its image $U \times F$ is an open subset of $M \times F$.

Let $V \subset U$ and $W \subset F$ be open subsets. It follows from the definition of the topology on E that $\Phi_U^{-1}(U \times W)$ is open. This shows that Φ_U is continuous. Also, if $O \subset E$ is open and $z \in O$, then by definition we have that $\Phi_U(O)$ contains some $V \times W$ which contains $\Phi_U(z)$, therefore $\Phi_U(O)$ is open, and Φ_U is a homeomorphism.

The transition maps are the maps:

$$\left\{ \begin{array}{ccc} (U \cap V) \times F & \to & (U \cap V) \times F \\ (x, y) & \mapsto & \left(x, \left(\theta_x^U\right)^{-1} \circ \theta_x^V(y)\right) \end{array} \right.$$

Their smoothness is one of the assumptions.

Fourth step: Show that (E,p,M,F) is a fibre bundle.

Since $p \circ \Theta^U(x,y) = x$, and Θ^U is a local diffeomorphism, we see that p is a submersion. It is surjective by definition of the disjoint union. The family $(\theta_x^U)_{x\in U}$ provide a trivialization of $p|_{p^{-1}(U)}$ with respect to F, which shows that (E,p,M,F) is a fibre bundle.

Fifth step: Prove uniqueness.

Because the maps Θ^U are smooth, the charts φ_U define an atlas, and the uniqueness follows from the uniqueness of a maximal atlas containing a given atlas.

□

1.1.3 Topological aspects of fibre bundles

Let us finish by discussing the gap between surjective submersions and locally trivial fibrations. The projection $p : \mathbb{R} \times \mathbb{R} \setminus \{(0;0)\} \to \mathbb{R}$ onto the first factor is a surjective submersion but not a locally trivial fibration, since the fibres are not all diffeomorphic to each other. There are also examples of surjective submersions whose fibres are all diffeomorphic to each other, but that are not locally trivial fibrations, however their constructions are much more involved.

In the specific case where the fibres are compact, things are more simple. If $\xi = (E,p,M,F)$ is a fibre bundle and F is compact, then p is proper. Reciprocally, proper surjective submersions are locally trivial fibrations.

Theorem 1.1.7 (Ehresmann)**.** *If $f : M \to N$ is a proper surjective submersion, then (M,f,N,F) is a fibre bundle (where $F = f^{-1}(x)$ for some $x \in N$).*

Note that if the total space of a fibre bundle is compact, then so are the base and the fibre. The other direction is also true.

Proposition 1.1.8. *Let $\xi = (E,p,M,F)$ be a fibre bundle. If M and F are compact, then so is E.*

Proof. Since the topology of a manifold is metrizable, we can carry a sequential proof. Let $(z_n)_{n\in\mathbb{N}} \in E^{\mathbb{N}}$, and set $x_n = p(z_n) \in M$. Since M is compact, there is a subsequence (z_{n_k}) such that x_{n_k} converges to some $x \in M$. Let (U,θ) be a trivialising chart around x, and set $y_n = \theta_{x_n}^{-1}(z_n) \in F$ when it is defined. Since F is compact, up to replacing n_k with another subsequence we can assume that y_{n_k} converges to some $y \in F$. Now set $z = \theta_x(y) \in E$, and notice that $z_{n_k} = \Theta(x_{n_k}, y_{n_k}) \to z$. □

1.1.4 Morphisms and subbundles

Definition 1.1.9. *If $\xi = (E, p, M, F)$ and $\xi' = (E', p', M, F')$ are fiber bundles, a* ***fibre bundle morphism*** *is a smooth map $\varphi : E \to E'$ such that $p' \circ \varphi = p$. It is a* ***fibre bundle isomorphism*** *if it is also a diffeomorphism.*

Remark. *Here we only define morphisms for bundles with the same base. There is also a definition for bundles over different bases, involving a map between the bases. A fibre bundle morphism as defined above is called a fibre bundle morphism over the identity in this general setting.*

Proposition 1.1.10. *Let $\xi = (E, p, M, F)$ be a fibre bundle, and $f : N \to M$ a smooth map. Define the set*

$$f^*E = \{(x,z) \in N \times E \,|\, p(z) = f(x)\}$$

and the map

$$f^*p : \left\{ \begin{array}{ccc} f^*E & \to & N \\ (x,z) & \mapsto & x \end{array} \right.$$

Then f^*E is a submanifold of $N \times E$, and $f^*\xi = (f^*E, f^*p, N, F)$ is a fibre bundle, called the ***pulled back bundle*** *of ξ by f.*

Remark. *The pulled back bundle should be thought of as satisfying $(f^*\xi)_x = \xi_{f(x)}$, i.e. it is the fibre bundle over N, for which the fibre over $x \in N$ is the fibre of ξ over $f(x) \in M$. This is how we will prove that it is a fibre bundle: given a local trivialisation (U, θ) of ξ, we show that $\left(f^{-1}(U), (\theta_{f(x)})_{x \in f^{-1}(U)}\right)$ is a local trivialisation of $f^*\xi$.*

Proof. The fact that f^*E is a submanifold of $N \times E$ is a basic example of transversality: consider the map $f \times p : N \times E \to M \times M$ defined by $f \times p(x,z) = (f(x), p(z))$. Then $f^*E = f \times p^{-1}(\Delta)$, where $\Delta \subset M \times M$ is the diagonal, and $f \times p$ is transverse to Δ because p is a submersion. It follows that f^*E is a submanifold of $N \times E$ and that $T_{(x,z)} f^*E = \{(v,w) \in T_xN \times T_zE \,|\, d_xf(v) = d_zp(w)\}$. It also follows from this characterization of the tangent space that f^*p is a submersion. It is surjective because p is surjective.

Finally, given a local trivialisation (U, θ) of ξ, set $\theta'_x(y) = (x, \theta_{f(x)}(y)) \in (f^*p)^{-1}(\{x\})$, so that $(f^{-1}(U), \theta')$ is a local trivialisation of $f^*\xi$. □

Definition 1.1.11. *Let $\xi = (E, p, M, F)$ be a fibre bundle. A* ***subbundle*** *of ξ is a fibre bundle $\xi' = (E', p', M, F')$ such that:*

- *$E' \subset E$ is a submanifold, and $p' = p|_{E'}$.*
- *$F' \subset F$ is a submanifold.*
- *For every $x \in M$, there is a local trivialisation (U, θ) with $x \in U$ such that:*

$$\forall y \in U \quad \theta_y(F') = p^{-1}(y) \cap E$$

Note that the other requirements imply that E' must be a submanifold of E.

Proposition 1.1.12. *Let $\xi = (E,p,M,F)$ be a fibre bundle. Consider a submanifold $F' \subset F$, and a collection of submanifolds $\xi'_x \subset \xi_x$ for all $x \in M$. Assume that for every $x \in M$, there is a local trivialisation (U,θ) with $x \in U$ such that:*

$$\forall y \in U \quad \theta_y(F') = p^{-1}(y) \cap E$$

Then $E' = \sqcup_{x\in M} \xi'_x$ is a submanifold of E, and $\xi' = (E',p',M,F')$ is a subbundle of ξ.

The proof is left as an exercise.

1.2 Reductions of the structural group

1.2.1 Transition functions

Definition 1.2.1. *Let $\xi = (E,p,M,F)$ be a fibre bundle, and (U,θ), (U',θ') two trivialising charts. The **transition function** is the map:*

$$\begin{cases} U \cap U' & \to & \mathrm{Diff}(F) \\ x & \mapsto & \theta_x^{-1} \circ \theta'_x \end{cases}$$

Given a subgroup $G \subset \mathrm{Diff}(F)$, two trivialising charts are called G-compatible if the transition function has values in G. A trivialising atlas is called G-compatible if it consists of trivialising charts that are pairwise G-compatible.
*A **reduction of the structural group of ξ to** G is a maximal G-compatible atlas (i.e. maximal amongst G-compatible atlases).*

The data of a reduction of the structural group of $\xi = (E,p,M,F)$ to $G \subset \mathrm{Diff}(F)$ means that fibres ξ_x for $x \in M$ inherit any algebraic or geometric structure of F which is invariant by G.

We will discuss two examples of reductions of the structure group: the case where F is a vector space, and G is the general linear group $\mathrm{GL}(F)$, known as vector bundles, and the case where F is a Lie group, and G is the group of right translations in F, known as principal bundles.

Definition 1.2.2. *Let H be a Lie group. A H-**principal bundle** (or principal bundle with structural group H) is the data of a fibre bundle $\xi = (E,p,M,H)$ and a reduction of the structural group to $\left\{R_g \,\middle|\, g \in H\right\}$.*

Remark. *Equivalently, one could consider a reduction of the structural group to $\left\{L_g \,\middle|\, g \in H\right\}$, but experts seem to prefer right multiplication.*

Definition 1.2.3. *Let M be a manifold, and $r \in \mathbb{N}$. A real (resp. complex) **vector bundle** of rank r over M is the data of a fibre bundle $\xi = (E,p,M,\mathbb{R}^r)$ (resp. $\xi = (E,p,M,\mathbb{C}^r)$) and a reduction of the structural group to $\mathrm{GL}(\mathbb{R}^n)$ (resp. $\mathrm{GL}(\mathbb{C}^n)$).*

1.2.2 Principal bundles

The archetypal principal bundle is a quotient G/H of a Lie group by an embedded Lie sugroup.

Proposition 1.2.4. *Let G be a Lie group, $H \subset G$ an embedded Lie subgroup and $\pi : G \to G/H$ the projection. Then $(G,\pi,G/H,H)$ is a H-principal bundle.*

We do not need to prove Proposition 1.2.4 as it is a consequence of the following characterisation of principal bundles in terms of group free proper actions of Lie groups.

Theorem 1.2.5 (Principal bundles)**.** *Let $H \curvearrowright X$ be a smooth, free and proper action of a Lie group H on a manifold X.*
There is a unique manifold structure on the quotient space X/H such that the projection $\pi : X \to X/H$ is a submersion.
Moreover, $(X,\pi,X/H,H)$ is a H-principal bundle.
Reciprocally, if $\xi = (E,p,M,H)$ is a H-principal fibre bundle, then there is a smooth, free and proper action $H \curvearrowright X$ whose orbits are the fibres of ξ.

Proof. We start with a smooth, proper and free action $H \curvearrowright X$ on the right. For the manifold structure on X/H, the reader can check that the proof of Theorem **??** can be carried out mutatis mutandis. It follows from the proof that $\ker d_x\pi = T_x(x.H)$ for all $x \in X$.

Let $b \in X/H$. Since π is a submersion, we can consider a neighbourhood $V \subset X/H$ of b and a smooth map $\sigma : V \to X$ such that $\pi \circ \sigma = \mathrm{Id}$ (this is a consequence of the Submersion Theorem).

For $x \in V$, we consider the orbit map $\varphi_{\sigma(x)} : H \to \pi^{-1}(\{x\})$, and we wish to show that the family $(\varphi_{\sigma(x)})_{x\in V}$ is a trivialisation of $\pi|_{\pi^{-1}(V)}$ with respect to H.

The map $(x,h) \mapsto \varphi_{\sigma(x)}(h) = \sigma(x).h$ is smooth because σ and the action are smooth. Given $x \in V$, the map $\varphi_{\sigma(x)}$ is injective because the action is free, and surjective by definition of an orbit. It is an immersion as shows Proposition **??**. It follows that it is a diffeomorphism.

For every $g \in H$, we also have that $(m_g \circ \varphi_{\sigma(x.g^{-1})})_{x\in V.g}$ is a trivialisation of $\pi|_{\pi^{-1}(V.g)}$ with respect to H, so $(X,\pi,X/H,H)$ is a fibre bundle. Since the transition maps are right translations in H, it is a H-principal bundle.

Given a H-principal bundle $\xi = (E,p,M,H)$, consider a trivialising atlas $\mathcal{A}$ of G-compatible trivialising charts. Define the (right) action of H on M in the following way: for $z \in E$ and $h \in H$, let $(U,\theta) \in \mathcal{A}$ be such that $x \in U$, and set $z.h = \Theta\left(p(z), \theta^{-1}_{p(z)}(z)h\right)$. On can easily (yet reluctantly) check that it is a smooth, proper and free action, and that the orbits are the fibres of ξ. □

Examples :

1. Principal bundles with 0-dimensional fibre are Galois coverings.

2. If M is a d-dimensional manifold, we denote by

$$\mathcal{R}(M) = \left\{(x,\varphi)\,\middle|\,\varphi : \mathbb{R}^d \to T_xM \text{ isomorphism}\right\}$$

the **frame bundle** of M. The projection $\pi : \mathcal{R}(M) \to M$ is the quotient map by the right action of $\mathrm{GL}(\mathbb{R}^d)$ on $\mathcal{R}(M)$ given by $f.(x,\varphi) = (x, \varphi \circ f)$.
It is a principal bundle with structural group $\mathrm{GL}(\mathbb{R}^d)$. The fibre $\mathcal{R}(M)_x$ over $x \in M$ can be identified with the set of frames (vector bases) of T_xM.

3. The **Hopf fibration** of $\mathbb{S}^3 = \{(z,w) \in \mathbb{C}^2 |\ |z|^2 + |w|^2 = 1\}$. Consider the $\mathbb{S}^1$ action on $\mathbb{S}^3$ given by $e^{i\theta}.(z,w) = (e^{i\theta}z, e^{i\theta}w)$. It is free and proper. The quotient is diffeomorphic to $\mathbb{S}^2$ (because the projection is the restriction of the canonical projection $\mathbb{C}^2 \setminus \{0\} \to \mathbb{CP}^1 \approx \mathbb{S}^2$).

1.3 Sections of fibre bundles

Definition 1.3.1. *Let $\xi = (E,p,M,F)$ be a fibre bundle. We set:*

$$\Gamma(\xi) = \{\sigma \in \mathcal{C}^\infty(M,E) \,|\, \forall x \in M\ \sigma(x) \in \xi_x\}$$

*A **section** of ξ is an element $\sigma \in \Gamma(\xi)$.*

Remark. *For people who are into commutative diagrams, $\sigma \in \Gamma(\xi)$ translates as $p \circ \sigma = \mathrm{Id}_M$.*

General fibre bundles need not have sections. For principal bundles, the existence of a section is equivalent to triviality.

Proposition 1.3.2. *A principal bundle $\xi = (X, \pi, X/H, H)$ is trivialisable if and only if it admits a section $\sigma : X/H \to X$.*

Proof. Let $\sigma : X/H \to X$ be a section. Then $(\varphi_{\sigma(x)})_{x \in X/H}$ is a trivialisation of π with respect to H. □

Remarks.

- *Given a principal bundle, fibres can be identified with H up to choosing a point in the fibre that we associate with e (this is exactly what a section does).*
- *Given a closed Lie subgroup $H \subset G$, the H-principal bundle $(G, \pi, G/H, H)$ is not necessarily trivial.*

Vector bundles, on the other end of the sprectrum, have many sections. For a start, you can always define the **zero section**. However trivialisability can still be expressed in terms of sections.

Proposition 1.3.3. *Let $\xi = (E, p, M, \mathbb{K}^r)$ be a vector bundle of rank r (where $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$). If there are r sections $\sigma_1, \ldots, \sigma_r$ of ξ such that $(\sigma_1(x), \ldots, \sigma_r(x))$ is a vector basis of ξ_x for all $x \in M$, then ξ is trivialisable.*

Proof. For $x \in M$, consider the linear isomorphism

$$\varphi_x : \begin{cases} \mathbb{K}^r & \to & \xi_x \\ (v^1, \ldots, v^r) & \mapsto & \sum_{i=1}^r v^i \sigma_i(x) \end{cases}$$

Then $(\varphi_x)_{x \in M}$ is a trivialisation of ξ. $\square$

We will see later that this is an equivalence.

Chapter 2

Vector bundles

Remarks.

- *We only consider smooth vector bundles over real manifolds (the fibres can be complex vector spaces), not holomorphic vector bundles over complex manifolds, which are a science apart.*
- *A vector bundle will just be denoted by $\xi = (E, p, M)$.*
- *Given a vector bundle $\xi = (E, p, M)$ of rank r and $x \in M$, the fibre ξ_x is a vector space of dimension r (every trivialisation chart gives an isomorphism with $\mathbb{R}^r$ or $\mathbb{C}^r$).*

2.1 Morphisms of vector bundles and vector subbundles

All the operations described on fibre bundles make sense within vector bundles.

Definition 2.1.1. *Let M be a manifold, and $\xi = (E, p, M), \xi' = (E', p', M)$ be vector bundles over M. A* ***vector bundle morphism*** *from ξ to ξ' is a smooth map $\varphi : E \to E'$ such that $p' \circ \varphi = p$, and for all $x \in M$ the restriction $\varphi_x : \xi_x \to \xi'_x$ is linear.*
It is a ***vector bundle isomorphism*** *if moreover the restrictions to fibres are isomorphisms.*

One can easily check that a vector bundle isomorphism is a diffeomorphism between the total spaces (e.g. by using Proposition 1.1.2). We want to call a a vector bundle **trivialisable** if it is isomorphic as a vector bundle to the trivial vector bundle $(M \times \mathbb{K}^n, \pi_1, M)$. This happens to be equivalent to being trivialisable as a fibre bundle.

Proposition 2.1.2. *Let $\xi = (E, p, M)$ be a vector bundle of rank r over $\mathbb{K}$ ($= \mathbb{R}$ or $\mathbb{C}$). There is a vector bundle isomorphism between ξ and the trivial vector bundle $(M \times \mathbb{K}^r, \pi_1, M)$ if and only if the fibre bundle $(E, p, M, \mathbb{K}^r)$ is trivialisable.*

Proof. It is a straightforward consequence of the definitions that the trivialisability as a vector bundle implies the trivialisability as a fibre bundle. If the fibre bundle $(E, p, M, \mathbb{K}^r)$ is trivialisable, then consider a trivialisation $(\varphi_x)_{x \in M}$ of p with respect to $\mathbb{K}^r$. Then $(d_0\varphi_x)_{x\in M}$ is also a trivialisation of p with respect to $\mathbb{K}^r$, made of linear maps, so the map $\psi : M \times \mathbb{K}^r \to E$ defined by $\psi(x, v) = d_0\varphi_x(v)$ is a vector bundle isomorphism between ξ and the trivial vector bundle $(M \times \mathbb{K}^r, \pi_1, M)$. □

This means that there is no possible confusion on what we mean by a trivialisable vector bundle.

Pulling back a vector bundle also yields a vector bundle.

Proposition 2.1.3. *Let $\xi = (E, p, M)$ be a vector bundle of rank r, and $f : N \to M$ a smooth map. There is a unique vector bundle structure on the pulled-back bundle $f^*\xi$ such that the vector space operations on $(f^*\xi)_x$ for $x \in N$ coincide with those on $\xi_{f(x)}$.*

Definition 2.1.4. *Let $\xi = (E, p, M)$ be a vector bundle. A* ***vector subbundle*** *of ξ is a fibre subbundle ξ' of ξ such that ξ'_x is a vector subspace of ξ_x for all $x \in M$.*

2.2 Sections of vector bundles

2.2.1 Frame fields

Instead of working with vector bundle isomorphisms or some more or less sophisticated types of charts, it is more practical to deal with vector bundles through frame fields, i.e. a vector basis of each fibre that depends smoothly on the base point.

Definition 2.2.1. *Let $\xi = (E, p, M)$ be a vector bundle of rank r. A* ***frame field*** *of ξ is a r-tuple $(\varepsilon_1, \ldots, \varepsilon_r) \in \Gamma(\xi)^r$ such that, for any $x \in M$, the family $(\varepsilon_1(x), \ldots, \varepsilon_r(x))$ is a basis of ξ_x.*

Frame fields need not exist globally, but only locally, since they are directly related to the trivialisability of the vector bundle.

Proposition 2.2.2. *Let $\xi = (E, p, M)$ be a vector bundle of rank r. Then ξ is trivialisable if and only if it possesses a frame field.*

Proof. If φ is a vector bundle isomorphism from the trivial bundle to ξ and $(e_1, \ldots, e_r)$ is a vector basis of $\mathbb{K}^r$, then $\varepsilon_i(x) = \varphi(x, e_i)$ defines a frame field.

Reciprocally, if $(\varepsilon_1, \ldots, \varepsilon_r)$ is a frame field, we set $\theta_x(v) = v^1\varepsilon_1(x) + \cdots +$

$v^r \varepsilon_r(x)$ for $x \in M$ and $v = (v^1, \dots, v^r) \in \mathbb{K}^r$. For each $x \in M$, the map $\theta_x : \mathbb{K}^r \to \xi_x$ is a linear isomorphism, in particular a diffeomorphism. Since $\varepsilon_1, \dots, \varepsilon_r$ are smooth maps, the map $(x, v) \mapsto \theta_x(v)$ is smooth, and it is a trivialisation of ξ. □

Most of the local computations on vector bundles will be done through the choice of a local frame field, i.e. a frame field for a restriction $\xi|_U$ to an open set $U \subset M$. This is practical because we then treat sections of ξ as a r-tuple of smooth functions.

Lemma 2.2.3. *Let $\xi = (E, p, M)$ be a vector bundle of rank r, and let $(\varepsilon_1, \dots, \varepsilon_r)$ be a frame field of ξ. Any section $\sigma \in \Gamma(\xi)$ decomposes uniquely as $\sigma = \sum_{i=1}^r \sigma^i \varepsilon_i$ where $\sigma_1, \dots, \sigma_r \in \mathcal{C}^\infty(M)$.*

Proof. The uniqueness is a consequence of the uniqueness of the decomposition of a vector in a vector basis.

The existence of functions $\sigma_1, \dots, \sigma_r$ follows from the same fibrewise consideration. To prove their smoothness, consider the trivialisation $(\theta_x)_{x \in M}$ defined in the proof of Proposition 2.2.2. The map $\Theta : M \times \mathbb{K}^r \to E$ defined by $\Theta(x, v) = \theta_x(v)$ is a diffeomorphism according to Proposition 1.1.2. Now notice that $(x, (\sigma_1(x), \dots, \sigma_r(x))) = \Theta^{-1}(x, \sigma(x))$, the smoothness of the functions $\sigma_1, \dots, \sigma_r$ follows. □

2.2.2 The space of sections of a vector bundle

Given a vector bundle ξ, the space of sections $\Gamma(\xi)$ is a vector space, as one can add and multiply by scalars on each fibre. First, notice that sections of vector bundles are plentiful (which is the main difference with the holomorphic setting).

Lemma 2.2.4. *Si $\xi = (E, p, M)$ be a vector bundle of rank r. For all $x \in M$ and $v \in \xi_x$, there is a section $\sigma \in \Gamma(\xi)$ such that $\sigma(x) = v$.*

Proof. Let $U \subset M$ be a trivializing domain containing x and $(\varepsilon_1, \dots, \varepsilon_r)$ a frame field of $\xi|_U$. Every $v \in \xi_x$, decomposes as $v = \sum_{i=1}^r v^i \varepsilon_i(x)$. Using a bump function $\varphi \in \mathcal{C}^\infty(M)$ such that $\varphi(x) = 1$ and $\varphi = 0$ outside U, we can define $\sigma_i = \varphi \varepsilon_i \in \Gamma(\xi)$, and $\sigma = \sum_{i=1}^r v^i \sigma_i \in \Gamma(\xi)$ satisfies $\sigma(x) = v$. □

Of course, such a section is far from being unique (it depends strongly on all the choices made). One can easily check that the vector space $\Gamma(\xi)$ of all sections is infinite dimensional.

The space $\Gamma(\xi)$ has an additional algebraic structure: given a smooth function $f \in \mathcal{C}^\infty(M)$ and a section $\sigma \in \Gamma(\xi)$, we can define the fibrewise product $f\sigma \in \Gamma(\xi)$. This means that $\Gamma(\xi)$ is a $\mathcal{C}^\infty(M)$-module. Proposition 2.2.2 can be restated as saying that ξ is trivialisable if and only if $\Gamma(\xi)$ is a free $\mathcal{C}^\infty(M)$-module (necessarily of rank r).

2.2.3 Constructing vector bundles from frame fields

It is actually possible to construct vector bundles via frame fields, and this is what we will use for all algebraic constructions.

Theorem 2.2.5. *Let M be a manifold, $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, and $r \in \mathbb{N}$. Consider a collection of $\mathbb{K}$-vector spaces $(\xi_x)_{x\in M}$ of dimension r, an open cover $\mathcal{U}$ of M, and for each $U \in \mathcal{U}$ and $x \in U$ a vector basis $(\sigma_{U,1}(x),\dots,\sigma_{U,r}(x))$ of ξ_x.*

Assume that for every $U,V \in \mathcal{U}$, the maps $\tau_{U,i}^{V,j} : U \cap V \to \mathbb{K}$ defined by

$$\sigma_{U,i}(x) = \sum_{j=1}^{r} \tau_{U,i}^{V,j}(x)\sigma_{V,j}(x)$$

are smooth. Then there is a unique manifold structure on the disjoint union $E = \sqcup_{x\in M}\xi_x$ satisfying these two conditions:

- *(E,p,M) is a vector bundle of rank r, where $p : E \to M$ is defined by $p(z) = x$ when $z \in \xi_x$.*
- *The functions $\sigma_{U,i}$ are smooth for all $U \in \mathcal{U}$ and $1 \le i \le r$.*

Proof. The existence and uniqueness of a manifold structure on E for which $(E,p,M,\mathbb{K}^n)$ is a fibre bundle is given by Theorem 1.1.6, when considering the diffeomorphisms $\theta_x^U : \mathbb{K}^r \to \xi_x$ defined by $\theta_x^U(v) = \sum_{i=1}^r v^i\sigma_{U,i}(x)$. Since the transition functions are linear, with matrices $\left(\tau_{U,i}^{V,j}(x)\right)_{1\le i,j\le r}$ that depend smoothly on x, it is a vector bundle. $\square$

Vector subbundles can be defined in terms of frame fields.

Proposition 2.2.6. *Let $\xi = (E,p,M)$ be a vector bundle of rank r. Consider a vector subbundle $\xi' = (E',p',M,\mathbb{K}^{r'})$ of ξ. Then around every $x \in M$ there is a local frame field $(\varepsilon_1,\dots,\varepsilon_r)$ of ξ such that $(\varepsilon_1,\dots,\varepsilon_{r'})$ is a local frame field of ξ'.*

Proof. The idea is very similar to Proposition 2.1.2. By definition of a vector subbundle, for every $x \in M$ the submanifold $\xi'_x \subset \xi_x$ is a vector subspace.

Consider a local trivialisation $(\theta_x)_{x\in U}$ of ξ (as a fibre bundle, i.e. each $\theta_x : \mathbb{K}^r \to \xi_x$ is a diffeomorphism, but not necessarily linear) such that $\xi'_x = \theta_x(\mathbb{K}^{r'} \times \{0\})$ (the existence of such trivialisations around every point of M comes from the definition of a fibre subbundle).

Now let $(e_1,\dots,e_r)$ be the canonical basis of $\mathbb{K}^r$, and simply check that $(\varepsilon_1,\dots,\varepsilon_r)$ is a local frame field of ξ satisfying the requirements where $\varepsilon_i(x) = d_0\theta_x(e_i)$. $\square$

Remark. *If $\xi = (E,p,M)$ is a a vector bundle, and $E' \subset E$ is a subset such that for all $x \in M$, the intersection $E' \cap \xi_x$ is a vector subspace of dimension r', and if around every $x \in M$ there are sections $(\varepsilon_1,\dots,\varepsilon_{r'})$ of ξ such that $(\varepsilon_1(y),\dots,\varepsilon_{r'}(y))$ is a vector basis of $E' \cap \xi_y$ for all y near x, then $\xi' = (E',p|_{E'},M)$ is a vector bundle, and a vector sub-bundle of ξ.*

2.3 Examples of vector bundles

2.3.1 The tangent bundle

Recall that given a manifold M and $x \in M$, the tangent space T_xM is defined as:

$$T_xM = \left\{ D \in \mathcal{L}(\mathcal{C}^\infty(M), \mathbb{R}) \,\middle|\, \forall (f,g) \in \mathcal{C}^\infty(\mathbb{R})^2 \; D(fg) = D(f)g + fD(g) \right\}$$

Theorem 2.2.5 provides a vector bundle structure on $TM = \bigcup_{x \in M} T_xM$ by requiring that for every chart (U, φ) the local sections $\partial_1, \dots, \partial_d$ given by $\partial_i(x).f = \frac{\partial f \circ \varphi^{-1}}{\partial x^i}(\varphi(x))$ are smooth.

A section of TM is called a **vector field**. We write $\mathcal{X}(M) = \Gamma(TM)$. A manifold M is called **parallelisable** if its tangent bundle TM is trivialisable.

2.3.2 Tautological bundles

Definition 2.3.1. *Let V be a finite dimensional vector space over $\mathbb{K}$ (= $\mathbb{R}$ of $\mathbb{C}$), and let $k \in \{0, \dots, \dim V\}$. The **tautological bundle** of the Grassmannian $\mathcal{G}_k(V) = \{W \subset V \,|\, \dim W = k\}$ is the vector subbundle $\tau_k(V)$ of the trivial bundle $\mathcal{G}_k(V) \times V$ with total space $\{(x,v) \in \mathcal{G}_k(V) \,|\, v \in x\}$.*

Proposition 2.3.2. *Up to vector bundle isomorphism, the only vector bundles of rank 1 over the circle are the trivial bundle and $\tau_1(\mathbb{R}^2)$.*

2.4 Vector subbundles and direct sums of vector bundles

2.4.1 Supplementary vector subbundles

Definition 2.4.1. *Let $\xi = (E,p,M)$ be a vector bundle, and $\xi_1 = (E_1,p_1,M), \xi_2 = (E_2,p_2,M)$ be vector subbundles of ξ. We say that ξ_1 and ξ_2 are **supplementary** in ξ if $\xi_x = (\xi_1)_x \oplus (\xi_2)_x$ for all $x \in M$.*

The fibrewise projection is a vector bundle morphism.

Proposition 2.4.2. *Let $\xi = (E,p,M)$ be a vector bundle, and $\xi_1 = (E_1,p_1,M), \xi_2 = (E_2,p_2,M)$ be supplementary vector subbundles of ξ. The map $\pi_1 : E \to E_1$ such that $(\pi_1)|_{\xi_x}$ is the projection of ξ_x onto $(\xi_1)_x$ parallel to $(\xi_2)_x$ is a vector bundle morphism.*

2.4.2 The direct sum of vector bundles

Definition 2.4.3. *Let $\xi = (E,p,M)$ and $\eta = (F,q,M)$ be vector bundles over a same manifold. The **direct sum** $\xi \oplus \eta$ is the vector bundle defined by $(\xi \oplus \eta)_x =$*

$\xi_x \oplus \eta_x$ for all $x \in M$, and such that for local frame fields $(\varepsilon_1,\dots,\varepsilon_r)$ of ξ and $(\zeta_1,\dots,\zeta_s)$ of η over a same open set $U \subset M$, the maps $(\varepsilon_1 \oplus 0,\dots,\varepsilon_r \oplus 0, 0 \oplus \zeta_1,\dots,0 \oplus \zeta_s)$ form a frame field.

The existence and uniqueness of such a vector bundle is given by Theorem 2.2.5.

One can also recover its total space $E \oplus F$ as follows:

$$E = \{(v,w) \in E \times F \,|\, p(v) = q(w)\}$$

Given sections $\sigma \in \Gamma(\xi)$ and $\tau \in \Gamma(\eta)$, we can define $\sigma \oplus \tau \in \Gamma(\xi \oplus \eta)$.

2.5 Algebraic operations on vector bundles

2.5.1 The dual bundle

Recall the definition of a dual basis.

Proposition 2.5.1. *Let V be a vector space and $e = (e_1,\dots,e_r)$ be a basis of V. There is a unique basis $e^* = (e_1^*,\dots,e_r^*)$ of V^* such that $e_i^*(e_j) = \delta_{i,j}$ for all $1 \le i,j \le r$.*

The basis e^* is called the **dual basis** of e.

Definition 2.5.2. *Let $\xi = (E,p,M)$ be a vector bundle of rank r. The **dual bundle** ξ^* of ξ is the vector bundle defined by $(\xi^*)_x = (\xi_x)^*$ for all $x \in M$, and such that for a local frame $(\varepsilon_1,\dots,\varepsilon_r)$, the family $(\varepsilon_1^*,\dots,\varepsilon_r^*)$ is a local frame of ξ^*, where $(\varepsilon_1^*(x),\dots,\varepsilon_r^*(x))$ is the dual basis of $(\varepsilon_1(x),\dots,\varepsilon_r(x))$ when defined.*

If $\lambda \in \Gamma(\xi^*)$ and $\sigma \in \Gamma(\xi)$, we can define $\lambda(\sigma) \in \mathcal{C}^\infty(M)$. Note that while a section $\sigma \in \Gamma(\xi)$ is usually written with a functional notation, i.e. $\sigma(x) \in \xi_x$ for $x \in M$, for a section $\lambda \in \Gamma(\xi^*)$ it is more convenient to use a subscript notation $\lambda_x \in \xi_x^*$ for $x \in M$, so that given $v \in \xi_x$ we can write $\lambda_x(v) \in \mathbb{K}$ instead of $\lambda(x)(v)$.

The dual bundle T^*M of the tangent bundle TM of a manifold M is called the **cotangent bundle**.

Since we are working with finite dimensional vector spaces, there is a natural identification between as space V and its bidual $V^{**} = (V^*)^*$, sending $x \in V$ to the evaluation map $\lambda \mapsto \lambda(x)$. We can also define the transpose ${}^tA \in \Gamma(\mathcal{L}(\eta^*,\xi^*))$ by considering the fibrewise transpose (recall that given $u \in \mathcal{L}(V,W)$, the transpose ${}^tu \in \mathcal{L}(W^*,V^*)$ is defined by ${}^tu(\lambda) = \lambda \circ u$).

2.5.2 The homomorphism bundle

Given two vector spaces V, W and elements $\lambda \in V^*, w \in W$, we can define the linear map $\lambda \otimes w \in \mathcal{L}(V, W)$ by $\lambda \otimes w(v) = \lambda(v)w$ for all $v \in V$.

If $(e_1, \dots, e_r)$ is a basis of V and $(f_1, \dots, f_s)$ is a basis of W, then $(e_i^* \otimes f_j)$ is a basis of $\mathcal{L}(V, W)$.

Definition 2.5.3. *Let $\xi = (E, p, M)$ and $\eta = (F, q, M)$ be vector bundles over the same manifold. The homomorphism bundle $\mathcal{L}(\xi, \eta)$ is the vector bundle defined by $\mathcal{L}(\xi, \eta)_x = \mathcal{L}(\xi_x, \eta_x)$ for all $x \in M$, and such that for local frames $(\varepsilon_1, \dots, \varepsilon_r)$ of ξ and $(\zeta, \dots, \zeta)$ of η over a same open set $U \subset M$, the maps $(\varepsilon_i^* \otimes \zeta)$ form a frame field.*

Sections $\sigma \in \Gamma(\xi)$ and $A \in \Gamma(\mathcal{L}(\xi, \eta))$ produce a section $A(\sigma) \in \Gamma(\eta)$.

2.5.3 The tensor product bundle

Given vector spaces $V_1, \dots, V_k$ and W, we denote by $\mathcal{L}(V_1, \dots, V_k; W)$ the space of multi-linear maps from $V_1 \times \cdots \times V_k$ to W.

Note that for $j \in \{1, \dots, k\}$, there is a natural isomorphism

$$f : \mathcal{L}(V_1, \dots, V_k; W) \to \mathcal{L}\Big(V_1, \dots, V_j; \mathcal{L}(V_{j+1}, \dots, V_k; W)\Big)$$

defined by $f(u)(x_1, \dots, x_j)(x_{j+1}, \dots, x_k) = u(x_1, \dots, x_k)$.

Since we will only work with finite dimensional vector spaces, we can define the tensor product as:

$$V_1 \otimes \cdots \otimes V_k = \mathcal{L}(V_1^*, \dots, V_k^*; \mathbb{K})$$

Given elements $(v_1, \dots, v_k) \in V_1 \times \cdots \times V_k$, we can define the pure tensor $v_1 \otimes \cdots \otimes v_k$ as the map:

$$v_1 \otimes \cdots \otimes v_k(\lambda_1, \dots, \lambda_k) = \lambda_1(v_1) \cdots \lambda_k(v_k)$$

Given a basis $(e_1^j, \dots, e_{r_j}^j)$ of each V_j, the family of all $(e_{i_1}^1 \otimes \cdots \otimes e_{i_k}^k)$ is a basis of $V_1 \otimes \cdots \otimes V_k$.

The identification between V and V^{**} yields an identification between $V^* \otimes W$ and $\mathcal{L}(V, W)$, and the notation $\lambda \otimes w$ for $\lambda \in V^*$ and $w \in W$ given in 2.5.2 is consistent with this identification.

More generally, we can identify $V_1^* \otimes \cdots \otimes V_k^* \otimes W$ with $\mathcal{L}(V_1, \dots, V_k; W)$.

Definition 2.5.4. *Let $\xi_1 = (E_1, p_1, M), \dots, \xi_k = (E_k, p_k, M)$ be vector bundles over the same manifold M. The **tensor product bundle** $\xi_1 \otimes \cdots \otimes \xi_k$ is the vector bundle defined by $(\xi_1 \otimes \cdots \otimes \xi_k)_x = (\xi_1)_x \otimes \cdots \otimes (\xi_k)_x$ for all $x \in M$, and such that for local frames $(\varepsilon_1^j, \dots, \varepsilon_{r_j}^j)$ of each ξ_j defined over a same open set $U \subset M$, the maps $(\varepsilon_{i_1}^1 \otimes \cdots \otimes \varepsilon_{i_k}^k)$ for a local frame field.*

Given a finite dimensional space V, we set $V^{\otimes k} = \underbrace{V \otimes \cdots \otimes V}_{k \text{ times}}$. Similarly, given a vector bundle ξ we can define $\xi^{\otimes k}$.

Most of the vector bundles that we will be interested in will be tensor powers of the tangent bundle of a manifold, or subbundles of these.

Definition 2.5.5. *Let M be a manifold, and $p, q \in \mathbb{N}$. A* ***tensor of type*** (p,q) *on M is a section of $(T^*M)^{\otimes p} \otimes TM^{\otimes q}$. We denote by $\mathcal{T}^{p,q}(M) = \Gamma\left((T^*M)^{\otimes p} \otimes TM^{\otimes q}\right)$ the space of tensors of type (p,q) on M.*

A tensor of type $(p,0)$ is called **covariant** and a tensor of type $(0,q)$ is called **contravariant**.

There is a widely accepted consensus as to how one should write a tensor in coordinates. Given $T \in \mathcal{T}^{p,q}(M)$, we always write its expression in coordinates with indices relating to the covariant part on the bottom, and indices for the contravariant part on top, i.e.

$$T = \sum_{1 \leq i_1, \ldots, i_p, j_1, \ldots, j_q \leq d} T_{i_1, \ldots, i_p}^{j_1, \ldots, j_q} dx^{i_1} \otimes \cdots \otimes dx^{i_p} \otimes \partial_{j_1} \otimes \partial_{j_q}$$

For example, a vector field $X \in \mathcal{X}(M) = \mathcal{T}^{0,1}(M)$, which is a contravariant tensor, is always written locally as $X = \sum_{i=1}^d X^i \partial_i$. A differential 1-form $\omega \in \Omega^1(M) = \mathcal{T}^{1,0}(M)$, on the other hand, is a covariant tensor, and is written locally as $\omega = \sum_{i=1}^d \omega_i dx^i$.

2.5.4 Exterior and symmetric powers of a vector bundle

The **exterior power** $\Lambda^k V^*$ is the subspace of $(V^*)^{\otimes k}$ composed of skew-symmetric forms (i.e. $u(x_{\sigma(1)}, \ldots, x_{\sigma(k)}) = \varepsilon(\sigma) u(x_1, \ldots, x_k)$ for $(x_1, \ldots, x_k) \in V^k$ and $\sigma \in \mathfrak{S}_k$).

The **symmetric power** $S^k V^*$ is the subspace of $(V^*)^{\otimes k}$ composed of symmetric forms (i.e. $u(x_{\sigma(1)}, \ldots, x_{\sigma(k)}) = u(x_1, \ldots, x_k)$ for $(x_1, \ldots, x_k) \in V^k$ and $\sigma \in \mathfrak{S}_k$).

Note that $(V^*)^{\otimes k} = \Lambda^k V^* \oplus S^k V^*$. Given $(\lambda_1, \ldots, \lambda_k) \in (V^*)^k$, we denote by $\lambda_1 \wedge \cdots \wedge \lambda_k$ (resp. $\lambda_1 \vee \cdots \vee \lambda_k$) the projection of $\lambda_1 \otimes \cdots \otimes \lambda_k$ on $\Lambda^k V^*$ (resp. $S^k V^*$) given by this direct sum. The explicit formula is:

$$\begin{aligned} \lambda_1 \wedge \cdots \wedge \lambda_k(x_1, \ldots, x_k) &= \frac{1}{k!} \sum_{\sigma \in \mathfrak{S}_k} \varepsilon(\sigma) \lambda_1(x_{\sigma(1)}) \cdots \lambda_k(x_{\sigma(k)}) \\ &= \det\left(\lambda_i(x_j)\right)_{1 \leq i,j \leq k} \end{aligned}$$

If $(e_1, \ldots, e_r)$ is a basis of V, then $(e_{i_1}^* \wedge \cdots \wedge e_{i_k}^*)_{i_1 < \cdots < i_k}$ is a basis of $\Lambda^k V^*$.

Given $j \in \{1,\ldots,k\}$, $\lambda \in \Lambda^j V^*$ and $\mu \in \Lambda^{k-j} V^*$, we define $\lambda \wedge \mu \in \Lambda^k V^*$ as the projection of $\lambda \otimes \mu$.

Definition 2.5.6. *Let $\xi = (E,p,M)$ be a vector bundle of rank r, and k an integer. The k^{th}-**exterior power** of ξ^* is the vector subbundle $\Lambda^k \xi^*$ of $(\xi^*)^{\otimes k}$ defined by $\left(\Lambda^k \xi^*\right)_x = \Lambda^k(\xi^*)_x$ for all $x \in M$.*

A **differential** k**-form** on a manifold M is a section of $\Lambda^k T^*M$. We use the notation $\Omega^k(M) = \Gamma\left(\Lambda^k T^*M\right)$. There are three main operations on differential forms.

The wedge product: Given $\omega_1 \in \Omega^{k_1}(M)$ and $\omega_2 \in \Omega^{k_2}(M)$ we can define $\omega_1 \wedge \omega_2 \in \Omega^{k_1+k_2}(M)$ fibrewise.

Interior product: Given $X \in \mathcal{X}(M)$ and $\omega \in \Omega^{k+1}(M)$, we define $\iota_X \omega \in \Omega^k(M)$ by:

$$\forall x \in M \ \forall (v_1,\ldots,v_k) \in T_x M^k \quad (\iota_X \omega)_x(v_1,\ldots,v_k) = \omega_x(X(x),v_1,\ldots,v_k)$$

The exterior differential : Given $\omega \in \Omega^k(M)$, we define $d\omega \in \Omega^{k+1}(M)$, this will be discussed later.

We have the following rule for the exterior differential of a wedge product.

$$\forall \omega_1 \in \Omega^{k_1}(M) \ \forall \omega_2 \in \Omega^{k_2}(M) \quad d(\omega_1 \wedge \omega_2) = d\omega_1 \wedge \omega_2 + (-1)^{k_1} \omega_1 \wedge \omega_2$$

The exterior differential is also related to the Lie derivative through Cartan's formula.

$$\forall X \in \mathcal{X}(M) \ \forall \omega \in \Omega^k(M) \quad \mathcal{L}_X \omega = d(\iota_X \omega) + \iota_X d\omega$$

Definition 2.5.7. *Let $\xi = (E,p,M)$ be a vector bundle of rank r, and k an integer. The k^{th}-**symmetric power** of ξ^* is the vector subbundle $S^k \xi^*$ of $(\xi^*)^{\otimes k}$ defined by $\left(\Lambda^k \xi^*\right)_x = S^k(\xi^*)_x$ for all $x \in M$.*

2.5.5 Vector bundle valued differential forms

Definition 2.5.8. *Let $\xi = (E,p,M)$ be a vector bundle and $p \in \mathbb{N}$. A ξ**-valued differential** k**-form** ω is a section $\omega \in \Gamma\left(\Lambda^k T^*M \otimes \xi\right)$.*

We will use the notation $\Omega^k(\xi) = \Gamma\left(\Lambda^k T^*M \otimes \xi\right)$. Note that for $\xi = (M \times \mathbb{R}, \pi_1, M)$, we get usual differential forms.

In general, given $\omega_1 \in \Omega^{k_1}(\xi)$ and $\omega_2 \in \Omega^{k_2}(\xi)$, there is no natural way of defining some element of $\Omega^{k_1+k_2}(\xi)$ (the definition of $\omega_1 \wedge \omega_2 \in \Omega^{k_1+k_2}(M)$

for $\xi = (M \times \mathbb{R}, \pi_1, M)$ involves multiplication).

It is however possible to define a product between a differential form on M and a section of ξ. We will only use it for 1-forms: given $\lambda \in \Omega^1(M) = \Gamma(T^*M)$ and $\sigma \in \Gamma(\xi)$, we define $\lambda \otimes \sigma \in \Gamma(T^*M \otimes \xi) = \Omega^1(\xi)$. The formula is:

$$\forall x \in M \ \forall v \in T_x M \quad (\lambda \otimes \sigma)_x(v) = \lambda_x(v)\sigma(x)$$

2.5.6 Quotients of vector bundles

Given a vector space V and a subspace $W \subset V$, we let $\overline{x} \in V/W$ be the image of $x \in V$ as long as there is no possible confusion.

Proposition 2.5.9. *Let $\xi = (E, p, M)$ be a vector bundle, and $\eta = (F, q, M)$ be a vector subbundle of ξ. The **quotient bundle** ξ/η is the vector bundle defined by $(\xi/\eta)_x = \xi_x/\eta_x$ for all $x \in M$, and such that for any local frame $(\varepsilon_1, \ldots, \varepsilon_r)$ of ξ for which $(\varepsilon_1, \ldots, \varepsilon_k)$ is a local frame of η, the maps $(\overline{\varepsilon}_{k+1}, \ldots, \overline{\varepsilon}_r)$ form a frame field.*

2.5.7 The conjugate of a complex vector bundle

Complex vector spaces come in pair. Indeed, recall that a vector space V over a field $\mathbb{K}$ is not just a set, but a triple $V = (E, +, .)$ where E is a set, the addition map $+ : E \times E \to E$ is such that $(E, +)$ is an abelian group, and the exterior multiplication $. : \mathbb{K} \times E \to E$ satisfies a rather long list of axioms.

In the case where $\mathbb{K} = \mathbb{C}$, we can associate to V its **conjugate space** $\overline{V}$ defined by $\overline{V} = (E, +, \overline{\ .\ })$, where $\lambda \overline{\ .\ } v = \overline{\lambda}.v$. One can check that it is also a complex vector space.

Given a complex vector bundle ξ, we can define its conjugate $\overline{\xi}$ as we have for all other algebraic operations. This allows us to define the vector bundle of sesquilinear forms over ξ as $(\overline{\xi})^* \otimes \xi^*$.

2.5.8 Grassmannian bundles

Up to now, we saw that vector bundles produce more vector bundles. But they also produce fibre bundles that are not vector bundle. The main examples are Grassmannians.

Let $\xi = (E, p, M)$ be a vector bundle of rank r, and $k \in \{1, \ldots, r\}$. We wish to define a fibre bundle $\mathcal{G}_k(\xi)$ over M with fibres $\mathcal{G}_k(\xi)_x = \mathcal{G}_k(\xi_x)$ for $x \in M$. For this matter, we consider an open set $U \subset M$ and a frame field $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_r)$ of $\xi|_U$.

Recall that the differentiable structure on $\mathcal{G}_k(\mathbb{R}^r)$ is given by that of the homogeneous space $\mathrm{GL}(r,\mathbb{R})/P_{k,r}$ where:

$$P_{k,r} = \left\{ \left(\begin{array}{cc} A & B \\ 0 & C \end{array} \right) \middle| A \in \mathrm{GL}(k,\mathbb{R}), B \in \mathcal{M}_{k,r-k}(\mathbb{R}), C \in \mathrm{GL}(r-k,\mathbb{R}) \right\}$$

For any $x \in U$, the basis $\varepsilon(x)$ of ξ_x induces an isomorphism between $\mathrm{GL}(r,\mathbb{R})$ and $\mathrm{GL}(\xi_x)$, sending $P_{k,r}$ to some group $P_\varepsilon(x) \subset \mathrm{GL}(\xi_x)$. We therefore get a diffeomorphism between the homogeneous spaces $\mathrm{GL}(r,\mathbb{R})/P_{k,r}$ and $\mathrm{GL}(\xi_x)/P_\varepsilon(x)$, hence a diffeomorphism $\theta_x^\varepsilon : \mathcal{G}_k(\mathbb{R}^r) \to \mathcal{G}_k(\xi_x)$.

Given another frame field δ, the transition map $(\theta_x^\delta)^{-1} \circ \theta_x^\varepsilon : \mathcal{G}(k,\mathbb{R}^r) \to \mathcal{G}(k,\mathbb{R})$ lifts to a diffeomorphism $\mathrm{GL}(k,\mathbb{R}^r) \to \mathrm{GL}(k,\mathbb{R}^r)$ which is the conjugation by the change-of-base matrix between $\varepsilon(x)$ and $\delta(x)$, so it depends smoothly on x. By Theorem 1.1.6, we have found a fibre bundle structure on $\mathcal{G}_k(\xi) = (\sqcup_{x\in M}\mathcal{G}_k(\xi_x), p, M, \mathcal{G}_k(\mathbb{R}^d))$.

Definition 2.5.10. *Let $\xi = (E,p,M)$ be a vector bundle of rank r, and $k \in \{1,\dots,r\}$. The fibre bundle $\mathcal{G}_k(\xi)$ constructed above is called the k^{th}-**Grassmannian bundle** of ξ.*

Note that since Grassmannian spaces are compact, the total space of a Grassmannian bundle of a vector bundle over a compact manifold is compact, due to Proposition 1.1.8.

2.6 Classification of vector bundles

The results of this section will only be stated, not proved.

Two continuous maps $f, g : B \to M$ are called **homotopic** if there is a continuous map $h : [0,1] \times B \to M$ such that $h(0,\cdot) = f$ and $h(1,g)$. If f and g are smooth, it is always possible to choose a smooth h.

We say that B is **contractible** if $\mathrm{Id} : B \to B$ is homotopic to a constant.

Theorem 2.6.1. *Any vector bundle over a contractible manifold is trivialisable.*

Theorem 2.6.2. *Let M be a compact manifold.*

1. *If ξ is a vector bundle of rank r over M, there are an integer N and a smooth map $f : M \to \mathcal{G}_r(\mathbb{K}^N)$ such that ξ is isomorphic to $f^*\tau_r(\mathbb{K}^N)$ where $\tau_r(\mathbb{K}^N)$ is the tautological bundle.*

2. *Given two smooth maps $f, g : M \to \mathcal{G}_r(\mathbb{K}^N)$, the vector bundles $f^*\tau_n(\mathbb{K}^N)$ and $g^*\tau_n(\mathbb{K}^N)$ are isomorphic if and only if there is an integer N' such that f and g are homotopic as maps from M to $\mathcal{G}_r(\mathbb{K}^{N'})$.*

Chapter 3

Covariant derivatives

In order to make sense of the definition of a connection, we need to understand how central the Leibniz rule is in calculus, and that it is a defining rule of differentiation.

Proposition 3.0.1.

- *Let $D : \mathcal{C}^\infty(\mathbb{R}) \to \mathcal{C}^\infty(\mathbb{R})$ be a linear map such that $D(fg) = D(f)g + fD(g)$ for all $f, g \in \mathcal{C}^\infty(\mathbb{R})$. There is $\lambda \in \mathcal{C}^\infty(\mathbb{R})$ such that $D(f) = \lambda \dot{f}$ for all $f \in \mathcal{C}^\infty(\mathbb{R})$.*

- *Let $D : \mathcal{C}^\infty(\mathbb{R}, \mathbb{R}^d) \to \mathcal{C}^\infty(\mathbb{R}, \mathbb{R}^d)$ be a linear map such that $D(f\sigma) = f'\sigma + fD(\sigma)$ for all $f \in \mathcal{C}^\infty(\mathbb{R})$ and $\sigma \in \mathcal{C}^\infty(\mathbb{R}, \mathbb{R}^d)$. There is $A \in \mathcal{C}^\infty(\mathbb{R}, \mathrm{End}(\mathbb{R}^d))$ such that $D(\sigma)(t) = \dot{\sigma}(t) + A(t)\sigma(t)$ for all $\sigma \in \mathcal{C}^\infty(\mathbb{R}, \mathbb{R}^d)$ and $t \in \mathbb{R}$.*

3.1 Vector bundles over the line

3.1.1 Existence of frame fields

The study of vector bundles over an interval is made easy by the fact that they are all trivialisable.

Theorem 3.1.1. *Let $I \subset \mathbb{R}$ be an interval, and $\xi = (E, p, I)$ be a vector bundle of rank r. Then ξ is trivialisable.*

This is a consequence of Theorem 2.6.1, however we will give a proof of Theorem 3.1.1 as we will use it repeatedly (actually Theorem 2.6.1 can be deduced from Theorem 3.1.1).

The key is that if there is only one dimension, then there is only one way of "gluing" local trivialisations. This is because the intersection of two intervals is an interval.

Lemma 3.1.2. *Let $I \subset \mathbb{R}$ be an interval, and $\xi = (E, p, I)$ be a vector bundle of rank r. Assume that there are two subintervals $I_0, I_1 \subset I$ such that $\xi|_{I_0}$ and $\xi|_{I_1}$*

are trivialisable, and $I_0 \cup I_1 = I$. Then ξ is trivialisable.
Moreover, for any compact subinterval $J \subset I_0$ and any frame field $(\varepsilon_1, \dots, \varepsilon_r)$ of $\xi|_{I_0}$, there is a frame field $(\overline{\varepsilon_1}, \dots, \overline{\varepsilon_r})$ of ξ such that $\overline{\varepsilon_i}(t) = \varepsilon_i(t)$ for all $t \in J$ and $i \in \{1, \dots, r\}$.

Proof. Notice that the second point implies the first. Up to enlarging J, we can assume that $J \cap I_1 \neq \emptyset$.

Let $(\varepsilon_1, \dots, \varepsilon_r)$ be a frame field of $\xi|_{I_0}$, and $(\varepsilon'_1, \dots, \varepsilon'_r)$ a frame field of $\xi|_{I_1}$.

For $t \in I_0 \cap I_1$, we can decompose:

$$\varepsilon_i(t) = \sum_{j=1}^{r} A_i^j(t)\varepsilon'_j(t)$$

This defines a smooth curve $A = (A_i^j)_{1 \le i,j \le r} : I_0 \cap I_1 \to \mathrm{GL}(r, \mathbb{R})$. Consider a smooth curve $\overline{A} : I_1 \to \mathrm{GL}(r, \mathbb{R})$ such that $\overline{A}(t) = A(t)$ for all $t \in J \cap I_1$ (this is possible because $\mathrm{GL}(r, \mathbb{R})$ is a manifold).

Now define $(\overline{\varepsilon_1}, \dots, \overline{\varepsilon_r})$ in the following way:

$$\overline{\varepsilon_i}(t) = \begin{cases} \varepsilon_i(t) & \text{if } t \in I_0 \setminus I_1 \\ \sum_{j=1}^{r} \overline{A}_i^j(t)\varepsilon'_j(t) & \text{if } t \in I_1 \end{cases}$$

One easily checks that $(\overline{\varepsilon_1}, \dots, \overline{\varepsilon_r})$ is a frame field with the required property. □

Proof of Theorem 3.1.1. Using the topological properties of $\mathbb{R}$, given $t_0 \in I$, we can find a sequence $(I_i)_{i \in \mathbb{Z}}$ of open intervals such that:

- $t_0 \in I_0$.
- For all $i \in \mathbb{Z}$, I_i is a trivialising domain of ξ.
- $I_i \cap I_j = \emptyset$ whenever $|i - j| \ge 2$.
- For all $i \in \mathbb{Z}$, $I_i \cap I_{i+1}$ is an interval.
- $\bigcup_{i \in \mathbb{Z}} I_i = I$.

Using Lemma 3.1.2, we can construct a sequence of frame fields $(\varepsilon_1^k, \dots, \varepsilon_n^k)$ over $J_k = \bigcup_{-k \le j \le k} I_j$ such that
e_i^{k+1} restricts to ε_i^k on J_k, which leads to a trivialization of ξ.

□

If frame fields exist, they are however far from being unique. One can check that given $t_0 \in I$ and a vector basis $(e_1, \dots, e_r)$ of ξ_{t_0}, there exists a trivializing frame $(\varepsilon_1, \dots, \varepsilon_r)$ of ξ such that $\varepsilon_i(t_0) = e_i$ for all $i \in \{1, \dots, r\}$. But this frame field is still far from being unique.

3.1.2 The archetypal vector bundle over the line

Since frame fields of vector bundles over the line exist but are not unique, we can ask whether there is a canonical choice. In order to understand this, let us understand the archetypal construction of a vector bundle over the line. For this, consider a manifold M, and a smooth curve $\gamma : I \to M$. Then γ^*TM is a vector bundle over I. If M is not parallelisable, then there is really no reason for a "canonical" frame field to exist (the fact that γ^*TM is trivial is a property of the interval, not of M or γ).

In a trivial bundle $\xi = (I \times \mathbb{R}^r, \pi_1, I)$, there is a canonical way of choosing such a frame fields: picking sections that are constant maps $I \to \mathbb{R}^r$. But for a pulled back bundle γ^*TM there is no natural definition of a constant frame field. Even when M is a submanifold of $\mathbb{R}^d$, there is no reason for a section of γ^*TM with constant coordinates in $\mathbb{R}^d$ to exist.

3.1.3 Intrinsic derivatives

Definition 3.1.3. *Let $I \subset \mathbb{R}$ be an interval, and $\xi = (E, p, I)$ a vector bundle of rank r. An* ***intrinsic derivative*** *on ξ is a linear map $\frac{D}{dt} : \Gamma(\xi) \to \Gamma(\xi)$ such that:*

$$\forall \sigma \in \Gamma(\xi)\ \forall f \in \mathcal{C}^\infty(I) \quad \frac{D}{dt}(f\sigma) = f\frac{D}{dt}\sigma + \dot{f}\sigma$$

A section $\sigma \in \Gamma(\xi)$ is ***parallel*** *if $\frac{D}{dt}\sigma = 0$. A frame field $(\varepsilon_1, \dots, \varepsilon_r)$ is called parallel if $\varepsilon_1, \dots, \varepsilon_r$ are parallel.*

A frame field allows us to define an intrinsic derivative.

Proposition 3.1.4. *Let $I \subset \mathbb{R}$ be an interval, $\xi = (E, p, I)$ a vector bundle of rank r and $\varepsilon = (\varepsilon_1, \dots, \varepsilon_r)$ a frame field of ξ. There is a unique intrinsic derivative on ξ for which ε is parallel.*

Proof. Any section $\sigma \in \Gamma(\xi)$ decomposes uniquely as $\sigma = \sum_{i=1}^r \sigma^i \varepsilon_i$ for $\sigma^1, \dots, \sigma^r \in \mathcal{C}^\infty(I)$. Define:

$$\frac{D}{dt} : \begin{cases} \Gamma(\xi) & \to & \Gamma(\xi) \\ \sigma = \sum_{i=1}^r \sigma^i \varepsilon_i & \mapsto & \sum_{i=1}^r \dot{\sigma}^i \varepsilon_i \end{cases}$$

It is linear, and the product rule for derivation of functions shows that it is an intrinsic derivative. By definition, ε is parallel.

Reciprocally, if $\frac{D}{dt}$ is an intrinsic derivative for which ε is parallel, then:

$$\frac{D}{dt}\left(\sum_{i=1}^{r}\sigma^i\varepsilon_i\right) = \sum_{i=1}^{r}\sigma^i \underbrace{\frac{D}{dt}\varepsilon_i}_{=0} + \sum_{i=1}^{r}\dot{\sigma}^i\varepsilon_i = \sum_{i=1}^{r}\dot{\sigma}^i\varepsilon_i$$

This yields uniqueness. □

Another situation in which we can define a "canonical" intrinsic derivative is for pullbacks of tangent bundles of submanifolds of $\mathbb{R}^n$ by curves.

Proposition 3.1.5. *Let $M \subset \mathbb{R}^n$ be a submanifold, and let $\gamma : I \to M$ be a smooth curve. For $x \in M$, we let $p_x \in \mathrm{End}(\mathbb{R}^n)$ be the orthogonal projection onto T_xM for the canonical inner product.*
The map $\frac{D}{dt} : \Gamma(\gamma^*TM) \to \Gamma(\gamma^*TM)$ defined by $\frac{D}{dt}\sigma(t) = p_{\gamma(t)}(\dot{\sigma}(t))$ is an intrinsic derivative.

Proof. Firstly, since M is a smooth manifold, the linear map $p_x \in \mathrm{End}(\mathbb{R}^n)$ depends smoothly on x, so $\frac{D}{dt}\sigma$ is smooth for any $\sigma \in \Gamma(\gamma^*TM)$, and $\frac{D}{dt}$: $\Gamma(\gamma^*TM) \to \Gamma(\gamma^*TM)$ is well defined. It is linear because the orthogonal projections are linear.
Let $\sigma \in \Gamma(\gamma^*TM)$ and $f \in \mathcal{C}^\infty(I)$. Since $\sigma(t) \in T_{\gamma(t)}M$ for all $t \in I$, we have that $p_{\gamma(t)}(\sigma(t)) = \sigma(t)$, and it follows that:

$$\frac{D}{dt}(f\sigma)(t) = p_{\gamma(t)}(f(t)\dot{\sigma}(t) + \dot{f}(t)\sigma(t)) = f(t)\frac{D}{dt}\sigma(t) + \dot{f}(t)\sigma(t)$$

□

We now wish to show that any intrinsic derivative possesses a parallel frame field. The whole point is that seeking parallel sections amounts to solving a linear ODE.

Proposition 3.1.6. *Let $I \subset \mathbb{R}$ be an interval, $\xi = (E,p,I)$ a vector bundle of rank r, and $\frac{D}{dt}$ an intrinsic derivative on ξ. For all $t_0 \in I$ and $v \in \xi_{t_0}$, there is a unique parallel section $\sigma \in \Gamma(\xi)$ such that $\sigma(t_0) = v$.*

Proof. Since ξ is trivialisable according to Theorem 3.1.1, consider a frame field $(\varepsilon_1, \ldots, \varepsilon_r)$ of ξ. Any section σ can be written as $\sigma = \sum_{i=1}^{r}\sigma^i\varepsilon_i$ for some functions $\sigma^1, \ldots, \sigma^r \in \mathcal{C}^\infty(I)$.

Consider $A = \left(A_i^j\right)_{1\leq i,j\leq r} : I \to \mathcal{M}(r,\mathbb{R})$ such that:

$$\frac{D}{dt}\varepsilon_i = \sum_{j=1}^{r}A_i^j\varepsilon_j$$

Since $\frac{D}{dt}\sigma = \sum_{i=1}^{r} \sigma^i \frac{D}{dt}\varepsilon_i + \dot{\sigma}_i \varepsilon_i$, we find:

$$\frac{D}{dt}\sigma = 0 \iff \forall j \in \{1,\ldots,r\}\ \dot{\sigma}^j + \sum_{i=1}^{r} A_i^j \sigma^i = 0$$

Existence and uniqueness follow from the Cauchy-Lipschitz Theorem. □

The existence of parallel sections allows us to define linear maps between fibres (i.e. we connect fibres with each other).

Definition 3.1.7. *Let $I \subset \mathbb{R}$ be an interval, $\xi = (E,p,I)$ a vector bundle of rank r, and $\frac{D}{dt}$ an intrinsic derivative on ξ. For $t_0, t_1 \in I$, the map $\|_{t_0}^{t_1} : \xi_{t_0} \to \xi_{t_1}$ defined by $\|_{t_0}^{t_1} v = \sigma(t_1)$, where $\sigma \in \Gamma(\xi)$ is parallel and $\sigma(t_0) = v$, is called the* ***parallel transport****.*

The parallel transport has a semi-group type property (which is not surprising since it is more or less defined as the flow of an ODE).

Proposition 3.1.8. *Let $I \subset \mathbb{R}$ be an interval, $\xi = (E,p,I)$ a vector bundle of rank r, and $\frac{D}{dt}$ an intrinsic derivative on ξ.*

- *For $t_0, t_1 \in I$, the parallel transport $\|_{t_0}^{t_1} : \xi_{t_0} \to \xi_{t_1}$ is an isomorphism, with inverse $\left(\|_{t_0}^{t_1}\right)^{-1} = \|_{t_1}^{t_0}$.*
- *For $t_0, t_1, t_2 \in I$ we have $\|_{t_1}^{t_2} \circ \|_{t_0}^{t_1} = \|_{t_0}^{t_2}$.*

Proof. Linearity of the parallel transport is because the space of parallel vector fields is a vector space. Everything else is a consequence of the uniqueness in Proposition 3.1.6. □

This can be used to show the existence of parallel frame fields

Proposition 3.1.9. *Let $I \subset \mathbb{R}$ be an interval, $\xi = (E,p,I)$ a vector bundle of rank r, and $\frac{D}{dt}$ an intrinsic derivative on ξ.*
Given $t_0 \in I$ and a basis $(e_1,\ldots,e_r)$ of ξ_{t_0}, there is a unique parallel frame field $(\varepsilon_1,\ldots,\varepsilon_r)$ such that $(\varepsilon_1(t_0),\ldots,\varepsilon_r(t_0)) = (e_1,\ldots,e_r)$.

Proof. Uniqueness follows directly from the uniqueness in Proposition 3.1.6. For the existence, we only need to check that if $e_1,\ldots,\varepsilon_r \in \Gamma(\xi)$ are parallel and $(e_1(t_0),\ldots,e_r(t_0))$ is linearly independent, then $(e_1(t),\ldots,e_r(t))$ is linearly independent for all $t \in I$. This is true because the parallel transport $\|_{t_0}^{t}$ is an isomorphism (Proposition 3.1.8). □

The data of a parallel frame field allows for easy computations of everything that involves the intrinsic derivative.

Lemma 3.1.10. *Let $I \subset \mathbb{R}$ be an interval, $\xi = (E,p,I)$ a vector bundle of rank r, $\frac{D}{dt}$ an intrinsic derivative on ξ and $(\varepsilon_1,\ldots,\varepsilon_r)$ a parallel frame field.*

- *Given* $\sigma_1, \ldots, \sigma_r \in \mathcal{C}^\infty(I)$, *if* $\sigma = \sum_{i=1}^r \sigma^i \varepsilon_i \in \Gamma(\xi)$, *then* $\frac{D}{dt}\sigma = \sum_{i=1}^r \dot\sigma^i \varepsilon_i$.
- *Given* $t_0, t_1 \in I$ *and* $v = \sum_{i=1}^r v^i \varepsilon_i(t_0) \in \xi_{t_0}$, *we find* $\|_{t_0}^{t_1} v = \sum_{i=1}^r v^i \varepsilon_i(t_0)$.

Proof. The first point is a consequence of the Leibniz rule and the fact that $\frac{D}{dt}\varepsilon_i = 0$. The second point is a consequence of the linearity of the parallel transport, and the fact that $\|_{t_0}^{t_1}\varepsilon_i(t_0) = \varepsilon_i(t_1)$. □

Luckily for us, in many situations we will not need to know an explicit parallel frame field, but it will be enough to know that they exist.

3.2 The Tensoriality Lemma

Since we are going to work with vector bundles through calculus on spaces of sections, we need to have ways of constructing sections of vector bundles. We will use a technical result relating sections that are equal at one point.

Lemma 3.2.1. *Let* $\xi = (E, p, M)$ *be a vector bundle of rank* r. *If* $\sigma, \sigma' \in \Gamma(\xi)$ *satisfy* $\sigma(x) = \sigma'(x)$ *for some* $x \in M$, *then there are* $s_1, \ldots, s_r \in \Gamma(\xi)$ *and* $f_1, \ldots, f_r \in \mathcal{C}^\infty(M)$ *such that* $f_i(x) = 0$ *and* $\sigma = \sigma' + \sum_{i=1}^r f_i s_i$ *on a neighbourhood of* x.

Proof. Let $U \subset M$ be a trivializing domain containing x and $(\varepsilon_1, \ldots, \varepsilon_r)$ a frame field of $\xi|_U$.

If $\varphi \in \mathcal{C}^\infty(M)$ is a plateau function, equal to 1 on a neighbourhood of x and 0 outside of U, then the sections $s_i = \varphi\varepsilon_i \in \Gamma(\xi)$ are well defined.

There are smooth functions $g_1, \ldots, g_r \in \mathcal{C}^\infty(U)$ such that $\sigma - \sigma' = \sum_{i=1}^r g_i \varepsilon_i$ on U.

The functions $f_i = \varphi g_i \in \mathcal{C}^\infty(M)$ are well defined, and we have that $\sigma = \sigma' + \sum_{i=1}^r f_i s_i$ near x. □

Let us now see how to construct sections of a dual bundle. If $\xi = (E, p, M)$ is a vector bundle of rank r, then a section $\lambda \in \Gamma(\xi^*)$ defines a linear map $\Lambda : \Gamma(\xi) \to \mathcal{C}^\infty(M)$ by setting $\Lambda(\sigma)(x) = \lambda_x(\sigma(x))$. However, not all linear maps $\Lambda : \Gamma(\xi) \to \mathcal{C}^\infty(M)$, as the value of $\Lambda(\sigma)(x)$ could also depend on the values of σ at other points of M, or on derivatives of σ.

Lemma 3.2.2. *Let* $\xi = (E, p, M)$ *be a vector bundle of rank* r, *and let* $\Lambda : \Gamma(\xi) \to \mathcal{C}^\infty(M)$ *be a linear map. The following are equivalent:*

1. $\forall \sigma \in \Gamma(\xi)\ \forall f \in \mathcal{C}^\infty(M)\ \ \Lambda(f\sigma) = f\Lambda(\sigma)$
2. $\exists \lambda \in \Gamma(\xi^*)\ \ \forall \sigma \in \Gamma(\xi)\ \forall x \in M\ \ \Lambda(\sigma)(x) = \lambda_x(\sigma(x))$

Proof. The implication *(2)⇒(1)* is straightforward, let us prove *(1)⇒(2)*. Consider $\Lambda : \Gamma(\xi) \to \mathcal{C}^\infty(M)$ which is $\mathcal{C}^\infty(M)$-linear. For $x \in M$ and $v \in \xi_x$, consider a section $\sigma \in \Gamma(\xi)$ such that $\sigma(x) = v$ (the existence being given by Lemma 2.2.4). Set $\lambda_x(v) = \Lambda(\sigma)(x)$.

Let us first check that $\lambda_x(v)$ does not depend on the choice of σ.

Let $\sigma, \sigma' \in \Gamma(\xi)$ be such that $\sigma(x) = \sigma'(x) = v$. According to Lemma 3.2.1, there are sections $s_1, \ldots, s_r \in \Gamma(\xi)$ and functions $f_1, \ldots, f_r \in \mathcal{C}^\infty(M)$ such that $\sigma = \sigma' + \sum_{i=1}^r f_i s_i$ on a neighbourhood V of x.

If $\varphi \in \mathcal{C}^\infty(M)$ is a plateau function such that $\varphi = 1$ on a neighbourhood of x and $\varphi = 0$ outside V, then we find:

$$\varphi\sigma = \varphi\sigma' + \sum_{i=1}^r \varphi f_i s_i \tag{3.1}$$

This equality stands on all of M. Evaluating Λ on the left hand side of 3.1, we find:

$$\begin{aligned} \Lambda(\varphi\sigma)(x) &= \underbrace{\varphi(x)}_{=1} \Lambda(\sigma)(x) \\ &= \Lambda(\sigma)(x) \end{aligned}$$

Evaluating Λ on the right hand of 3.1 side yields:

$$\begin{aligned} \Lambda\left(\varphi\sigma' + \sum_{i=1}^r \varphi f_i s_i\right)(x) &= \underbrace{\varphi(x)}_{=1} \Lambda(\sigma')(x) + \sum_{i=1}^r \varphi(x) \underbrace{f_i(x)}_{=0} \Lambda(s_i)(x) \\ &= \Lambda(\sigma')(x) \end{aligned}$$

In the end, we do find that $\Lambda(\sigma)(x) = \Lambda(\sigma')(x)$.

In order to prove the linearity and regularity of λ, consider a trivializing domain $U \subset M$ containing x, and $(\varepsilon_1, \ldots, \varepsilon_r)$ a frame field of $\xi|_U$.

We once again consider a plateau function $\psi \in \mathcal{C}^\infty(M)$ such that $\psi = 1$ on a neighbourhood W of x and $\psi = 0$ outside of U.

On W, we find:

$$\lambda_y = \sum_{i=1}^r \Lambda(\psi\varepsilon_i)\varepsilon_i^*$$

This shows both the linearity and the smoothness of λ, i.e. $\lambda \in \Gamma(\xi^*)$.

□

Remark. *We do not require any regularity of the functional Λ (we do not even need to consider topologies of the vector spaces $\Gamma(\xi)$ and $\mathcal{C}^\infty(M)$). The regularity is hidden in the fact that given a smooth section $\sigma \in \Gamma(\xi)$, the resulting function $\Lambda(\sigma)$ is smooth.*

We can generalize this result to multi-linear functionals on spaces of sections, with values in a space of sections.

Definition 3.2.3. *Consider vector bundles $\xi, \xi_1, \dots, \xi_m$ over the same basis M. A multi-linear map $A : \Gamma(\xi_1) \times \cdots \times \Gamma(\xi_m) \to \Gamma(\xi)$ is called* ***tensorial*** *if it is $\mathcal{C}^\infty(M)$-multi-linear, i.e.*

$$\forall (f_1, \dots, f_m) \in \mathcal{C}^\infty(M)^m \ \forall (\sigma_1, \dots, \sigma_m) \in \Gamma(\xi_1) \times \cdots \times \Gamma(\xi_m)$$

$$A(f_1\sigma_1, \dots, f_m\sigma_m) = f_1 \cdots f_m A(\sigma_1, \dots, \sigma_m)$$

Theorem 3.2.4 (Tensoriality Lemma)**.** *Consider vector bundles $\xi, \xi_1, \dots, \xi_m$ over the same basis M, and a multi-linear map*

$$A : \Gamma(\xi_1) \times \cdots \times \Gamma(\xi_m) \to \Gamma(\xi)$$

The following are equivalent:

1. *A is tensorial*
2. *There is $\alpha \in \Gamma(\xi_1^* \otimes \cdots \otimes \xi_m^* \otimes \xi)$ such that:*

$$\forall x \in M \ \forall (\sigma_i)_{1 \le i \le m} \in \prod_{i=1}^{m} \Gamma(\xi_i) \quad A(\sigma_1, \dots, \sigma_m)(x) = \alpha_x(\sigma_1(x), \dots, \sigma_m(x))$$

Remark. *Such a section α is necessarily unique.*

Proof. Lemma 3.2.2 is exactly the case where $m = 1$ and ξ is the trivial line bundle $(M \times \mathbb{R}, \pi_1, M)$. The proof of Lemma 3.2.2 can be carried out mutatis mutandis to obtain the case where $m = 1$, and an induction process gives the result for any integer m.

$\square$

Remark. *The Tensoriality Lemma can be used to define the exterior differential of a differential form, using Cartan's formula as a definition instead of a property. If α is a differential p-form on a manifold M (i.e. $\alpha \in \Gamma(\Lambda^p T^*M) = \Omega^p(M)$), then $d\alpha$ is the unique differential $p+1$-form such that, for any vector fields $X_0, \dots, X_p \in \mathcal{X}(M)$, we have:*

$$\begin{aligned} d\alpha(X_0, \dots, X_p) = &\sum_{i=0}^{p} (-1)^i X_i \cdot \alpha(X_0, \dots, \widehat{X_i}, \dots, X_p) \\ &+ \sum_{0 \le i < j \le p} (-1)^{i+j} \alpha([X_i, X_j], X_0, \dots, \widehat{X_i}, \dots, \widehat{X_j}, \dots, X_p). \end{aligned}$$

The Tensoriality Lemma shows that this expression defines $d\alpha \in \Gamma\left(T^*M^{\otimes p+1}\right)$, then one can check that it is skew-symmetric, i.e. $d\alpha \in \Gamma\left(\Lambda^{p+1} T^*M\right) = \Omega^{p+1}(M)$.

3.3 Koszul connections

Given a function $f \in \mathcal{C}^\infty(M)$, a point $x \in M$ and a tangent vector $v \in T_xM$, we can define $d_xf(v) \in \mathbb{R}$. If we replace f with a section $\sigma \in \Gamma(\xi)$ of a vector bundle ξ, then $d_x\sigma(v) \in T_xE$. The whole point of vector bundles (and more generally fibre bundles) is that we do not want to consider the manifold structure of the total space, but rather work either in a given fibre, or on the base. More simply put, TE is a vector bundle over a vector bundle, so it is meant to mess with your head.

To define connections on vector bundles with arbitrary basis, we first notice that differentiating a function $f \in \mathcal{C}^\infty(M)$ does not yield another function, but a 1-form $df \in \Gamma(T^*M) = \Omega^1(M)$. For sections of bundles, we wish to define a differential $\nabla_x\sigma(v) \in \xi_x$ for $v \in T_xM$, hence a linear map $\nabla_x\sigma : T_xM \to \xi_x$, i.e. a ξ-valued 1-form $\nabla\sigma \in \Gamma(\mathcal{L}(TM,\xi)) = \Gamma(T^*M \otimes \xi) = \Omega^1(\xi)$.

Definition 3.3.1. *Let $\xi = (E,p,M)$ be a vector bundle. A* ***connection*** *on ξ is a linear map $\nabla : \Gamma(\xi) \to \Gamma(T^*M \otimes \xi) = \Omega^1(\xi)$ satisfying the* ***Leibniz rule****:*

$$\forall \sigma \in \Gamma(\xi)\ \forall f \in \mathcal{C}^\infty(M) \quad \nabla(f\sigma) = df \otimes \sigma + f\nabla\sigma$$

Remark. *By definition, $(df \otimes \sigma)_x(v) = d_xf(v)\sigma(x)$ for $x \in M$ and $v \in T_xM$. We will write $\nabla_x\sigma(v)$, however the notation $\nabla_v\sigma$ is more frequent in the literature.*

We will only use the notation with the vector as a subscript for vector fields: given $X \in \mathcal{X}(M)$ define the map $\nabla_X : \left\{ \begin{array}{ccc} \Gamma(\xi) & \to & \Gamma(\xi) \\ \sigma & \mapsto & \nabla\sigma(X) \end{array} \right.$.

On a trivial bundle $\xi = (M \times \mathbb{R}^r, \pi_1, M)$, sections can be identified with smooth functions $f \in \mathcal{C}^\infty(M,\mathbb{R}^r)$. One can check that $D_xf(v) = d_xf(v)$ then defines a connection D on ξ, called the trivial connection. Let us see how to treat this in terms of frame fields.

Proposition 3.3.2. *Let $\xi = (E,p,M)$ be a vector bundle of rank r, and $(\varepsilon_1,\dots,\varepsilon_r)$ a trivializing frame of ξ. There is a unique connection D on ξ, called the* ***trivial connection****, such that:*

$$\forall i \in \{1,\dots,r\} \quad D\varepsilon_i = 0$$

Proof. Any section $\sigma \in \Gamma(\xi)$ decomposes uniquely as $\sigma = \sum_{i=1}^r \sigma^i\varepsilon_i$. Define $D : \Gamma(\xi) \to \Omega^1(\xi)$ by:

$$D\sigma = \sum_{i=1}^r d\sigma^i \otimes \varepsilon_i$$

It is linear and satisfies the Leibniz rule for connections because of the usual Leibniz rule for functions. We also have $D\varepsilon_i = 0$ for all i.

If ∇ is another connection with the same property, then the Leibniz rule ensures that $\nabla = D$. □

Definition 3.3.3. *Let $\xi = (E, p, M)$ be a vector bundle, and ∇ a connection on ξ. A section $\sigma \in \Gamma(\xi)$ is called* ***parallel*** *if $\nabla\sigma = 0$.*

Proposition 3.3.4. *Let $M \subset \mathbb{R}^n$ be a submanifold. For $x \in M$, we let $p_x \in \mathrm{End}(\mathbb{R}^n)$ be the orthogonal projection onto T_xM for the canonical inner product. The map $\nabla : \mathcal{X}(M) \to \Omega^1(TM) = \mathcal{T}^{1,1}(M) = \Gamma(\mathrm{End}(TM))$ defined by $\nabla_x\sigma(v) = p_x(d_x\sigma(v))$ is a connection.*

The proof is very similar to 3.1.5.

If $M = I \subset \mathbb{R}$ is an interval, then connections on vector bundles over I are essentially the same as intrinsic derivatives.

Recall that $\frac{d}{dt} \in \Gamma(TI)$ is defined by $\frac{d}{dt} \cdot f = \dot{f}$ for every $f \in \mathcal{C}^\infty(I)$, and $dt \in \Gamma(T^*I)$ satisfies $dt\left(\frac{d}{dt}\right) = 1$.

Proposition 3.3.5. *Set $I \subset \mathbb{R}$ be an interval, and $\xi = (E, p, I)$ a vector bundle. If $\frac{D}{dt}$ is an intrinsic derivative on ξ, then $\nabla : \sigma \mapsto dt \otimes \frac{D}{dt}\sigma$ is a connection. Reciprocally, if ∇ is a connection on ξ, then $\frac{D}{dt} : \sigma \mapsto \nabla\sigma\left(\frac{d}{dt}\right)$ is an intrinsic derivative.*

Proof. One just has to check that the Leibniz rules are equivalent. Note that for $f \in \mathcal{C}^\infty(I)$, we have $df = \dot{f}dt$ and $df\left(\frac{d}{dt}\right) = \dot{f}$.
Given an intrinsic derivative $\frac{D}{dt}$, we check:

$$\nabla(f\sigma) = dt \otimes \frac{D}{dt}(f\sigma) = dt \otimes (\dot{f}\sigma) + dt \otimes f\frac{D}{dt}\sigma = dt \otimes \nabla\sigma + f\nabla\sigma$$

Given a connection ∇, we check:

$$\frac{D}{dt}(f\sigma) = \nabla(f\sigma)\left(\frac{d}{dt}\right) = df \otimes \sigma\left(\frac{d}{dt}\right) + f\nabla\sigma\left(\frac{d}{dt}\right) = \dot{f}\sigma + f\frac{D}{dt}\sigma$$

□

Proposition 3.3.6. *Let ξ, ξ' be vector bundles with the same base M, and ∇, ∇' connections on ξ, ξ'.*
There are unique connections ∇^*, $\nabla \oplus \nabla'$ and $\nabla \otimes \nabla'$ on ξ^*, $\xi \oplus \xi'$ and $\xi \otimes \xi'$ such that:

$$\begin{aligned} \forall\lambda \in \Gamma(\xi^*)\ \forall\sigma \in \Gamma(\xi)\ \forall\sigma' \in \Gamma(\xi') \\ \nabla^*\lambda(\sigma) = d(\lambda(\sigma)) - \lambda(\nabla\sigma) \\ (\nabla \oplus \nabla')(\sigma + \sigma') = \nabla\sigma + \nabla'\sigma' \\ (\nabla \otimes \nabla')(\sigma \otimes \sigma') = \nabla\sigma \otimes \sigma' + \sigma \otimes \nabla'\sigma' \end{aligned}$$

More generally, ∇ induces a connection, still denoted by ∇, on the vector bundles $(\xi^*)^{\otimes p} \otimes \xi^{\otimes q}$. For $\omega \in \mathcal{T}^{p,0}(TM)$ a type-$(p,0)$ tensor, we get:

$$\nabla\omega(X_1,\ldots,X_p) = d(\omega(X_1,\ldots,X_p)) - \sum_{i=1}^{p} \omega(X_1,\ldots,\nabla X_i,\ldots,X_p)$$

In other words, we can differentiate $\omega(X_1,\ldots,X_p)$ using a Leibniz rule:

$$d(\omega(X_1,\ldots,X_p)) = \nabla\omega(X_1,\ldots,X_p) + \sum_{i=1}^{p} \omega(X_1,\ldots,\nabla X_i,\ldots,X_p)$$

Note that this formula also preserves skew-symmetry, so a connection ∇ on TM also induces a connection on $\Lambda^k T^*M$.

For $R \in \mathcal{T}^{p,1}(TM)$ a type-$(p,1)$ tensor, we get:

$$\nabla R(X_1,\ldots,X_p) = \nabla(R(X_1,\ldots,X_p)) - \sum_{i=1}^{p} R(X_1,\ldots,\nabla X_i,\ldots,X_p)$$

Proposition 3.3.7. *Let $\xi = (E,p,M)$ be a vector bundle. The set of connections on ξ has the structure of an affine space, with underlying vector space $\Gamma(\xi^* \otimes T^*M \otimes \xi) = \Gamma(T^*M \otimes \mathrm{End}(\xi)) = \Omega^1(\mathrm{End}(\xi))$.*

Proof. Simply notice that if ∇ and ∇' are connections on ξ, then $\nabla - \nabla'$ is tensorial. □

This idea can be used to show the existence of connections.

Proposition 3.3.8. *Every vector bundle has a connection.*

Proof. Let $\xi = (E,p,M)$ be a vector bundle, and consider a locally finite open cover $M = \bigcup_{U\in\mathcal{U}} U$ such that every restriction $\xi|_U$ is trivialisable. Consider a connection ∇^U on $\xi|_U$ (which exists because of Proposition 3.3.2).

We let $(\varphi_U)_{U\in\mathcal{U}}$ be a partition of unity associated to $\mathcal{U}$. For $\sigma \in \Gamma(\xi)$ and $x \in M$, we set:

$$\nabla\sigma(x) = \sum_{x\in U} \varphi_U(x)\nabla^U(\sigma|_U)(x)$$

Since $\mathcal{U}$ is locally finite, $V = \bigcap_{x\in U} U$ is a neighbourhood of x, and the indexes in the sum for $\nabla\sigma(y)$ for $y \in V$ are the same as for x, which shows that $\nabla\sigma$ is a smooth map, hence an element of $\Gamma(T^*M \otimes \xi)$.

For $f \in \mathcal{C}^\infty(M)$, we find:

$$\begin{aligned}\nabla(f\sigma) &= \sum_{U\in\mathcal{U}} \varphi_U \nabla^U (f\sigma|_U) \\ &= \sum_{U\in\mathcal{U}} \varphi_U \left(df\otimes\sigma + f\nabla^U(\sigma|_U)\right) \\ &= \left(\sum_{U\in\mathcal{U}} \varphi_U\right) df\otimes\sigma + f\sum_{U\in\mathcal{U}} \varphi_U\nabla^U(\sigma|_U) \\ &= df\otimes\sigma + f\nabla\sigma\end{aligned}$$

□

3.3.1 Local description of a connection

Even though the definition of a connection is through global sections, it still allows us to differentiate local sections.

Lemma 3.3.9. *Let $\xi = (E,p,M)$ be a vector bundle, and ∇ a connection on ξ. Consider $\sigma,\sigma' \in \Gamma(\xi)$. Let $U \subset M$ be an open set such that $\sigma(x) = \sigma'(x)$ for all $x \in U$. Then $\nabla_x\sigma = \nabla_x\sigma'$ for all $x \in U$.*

Proof. Fix $x \in U$, and consider a plateau function $\varphi \in \mathcal{C}^\infty(M)$ such that $\varphi = 1$ on a neighbourhood of x and $\varphi = 0$ outside of U. Since $\varphi(x) = 1$ and $d_x\varphi = 0$, we find:

$$\begin{aligned}\nabla_x(\varphi\sigma) &= d_x\varphi\otimes\sigma(x) + \varphi(x)\nabla_x\sigma \\ &= \nabla_x\sigma\end{aligned}$$

As $\varphi\sigma = \varphi\sigma'$, it follows that $\nabla_x\sigma = \nabla_x\sigma'$ □

An immediate consequence is that $\nabla_x\sigma$ only depends on the 1-jet of σ at x.

Proposition 3.3.10. *Let $\xi = (E,p,M)$ be a vector bundle, and ∇ a connection on ξ. For all $\sigma \in \Gamma(\xi)$, $x \in M$ and $v \in T_xM$, the value of $\nabla_x\sigma(v)$ only depends on $\sigma(x) \in \xi_x$ and $d_x\sigma(v) \in T_{\sigma(x)}E$.*

Proof. If $\sigma' \in \Gamma(\xi)$ satisfies $\sigma'(x) = \sigma(x)$ then according to Lemma 3.2.1, there are sections $s_1,\dots,s_r \in \Gamma(\xi)$ and functions $f_1,\dots,f_r \in \mathcal{C}^\infty(M)$ such that $\sigma' = \sigma + \sum f_i s_i$ on a neighbourhood of x and $f_i(x) = 0$.
If $d_x\sigma'(v) = d_x\sigma(v)$, then we also get $d_xf_i(v) = 0$, and Lemma 3.3.9 yields:

$$\begin{aligned}\nabla_x\sigma'(v) &= \nabla_x\left(\sigma + \sum f_is_i\right)(v) \\ &= \nabla_x\sigma(v) + \sum\left(d_xf_i(v)s_i(x) + f_i(x)\nabla_xs_i(v)\right) \\ &= \nabla_x\sigma(v)\end{aligned}$$

□

Proposition 3.3.11. *Let $\xi = (E, p, M)$ be a vector bundle, and ∇ a connection on ξ. For any open set $U \subset M$, there is a unique connection $\nabla|_U$ on $\xi|_U$ such that:*

$$\forall \sigma \in \Gamma(\xi) \ \ (\nabla|_U)\sigma|_U = (\nabla\sigma)|_U.$$

Proof. Lemma 3.3.9 and a tiny bit of thinking guarantee the uniqueness. To prove the existence, consider $\sigma \in \Gamma(\xi|_U)$.
For $x \in x \in U$, consider a plateau function $\varphi \in \mathcal{C}^\infty(M)$ such that $\varphi = 1$ on a neighbourhood of x and $\varphi = 0$ outside of U. This allows us to define $\varphi\sigma \in \Gamma(\xi)$, and set $(\nabla|_U)_x\sigma = \nabla_x(\varphi\sigma)$. Following Lemma 3.3.9, $(\nabla|_U)_x\sigma(x)$ does not depend on the choice of φ. Since the same function φ works for any point in a neighbourhood of x, we do obtain a smooth section of $T^*U \otimes \xi|_U$. We have defined a map $\nabla|_U : \Gamma(TU) \to \Gamma(T^*U \otimes \xi|_U)$. Its linearity and the Leibniz rule follow from the same properties for ∇. □

Most of the time, we will use the notation ∇ for $\nabla|_U$ (so Proposition 3.3.11 means that $\nabla\sigma$ can be defined even if σ is only defined on an open subet of M).

Reciprocally, knowing the local expressions of a connection is enough to recover the whole connection.

Lemma 3.3.12. *Let $\xi = (E, p, M)$ be a vector bundle. Consider an open cover $M = \bigcup_{i \in I} U_i$ of M and connections ∇^i on $\xi|_{U_i}$. If*

$$\forall i, j \in I \quad \nabla^i\big|_{U_i \cap U_j} = \nabla^j\big|_{U_i \cap U_j}$$

then there is a unique connection ∇ on ξ such that

$$\forall i \in I \ \ \nabla|_{U_i} = \nabla^i$$

Remark. *Readers who have taken a walk down the algebraic side of things will recognize that this means that a connection is well defined on the sheaf of local sections of a vector bundle.*

Proof. Uniqueness is a consequence of Lemma 3.3.9. For the existence, simply set $\nabla_x\sigma = \nabla^i_x\left(\sigma|_{U_i}\right)$ whenever $x \in U_i$. Because of the assumption on the restriction to intersections, it does not depend on the choice of U_i, and one easily checks that it defines a connection.

□

The restriction of a connection is what allows us to study a connection from a local point of view, i.e. in coordinates.

Consider a trivializing domain $U \subset M$, and a frame field $(\varepsilon_1, \ldots, \varepsilon_r)$ of $\xi|_U$.

According to Proposition 3.3.2, there is a unique connection D on $\xi|_U$ for which $\varepsilon_1, \ldots, \varepsilon_r$ are parallel. According to Proposition 3.3.7, $\nabla - D$ is a represented by a section $A \in \Gamma(T^*U \otimes \mathrm{End}(\xi|_U)) = \Omega^1(\mathrm{End}(\xi|_U))$, called the **connection form** of (U, Φ) (or the **connection 1-form**). We find:

$$\nabla_x \sigma(v) = D_x \sigma(v) + A_x(v)(\sigma(x)).$$

Every section $\sigma \in \Gamma(\xi|_U)$ decomposes as $\sigma = \sum_{\alpha=1}^r \sigma^\alpha \varepsilon_\alpha$.

Now consider also a coordinate system $(x^1, \ldots, x^d)$ on U. We let $\partial_i = \frac{\partial}{\partial x^i}$ be the fundamental vector fields, and dx^i the differentials of the functions x^i.

$$\begin{aligned}
\nabla \sigma &= \sum_{\alpha=1}^{r} \left[D(\sigma^\alpha \varepsilon_\alpha) + A(\sigma^\alpha \varepsilon_\alpha) \right] \\
&= \sum_{\alpha=1}^{r} d\sigma^\alpha \otimes \varepsilon_\alpha + \sigma^\alpha A(\varepsilon_\alpha)
\end{aligned}$$

Since $D\varepsilon_\alpha = 0$, we find:

$$\nabla \varepsilon_\alpha = A(\varepsilon_\alpha) = \sum_{\beta=1}^{r} A_{i,\alpha}^{\beta} dx^i \otimes \varepsilon_\beta$$

For $v = \sum v^i \partial_i$ and $\sigma = \sum \sigma^\alpha \varepsilon_\alpha$, we find:

$$\nabla_x \sigma(v) = \underbrace{\sum_{\substack{1 \le i \le d \\ 1 \le \alpha \le r}} v^i \partial_i \sigma^\alpha(x) \varepsilon_\alpha(x)}_{= D_x \sigma(v)} + \underbrace{\sum_{\substack{1 \le i \le d \\ 1 \le \alpha, \beta \le r}} A_{i,\alpha}^{\beta}(x) v^i \sigma^\alpha(x) \varepsilon_\beta(x)}_{= A_x(v)(\sigma(x))}$$

When calculations start to involve sums on many indexes, it is more practical to use Einstein's convention: we do not write the sum symbol $\sum$, but whenever an index is repeated once as a subscript and once as a superscript, we implicitely consider the sum over all possible values of this index. Here, we find:

$$\nabla_x \sigma(v) = v^i \partial_i \sigma^\alpha \varepsilon_\alpha + A_{i,\alpha}^{\beta} v^i \sigma^\alpha \varepsilon_\beta$$

There will always be a warning before using Einstein's convention.

One way of interpreting Lemma 3.3.12 is that it is possible to define a connection by using local coordinates, as long as the expression is invariant under a change of coordinates. For this, we need to know how the connection form behaves under a change of coordinates. On an arbitrary vector bundle, we need to choose local coordinates on the base manifold and a local frame field. Here we just give the case of a connection on the tangent bundle, where the frame field is made of the coordinate vector fields.

Lemma 3.3.13. *Let M be a manifold, and ∇ a connection on TM. Consider two local coordinate systems $(x^1,\dots,x^d)$ and $(y^1,\dots,y^d)$ defined on the same open set $U \subset M$.*

The components of the connection form $A^k_{i,j}$ (resp. $\overline{A}^k_{i,j}$) of ∇ with respect to the coordinates $(x^1,\dots,x^d)$ (resp. $(y^1,\dots,y^d)$) are related by:

$$\overline{A}^k_{i,j} = \frac{\partial^2 x^p}{\partial y^j \partial y^i}\frac{\partial y^k}{\partial x^p} + A^r_{p,q}\frac{\partial x^p}{\partial y^i}\frac{\partial x^q}{\partial y^j}\frac{\partial y^k}{\partial x^r}$$

Remark. *We use Einstein's convention.*

Proof. The coefficients $A^k_{i,j}$ are defined by:

$$\nabla\frac{\partial}{\partial x^i} = A^k_{i,j}dx^j \otimes \frac{\partial}{\partial x^k}$$

Since $\frac{\partial}{\partial y^i} = \frac{\partial x^p}{\partial y^i}\frac{\partial}{\partial x^p}$ and $\frac{\partial}{\partial x^p} = \frac{\partial y^k}{\partial x^p}\frac{\partial}{\partial y^k}$, we find:

$$\begin{aligned}\nabla\frac{\partial}{\partial y^i} &= \frac{\partial^2 x^p}{\partial y^j \partial y^i}dy^j \otimes \frac{\partial}{\partial x^p} + \frac{\partial x^p}{\partial y^i}\nabla\frac{\partial}{\partial x^p}\\ &= \frac{\partial^2 x^p}{\partial y^j \partial y^i}\frac{\partial y^k}{\partial x^p}dy^j \otimes \frac{\partial}{\partial y^k} + \frac{\partial x^p}{\partial y^i}A^r_{p,q}dx^q \otimes \frac{\partial}{\partial x^r}\\ &= \frac{\partial^2 x^p}{\partial y^j \partial y^i}\frac{\partial y^k}{\partial x^p}dy^j \otimes \frac{\partial}{\partial y^k} + \frac{\partial x^p}{\partial y^i}A^r_{p,q}\frac{\partial x^q}{\partial y^j}\frac{\partial y^k}{\partial x^r}dy^j \otimes \frac{\partial}{\partial y^k}\end{aligned}$$

The formula follows from the identification of each term with the definition of $\overline{A}^k_{i,j}$:

$$\nabla\frac{\partial}{\partial y^i} = \overline{A}^k_{i,j}dy^j \otimes \frac{\partial}{\partial y^k}$$

□

3.3.2 Connections and curves

We have seen that intrinsic derivatives on vector bundles over an interval are related to choices of trivializations of these bundles. However, connections on higher dimensional manifolds do not lead to trivializations. One reason is that vector bundles need to be trivializable, but there are also local obstructions, which we will study later.

However, connections can be studied by considering curves on the base.

Definition 3.3.14. *Let $\xi = (E,p,M)$ be a vector bundle. Given a smooth curve $c : I \to \mathbb{R}$, a **ξ-valued vector field along** c is a section of $c^*\xi$, i.e. a smooth function $\sigma : I \to E$ such that:*

$$\forall t \in I \quad \sigma(t) \in \xi_{c(t)}$$

Note that every $\sigma \in \Gamma(\xi)$ defines a ξ-valued vector field $\sigma \circ c$ along c. However, not all ξ-valued vector fields along c can be obtained in this way (the curve c is not necessarily injective).

A vector field along c (without precising the vector bundle) is a TM-valued vector field along c.

Proposition 3.3.15. *Let $\xi = (E,p,M)$ be a vector bundle, ∇ a connection on ξ, and $c : I \to M$ a smooth curve.*
There is a unique intrinsic derivative $\frac{D}{dt}$ on $c^*\xi$, called the ***intrinsic derivative along*** *c, such that:*

$$\forall \sigma \in \Gamma(\xi)\ \forall t \in I \quad \frac{D}{dt}(\sigma \circ c)(t) = \nabla_{c(t)}\sigma(\dot{c}(t))$$

Remark. *For $\sigma \in \Gamma(\xi)$, we will usually write $\frac{D}{dt}\sigma$ instead of $\frac{D}{dt}(\sigma \circ c)$.*

Proof. Let us start with the case where ξ is trivializable. Consider a frame field $(\varepsilon_1,\ldots,\varepsilon_r)$ of ξ, the associated trivial connection D, and the connection form $\Gamma \in \Gamma(T^*M \otimes \mathrm{End}(\xi))$.

If $\frac{D}{dt}$ is an intrinsic derivative on $c^*\xi$ satisfying the requirement, then:

$$\forall \alpha \in \{1,\ldots,r\}\ \forall t \in I \quad \frac{D}{dt}(\varepsilon_\alpha \circ c)(t) = \nabla_{c(t)}\varepsilon_\alpha(\dot{c}(t)) = A_{c(t)}(\dot{c}(t))(\varepsilon_\alpha(c(t)))$$

Since any ξ-valued vector field along c decomposes as $\sigma = \sum_{\alpha=1}^r \sigma^\alpha \varepsilon_\alpha \circ c$ for some functions $\sigma^\alpha \in \mathcal{C}^\infty(I)$, we get:

$$\frac{D}{dt}\sigma = \sum_{\alpha=1}^r \left(\dot{\sigma}^\alpha \varepsilon_\alpha + \sigma^\alpha A(\dot{c})(\varepsilon_\alpha)\right)$$

This formula not only guarantees uniqueness, but can also be used to prove existence.

In order to move on to the general case, we use the fact that for every $t \in I$, we can choose an open interval $I_t \subset I$ containing I and a trivializing domain $U_t \subset M$ such that $c(I_t) \subset U_t$. The previous discussion shows that the intrinsic derivative along $c|_{I_t}$ is uniquely defined. Because of the uniqueness, they coincide on $I_t \cap I_s$ for $t,s \in I$, so Lemma 3.3.12 allows us to define a unique intrinsic derivative with the same property for the whole curve c (actually Lemma 3.3.12 only concerns connections on $c^*\xi$, but the relationship with intrinsic derivatives is given by Proposition 3.3.5). $\square$

The proof actually gave us the local expression for $\frac{D}{dt}$, given a local frame field $(\varepsilon_1,\ldots,\varepsilon_r)$ of ξ:

$$\frac{D}{dt}\sigma = \sum_{\alpha=1}^{r} \Big(\dot{\sigma}^\alpha \varepsilon_\alpha + \sigma^\alpha A(\dot{c})(\varepsilon_\alpha) \Big)$$

If c has values inside a chart domain of M, and $(x^1, \ldots, x^d)$ are local coordinates, then we decompose $\dot{c}(t) = \sum_{i=1}^{d} \dot{c}^i(t)\partial_i$. We now find:

$$A(\dot{c})(\varepsilon_\alpha) = \sum_{\substack{1\le i\le d\\ 1\le\beta\le r}} \left(A_{i,\alpha}^{\beta} \circ c\right) \dot{c}^i \varepsilon_\beta$$

Which leads to:

$$\frac{D}{dt}\sigma = \sum_{1\le\alpha\le r} \dot{\sigma}^\alpha \varepsilon_\alpha + \sum_{\substack{1\le i\le d\\ 1\le\alpha,\beta\le r}} \left(A_{i,\alpha}^{\beta} \circ c\right) \sigma^\alpha \dot{c}^i \varepsilon_\beta$$

Example: Keeping the same notations for local coordinates and frame field, through any point $x \in U$ we can consider the curves obtained by varying one of the coordinates, i.e. $t \mapsto (x^1(x), \ldots, x^{i-1}(x), t, x^{i+1}(x), \ldots, x^d(x))$. We will denote by $\frac{D}{\partial x^i}$ the intrinsic derivative along this curve. Note that for $\sigma \in \Gamma(\xi)$, we have $\frac{D}{\partial x^i}\sigma = \nabla\sigma(\partial_i)$.

Since a connection defines an intrinsic derivative for any curve in M, it also defines a parallel transport. Since $(c^*\xi)_t = \xi_{c(t)}$, it defines linear isomorphisms between different fibres of ξ.

Definition 3.3.16. *Let $\xi = (E, p, M)$ be a vector bundle, ∇ a connection on ξ, and $c : I \to M$ a smooth curve. Given $t_0, t_1 \in I$, the parallel transport $\|_{t_0}^{t_1} : \xi_{c(t_0)} \to \xi_{c(t_1)}$ for the intrinsic derivative along c is called the* ***parallel transport along*** *c.*

Note that because $\|_{t_1}^{t_2} \circ \|_{t_0}^{t_1} = \|_{t_0}^{t_2}$, we can define the parallel transport along any **piecewise smooth** curve in a consistent way, so that we still have the same property for composition (we use the convention that piecewise smooth curves are continuous).

Note that the parallel transport usually depends heavily on the choice of a curve. In particular, if $c(t_0) = c(t_1)$, then there is no reason that $\|_{t_0}^{t_1}$ should be the identity map.

Definition 3.3.17. *Let $\xi = (E, p, M)$ be a vector bundle, ∇ a connection on ξ, and $x \in M$. The* ***holonomy group*** *of ∇ at x is the subgroup $\mathrm{Hol}_x \subset \mathrm{GL}(\xi_x)$ of parallel transports $\|_0^1$ along piecewise smooth curves $c : [0,1] \to M$ such that $c(0) = c(1) = x$.*
The ***restricted holonomy group*** *is the subgroup $\mathrm{Hol}_x^\circ \subset \mathrm{Hol}_x$ of parallel transporst along null homotopic piecewise smooth curves.*

Note that if M is connected, then Hol_x does not really depend on x, as for $x, y \in M$ there is a linear isomorphism $\varphi : \xi_x \to \xi_y$ such that $\mathrm{Hol}_x = \varphi^{-1}\mathrm{Hol}_y\varphi$ (simply take φ to be the parallel transport along any curve from x to y).

Not only can the parallel transport along curves give many informations about a connection, it actually allows to retrieve the whole connection.

Proposition 3.3.18. *Let $\xi = (E, p, M)$ be a vector bundle and ∇ a connection on ξ. Consider $x \in M$, $v \in T_xM$, and a smooth curve $c :]-\varepsilon, \varepsilon[\to M$ such that $c(0) = x$ and $\dot{c}(0) = v$. For all $\sigma \in \Gamma(\xi)$, we get:*

$$\nabla_x\sigma(v) = \left.\frac{d}{dt}\right|_{t=0} \|_t^0 \sigma(c(t))$$

Proof. Consider a parallel frame $(\varepsilon_1, \ldots, \varepsilon_r)$ along c, and decompose $\sigma(c(t)) = \sum_{\alpha=1}^r \sigma^\alpha(t)\varepsilon_\alpha(t)$. We have that:

$$\nabla_x\sigma(v) = \frac{D}{dt}(\sigma \circ c)(0) = \sum_{\alpha=1}^r \dot{\sigma}^\alpha(0)\varepsilon_\alpha(0)$$

We also have:

$$\|_t^0 \sigma(c(t)) = \sum_{\alpha=1}^r \sigma^\alpha(t)\varepsilon_\alpha(0)$$

Differentiating this expression at $t = 0$ offers the conclusion. □

3.3.3 Pulling back connections

Recall that the pulled back bundle $f^*\xi$ of a vector bundle $\xi = (E, p, M)$ by a smooth map $f : N \to M$ is defined by $f^*\xi = (f^*E, N, f^*p)$ where

$$f^*E = \{(x, v) \in N \times E | p(v) = f(x)\}$$

and $f^*p(x, v) = x$.

In other words, the fibre above $x \in N$ is the fibre above $f(x) \in M$:

$$(f^*\xi)_x = \xi_{f(x)}$$

Note that pulling back vector bundles is compatible with algebraic operations on vector bundles, i.e. $f^*(\xi \oplus \xi') = (f^*\xi) \oplus (f^*\xi')$, $f^*(\xi \oplus \xi') = (f^*\xi) \oplus (f^*\xi')$ and $f^*(\xi^*) = (f^*\xi)^*$.

Any section $\sigma \in \Gamma(\xi)$ can be pulled back to a section $f^*\sigma = \sigma \circ f \in \Gamma(f^*\xi)$. Not all sections of $f^*\xi$ can be obtained in this way (indeed, f could be constant), but sections can all be recovered from these.

Lemma 3.3.19. *Let $\xi = (E, p, M)$ be a vector bundle of rank r, $f : N \to M$ a smooth map, and $\sigma \in \Gamma(f^*\xi)$. For every $x \in N$, there are sections $s_1, \dots, s_r \in \Gamma(\xi)$ and functions $u^1, \dots, u^r \in \mathcal{C}^\infty(N)$ such that $\sum_{i=1}^r u^i f^* s_i$ is equal to σ on a neighbourhood of x.*

Proof. Consider a trivialising domain $U \subset M$ of ξ that contains $f(x)$, and a frame field $(\varepsilon_1, \dots, \varepsilon_r)$ of $\xi|_U$. Then $(f^*\varepsilon_1, \dots, f^*\varepsilon_r)$ is a frame field of $f^*\xi|_{f^{-1}(U)}$. Write $\sigma|_U = \sum_{i=1}^r \sigma^i f^*\varepsilon_i$, and consider a plateau function φ on M such that $\varphi = 1$ on a neighbourhood of $f(x)$ and $\varphi = 0$ outside of U. Then $s_i = \varphi\varepsilon_i$ and $u^i = (\varphi \circ f)\sigma^i$ are the required functions and sections. □

In order to define a connection on the pulled back bundle, notice that we can also pull back bundle-valued differential forms: for $\omega \in \Gamma(T^*M \otimes \xi)$, define $f^*\omega$ by:

$$\forall x \in N \ \forall v \in T_x N \quad (f^*\omega)_x(v) = \omega_{f(x)}(d_x f(v)) \in \xi_{f(x)} = (f^*\xi)_x$$

Proposition 3.3.20. *Let $\xi = (E, p, M)$ be a vector bundle of rank r, ∇ a connection on M, and $f : N \to M$ a smooth map. There is a unique connection $f^*\nabla$ on $f^*\xi$ such that:*

$$\forall \sigma \in \Gamma(\xi) \quad (f^*\nabla)(f^*\sigma) = f^*(\nabla\sigma)$$

Proof. Let us start with uniqueness: let ∇^1, ∇^2 be connections on $f^*\xi$ such that $\nabla^1 f^*\sigma = \nabla^2 f^*\sigma = f^*(\nabla\sigma)$ for every $\sigma \in \Gamma(\xi)$.

Let $\sigma \in \Gamma(f^*\xi)$ and $x \in N$. According to Lemma 3.3.19, there are functions $u^1, \dots, u^r \in \mathcal{C}^\infty(N)$ and sections $s_1, \dots, s_r \in \Gamma(\xi)$ such that $\sum_{i=1}^r u^i f^* s_i$ is equal to σ on a neighbourhood of x. From Lemma 3.3.9 we know that $\nabla^j_x \sigma = \nabla^j_x \left(\sum_{i=1}^r u^i f^* s_i\right)$ for $j = 1, 2$.

$$\begin{aligned}
\nabla^j_x \sigma &= \nabla^j_x \left(\sum_{i=1}^r u^i f^* s_i\right) \\
&= \sum_{i=1}^r \left(d_x u^i \otimes f^* s_i(x) + u^i(x) \nabla^j_x f^* s_i\right) \\
&= \sum_{i=1}^r \left(d_x u^i \otimes f^* s_i(x) + u^i(x) (f^*\nabla s_i)_x\right)
\end{aligned}$$

Since the last line foes not depend on j, we find that $\nabla^1 = \nabla^2$.

In order to show the existence, we simply need to check that the expression $\sum_{i=1}^r \left(du^i \otimes f^* s_i + u^i (f^*\nabla s_i)_x\right)$ defines a connection on $f^*\xi$.

□

Let us compute the components of the connection form of the pulled back connection. Consider an open set $U \subset M$ and a frame field $(\varepsilon_1, \dots, \varepsilon_r)$ of $\xi|_U$. Then $(f^*\varepsilon_1, \dots, f^*\varepsilon_r)$ is a frame field of $f^*(\xi|_U) = (f^*\xi)|_{f^{-1}(U)}$.

Consider also coordinates $(x^1, \dots, x^d)$ on U and coordinates $(y^1, \dots, y^{d'})$ on an open subset of $f^{-1}(U)$.

For $\omega \in \Gamma(T^*M \otimes \xi)$, we write locally $\omega = \sum_{\substack{1\le i\le d\\ 1\le\alpha\le r}} \omega_i^\alpha dx^i \otimes \varepsilon_\alpha$. Then $f^*\omega$ is locally given by:

$$f^*\omega = \sum_{\substack{1\le j\le d'\\ 1\le i\le d\\ 1\le\alpha\le r}} \frac{\partial f^i}{\partial y^j} \omega_i^\alpha dy^j \otimes (f^*\varepsilon_\alpha)$$

Since $(f^*\nabla)(f^*\varepsilon_\alpha) = f^*(\nabla\varepsilon_\alpha)$, we find the components of the connection form f^*A of $f^*\nabla$.

$$(f^*A)_{j,\alpha}^\beta(y) = \sum_{i=1}^d A_{i,\alpha}^\beta(f(y)) \frac{\partial f^i}{\partial y^j}(y)$$

Note that the formula justifies the notation f^*A, as it is also the formula for the pull-back of A.

The pulled back connection can also be described in terms of parallel transport.

Proposition 3.3.21. *Let $\xi = (E, p, M)$ be a vector bundle, ∇ a connection on M, $f : N \to M$ a smooth map, and $f^*\nabla$ the pulled back connection on $f^*\xi$. For every smooth curve $c : I \to N$ and $t_0, t_1 \in I$, the parallel transport along c from t_0 to t_1*

$$\|_{t_0}^{t_1} : (f^*\xi)_{c(t_0)} = \xi_{f(c(t_0))} \to (f^*\xi)_{c(t_1)} = \xi_{f(c(t_1))}$$

is equal to the parallel transport along $f \circ c$ from t_0 to t_1.

Proof. If σ is a ξ-valued vector field along $f \circ c$, then $f^*\sigma$ is a $f^*\xi$-valued vector field along c. By working locally, we can see that $\frac{D}{dt}(f^*\sigma) = f^*\left(\frac{D}{dt}\sigma\right)$, so σ is parallel along $f \circ c$ if and only if $f^*\sigma$ is parallel along c. □

We will mostly deal with pulled back connections in two situations: constant maps and the inclusion of a submanifold.

Pulled back connection by a constant map

Assume that the map $f : N \to M$ is constant, say $f(x) = y_0$ for all $x \in N$. Then the vector bundle $f^*\xi$ is trivialisable: the fibre over $x \in N$ is always the same vector space ξ_{y_0}. Given a vector basis $(e_1, \dots, e_r)$ of ξ_{y_0}, the sections $(\varepsilon_1, \dots, \varepsilon_r)$ of $f^*\xi$ given by $\varepsilon_\alpha(x) = e_\alpha$ for all $x \in N$ form a frame field of $f^*\xi$.

The connection $f^*\nabla$ is equal to the trivial connection D associated to

$(\varepsilon_1,\ldots,\varepsilon_r)$. In other words, sections of $f^*\xi$ can be identified with smooth maps from N to the vector space ξ_{y_0}, the connection $f^*\nabla$ is then mapped to the usual differential.

Restriction of a connection to a submanifold

Let $\xi = (E,p,M)$ be a vector bundle equipped with a connection ∇, and consider an immersed submanifold $N \subset M$. Recall that by calling $N \subset M$ an immersed submanifold, we have implicitely chosen a differentiable structure on N for which the inclusion $i : N \to M$ is an immersion.

The pulled back bundle $i^*\xi$ is the vector bundle over N whose fibres are given by $(i^*\xi)_x = \xi_x$ for $x \in N$, i.e. $i^*\xi$ should be seen as the **restriction** $\xi|_N$ of ξ to N.

The connection $i^*\nabla$ should also be considered as the restriction of ∇ to $\xi|_N$. This means that given a section $\sigma \in \Gamma(i^*\xi)$, which should be seen as a section of ξ which is only defined on the submanifold N, we can define $\nabla_v\sigma \in \xi_x$ for $x \in N$ and $v \in T_xN$ (but not necessarily for a general $v \in T_xM$). This is exactly the same as for smooth real valued functions defined on submanifolds: a function defined on N can be differentiated along a direction tangent to N.

3.4 Tensorial invariants of a connection

We will now see how the non triviality of a connection can be encoded in two fields. A first candidate would be the connection form. However, it depends on the choice of a trivialisation, and its value at one point cannot help in the definition of an invariant.

Proposition 3.4.1. *Let $\xi = (E,p,M)$ be a vector bundle of rank r, ∇ a connection on ξ and $x \in M$. There is a neighbourhood $U \subset M$ and a frame field $(\varepsilon_1,\ldots,\varepsilon_r)$ of $\xi|_U$ such that $\nabla\varepsilon_\alpha$ vanishes at x for all $\alpha \in \{1,\ldots,r\}$.*

Remark. *It follows that for every coordinate system $(x^1,\ldots,x^d)$ of M around x, we have $A^\beta_{i,\alpha}(x) = 0$ for all $i \in \{1,\ldots,d\}$ and $\alpha,\beta \in \{1,\ldots,r\}$.*

Proof. Start with a frame field $(\overline{\varepsilon}_1,\ldots,\overline{\varepsilon}_n)$ of $\xi|_U$ for some neighbourhood U of x, and decompose $\nabla_x\overline{\varepsilon}_\alpha = \sum_{\beta=1}^r \omega^\beta_\alpha \otimes \overline{\varepsilon}_\beta(x)$ where $\omega^\beta_\alpha \in T^*_xM$ (given local coordinates $(x^1,\ldots,x^d)$ on U, the formula is $\omega^\beta_\alpha = \sum_{i=1}^d A^\beta_{i,\alpha} dx^i$).

Choose functions $f^\beta_\alpha \in \mathcal{C}^\infty(M)$ such that $f^\beta_\alpha(x) = 0$ and $d_x f^\beta_\alpha = -\omega^\beta_\alpha$. Let $\varepsilon_\alpha = \overline{\varepsilon}_\alpha + \sum_{\beta=1}^r f^\beta_\alpha \overline{\varepsilon}_\beta$.

Since $\varepsilon_\alpha(x) = \overline{\varepsilon}_\alpha(x)$, up to making U smaller, we can assume that $(\varepsilon_1,\ldots,\varepsilon_n)$

is a trivialising frame of $\xi|_U$.

$$\begin{aligned}
\nabla_x \varepsilon_\alpha &= \nabla_x \left(\overline{\varepsilon}_\alpha + \sum_{\beta=1}^{r} f_\alpha^\beta \overline{\varepsilon}_\beta \right) \\
&= \sum_{\beta=1}^{r} \underbrace{\left(\omega_\alpha^\beta + d_x f_\alpha^\beta \right)}_{=0} \otimes \overline{\varepsilon}_\beta(x) + \sum_{\beta=1}^{r} \underbrace{f_\alpha^\beta(x)}_{=0} \nabla_x \overline{\varepsilon}_\beta \\
&= 0
\end{aligned}$$

□

Because of this, we have two options. The first one is to consider derivatives of the connection form, this leads to the notion of **curvature**. The second one is to only consider some specific trivialising frames, which we can do on the tangent bundle of a manifold, this will lead to the notion of **torsion**.

Even though curvature and torsion are two very different types of invariants, they share a common interpretation as the failure of generalisations of the Schwarz Lemma for second order covariant derivatives. Curvature is the failure of the Schwarz Lemma when considering second order covariant derivatives of sections of a vector bundle, and torsion is the failure of the Schwarz Lemma for second order derivatives of functions on a manifold (we will see how a connection on the tangent bundle allows us to define second order differentials of functions).

3.4.1 The curvature of a connection

When dealing with connections on vector bundles over the line, we saw how useful it was to be able to consider parallel frame fields. So it is natural to ask whether these exist in higher dimensions.

Definition 3.4.2. *Let $\xi = (E, p, M)$ be a vector bundle of rank r and ∇ a connection on M. We say that ∇ is **locally trivial** if every $x \in M$ has an open neighbourhood $U \subset M$ on which there is a frame field $(\varepsilon_1, \dots, \varepsilon_r)$ such that $\nabla|_U$ is equal to the corresponding trivial connection (i.e. $\forall \alpha \in \{1, \dots, r\}$ $\nabla \varepsilon_\alpha = 0$).*

In order to know whether a connection is locally trivial, a starting point would be to ask if given a point $x \in M$ and a vector $v \in \xi_x$, we can find a parallel section $\sigma \in \Gamma(\xi)$ such that $\sigma(x) = v$. First, we notice that uniqueness still holds.

Proposition 3.4.3. *Let $\xi = (E, p, M)$ be a vector bundle and ∇ a connection on ξ. Let $\sigma \in \Gamma(\xi)$ be parallel. If σ vanishes a some point, and if M is connected, then σ vanishes identically on M.*

Proof. Consider $x \in M$ such that $\sigma(x) = 0$, and $y \in M$. Since M is connected, we can consider a smooth path c joining x and y. Then $\sigma \circ c$ is a parallel section of $c^*\xi$ that vanishes at some point, so the uniqueness in Proposition 3.1.6 guarantees that $\sigma \circ c = 0$, therefore $\sigma(y) = 0$. □

This tells us that parallel section should not always exist, because nowhere vanishing sections do not always exist (e.g. the tangent bundle of $\mathbb{S}^2$). However, one should not expect local existence to hold either, since the equation $\nabla\sigma = 0$ is a partial differential equation, and a generic partial differential equation tends to not have solutions. What stands out is that the obstruction to the existence of parallel section is encoded in a field, called the curvature.

Lemma 3.4.4. *Let $\xi = (E, p, M)$ be a vector bundle, and ∇ a connection on M. The map $F : \mathcal{X}(M) \times \mathcal{X}(M) \times \Gamma(\xi) \to \Gamma(\xi)$ defined by*

$$F(X,Y)\sigma = \nabla[\nabla\sigma(Y)](X) - \nabla[\nabla\sigma(X)](Y) - \nabla\sigma([X,Y])$$

is tensorial, and skew-symmetric in the first two variables.

Proof. Skew-symmetry is straightforward. For tensoriality with respect to σ, we consider $f \in \mathcal{C}^\infty(M)$, and we first compute $\nabla[\nabla(f\sigma)(Y)](X)$:

$$\begin{aligned}\nabla\Big(\nabla(f\sigma)(Y)\Big)(X) &= \nabla\Big(df(Y)\sigma + f\nabla\sigma(Y)\Big)(X)\\ &= d\Big(df(Y)\Big)(X)\sigma + df(Y)\nabla\sigma(X) + df(X)\nabla\sigma(Y) + f\nabla\Big(\nabla\sigma(Y)\Big)(X)\end{aligned}$$

Similarly, we get:

$$\nabla\Big(\nabla(f\sigma)(X)\Big)(Y) = d\Big(df(X)\Big)(Y)\sigma + df(X)\nabla\sigma(Y) + df(Y)\nabla\sigma(X) + f\nabla\Big(\nabla\sigma(X)\Big)(Y)$$

Since $d\Big(df(Y)\Big)(X) - d\Big(df(X)\Big)(Y) = df\Big([X,Y]\Big)$, we find:

$$\nabla\Big(\nabla(f\sigma)(Y)\Big)(X) - \nabla\Big(\nabla(f\sigma)(X)\Big)(Y) = df\Big([X,Y]\Big)\sigma + f\Big(\nabla(\nabla\sigma(Y))(X) - \nabla(\nabla\sigma(X))(Y)\Big)$$

This simplifies to:

$$\begin{aligned}F(X,Y)(f\sigma) &= -\nabla(f\sigma)\Big([X,Y]\Big) + df\Big([X,Y]\Big)\sigma + f\Big(F(X,Y)\sigma + \nabla\sigma\Big([X,Y]\Big)\Big)\\ &= fF(X,Y)\sigma\end{aligned}$$

Let us now prove tensoriality with respect to X. First, recall that:

$$[fX,Y] = f[X,Y] - df(Y)X$$

It follows that:

$$\nabla\sigma\left([fX,Y]\right) = f\nabla\sigma\left([X,Y]\right) - df(Y)\nabla\sigma(X)$$

We also find:

$$\begin{aligned}\nabla\left(\nabla\sigma(Y)\right)(fX) - \nabla\left(\nabla\sigma(fX)\right)(Y) &= f\nabla\left(\nabla\sigma(Y)\right)(X) - \nabla\left(f\nabla\sigma(X)\right)(Y) \\ &= f\left(\nabla\left(\nabla\sigma(Y)\right)(X) - \nabla\left(\nabla\sigma(X)\right)(Y)\right) - df(Y)\nabla\sigma(X)\end{aligned}$$

Combining with the previous computation, we get:

$$F(fX,Y)\sigma = fF(X,Y)\sigma$$

Tensoriality with respect to Y follows from the skew-symmetry. □

Definition 3.4.5. *Let $\xi = (E,p,M)$ be a vector bundle, and ∇ a connection on M. The* ***curvature*** *of ∇ is the field $F \in \Gamma(\Lambda^2 T^*M \otimes \mathrm{End}(\xi))$ such that, for all $X,Y \in \mathcal{X}(M)$ and $\sigma \in \Gamma(\xi)$, we have:*

$$F(X,Y)\sigma = \nabla[\nabla\sigma(Y)](X) - \nabla[\nabla\sigma(X)](Y) - \nabla\sigma([X,Y])$$

We say that ∇ is ***flat*** *if F vanishes identically on M.*

Let us describe the curvature tensor in coordinates. Let $(x^1,\dots,x^d)$ be a local coordinate system on M, and $(\varepsilon_1,\dots,\varepsilon_r)$ a local frame field of ξ.

For $X = \sum_{i=1}^d X^i\partial_i, Y = \sum_{i=1}^d Y^i\partial_i \in \mathcal{X}(M)$ and $\sigma = \sum_{\alpha=1}^r \sigma^\alpha\varepsilon_\alpha$, we get:

$$\begin{aligned}F(X,Y)\sigma &= \sum_{\substack{1\le i,j\le d\\ 1\le\alpha\le r}} X^iY^j\sigma^\alpha F(\partial_i,\partial_j)\varepsilon_\alpha \\ &= \sum_{\substack{1\le i<j\le d\\ 1\le\alpha\le r}} \left(X^iY^j - X^jY^i\right)\sigma^\alpha F(\partial_i,\partial_j)\varepsilon_\alpha\end{aligned}$$

We can compute $F(\partial_i,\partial_j)\varepsilon_\alpha$ by using the components of the connection form:

$$\begin{aligned}F(\partial_i,\partial_j)\varepsilon_\alpha &= \nabla_{\partial_i}\nabla_{\partial_j}\varepsilon_\alpha - \nabla_{\partial_j}\nabla_{\partial_i}\varepsilon_\alpha - \underbrace{\nabla_{[\partial_i,\partial_j]}\varepsilon_\alpha}_{=0} \\ &= \nabla_{\partial_i}\left(\sum_{\beta=1}^r A^\beta_{j,\alpha}\varepsilon_\beta\right) - \nabla_{\partial_j}\left(\sum_{\beta=1}^r A^\beta_{i,\alpha}\varepsilon_\beta\right) \\ &= \sum_{1\le\beta\le r}\left(\partial_iA^\beta_{j,\alpha} - \partial_jA^\beta_{i,\alpha}\right)\varepsilon_\beta + \sum_{1\le\beta,\gamma\le r}\left(A^\beta_{j,\alpha}A^\gamma_{i,\beta} - A^\beta_{i,\alpha}A^\gamma_{j,\beta}\right)\varepsilon_\gamma\end{aligned}$$

So we find $F(\partial_i, \partial_j)\varepsilon_\alpha = \sum_{\beta=1}^r F_{i,j,\alpha}^\beta \varepsilon_\beta$ where:

$$F_{i,j,\alpha}^\beta = \partial_i A_{j,\alpha}^\beta - \partial_j A_{i,\alpha}^\beta + \sum_{\gamma=1}^r \left(A_{j,\alpha}^\gamma A_{i,\gamma}^\beta - A_{i,\alpha}^\gamma A_{j,\gamma}^\beta \right)$$

Reformulation: This formula can be written as $F = dA + [A,A]$ where dA is the exterior differential of the connection 1-form A (it only makes sense for a trivial bundle, but A already depends on the choice of a trivialization), and $[A,A]$ is the $\mathrm{End}(\xi)$-valued 2-form defined by $[A,A](u,v) = [A(u), A(v)]$.

Proposition 3.4.6. *Let $\xi = (E,p,M)$ be a vector bundle, ∇ a connection on ξ, and F its curvature.*
Given a smooth map $f : U \to M$ where $U \subset \mathbb{R}^2$ is open, we denote by:

- *$\frac{D}{\partial t}$ the intrinsic derivatives along the curves $t \mapsto f(t, s_0)$ for fixed s_0.*
- *$\frac{D}{\partial s}$ the intrinsic derivatives along the curves $s \mapsto f(t_0, s)$ for fixed t_0.*

Then for all $\sigma \in \Gamma(f^*\xi)$, we have that:

$$\frac{D}{\partial t}\frac{D}{\partial s}\sigma - \frac{D}{\partial s}\frac{D}{\partial t}\sigma = F\left(\frac{\partial f}{\partial t}, \frac{\partial f}{\partial s}\right)\sigma$$

Remark. *The result can be abbreviated as $\left[\frac{D}{\partial t}, \frac{D}{\partial s}\right] = F\left(\frac{\partial f}{\partial t}, \frac{\partial f}{\partial s}\right)$.*

Proof. Since both sides of the equation can be computed locally, consider a local coordinate system $(x^1, \dots, x^d)$ on M, and a local frame field $(\varepsilon_1, \dots, \varepsilon_r)$ of ξ.

Write $f^i(t,s) = x^i(f(t,s))$ for $i \in \{1, \dots, d\}$ and $(t,s) \in U$, and decompose:

$$\frac{\partial f}{\partial t} = \sum_{i=1}^d \frac{\partial f^i}{\partial t}\partial_i \quad ; \quad \frac{\partial f}{\partial s} = \sum_{i=1}^d \frac{\partial f^i}{\partial s}\partial_i$$

We also decompose $\sigma = \sum_{\alpha=1}^r \sigma^\alpha \varepsilon_\alpha$ (here ε_α stands for an abbreviation of $\varepsilon_\alpha \circ f$).

First, we compute $\frac{D}{\partial t}\varepsilon_\alpha$:

$$\frac{D}{\partial t}\varepsilon_\alpha = \nabla_{\frac{\partial f}{\partial t}}\varepsilon_\alpha = A\left(\frac{\partial f}{\partial t}, \varepsilon_\alpha\right) = \sum_{i=1}^d \frac{\partial f^i}{\partial t} A(\partial_i, \varepsilon_\alpha) = \sum_{\substack{1 \le i \le d \\ 1 \le \beta \le r}} \frac{\partial f^i}{\partial t} A_{i,\alpha}^\beta \varepsilon_\beta$$

Similarly, we find:

$$\frac{D}{\partial s}\varepsilon_\alpha = \sum_{\substack{1 \le i \le d \\ 1 \le \beta \le r}} \frac{\partial f^i}{\partial s} A_{i,\alpha}^\beta \varepsilon_\beta$$

We can compute $\frac{D}{\partial t}\sigma$ and $\frac{D}{\partial s}\sigma$:

$$\begin{aligned}
\frac{D}{\partial t}\sigma &= \sum_{\alpha=1}^{r}\left(\frac{\partial \sigma^\alpha}{\partial t}\varepsilon_\alpha + \sigma^\alpha \frac{D}{\partial t}\varepsilon_\alpha\right) \\
&= \sum_{\alpha=1}^{r}\left(\frac{\partial \sigma^\alpha}{\partial t}\varepsilon_\alpha + \sum_{\substack{1\le i\le d\\ 1\le\beta\le r}} \sigma^\alpha \frac{\partial f^i}{\partial t} A^{\beta}_{i,\alpha}\varepsilon_\beta\right) \\
&= \sum_{\alpha=1}^{r}\left(\frac{\partial \sigma^\alpha}{\partial t} + \sum_{\substack{1\le i\le d\\ 1\le\beta\le r}} \sigma^\beta \frac{\partial f^i}{\partial t} A^{\alpha}_{i,\beta}\right)\varepsilon_\alpha
\end{aligned}$$

In slightly simpler terms, $\frac{D}{\partial t}\sigma = \sum_{\alpha=1}^{r}\left(\frac{D}{\partial t}\sigma\right)^\alpha \varepsilon_\alpha$ where:

$$\left(\frac{D}{\partial t}\sigma\right)^\alpha = \frac{\partial \sigma^\alpha}{\partial t} + \sum_{\substack{1\le i\le d\\ 1\le\beta\le r}} \sigma^\beta \frac{\partial f^i}{\partial t} A^{\alpha}_{i,\beta}$$

Similarly, we find:

$$\left(\frac{D}{\partial s}\sigma\right)^\alpha = \frac{\partial \sigma^\alpha}{\partial s} + \sum_{\substack{1\le i\le d\\ 1\le\beta\le r}} \sigma^\beta \frac{\partial f^i}{\partial s} A^{\alpha}_{i,\beta}$$

We now have everything in place to (reluctantly) compute $\frac{D}{\partial t}\frac{D}{\partial s}\sigma$:

$$\begin{aligned}
\left(\frac{D}{\partial t}\frac{D}{\partial s}\sigma\right)^\alpha = \frac{\partial^2 \sigma^\alpha}{\partial t \partial s} &+ \sum_{\substack{1\le i\le d\\ 1\le\beta\le r}}\left(\frac{\partial \sigma^\beta}{\partial t}\frac{\partial f^i}{\partial s} + \sigma^\beta \frac{\partial^2 f^i}{\partial t\partial s}\right)A^{\alpha}_{i,\beta} + \sum_{\substack{1\le i,j\le d\\ 1\le\beta\le r}} \sigma^\beta \frac{\partial f^i}{\partial s}\frac{\partial f^j}{\partial t}\partial_j A^{\beta}_{i,\alpha} \\
&+ \sum_{\substack{1\le i\le d\\ 1\le\beta\le r}}\left(\frac{\partial \sigma^\beta}{\partial s} + \sum_{\substack{1\le j\le d\\ 1\le\gamma\le r}} \sigma^\gamma \frac{\partial f^j}{\partial s} A^{\beta}_{j,\gamma}\right)\frac{\partial f^i}{\partial t}A^{\alpha}_{i,\beta}
\end{aligned}$$

It becomes slightly tidier when using Einstein's convention:

$$\begin{aligned}
\left(\frac{D}{\partial t}\frac{D}{\partial s}\sigma\right)^\alpha = \frac{\partial^2 \sigma^\alpha}{\partial t \partial s} &+ \left(\frac{\partial \sigma^\beta}{\partial t}\frac{\partial f^i}{\partial s} + \sigma^\beta \frac{\partial^2 f^i}{\partial t\partial s}\right)A^{\alpha}_{i,\beta} + \sigma^\beta \frac{\partial f^i}{\partial s}\frac{\partial f^j}{\partial t}\partial_j A^{\beta}_{i,\alpha} \\
&+ \frac{\partial \sigma^\beta}{\partial s}\frac{\partial f^i}{\partial t}A^{\alpha}_{i,\beta} + \sigma^\beta \frac{\partial f^i}{\partial s}A^{\gamma}_{i,\beta}\frac{\partial f^j}{\partial t}A^{\alpha}_{j,\gamma}
\end{aligned}$$

We have developed the last term and swapped the indexes β and γ as well as i and j in the very last sum, so that it will be easier to see what cancels out with $\frac{D}{\partial s}\frac{D}{\partial t}\sigma$.

For that purpose, we label the indexes differently for $\frac{D}{\partial s}\frac{D}{\partial t}\sigma$: whenever there is a sum over both i and j, we swap them. Other sums keep the same labelling.

$$\begin{aligned}\left(\frac{D}{\partial s}\frac{D}{\partial t}\sigma\right)^\alpha = \frac{\partial^2\sigma^\alpha}{\partial s\partial t} + \left(\frac{\partial\sigma^\beta}{\partial s}\frac{\partial f^i}{\partial t} + \sigma^\beta\frac{\partial^2 f^i}{\partial s\partial t}\right)A^\alpha_{i,\beta} + \sigma^\beta\frac{\partial f^j}{\partial t}\frac{\partial f^i}{\partial s}\partial_i A^\beta_{j,\alpha} \\ + \frac{\partial\sigma^\beta}{\partial t}\frac{\partial f^i}{\partial s}A^\alpha_{i,\beta} + \sigma^\beta\frac{\partial f^j}{\partial t}A^\gamma_{j,\beta}\frac{\partial f^i}{\partial s}A^\alpha_{i,\gamma}\end{aligned}$$

Out of the six terms in each sum, four will cancel out, either directly or by permuting the order of partial derivatives with respect to s and t thanks to the Schwarz Lemma. We are left with:

$$\begin{aligned}\left(\frac{D}{\partial t}\frac{D}{\partial s}\sigma - \frac{D}{\partial s}\frac{D}{\partial t}\sigma\right)^\alpha &= \sigma^\beta\frac{\partial f^i}{\partial s}\frac{\partial f^j}{\partial t}\left(\partial_j A^\beta_{i,\alpha} - \partial_i A^\beta_{j,\alpha} + A^\gamma_{i,\beta}A^\alpha_{j,\gamma} - A^\gamma_{j,\beta}A^\beta_{i,\gamma}\right) \\ &= \sigma^\beta\frac{\partial f^i}{\partial s}\frac{\partial f^j}{\partial t}F^\beta_{j,i,\alpha} \\ &= \left(F\left(\frac{\partial f}{\partial t},\frac{\partial f}{\partial s}\right)\sigma\right)^\alpha\end{aligned}$$

□

Remark. *This kind of calculation can seem a daunting task when first approaching the subject, but with time and practice they become rather easy. Clever labelling of indices is the key (e.g. note that we used Greek letters for coordinates in the fibres and Roman letters for coordinates on the base manifold).*

Note that the computation can be made much simpler if f is assumed to be an immersion. In this case, we can find local coordinates $(x^1,\dots,x^d)$ on M such that $x^1(f(t,s)) = t$ and $x^2(f(t,s)) = s$, as well as $x^i(f(t,s)) = 0$ for $i > 2$. It follows that for $\sigma \in \Gamma(\xi)$, we have $\frac{D}{\partial t}\sigma = \nabla\sigma(\partial_1)$ and $\frac{D}{\partial s}\sigma = \nabla\sigma(\partial_2)$, so using the fact that $[\partial_1,\partial_2] = 0$ we easily find $\frac{D}{\partial t}\frac{D}{\partial s}\sigma - \frac{D}{\partial s}\frac{D}{\partial t}\sigma = F(\partial_1,\partial_2)\sigma = F\left(\frac{\partial f}{\partial t},\frac{\partial f}{\partial s}\right)\sigma$. A double use of the Leibniz rule allows to replace $\sigma \in \Gamma(\xi)$ with $\sigma \in \Gamma(f^*\xi)$ by decomposing σ in a local trivializing frame of ξ (which amounts to showing that $\left[\frac{D}{\partial t},\frac{D}{\partial s}\right]$ is tensorial on $\Gamma(f^*\xi)$).

Theorem 3.4.7. *Let $\xi = (E,p,M)$ be a vector bundle. A connection ∇ on ξ is flat if and only if it is locally trivial.*

Lemma 3.4.8. *Let $\xi = (E, p, M)$ be a vector bundle and ∇ a flat connection on M. Then for any $x \in M$ and $v \in \xi_x$, there is a local parallel section σ around x such that $\sigma(x) = v$.*

Proof. To understand the idea, let us start with the two-dimensional case. Consider a local parametrization $f : \mathbb{R}^2 \to M$ such that $f(0,0) = x$. Denote by $\frac{D}{\partial t}$ (resp. $\frac{D}{\partial s}$) the intrinsic derivatives along the curves $t \mapsto f(t, s_0)$ for fixed s_0 (resp. $t \mapsto f(t_0, s)$ for fixed t_0).

Define σ in two steps:

- First define $\sigma(f(t,0))$ such that $\sigma(x) = v$ and $\frac{D}{\partial t}\sigma((f(t,0)) = 0$ for all $t \in \mathbb{R}$ (recall that from Proposition 3.1.6, there is a unique way of doing so).
- Then define $\sigma(f(t,s))$ with the value at $s = 0$ given by the first step, and so that $\frac{D}{\partial s}\sigma(f(t,s)) = 0$ for all $(t,s) \in \mathbb{R}^2$.

Smoothness of σ comes from the Cauchy-Lipschitz Theorem and the fact that $\frac{D}{\partial t}\sigma = 0$ and $\frac{D}{\partial s}\sigma = 0$ are ordinary differential equations (for exemple, in coordinates, $\frac{D}{\partial s}\sigma = 0$ writes as $\frac{\partial \sigma^\alpha}{\partial s} + A^\alpha_{2,\beta}\sigma^\beta = 0$).

We have that $\frac{D}{\partial s}\sigma = 0$, and according to Proposition 3.4.6:

$$\frac{D}{\partial s}\frac{D}{\partial t}\sigma = \frac{D}{\partial t}\frac{D}{\partial s}\sigma + F(\partial_t, \partial_s)\sigma = 0$$

For all $t_0 \in \mathbb{R}$, the ξ-valued vector field $s \mapsto \frac{D}{\partial t}\sigma(f(t_0,s))$ along the curve $s \mapsto f(t_0, s)$ is parallel, and vanishes at $s = 0$, so it must vanish identically because of the uniqueness in Proposition 3.1.6.

We now know that $\frac{D}{\partial t}\sigma = \frac{D}{\partial s}\sigma = 0$, hence $\nabla_{\partial_t}\sigma = \nabla_{\partial_s}\sigma = 0$. Tensoriality with respect to the vector field shows that $\nabla\sigma = 0$.

To proceed in arbitrary dimension, we use an induction process. For $d \geq 0$, let $\mathcal{A}(d)$ be the assertion: "Lemma 3.4.8 is true for d-dimensional manifolds". We have proved $\mathcal{A}(2)$, but note that $\mathcal{A}(0)$ and $\mathcal{A}(1)$ trivially hold.

Assume that $\mathcal{A}(d)$ is true. Consider that M has dimension $d+1$, and let $f : \mathbb{R}^{d+1} \to M$ be a local parametrization such that $f(0) = x$. We now denote by $\frac{D}{\partial x_i}$ the intrinsic derivative along the curves $t \mapsto f(x_1, \ldots, x_{i-1}, t, x_{i+1}, \ldots, x_{d+1})$ for fixed $(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_{d+1}) \in \mathbb{R}^d$.

Using the assumption $\mathcal{A}(d)$ applied to the manifold $N = f\left(\mathbb{R}^d \times \{0\}\right)$, and the pulled-back connection (which is still flat because of Proposition 3.4.6), we can define σ on N such that $\sigma(x) = v$ and $\frac{D}{\partial x_i}\sigma(y) = 0$ for all $y \in N$ and $i \in \{1, \ldots, d\}$.

We now defined σ on $f\left(\mathbb{R}^{d+1}\right)$ with given value on N and such that $\frac{D}{\partial x_{d+1}}\sigma = 0$.

Once again, the theory of ordinary diffential equations guarantees the smoothness of σ.

Just as in the 2-dimensional case, we have that $\frac{D}{\partial x_i}\sigma$ is parallel along all curves $t \mapsto f(x_1,\dots,x_d,t)$ for $(x_1,\dots,x_d) \in \mathbb{R}^d$, and vanishes at $t=0$, so it must be identically 0.

Tensoriality of ∇ with respect to the vector field once again leads to $\nabla\sigma = 0$. □

Remark. *We just solved a partial differential equation. The method we used applies to a family of equations called transport equations, or conservation laws.*

Proof of Theorem 3.4.7. Notice that if $\sigma \in \Gamma(\xi)$ is parallel, the definition of the curvature tensor directly leads to $F(X,Y)\sigma = 0$ for all $X,Y \in \mathcal{X}(M)$. Added to the tensoriality of the curvature, this shows that locally trivial connections are flat.

We now assume that ∇ is flat. Consider $x \in M$ and a basis $(e_1,\dots,e_r)$ of ξ_x. According to Lemma 3.4.8, we can define local parallel sections $\varepsilon_1,\dots,\varepsilon_r$ of ξ such that $\varepsilon_\alpha(x) = e_\alpha$ for $\alpha \in \{1,\dots,r\}$ (note that they can be defined on the same neighbourhood of x because a finite intersection of open sets is open).

According to the uniqueness in Proposition 3.3.2, we only need to check that $(\varepsilon_1,\dots,\varepsilon_r)$ is a frame field on a neighbourhood of x. For this consider a local frame field $(\delta_1,\dots,\delta_r)$, and write $\varepsilon_\alpha = \sum_{\beta=1}^r A_\alpha^\beta \delta_\beta$. Since $(A_\alpha^\beta(x))_{1\le\alpha,\beta\le r} \in \mathrm{GL}(r,\mathbb{R})$ and $\mathrm{GL}(r,\mathbb{R})$ is open in $\mathcal{M}(r,\mathbb{R})$, we have that $(A_\alpha^\beta(y))_{1\le\alpha,\beta\le r} \in \mathrm{GL}(r,\mathbb{R})$ for y sufficiently close to x, so $(\varepsilon_1(y),\dots,\varepsilon_r(y))$ is a vector basis of ξ_y. □

3.4.2 Flat connections and representations of the fundamental group

The parallel transport of a flat connection only depends on the homotopy class.

Proposition 3.4.9. *Let $\xi = (E,p,M)$ be a vector bundle, and ∇ a flat connection on ξ. Let $x,y \in M$, and consider smooth curves $c_1,c_2 : [0,1] \to M$ such that $c_0(0) = c_1(0) = x$ and $c_0(1) = c_1(1) = y$. If c_1 and c_2 are homotopic, then their parallel transports from ξ_x to ξ_y are equal.*

Proof. Since c_1 and c_2 are homotopic, there is a smooth map $f : U \to M$ where $U \subset \mathbb{R}^2$ is open and contains $[0,1]^2$, such that $f(\cdot,0) = c_0$, $f(\cdot,1) = c_1$, $f(0,s) = x$ and $f(1,s) = y$ for all s. We use the notations of Proposition 3.4.6.

Consider $v \in \xi_x$, and define $\sigma \in \Gamma(f^*\xi)$ in the following way:

- First define $\sigma(t,0)$ that is parallel along c_0 such that $\sigma(0,0) = v$.
- Then define $\sigma(t,s)$ that is parallel along the curves $f(t,\cdot)$ for fixed t, with the value $\sigma(t,0)$ given in the first step.

Smoothness of σ comes from the Cauchy-Lipschitz Theorem and the fact that $\frac{D}{\partial t}\sigma = 0$ and $\frac{D}{\partial s}\sigma = 0$ are ordinary differential equations (for exemple, in coordinates, $\frac{D}{\partial s}\sigma = 0$ writes as $\frac{\partial \sigma^\alpha}{\partial s} + A^\alpha_{2,\beta}\sigma^\beta = 0$).

By definition, we have that $\frac{D}{\partial s}\sigma = 0$ on U and $\frac{D}{\partial t}\sigma(t,0) = 0$. According to Proposition 3.4.6, we find:

$$\frac{D}{\partial s}\frac{D}{\partial t}\sigma = \frac{D}{\partial t}\frac{D}{\partial s}\sigma + F\left(\frac{\partial f}{\partial t}, \frac{\partial f}{\partial s}\right)\sigma = 0$$

It follows that $\frac{D}{\partial t}\sigma$ is parallel along the curves $f(t,\cdot)$. Since it vanishes at $s = 0$, it must be identically 0, i.e. $\frac{D}{\partial t}\sigma = 0$ on U. Therefore $\sigma(\cdot,1)$ is parallel along c_1, so the parallel transport of v along c_1 is equal to $\sigma(1,1)$. By definition of σ, the parallel transport of v along c_0 is equal to $\sigma(1,0)$.

Since $\sigma(1,s) \in T_yM$ and $\frac{D}{\partial s}\sigma = 0$, we find that $\sigma(1,\cdot)$ is constant, hence $\sigma(1,0) = \sigma(1,1)$.

□

A first consequence is that flat connections on vector bundles over simply connected manifolds only exist on trivialisable vector bundles.

Corollary 3.4.10. *Let $\xi = (E,p,M)$ be a vector bundle, and ∇ a flat connection on ξ. If M is simply connected, then ξ is trivialisable.*

Proof. For $x,y \in M$, we let $\|_x^y : \xi_x \to \xi_y$ be the parallel transport along any curve joining x and y (according to Proposition 3.4.9, it does not depend on the choice of such a curve because M is simply connected). Fix some $o \in M$, and let $(e_1,\dots,e_r)$ be a vector basis of ξ_o. For $x \in M$, we write $\varepsilon_i(x) = \|_o^x(e_i)$, so that $(\varepsilon_1(x),\dots,\varepsilon_r(x))$ is a vector basis of ξ_x.

We need to check that the sections ε_i are smooth (so that $(\varepsilon_1,\dots,\varepsilon_r)$ is a frame field of M, and ξ is trivialisable). The same arguments as in the proofs of Proposition 3.4.9, Theorem 3.4.7 and Lemma 3.4.8, which rely on locally choosing curves that depend smoothly on x, apply. □

A straightforward consequence of Corollary 3.4.10 is that $T\mathbb{S}^2$ admits no flat connection.

We now wish to discuss the relationship between flat connections on vector bundles over a manifold M and linear representations of the fundamental group $\pi_1(M)$, i.e. group homomorphisms $\rho : \pi_1(M) \to \mathrm{GL}(r,\mathbb{R})$ (or $\mathrm{GL}(r,\mathbb{C})$ for complex vector bundles).

Start with a vector bundle $\xi = (E,p,M)$ and a flat connection ∇ on ξ. Fix some $x \in M$. For $\gamma \in \pi_1(M)$, we set $\rho(\gamma) \in \mathrm{GL}(\xi_x)$ to be the parallel transport along a closed curve based at x representing γ (according to Proposition 3.4.9, it only depends on the homotopy class γ). Because of the semi-group property of parallel transport, this gives a group representation $\rho : \pi_1(M) \to \mathrm{GL}(\xi_x)$, called the **holonomy representation**. Note it is well defined up to conjugacy: if we choose a different point $y \in M$, then the representations are conjugated by the parallel transport along any curve from x to y.

Any representation of $\rho : \pi_1(M) \to \mathrm{GL}(V)$ can be obtained in this way. Consider the action of $\pi_1(M)$ on $\widetilde{M} \times V$ defined by $\gamma.(x,v) = (\gamma.x, \rho(\gamma)v)$. It is free and properly discontinuous (because $\pi_1(M) \curvearrowright \widetilde{M}$ is). We can consider the quotient manifold $E = (\widetilde{M} \times V)/\pi_1(M)$, and the map $p : E \to M$ defined by $p((x,v) \bmod \pi_1(M)) = \pi(x)$ (where $\pi : \widetilde{M} \to M$ is the universal covering map). One can check that $\xi = (E,p,M)$ is a vector bundle. Since the canonical flat connection on the trivial bundle $\widetilde{M} \times V$ is invariant under $\pi_1(M)$, it descends to a flat connection on ξ, whose holonomy representation is ρ.

3.4.3 Torsion of a connection on a manifold

So far, we have worked on arbitrary tangent bundles, and done most computations using local trivializing frames. In the specific case of the tangent bundle of a manifold, it is often convenient to choose frame fields coming from local coordinates, i.e. $(\partial_1, \dots, \partial_d)$.

For example, given a flat connection on TM, one can ask if there are local coordinates for which the associated frame field is parallel. Once again, the answer is no, and the obstruction is encoded in a tensor.

In terms of calculation rules, a connection ∇ on the tangent bundle TM of M allows to compare $\nabla_X Y$ and $\nabla_Y X$ for vector fields $X, Y \in \mathcal{X}(M)$. Let us calculate the difference for the trivial connection.

If $(x^1, \dots, x^d)$ are local coordinates and D is the trivial connection associated to the frame field $(\partial_1, \dots, \partial_d)$ of TM, then for $X = \sum_{i=1}^n X^i \partial_i$ and $Y = \sum_{i=1}^n Y^i \partial_i$, the formula for D is:

$$DY(X) = \sum_{1 \le i,j \le d} X^j \partial_j Y^i \partial_i$$

We get:

$$DY(X) - DX(Y) = \sum_{1 \le i,j \le d} \left(X^j \partial_j Y^i - Y^j \partial_j X^i \right) \partial_i = [X, Y]$$

This formula, $DY(X) - DX(Y) = [X,Y]$, does not hold for an arbitrary connection, but the failure of this formula defines a tensor, called the torsion.

Lemma 3.4.11. *Let M be a manifold, and ∇ a connection on TM. The map $T : \mathcal{X}(M) \times \mathcal{X}(M) \to \mathcal{X}(M)$ defined by:*

$$T(X,Y) = \nabla Y(X) - \nabla X(Y) - [X,Y]$$

is tensorial and skew-symmetric.

Proof. Skew-symmetry is a consequence of skew-symmetry of the Lie bracket. Consider $f \in \mathcal{C}^\infty(M)$. Recall that:

$$[fX, Y] = f[X, Y] - (Y \cdot f)X$$

We get:

$$\begin{aligned} T(fX, Y) &= \nabla Y(fX) - \nabla[fX](Y) - [fX, Y] \\ &= f\nabla Y(X) - f\nabla X(Y) - (Y \cdot f)X - \Big(f[X, Y] - (Y \cdot f)X\Big) \\ &= f\nabla Y(X) - f\nabla X(Y) - f[X, Y] \\ &= fT(X, Y) \end{aligned}$$

Skew-symmetry implies tensoriality with respect to Y. □

Definition 3.4.12. *Let M be a manifold, and ∇ a connection on TM. The **torsion** of ∇ is the tensor $T \in \Gamma\left(\Lambda^2 T^*M \otimes TM\right)$ such that, for all $X, Y \in \mathcal{X}(M)$, we have:*

$$T(X, Y) = \nabla Y(X) - \nabla X(Y) - [X, Y]$$

*We say that ∇ is **torsion free** if T vanishes identically.*

The absence of torsion should be seen as a much weaker condition that flatness. One of the many reasons is the general existence (and abundance) of torsion free connections.

Proposition 3.4.13. *Let M be a manifold, ∇ a connection on TM, and T its torsion. The connection $\nabla' = \nabla - \frac{1}{2}T$ on TM is torsion free.*

Remark. *Since $T \in \Gamma\left(\Lambda^2 T^*M \otimes TM\right) \subset \Gamma(T^*M \otimes \mathrm{End}(TM))$, this expression does define a connection ∇' thanks to Proposition 3.3.7. The formula is:*

$$\forall X, Y \in \mathcal{X}(M) \quad \nabla' Y(X) = \nabla Y - \frac{1}{2}T(X, Y)$$

Proof. This is a simple computation, using the skew symmetry of T.

$$\begin{aligned} \nabla' Y(X) - \nabla' X(Y) &= \nabla Y(X) - \frac{1}{2}T(X, Y) - \nabla X(Y) + \frac{1}{2}T(Y, X) \\ &= \nabla Y(X) - \nabla X(Y) - T(X, Y) \\ &= [X, Y] \end{aligned}$$

□

Let us look at the expression of the torsion in local coordinates $(x^1, \ldots, x^d)$. For $X = \sum_{i=1}^n X^i \partial_i$ and $Y = \sum_{i=1}^n Y^i \partial_i$, we get:

$$\begin{aligned} T(X, Y) &= \sum_{1 \le i, j \le d} X^i Y^j T(\partial_i, \partial_j) \\ &= \sum_{1 \le i < j \le d} \left(X^i Y^j - X^j Y^i\right) T(\partial_i, \partial_j) \end{aligned}$$

We can compute $T(\partial_i, \partial_j)$ by using the connection form in the trivializing frame $(\partial_1, \dots, \partial_d)$ (i.e. $\nabla_{\partial_i} \partial_j = \sum_{k=1}^d A_{i,j}^k \partial_k$):

$$T(\partial_i, \partial_j) = \nabla \partial_j(\partial_i) - \nabla \partial_i(\partial_j) - \underbrace{[\partial_i, \partial_j]}_{=0}$$
$$= \sum_{k=1}^{d} \left(A_{i,j}^k - A_{j,i}^k \right) \partial_k$$

So we find $T(\partial_i, \partial_j) = \sum_{k=1}^d T_{i,j}^k \partial_k$ where:

$$T_{i,j}^k = A_{i,j}^k - A_{j,i}^k$$

Lemma 3.4.14. *Let M be a manifold, and ∇ a torsion free connection on TM. For every $x \in M$, there is a local coordinate system $(x^1, \dots, x^d)$ of M around x such that $A_{i,j}^k(x) = 0$ for all $i, j, k \in \{1, \dots, d\}$.*

Proof. Consider a local coordinate system $(y^1, \dots, y^d)$ on an open set $V \subset M$ centered at x (i.e. $y^1(x) = \cdots = y^d(x) = 0$). Write $\overline{A}_{i,j}^k$ the components of the connection form in these coordinates. Consider functions f_i^k on M for $i, k \in \{1, \dots, d\}$ such that $f_i^k(x) = 0$ and $d_x f_i^k = \sum_{j=1}^d \overline{A}_{i,j}^k(x) d_x y^j$.

Since ∇ is torsion free, the derivatives of the functions f_i^k satisfy the following formula:

$$\frac{\partial f_i^k}{\partial y^j} = \overline{A}_{i,j}^k = \overline{A}_{j,i}^k = \frac{\partial f_j^k}{\partial y^i}$$

This means that the differential 1-forms $\alpha^1, \dots, \alpha^d \in \Omega^1(U)$ defined by $\alpha^k = \sum_{i=1}^d f_i^k dy^i$ are closed. Let $U \subset V$ be an open set diffeomorphic to $\mathbb{R}^d$ that contains x. Poincaré's Lemma gives the existence of functions $z^1, \dots, z^d$ on U such that $dz^k = \alpha^k$.

Note that $\alpha_x^k = 0$, hence $d_x z^k = 0$. By setting $x^i = y^i + z^i$, the Local Inverse Function Theorem ensures that up to shrinking U, the map $(x^1, \dots, x^d)$ is a coordinate system on U.

Note that $\frac{\partial x^i}{\partial y^j}(x) = \frac{\partial y^i}{\partial x^j}(x) = \delta_i^j$. It follows from Lemma 3.3.13 that:

$$\overline{A}_{i,j}^k(x) = \frac{\partial^2 x^k}{\partial y^j \partial y^i}(x) + A_{i,j}^k(x) = \frac{\partial f_i^k}{\partial y^j}(x) + A_{i,j}^k(x) = \overline{A}_{i,j}^k(x) + A_{i,j}^k(x)$$

It follows that $A_{i,j}^k(x) = 0$. □

The torsion being a point-wise invariant, we can turn this into a point-wise result.

Proposition 3.4.15. *Let M be a manifold, ∇ a connection on TM and T its torsion. Given $x \in M$, the following are equivalent:*

1. *The torsion vanishes at x: $\forall u, v \in T_xM \ T_x(u,v) = 0$.*
2. *There is a local coordinate system $(x^1, \dots, x^d)$ of M around x such that $A^k_{i,j}(x) = 0$ for all $i, j, k \in \{1, \dots, d\}$.*

Proof. The formula given for the components of the torsion in coordinates show that *2.⇒1.*
If $T_x = 0$, we consider the torsion free connection ∇' given by Proposition 3.4.13. According to Lemma 3.4.14, there is a local coordinate system $(x^1, \dots, x^d)$ of M around x such that $A'^k_{i,j}(x) = 0$ for all $i, j, k \in \{1, \dots, d\}$, where the $A'^k_{i,j}$ are the components of the connection form of ∇'. The definition of ∇' gives the following relationship between the connection forms A of ∇ and that of ∇':

$$A'^k_{i,j}(x) = A^k_{i,j}(x) - \frac{1}{2} T^k_{i,j}(x)$$

It follows that $A^k_{i,j}(x) = 0$. □

Proposition 3.4.16. *Let M be a manifold, ∇ a connection on TM, and T its torsion.*
Given a smooth map $f : U \to M$ where $U \subset \mathbb{R}^2$, we define by:

- *$\frac{D}{\partial t}$ the intrinsic derivatives along the curves $t \mapsto f(t, s_0)$ for fixed s_0.*
- *$\frac{D}{\partial s}$ the intrinsic derivatives along the curves $s \mapsto f(t_0, s)$ for fixed t_0.*

We have that:

$$\frac{D}{\partial t}\frac{\partial f}{\partial s} - \frac{D}{\partial s}\frac{\partial f}{\partial t} = T\left(\frac{\partial f}{\partial t}, \frac{\partial f}{\partial s}\right)$$

Proof. Consider local coordinates $(x^1, \dots, x^d)$ on M. Write $f^i(t,s) = x^i(f(t,s))$ for $i \in \{1, \dots, d\}$ and $(t,s) \in U$, and decompose:

$$\frac{\partial f}{\partial t} = \sum_{i=1}^{d} \frac{\partial f^i}{\partial t} \partial_i \quad ; \quad \frac{\partial f}{\partial s} = \sum_{i=1}^{d} \frac{\partial f^i}{\partial s} \partial_i$$

The calculations made in the proof of Proposition 3.4.6 (replacing ε_α with ∂_i) translate as:

$$\frac{D}{\partial t}\partial_i = \sum_{1 \le j,k \le d} \frac{\partial f^j}{\partial t} A^k_{j,i} \partial_k \quad ; \quad \frac{D}{\partial s}\partial_i = \sum_{1 \le j,k \le d} \frac{\partial f^j}{\partial s} A^k_{j,i} \partial_k$$

We find:

$$\begin{aligned}
\frac{D}{\partial t}\frac{\partial f}{\partial s} &= \frac{D}{\partial t}\left(\sum_{i=1}^{d}\frac{\partial f^i}{\partial t}\partial_i\right)\\
&= \sum_{i=1}^{d}\left(\frac{\partial^2 f^i}{\partial t\partial s}\partial_i + \frac{\partial f^i}{\partial s}\frac{D}{\partial t}\partial_i\right)\\
&= \sum_{i=1}^{d}\left(\frac{\partial^2 f^i}{\partial t\partial s}\partial_i + \frac{\partial f^i}{\partial s}\sum_{1\le j,k\le d}\frac{\partial f^j}{\partial t}A^k_{j,i}\partial_k\right)\\
&= \sum_{i=1}^{d}\frac{\partial^2 f^i}{\partial t\partial s}\partial_i + \sum_{1\le i,j,k\le d}\frac{\partial f^i}{\partial s}\frac{\partial f^j}{\partial t}A^k_{j,i}\partial_k
\end{aligned}$$

We swap i and j in the second sum for the expression of $\frac{D}{\partial s}\frac{\partial f}{\partial t}$:

$$\frac{D}{\partial s}\frac{\partial f}{\partial t} = \sum_{i=1}^{d}\frac{\partial^2 f^i}{\partial s\partial t}\partial_i + \sum_{1\le i,j,k\le d}\frac{\partial f^j}{\partial t}\frac{\partial f^i}{\partial s}A^k_{i,j}\partial_k$$

Combining the two, we get:

$$\begin{aligned}
\frac{D}{\partial t}\frac{\partial f}{\partial s} - \frac{D}{\partial s}\frac{\partial f}{\partial t} &= \sum_{1\le i,j,k\le d}\frac{\partial f^i}{\partial s}\frac{\partial f^j}{\partial t}\left(A^k_{j,i} - A^k_{i,j}\right)\partial_k\\
&= \sum_{1\le i,j,k\le d}\frac{\partial f^i}{\partial s}\frac{\partial f^j}{\partial t}T^k_{j,i}\partial_k\\
&= T\left(\frac{\partial f}{\partial t},\frac{\partial f}{\partial s}\right)
\end{aligned}$$

□

Theorem 3.4.17. *Let M be a manifold and ∇ a connection on M. The following are equivalent:*

1. *∇ is flat and torsion free.*
2. *Around every point in M, there is a local coordinate system $(x^1,\dots,x^d)$ such that $\nabla\partial_i = 0$ for all $i \in \{1,\dots,d\}$.*

Proof. We have seen that the trivial connection associated to a coordinate system is torsion free and flat, so *2.⇒1.*

To show *1.⇒2.*, we fix $x \in M$ and use Theorem 3.4.7 that provides a local trivializing frame $(\varepsilon_1,\dots,\varepsilon_d)$ of TM around x such that $\nabla\varepsilon_i = 0$ for all

$i \in \{1,\ldots,d\}$.

Consider the flows $\varphi^t_{\varepsilon_i}$ of the vector fields ε_i, and define the map $f : U \to M$ on a small neighbourhood $U \subset \mathbb{R}^d$ of 0 by:

$$f(x^1,\ldots,x^d) = \varphi^{x^1}_{\varepsilon_1} \circ \cdots \circ \varphi^{x^d}_{\varepsilon_d}(x)$$

Since $d_0 f(v^1,\ldots,v^d) = v^1\varepsilon_1(x) + \cdots + v^d\varepsilon_d(x)$, the Inverse Function Theorem states that f is a diffeomorphism from a neighbourhood of 0 in $\mathbb{R}^d$ to a neighbourhood of x in M. Consider the local coordinates $(x^1,\ldots,x^d)$ on M defined by $f^{-1}(y) = (x^1(y),\ldots,x^d(y))$.

For $i,j \in \{1,\ldots,d\}$, the vector fields ε_i and ε_j commute:

$$[\varepsilon_i,\varepsilon_j] = \nabla_{\varepsilon_i}\varepsilon_j - \nabla_{\varepsilon_j}\varepsilon_i + T(\varepsilon_i,\varepsilon_j) = 0$$

It follows that the flows also commute, and therefore $\partial_i = \varepsilon_i$ is parallel. □

Remark. *If the flows do not commute, there is no reason for ∂_i to be equal to ε_i.*

3.4.4 Torsion and symmetry of the Hessian

Consider a manifold M. Recall that a connection ∇ on TM also defines a connection ∇^* on the cotangent bundle T^*M (the data of one or the other are equivalent). This allows us to define a second order differential for smooth functions of M.

Definition 3.4.18. *Let M be a manifold, ∇^* a connection on T^*M and $f \in \mathcal{C}^\infty(M)$. The **Hessian** of f with respect to ∇^* is the tensor $\mathrm{Hess}(f) = \nabla^* df \in \Omega^1(T^*M) = \Gamma(T^*M \otimes T^*M)$.*

Let us try to understand this definition. For $x \in M$ and $u \in T_xM$, we can define $\nabla^*_x(df)(u) \in T^*_xM$, and apply this linear form to some $v \in T_xM$ to get $\mathrm{Hess}(f)(u,v) = \left(\nabla^*_x df(u)\right)(v)$.

If ∇^* is the dual connection of ∇, then for $X, Y \in \mathcal{X}(M)$, we find:

$$\mathrm{Hess}(f)(X,Y) = d\left(df(X)\right)(Y) - df\left(\nabla X(Y)\right)$$

This formula can be understood as a product rule for differentiating the function $df(X)$: we add a term that differentiates df (the Hessian $\mathrm{Hess}(f)(X,Y)$) and a term that differentiates X (the second term $df\left(\nabla X(Y)\right)$).

Proposition 3.4.19. *Let M be a manifold, ∇ a connection on TM and ∇^* its dual connection on T^*M. The following assertions are equivalent:*

1. *The connection ∇ is torsion free.*

2. *For all $f \in \mathcal{C}^\infty(M)$, the Hessian of f with respect to ∇^* is symmetric.*

Proof. For $X, Y \in \mathcal{X}(M)$, we have:

$$\mathrm{Hess}(f)(X,Y) = d\Big(df(X)\Big)(Y) - df\Big(\nabla X(Y)\Big)$$

This leads to:

$$\mathrm{Hess}(f)(X,Y) - \mathrm{Hess}(f)(Y,X) = df\Big(T(X,Y)\Big)$$

The abundance of smooth functions allows to prove the equivalence. □

3.4.5 Bianchi identities

Most torsion free connections are not flat. However, vanishing of the torsion adds a type of symmetry on the curvature tensor.

Proposition 3.4.20 (First Bianchi identity)**.** *Let M be a manifold and ∇ a torsion free connection on TM. For $x \in M$ and $u, v, w \in T_xM$, we have that:*

$$F_x(u,v)w + F_x(v,w)u + F_x(w,u)v = 0$$

Remark. *The first Bianchi identity is also called the algebraic Bianchi identity.*

Proof. Consider a smooth map $f : \left\{ \begin{array}{ccc} \mathbb{R}^3 & \to & M \\ (t,s,r) & \mapsto & f(t,s,r) \end{array} \right.$ such that $f(0,0,0) = x$, $\frac{\partial f}{\partial t}(0,0,0) = u$, $\frac{\partial f}{\partial s}(0,0,0) = v$ and $\frac{\partial f}{\partial r}(0,0,0) = w$. This is always possible, e.g. by choosing a diffeomorphism $\varphi : \mathbb{R}^d \to U$ where $U \subset M$ is open, and $\varphi(0) = x$, then setting:

$$f(t,s,r) = \varphi\Big(td_x\varphi^{-1}(u) + sd_x\varphi^{-1}(v) + rd_x\varphi^{-1}(w)\Big)$$

According to Proposition 3.4.6, we have:

$$F\left(\frac{\partial f}{\partial t}, \frac{\partial f}{\partial s}\right)\frac{\partial f}{\partial r} = \frac{D}{\partial t}\frac{D}{\partial s}\frac{\partial f}{\partial r} - \frac{D}{\partial s}\frac{D}{\partial t}\frac{\partial f}{\partial r}$$

Since ∇ is torsion free, we can change the second term by means of Proposition 3.4.16:

$$F\left(\frac{\partial f}{\partial t}, \frac{\partial f}{\partial s}\right)\frac{\partial f}{\partial r} = \frac{D}{\partial t}\frac{D}{\partial s}\frac{\partial f}{\partial r} - \frac{D}{\partial s}\frac{D}{\partial r}\frac{\partial f}{\partial t}$$

Cyclically permuting the indexes and summing, each term on the right hand side appears twice, once with a positive sign and once with a negative sign, so they cancel each other out. In the end, we find:

$$F\left(\frac{\partial f}{\partial t}, \frac{\partial f}{\partial s}\right)\frac{\partial f}{\partial r} + F\left(\frac{\partial f}{\partial s}, \frac{\partial f}{\partial r}\right)\frac{\partial f}{\partial t} + F\left(\frac{\partial f}{\partial r}, \frac{\partial f}{\partial t}\right)\frac{\partial f}{\partial s} = 0$$

Evaluation at $(0,0,0)$ offers the conclusion. □

Recall that a connection ∇ on a vector bundle ξ defines connections, still denoted by ∇, on the dual bundle ξ^* and all tensor products (Proposition 3.3.6).

In particular, if ∇ is a connection on TM, then the curvature tensor $F \in \Omega^2(\mathrm{End}(TM))$ is a section of $\Lambda^2 T^*M \otimes \mathrm{End}(TM)$, so we can define $\nabla F \in \Omega^1(\Lambda^2 T^*M \otimes \mathrm{End}(TM))$.

Let us review the notation: given $x \in M$ and $u \in T_xM$, we can define $\nabla_x F(u)$ which is the same type of tensor as F_x, i.e. $\nabla_x F(u) \in \Lambda^2 T_x^*M \otimes \mathrm{End}(T_xM)$, so we use the same notation $\nabla_x F(u)(v,w) \in \mathrm{End}(T_xM)$ for $v,w \in T_xM$.

The very definition of the associated connection on $\Lambda^2 T^*M \otimes \mathrm{End}(TM)$ allows us to use of product rule for differentiating. Namely, given vector fields $X,Y,Z,W \in \mathcal{X}(M)$, if we want to differentiate the vector field $R(X,Y)Z \in \mathcal{X}(M)$ in the direction W, i.e. calculate

$$\nabla\Big(R(X,Y)Z\Big)(W)$$

then the result is a sum of terms where we differentiate each term of the expression $R(X,Y)Z$ at a time, i.e. consider $\nabla X(W)$, $\nabla Y(W)$, $\nabla Z(W)$ as well as $\nabla R(W)$:

$$\begin{aligned}\nabla\Big(R(X,Y)Z\Big)(W) = \nabla R(W)(X,Y)Z \\ + R(\nabla X(W),Y)Z + R(X,\nabla Y(W))Z \\ + R(X,Y)[\nabla Z(W)]\end{aligned}$$

Proposition 3.4.21 (Second Bianchi identity)**.** *Let M be a manifold and ∇ a torsion free connection on TM. For $x \in M$ and $u,v,w \in T_xM$, we have that:*

$$\nabla_x R(u)(v,w) + \nabla_x R(v)(w,u) + \nabla_x R(w)(u,v) = 0$$

Remarks.

- *The second Bianchi identity is also called the differential Bianchi identity.*
- *This is an equation in* $\mathrm{End}(T_xM)$.

Proof. Consider once again a smooth map $f : \left\{\begin{array}{ccc} \mathbb{R}^3 & \to & M \\ (t,s,r) & \mapsto & f(t,s,r)\end{array}\right.$ such that $f(0,0,0) = x$, $\frac{\partial f}{\partial t}(0,0,0) = u$, $\frac{\partial f}{\partial s}(0,0,0) = v$ and $\frac{\partial f}{\partial r}(0,0,0) = w$.

Let $z \in T_xM$, and consider a section $\sigma \in \Gamma(f^*TM)$ (i.e. a smooth map $\sigma : \mathbb{R}^3 \to TM$ such that $\sigma(t,s,r) \in T_{f(t,s,r)}M$ for all $(t,s,r) \in \mathbb{R}^3$) such that $\sigma(0,0,0) = z$ (existence of σ is guaranteed by Lemma 2.2.4).

$$
\begin{aligned}
\left(\frac{D}{\partial t}R\right)\left(\frac{\partial f}{\partial s},\frac{\partial f}{\partial r}\right)\sigma &= \frac{D}{\partial t}\left(R\left(\frac{\partial f}{\partial s},\frac{\partial f}{\partial r}\right)\sigma\right) - R\left(\frac{D}{\partial t}\frac{\partial f}{\partial s},\frac{\partial f}{\partial r}\right)\sigma \\
&\qquad - R\left(\frac{\partial f}{\partial s},\frac{D}{\partial t}\frac{\partial f}{\partial r}\right)\sigma - R\left(\frac{\partial f}{\partial s},\frac{\partial f}{\partial r}\right)\frac{D}{\partial t}\sigma \\
&= \frac{D}{\partial t}\left[\frac{D}{\partial s},\frac{D}{\partial r}\right]\sigma - R\left(\frac{D}{\partial t}\frac{\partial f}{\partial s},\frac{\partial f}{\partial r}\right)\sigma \\
&\qquad - R\left(\frac{\partial f}{\partial s},\frac{D}{\partial t}\frac{\partial f}{\partial r}\right)\sigma - \left[\frac{D}{\partial s},\frac{D}{\partial r}\right]\frac{D}{\partial t}\sigma \\
&= \left[\frac{D}{\partial t},\left[\frac{D}{\partial s},\frac{D}{\partial r}\right]\right]\sigma - R\left(\frac{D}{\partial t}\frac{\partial f}{\partial s},\frac{\partial f}{\partial r}\right)\sigma \\
&\qquad + R\left(\frac{D}{\partial r}\frac{\partial f}{\partial t},\frac{\partial f}{\partial s},\right)\sigma
\end{aligned}
$$

Cyclically permuting the indexes and summing, each term involving R on the right hand side appears twice, once with a positive sign and once with a negative sign, so they cancel each other out. The other terms sum up to zero because of the Jacobi identity (for linear endomorphisms of the vector space $\Gamma(f^*TM)$). Evaluating at $(0,0,0)$ offers the conclusion.

□

The second Bianchi identity is related to the fact that the curvature, when seen as a 2-form with values in $\mathrm{End}(TM)$, is closed. This is actually true for connections on general vector bundles (so torsion free is not necessary). In order to make sense of this, we need to work with differential forms with values in a vector bundle and consider the exterior covariant derivative.

Proposition 3.4.22. *Let $\xi = (E,p,M)$ be a vector bundle, and ∇ a connection on ξ. For $p \in \mathbb{N}$, and $\omega \in \Omega^p(\xi)$, the map $d^\nabla : \mathcal{X}(M)^{p+1} \to \Gamma(\xi)$ defined by:*

$$\forall (X_0,\dots,X_p) \in \mathcal{X}(M)^{p+1}$$

$$
\begin{aligned}
d^\nabla\omega(X_0,\dots,X_p) &= \sum_{0\le i\le p}(-1)^i\nabla\Big(\omega(X_0,\dots,\widehat{X_i},\dots,X_p)\Big)(X_i) \\
&\quad + \sum_{0\le i<j\le p}(-1)^{i+j}\omega([X_i,X_j],X_0,\dots,\widehat{X_i},\dots,\widehat{X_j},\dots,X_p)
\end{aligned}
$$

is tensorial and skew-symmetric

Definition 3.4.23. *Let $\xi = (E,p,M)$ be a vector bundle, and ∇ a connection on ξ. For $p \in \mathbb{N}$, the **exterior covariant derivative** is the map*

$$d^\nabla : \Omega^p(\xi) \to \Omega^{p+1}(\xi)$$

defined by:

$$\forall \omega \in \Omega^p(\xi)\ \forall X_0, \dots, X_p \in \mathcal{X}(M)$$

$$\begin{aligned} d^\nabla \omega(X_0, \dots, X_p) = &\sum_{0 \le i \le p} (-1)^i \nabla\left(\omega(X_0, \dots, \widehat{X_i}, \dots, X_p)\right)(X_i) \\ &+ \sum_{0 \le i < j \le p} (-1)^{i+j} \omega([X_i, X_j], X_0, \dots, \widehat{X_i}, \dots, \widehat{X_j}, \dots, X_p) \end{aligned}$$

For $p = 0$, i.e. $\omega \in \Gamma(\xi)$, we simply have $d^\nabla \omega = \nabla \omega$.
For a ξ-valued differential 1-form $\omega \in \Omega^1(\xi)$, the expression is:

$$d^\nabla \omega(X, Y) = \nabla(\omega(Y))(X) - \nabla(\omega(X))(Y) - \omega([X, Y])$$

Proposition 3.4.24. *Let $\xi = (E, p, M)$ be a vector bundle, ∇ a connection on M, and F its curvature.*

1. $\forall X, Y \in \mathcal{X}(M)\ \forall \sigma \in \Gamma(\xi)\quad F(X, Y)\sigma = d^\nabla\left(d^\nabla \sigma\right)(X, Y)$

2. $d^\nabla F = 0$

Remarks.

- *We will see later that 2. is equivalent to the second Bianchi identity.*

- *The exterior covariant derivative $d^\nabla F$ in 2. is the exterior covariant derivative for* End(ξ)*-valued differential forms, defined by the connection on* End(ξ) *induced by ∇ (i.e. $\forall u \in \Gamma(\mathrm{End}(\xi)) \forall \sigma \in \Gamma(\xi)\ (\nabla u)\sigma = \nabla(u(\sigma)) - u(\nabla \sigma)$).*

Proof.

1. Just apply the formulas given above for d^∇ on ξ-valued 0-forms and 1-forms.

2. Start by noticing that for $\omega \in \Omega^p(\mathrm{End}(\xi))$ and $\sigma \in \Gamma(\xi)$, we have that:

$$\left(d^\nabla \omega\right)(\sigma) = d^\nabla(\omega(\sigma)) - \omega\left(d^\nabla \sigma\right)$$

Applied to $F \in \Omega^2(\mathrm{End}(\xi))$, we find:

$$\begin{aligned} \left(d^\nabla F\right)(\sigma) &= d^\nabla(F(\sigma)) - F\left(d^\nabla \sigma\right) \\ &= d^\nabla\left(d^\nabla \circ d^\nabla(\sigma)\right) - d^\nabla \circ d^\nabla\left(d^\nabla \sigma\right) \\ &= 0 \end{aligned}$$

□

Let us now see how Proposition 3.4.24 gives another proof of Proposition 3.4.21.

Consider $X, Y, Z \in \mathcal{X}(M)$ such that $X(x) = u$, $Y(x) = v$ and $Z(x) = w$. Note that up to working locally, we can choose X, Y, Z that pairwise commute (say by choosing them to be constant in some local coordinates).

To make the expressions lighter, we fix $W \in \mathcal{X}(M)$, and set:

$$\mathcal{B}(X,Y,Z)W = \nabla F(X)(Y,Z)W + \nabla F(Y)(Z,X)W + \nabla F(Z)(X,Y)W$$

So we wish to show that $\mathcal{B}(X,Y,Z)W = 0$.

By definition of the exterior covariant derivative, the expression of the $\mathrm{End}(TM)$-valued differential 3-form $d^\nabla R$ is:

$$d^\nabla F(X,Y,Z) = \nabla\Big(F(Y,Z)\Big)(X) + \nabla\Big(F(Z,X)\Big)(Y) + \nabla\Big(F(X,Y)\Big)(Z)$$

Now $\nabla\Big(F(Y,Z)\Big)(X)$ is a section of $\mathrm{End}(TM)$. Its definition is:

$$\begin{aligned}\nabla\Big(F(Y,Z)\Big)(X)W &= \nabla\Big(F(Y,Z)W\Big)(X) - F(Y,Z)[\nabla W(X)]\\ &= \nabla F(X)(Y,Z)W + F(\nabla Y(X),Z)W + F(Y,\nabla Z(X))W\end{aligned}$$

Since $d^\nabla F = 0$, we find:

$$\begin{aligned}-\mathcal{B}(X,Y,Z)W = F(\nabla Y(X),Z)W &+ F(Y,\nabla Z(X))W\\ &+ F(\nabla Z(Y),X)W + F(Z,\nabla X(Y))W\\ &+ F(\nabla X(Z),Y)W + F(X,\nabla Y(Z))W\end{aligned}$$

Since ∇ is torsion free, we can simplify this expression:

$$-\mathcal{B}(X,Y,Z)W = F([X,Y],Z) + F([Z,X],Y) + F([Y,Z],X)$$

Since we have chosen X, Y, Z that pairwise commute, we get $\mathcal{B}(X,Y,Z)W = 0$.

3.5 Geodesics of a connection

3.5.1 The geodesic equation

Definition 3.5.1. *Let M be a manifold and ∇ a connection on TM. A **geodesic** of ∇ is a smooth curve $c : I \to M$ such that $\frac{D}{dt}\dot{c} = 0$.*

Remarks.

- *This notion is only well defined for connections on the tangent bundle TM.*
- *In local coordinates $(x^1,\ldots,x^d)$, the equation becomes:*

$$\forall k \in \{1,\ldots,d\} \;\; \ddot{c}^k(t) + \sum_{1\leq i,j\leq d} A^k_{i,j}(c(t))\dot{c}^i(t)\dot{c}^j(t) = 0$$

  It is a second order ordinary differential equation, non linear except in some exceptional situations.

- *The equation depends on the parameterization of the curve, not only on its image in M. For $M = \mathbb{R}^d$ and the trivial connection, geodesics are solution to $\ddot{c} = 0$, i.e. affinely parameterized straight lines.*

3.5.2 The exponential map of a connection

According to the theory of second order ordinary differential equations, a point $x \in M$ and a tangent vector $v \in T_xM$ define a unique maximal solution $c_v : I_v \to M$ such that $c_v(0) = x$ and $\dot{c}_v(0) = v$.

Because of uniqueness, we have that $I_{sv} = s^{-1}I_v$ for $s \in \mathbb{R}^*$, and $c_{sv}(t) = c_v(st)$.

Definition 3.5.2. *Let M be a manifold and ∇ a connection on TM. The* ***exponential map*** *of ∇ is the map*

$$\exp : \begin{cases} \{v \in TM \,|\, 1 \in I_v\} & \to & M \\ v & \mapsto & c_v(1) \end{cases}$$

We say that ∇ is ***complete*** *if* exp *is defined on all of TM.*
For $x \in M$ we set $\exp_x = \exp|_{T_xM}$.

According to the theory of second order ordinary differential equations, the domain on which exp is defined is an open subset of TM (containing the zero section), and exp is smooth.

Proposition 3.5.3. *Let M be a manifold and ∇ a connection on TM. For all $x \in M$, we have that* $d_0 \exp_x = \mathrm{Id}_{T_xM}$.

Remarks.

- *Since* $\exp_x$ *is defined on an open set of the vector space T_xM, its differential* $d_0 \exp_x$ *is defined on $T_0(T_xM) = T_xM$.*
- *According to the Local Inverse Function Theorem,* $\exp_x$ *defines a diffeomorphism from a neighbourhood of* 0 *in T_xM to a neighbourhood of x in M, hence local coordinates around x.*

Proof. Since $\exp_x(tv) = c_{tv}(1) = c_v(t)$, we get:

$$d_0 \exp_x(v) = \left.\frac{d}{dt}\right|_{t=0} \exp_x(tv) = \dot{c}_v(0) = v$$

□

For the trivial connection on $\mathbb{R}^d$, the exponential map $\exp : T\mathbb{R}^d = \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d$ is simply $\exp(x,v) = x + v$.

Proposition 3.5.4. *Let M be a manifold and ∇ a connection on TM. For all $x_0 \in M$, there exist an open neighbourhood $U \subset M$ of x_0, and a smooth map $\varphi : U \times U \to TM$ such that:*

- $\forall x, y \in U \ \varphi(x,y) \in T_xM$
- $\exp_x(\varphi(x,y)) = y$

Proof. Since the result is local, we may assume that M is an open subset of $\mathbb{R}^d$. Consider the map

$$F : \left\{ \begin{array}{ccc} M \times M \times \mathbb{R}^d & \to & \mathbb{R}^d \\ (x,y,v) & \mapsto & \exp_x(v) - y \end{array} \right.$$

The partial differential with respect to v at $(x_0, x_0, 0)$ is the identity map of $\mathbb{R}^d$, so there is an open set $U \subset M$ containing x_0 and a smooth map $\varphi : U \times U \to \mathbb{R}^d$ such that $F(x,y,\varphi(x,y)) = 0$, i.e. $\exp_x(\varphi(x,y)) = y$. □

3.5.3 Jacobi fields

We now wish to compute the differential of the exponential map at an arbitrary point. Since there is no explicit formula for the exponential map itself, we cannot expect an explicit formula for its differential. However, it can also be defined as the solution of a second order ordinary differential equation, which is linear (this is actually a general fact for the differential of the flow of a vector field).

Definition 3.5.5. *Let M be a manifold, ∇ a connection on TM and $c : I \to M$ be a smooth curve. A **Jacobi field along** c is a vector field J along c such that:*

$$\frac{D}{dt}\frac{D}{dt}J + F(J,\dot{c})\dot{c} = 0$$

where F is the curvature of ∇.

Remarks.

- *By vector field along c, we mean TM-valued vector field along c.*

- *This definition makes sense for any curve c, however it will only be useful when c is a geodesic.*

Since Jacobi fields are defined by a linear differential equation, solutions exist and are given by initial data consisting of the value and the derivative at a point.

Proposition 3.5.6. *Let M be a manifold, ∇ a connection on TM and $c : I \to M$ be a smooth curve. For every $t_0 \in I$ and $J_0, \dot{J}_0 \in T_{c(t_0)}M$, there is a unique Jacobi field J along c such that $J(t_0) = J_0$ and $\frac{D}{dt}J(t_0) = \dot{J}_0$.*

Proof. Let $(\varepsilon_1, \dots, \varepsilon_d)$ be a parallel frame along c. Decompose:

$$F(\varepsilon_i, \dot{c})\dot{c} = \sum_{j=1}^{d} f_i^j \varepsilon_j$$

for some functions $f_i^j \in \mathcal{C}^\infty(I)$. Also decompose $J_0 = \sum_{i=1}^d J_0^i \varepsilon_i(t_0)$ and $\dot{J}_0 = \sum_{i=1}^d \dot{J}_0^i \varepsilon_i(t_0)$.

Then $J = \sum_{i=1}^d J^i \varepsilon_i$ is a Jacobi field if and only if:

$$\forall i \in \{1, \dots, d\} \quad \ddot{J}^i + \sum_{j=1}^{d} f_j^i J^j = 0$$

According to the Cauchy-Lipschitz Theorem for linear equations, there is a unique solution $(J^i)_{1 \le i \le d} \in \mathcal{C}^\infty(I)^d$ such that $J^i(t_0) = J_0^i$ and $\dot{J}^i(t_0) = \dot{J}_0^i$ for all $i \in \{1, \dots, d\}$, i.e. a unique Jacobi field $J \in \Gamma(c^*TM)$ such that $J(t_0) = J_0$ and $\frac{D}{dt}J(t_0) = \dot{J}_0$. □

Let us now see how Jacobi fields are related to variations of geodesics.

Lemma 3.5.7. *Let M be a manifold and ∇ a torsion free connection on TM. Let $f : U \to M$ be a smooth map where $U \subset \mathbb{R}^2$. Assume that for all s_0, the curve $t \mapsto f(t, s_0)$ is a geodesic. Then for all s_0, the vector field $t \mapsto \frac{\partial f}{\partial s}(t, s_0)$ along the geodesic $t \mapsto f(t, s_0)$ is a Jacobi field.*

Proof. As usual we define:

- $\frac{D}{\partial t}$ the intrinsic derivatives along the curves $t \mapsto f(t, s_0)$ for fixed s_0.
- $\frac{D}{\partial s}$ the intrinsic derivatives along the curves $s \mapsto f(t_0, s)$ for fixed t_0.

Since ∇ is torsion free, Proposition 3.4.16 affirms that $\frac{D}{\partial t}\frac{\partial f}{\partial s} = \frac{D}{\partial s}\frac{\partial f}{\partial t}$. Now, we use Proposition 3.4.6 and the fact that $f(\cdot, s)$ is a geodesic (which translates as $\frac{D}{\partial t}\frac{\partial f}{\partial t} = 0$), to calculate:

$$\begin{aligned}\frac{D}{\partial t}\frac{D}{\partial t}\frac{\partial f}{\partial s} &= \frac{D}{\partial t}\frac{D}{\partial s}\frac{\partial f}{\partial t}\\ &= \frac{D}{\partial s}\underbrace{\frac{D}{\partial t}\frac{\partial f}{\partial t}}_{=0} + F\left(\frac{\partial f}{\partial t}, \frac{\partial f}{\partial s}\right)\frac{\partial f}{\partial t}\\ &= F\left(\frac{\partial f}{\partial t}, \frac{\partial f}{\partial s}\right)\frac{\partial f}{\partial t}\end{aligned}$$

□

Let us see how to use this to compute the differential of the exponential map.

Proposition 3.5.8. *Let M be a manifold and ∇ a torsion free connection on TM. Let $x \in M$ and $v, w \in T_xM$. If $\exp_x$ is defined at v, then:*

$$d_v \exp_x(w) = J(1)$$

where J is the Jacobi field along the geodesic c_v such that $J(0) = 0$ and $\frac{D}{dt}J(0) = w$.

Proof. Consider the map

$$f : \begin{cases} U & \to & M \\ (t,s) & \mapsto & \exp_x\big(t(v+sw)\big) \end{cases}$$

defined on an open set $U \subset \mathbb{R}^2$ which contains $[0,1] \times \{0\}$. Since $f(1,s) = \exp_x(v+sw)$, we wish to compute $d_v \exp_x(w) = \frac{\partial f}{\partial s}(1,0)$. In order to do so, we consider the vector field $t \mapsto \frac{\partial f}{\partial s}(t,0)$ along the curve $f(\cdot,0) = c_v$.

According to Lemma 3.5.7, it is a Jacobi field. Let us calculate its intial data. First, we have that $f(0,s) = x$ for all s, so $\frac{\partial f}{\partial s}(0,0) = 0$.

To get $\frac{D}{\partial t}\frac{\partial f}{\partial s}(0,0)$, we notice that $s \mapsto f(0,s)$ is a constant curve, so the intrinsic derivative $\frac{D}{\partial s}$ along this curve is just the usual derivative of functions to T_xM. Since $\frac{\partial f}{\partial t}(0,s) = \dot{c}_{v+sw}(0) = v + sw$, we get $\frac{D}{\partial s}\frac{\partial f}{\partial t}(0,0) = w$, hence $\frac{D}{\partial t}\frac{\partial f}{\partial s}(0,0) = w$. So $t \mapsto \frac{\partial f}{\partial s}(t,0)$ is the Jacobi field J defined above. □

3.6 Ehresmann connections

3.6.1 Vertical and horizontal bundles

So far we have only considered connections on vector bundles. It is possible to define a notion of connection on general fibre bundles. They always provide a way of differentiating sections, but do not always produce sections of the same bundle (there is no reason for this to happen if fibres are not vector spaces).

Definition 3.6.1. *Let $\xi = (E, p, M, F)$ be a fibre bundle. Denote by $\varphi_E : TE \to E$ and $\pi_M : TM \to M$ the canonical projections. The* ***vertical bundle*** *of ξ is the vector sub-bundle V of TE defined by $V_z = \ker d_z p = T_z \xi_{p(z)}$ for all $z \in E$.*
A ***horizontal bundle****, also called an* ***Ehresmann connection****, is a vector sub-bundle H of TE such that $H \oplus V = TE$.*

Remark. *The vector bundles V and H are vector bundles above the total space E of ξ.*

3.6.2 Covariant derivatives and Ehresmann connections

Let us see how a connection ∇ on a vector bundle $\xi = (E, p, M)$ defines an Ehresmann connection H^∇. First, consider $z \in E$, and let $x = p(z) \in M$. Define H_z^∇ to be the space of derivatives of parallel ξ-valued vector fields along curves passing though x with value z:

$$H_z^\nabla = \left\{ \dot\sigma(0) \middle| c \in \mathcal{C}^\infty(\mathbb{R}, M),\ c(0) = x,\ \sigma \in \Gamma(c^*\xi),\ \sigma(0) = z,\ \frac{D}{dt}\sigma = 0 \right\}$$

Consider local coordinates $(x^1, \dots, x^d)$ on a neighbourhood U of x and a local frame field $(\varepsilon_1, \dots, \varepsilon_r)$ for $\xi|_U$. This defines coordinates $(x^1, \dots, x^d, v^1, \dots, v^r)$ on $p^{-1}(U)$.

A vector W in H_z^∇ is the derivative at 0 of a curve $\gamma(t)$ in E whose coordinates $(c^1(t), \dots, c^d(t), \sigma^1(t), \dots, \sigma^r(t))$ satisfy $\dot\sigma^\alpha \varepsilon_\alpha + A_{i,\alpha}^\beta \circ c \sigma^\alpha \dot c^i \varepsilon_\beta = 0$. This means that $W = X^1 \frac{\partial}{\partial x^1} + \dots + X^d \frac{\partial}{\partial x^d} + V^1 \frac{\partial}{\partial v^1} + \dots + V^r \frac{\partial}{\partial v^r}$ satisfies the equations:

$$V^\alpha + \sum_{i,\beta} A_{i,\beta}^\alpha(x) v^\beta X^i = 0$$

Reciprocally, if $W \in T_z E$ satisfies this equation, then $W \in H_z^\nabla$. This shows that H^∇ is a sub-bundle of TE, and that $H \oplus V = TE$ (V is spanned by $\frac{\partial}{\partial v^1}, \dots, \frac{\partial}{\partial v^r}$). Therefore H^∇ is an Ehresmann connection.

However all the properties of an affine connection are not encoded in an Ehresmann connection, so we need to impose a condition on Ehresmann connection that is linked to the structural group of vector bundles.

Definition 3.6.2. *Let $\xi = (E, p, M)$ be a vector bundle. An Ehresmann connection H on ξ is called* ***linear*** *if $d_v m_\lambda(H_v) = H_{\lambda v}$ for all $v \in E$ and $\lambda \in \mathbb{R}$ (where $m_\lambda : E \to E$ is fibrewise multiplication by λ).*

Most of the litterature adds a condition of invariance by sum, however it is implied by the scaling invariance.

Theorem 3.6.3. *Let $\xi = (E, p, M)$ be a vector bundle. The map $\nabla \mapsto H^\nabla$ is a bijection from the set of affine connections to the set of linear Ehresmann connections.*

Proof. Linearity of H^∇ comes from the linearity of $\frac{D}{dt}$. A ξ-valued vector field σ along a curve c is parallel if and only if $\frac{d}{dt}\sigma \in H^\nabla$. It follows that H^∇ determines the parallel transport, hence the injectivity.

To prove the surjectivity, let H be a linear Ehresmann connection. Consider $P \in \Gamma(\mathrm{End}(TE))$ the projection from TE to V parallel to H. For $v \in E$, the vertical space $V_v = T_v\xi_{p(v)}$ is the tangent space of a vector space, so it identifies to $\xi_{p(v)}$. For $\sigma \in \Gamma(\xi)$, define $\nabla_x\sigma(v) = P_v(d_x\sigma(v))$. The Leibniz rule is a consequence of linearity of P, which follows from that of H (V is also linear). □

3.6.3 Curvature of Ehresmann connections

Note that an Ehresmann connection is always characterized by a field of projections $P \in \Gamma(\mathrm{End}(TE))$ on the vertical sub-bundle. Alternatively, one can see P as a TE valued 1-form, i.e. $P \in \Omega^1(TE)$.

A notion of curvature can be defined for Ehresmann connections, as a TE-valued 2-form on E. For $X, Y \in \Gamma(TE)$, define $F(X,Y) = P([X - P(X), Y - P(Y)])$. One can check that it is tensorial, so it defines $F \in \Omega^2(TE)$.

Notice that F is exactly the obstruction for H to be integrable in the Frobenius Theorem, so flat connections are exactly integrable ones.

For linear Ehresmann connections, this curvature is linked to the previously defined one by identifying TM with H.

3.6.4 Principal connections

For a fibre bundle with a reduction of the structure group, one can always define Ehresmann connections that are invariant under this restriction. For example, there is a notion of invariant Ehresmann connections on a H-principal bundle $\xi = (E, p, M, H)$, i.e. invariant under the action of H on E. Such a connection is also characterized by a simpler object, namely a **principal connection**, i.e. a $\mathfrak{h}$-valued 1-form $A \in \Omega^1(E, \mathfrak{h})$ that is equivariant for the actions of H (right-action on E, and Ad on $\mathfrak{h}$), and defines an isomorphism between the vertical sub-bundle and $\mathfrak{h}$. The curvature is then defined by $F = dA + \frac{1}{2}[A, A]$ where $[A, A]$ is the $\mathfrak{h}$-valued 2-form defined by $[A, A](u, v) = [A(u), A(v)]$.