Contents

Chapter 8

Pseudo-Riemannian manifolds

8.1 Metrics on vector bundles

8.1.1 Euclidean metrics

Definition 8.1.1. *Let $\xi = (E, p, M)$ be a real vector bundle of rank r. A* ***Euclidean metric*** *on ξ is a section $h \in \Gamma(S^2\xi^*)$ such that for all $x \in M$, the map $h_x : \xi_x \times \xi_x \to \mathbb{R}$ is an inner product.*

Proposition 8.1.2. *Euclidean metrics exist on every real vector bundle.*

Proof. Let $\xi = (E, p, M)$ be a real vector bundle of rank r, and consider an open cover $\mathcal{U}$ of M such that $\xi|_U$ is trivialisable for every $U \in \mathcal{U}$. Consider a partition of unity $(\varphi_U)_{U\in\mathcal{U}}$ subordinate to $\mathcal{U}$, and for each $U \in \mathcal{U}$ let $\varepsilon^U = (\varepsilon_1^U, \ldots, \varepsilon_r^U)$ be a frame field of $\xi|_U$. Now $h = \sum_{U\in\mathcal{U}} \varphi_U \sum_{\alpha=1}^r (e_\alpha^U)^* \otimes (e_\alpha^U)^*$ is a Euclidean metric on ξ. □

Definition 8.1.3. *Let $\xi = (E, p, M)$ be a real vector bundle of rank r, and h a Euclidean metric on ξ. An* ***orthonormal frame field*** *is a frame field $(\varepsilon_1, \ldots, \varepsilon_r)$ such that for all $x \in M$, the basis $(\varepsilon_1(x), \ldots, \varepsilon_r(x))$ of ξ_x is orthonormal for h_x.*

Proposition 8.1.4. *Let $\xi = (E, p, M)$ be a real vector bundle of rank r, and h a Euclidean metric on ξ. If ξ possesses a frame field, then it also possesses an orthonormal frame field.*

Proof. Consider an arbitrary frame field, and apply the Gram-Schmidt process on every fibre. Since the operations involved in this process are algebraic, they are smooth and the result is an orthonormal frame field. □

Proposition 8.1.5. *Let $\xi = (E, p, M)$ be a real vector bundle of rank r, and h a Euclidean metric on ξ. If η is a vector subbundle of ξ, then $\eta^\perp$ defined by $(\eta^\perp)_x = (\eta_x)^\perp$ for every $x \in M$ is a vector subbundle of ξ, supplementary to η.*

Proof. Consider a local frame field $(\varepsilon_1,\dots,\varepsilon_r)$ of ξ such that $(\varepsilon_1,\dots,\varepsilon_k)$ is a frame field of η. Applying the Gram-Schmidt process, we find an orthonormal frame field $(\delta_1,\dots,\delta_r)$ such that $(\delta_1,\dots,\delta_k)$ is still a frame field of η. It follows that $(\delta_{k+1},\dots,\delta_r)$ is a frame field of $\eta^\perp$. □

Proposition 8.1.6. *Let $\xi = (E,p,M)$ be a real vector bundle of rank r. The data of a Euclidean metric on ξ is equivalent to the data of a reduction of the structural group of ξ to* $\mathrm{O}(r,\mathbb{R})$.

Proof. Given a Euclidean metric on ξ, the trivialisations given by local orthonormal frame fields form a reduction of the structural group to $\mathrm{O}(r,\mathbb{R})$.

Given a reduction of the structural group to $\mathrm{O}(r,\mathbb{R})$, the inner product computed in a trivialisation belonging to this reduction of the structural group does not depend on said trivialisation, so it defines a Euclidean metric on ξ. □

Proposition 8.1.7. *Let $\xi = (E,p,M)$ be a real vector bundle of rank r, and h a Euclidean metric on ξ. For $x \in M$, we let $U\xi_x = \{v \in \xi_x \,|\, h_x(v,v) = 1\}$ and $UE = \bigcup_{x\in M} U\xi_x$. Then $(UE, p|_{UE}, M, \mathbb{S}^{r-1})$ is a fibre subbundle of ξ.*

Proof. Let $x \in M$, and let $(\varepsilon_1,\dots,\varepsilon_r)$ be a local orthonormal frame of ξ defined on some open set $U \subset M$ containing x. For $y \in U$, define

$$\theta_y : \left\{ \begin{array}{ccc} \mathbb{R}^r & \to & \xi_y \\ (v^1,\dots,v^r) & \mapsto & v^1\varepsilon_1(y) + \dots + v^r\varepsilon_r(y) \end{array} \right.$$

It is a local trivialisation of ξ sending $\mathbb{S}^{r-1} \subset \mathbb{R}^r$ diffeomorphically to $U\xi_y$ for all $y \in U$. □

Definition 8.1.8. *Let $\xi = (E,p,M)$ be a real vector bundle of rank r, and h a Euclidean metric on ξ. The fibre bundle $U\xi = (UE, p|_{UE}, M, \mathbb{S}^{r-1})$ is called the* ***unit bundle****.*

An inner product $\langle\cdot|\cdot\rangle$ on a finite dimension real vector space V induces an inner product on V^*, using the isomorphism between V and V^* obtained by sending $v \in V$ to $(w \mapsto \langle v|w\rangle) \in V^*$.

Inner products on two vector finite dimensional vector spaces $(V, \langle\cdot|\cdot\rangle)$ and $(W,(\cdot,\cdot))$ also induce a unique inner product $<< \cdot \| \cdot >>$ on $V \otimes W$ satisfying $<< v_1 \otimes w_1 \| v_2 \otimes w_2 >>= \langle v_1|v_2\rangle (w_1,w_2)$ for all pure tensors.

Put together, we find inner products on all tensor powers $(V^*)^{\otimes p} \otimes V^{\otimes q}$. The identification of $V^* \otimes V$ with $\mathrm{End}(V)$ gives the usual product $\langle f|g\rangle = \mathrm{Tr}(f^t g)$.

Similarly, a Euclidean metric h on a vector bundle ξ induces Euclidean metrics on all tensor powers $(\xi^*)^{\otimes p} \otimes \xi^{\otimes q}$.

A Euclidean metric h on a vector bundle $\xi = (E,p,M)$ allows us to define semi-norms on $\Gamma(\xi)$ (or a norm when M is compact): given a compact subset $K \subset M$, define $\|\sigma\|_{\infty,K} = \sup_{x\in K} \sqrt{h_x(\sigma(x),\sigma(x))}$. The topology associated to this family of semi-norms is the compact-open topology (it does not depend on the Euclidean metric h).

8.1.2 Pseudo-Euclidean metrics

Definition 8.1.9. *Let $\xi = (E,p,M)$ be a vector bundle of rank r, and $p,q \in \mathbb{N}$ be such that $p+q = r$. A **pseudo-Euclidean metric of signature** (p,q) on ξ is a section $h \in \Gamma(S^2\xi^*)$ such that h_x has signature (p,q) for every $x \in M$.*

Note that given a fixed signature, a pseudo-Euclidean metrics does not always exist.

A big difference with Euclidean metrics is that the restriction of a pseudo-Euclidean metric to a vector subbundle is not necessarily a pseudo-Euclidean metric. However, if it is, then we can still define the orthogonal complement.

Proposition 8.1.10. *Let $\xi = (E,p,M)$ be a vector bundle of rank r, and h a pseudo-Riemannian metric on ξ. Let η be a vector subbundle of ξ such that for all $x \in M$, the restriction of h_x to $\eta_x \times \eta_x$ is non degenerate. Then $\eta^\perp$ defined by $(\eta^\perp)_x = (\eta_x)^\perp$ for every $x \in M$ is a vector subbundle of ξ, supplementary to η.*

Proof. Use the corresponding statement for quadratic vector spaces and copy the proof of Proposition 8.1.5. □

Let us recall some basic facts on quadratic forms over real vector spaces.

Definition 8.1.11. *Let V, V' be vector spaces, and $\varphi : V \times V \to \mathbb{R}$, $\varphi' : V' \times V' \to \mathbb{R}$ be bilinear forms. We say that φ and φ' are equivalent if there is a linear isomorphism $f : V \to V'$ such that $\varphi'(f(x), f(y)) = \varphi(x,y)$ for all $x,y \in V$.*

Given integers $p,q,r \in \mathbb{N}$, we let $\langle\cdot|\cdot\rangle_{p,q,r}$ be the bilinear form on $\mathbb{R}^{p+q+r}$ defined by $\langle x|y\rangle_{p,q,r} = \sum_{i=1}^{p} x_i y_i - \sum_{i=p+1}^{p+q} x_i y_i$ for $x,y \in \mathbb{R}^{p+q+r}$.

Theorem 8.1.12 (Sylvester's inertia law)**.** *Let V be a finite dimensional real vector space, and $\varphi : V \times V \to \mathbb{R}$ a symmetric bilinear form. There is a unique triple (p,q,r), called the **signature** of φ, such that φ is equivalent to $\langle\cdot|\cdot\rangle_{p,q,r}$.*

Lemma 8.1.13. *Let V be a finite dimensional real vector space, and φ a symmetric bilinear form on V of signature (p,q,r). Then φ is non degenerate if and only if $r = 0$.*

For non degenerate forms, we will call (p,q) the signature.

Definition 8.1.14. *Let V be a finite dimensional vector space, and φ a symmetric bilinear form on V. A linear map $u \in \mathrm{End}(V)$ is called φ-self adjoint if $\varphi(x,u(y)) = \varphi(u(x),y)$ for all $x,y \in V$.*

Proposition 8.1.15. *Let V be a finite dimensional vector space, and φ a non degenerate symmetric bilinear form on V. If B is a symmetric bilinear form on V, then there is a unique φ-self adjoint operator $b \in \mathrm{End}(V)$ such that $B(x,y) = \varphi(x, b(y))$ for all $x, y \in V$.*

Definition 8.1.16. *Let V be a finite dimensional vector space, and φ a non degenerate symmetric bilinear form on V. If B is a symmetric bilinear form on V, then the* ***trace of** B **with respect to** φ *is $\mathrm{Tr}_\varphi(B) = \mathrm{Tr}(b)$ where $b \in \mathrm{End}(V)$ is the φ-self adjoint operator such that $B(x,y) = \varphi(x, b(y))$ for all $x, y \in V$.*

Recall that the matrix of a bilinear form φ in a basis $e = (e_1, \ldots, e_d)$ is $(\varphi(e_i, e_j))_{1 \le i,j \le d}$.

Proposition 8.1.17. *Let V be a finite dimensional vector space, and φ, B symmetric bilinear forms on V. Assume that φ is non degenerate. Let $e = (e_1, \ldots, e_d)$ be a basis of V, on consider the matrices P and Q respectively of φ and B in e. Then $\mathrm{Tr}_\varphi(B) = \mathrm{Tr}\left(QP^{-1}\right)$.*
In particular, if φ is positive definite and e is φ-orthonormal, then $\mathrm{Tr}_\varphi(B) = \mathrm{Tr}(Q)$.

8.1.3 Hermitian metrics on complex vector bundles

Definition 8.1.18. *Let $\xi = (E, p, M)$ be a complex vector space of rank r. A* ***Hermitian metric*** *on ξ is a section $h \in \Gamma(\overline{V}^* \otimes V^*)$ such that h_x is a Hermitian inner product on ξ_x for every $x \in M$.*

Proposition 8.1.19. *Every complex vector bundle possesses a Hermitian metric.*

Proof. Let $\xi = (E, p, M)$ be a complex vector bundle of rank r, and consider an open cover $\mathcal{U}$ of M such that $\xi|_U$ is trivialisable for every $U \in \mathcal{U}$. Consider a partition of unity $(\varphi_U)_{U \in \mathcal{U}}$ subordinate to $\mathcal{U}$, and for each $U \in \mathcal{U}$ let $\varepsilon^U = (\varepsilon^U_1, \ldots, \varepsilon^U_r)$ be a frame field of $\xi|_U$. Now $h = \sum_{U \in \mathcal{U}} \varphi_U \sum_{\alpha=1}^r \overline{(e^U_\alpha)^*} \otimes (e^U_\alpha)^*$ is a Hermitian metric on ξ. □

8.2 Metrics on manifolds

8.2.1 Pseudo-Riemannian metrics

Definition 8.2.1. *Let M be a manifold of dimension d.*
A ***Riemannian metric*** *on M is a Euclidean metric on TM, i.e. a section $g \in \Gamma(S^2 T^*M)$ such that for every $x \in M$, g_x is an inner product T_xM.*
A ***pseudo-Riemannian metric*** *of signature (p,q) (with $p + q = d$) is a pseudo-Euclidean metric of signature (p,q) on TM, i.e. a section $g \in \Gamma(S^2 T^*M)$ such that for every $x \in M$, g_x is non-degenerate and has signature (p,q).*
A ***Lorentzian metric*** *is a pseudo-Riemannian metric of signature $(d-1, 1)$.*

*A **(pseudo-)Riemannian manifold** is a pair (M,g) where M is a manifold and g is a (pseudo-)Riemannian metric on M.*

Notation: Given a pseudo-Riemannian metric g, we write $g_x(v,w) = \langle v|w\rangle_x = \langle v|w\rangle$ for $x \in M$ and $v,w \in T_xM$. If g is Riemannian, we write $\|v\| = \|v\|_x = \sqrt{\langle v|v\rangle_x}$.

Proposition 8.2.2. *Every manifold has a Riemannian metric.*

This is a consequence of the existence of Euclidean metrics on vector bundles (Proposition 8.1.2).

It does not hold for Lorentzian metrics: the sphere $\mathbb{S}^2$ has no Lorentzian metric (it is a consequence of the hairy ball theorem).

Before we move on any further with pseudo-Riemannian manifolds, let us start with the most basic example. On $\mathbb{R}^{p+q}$, we consider the bilinear form:

$$\langle x\big|y\rangle_{p,q} = \sum_{i=1}^{p} x_i y_i - \sum_{i=p+1}^{p+q} x_i y_i$$

Definition 8.2.3. *The **pseudo-Euclidean space** of signature (p,q) is the pseudo-Riemannian manifold $\mathbb{R}^{p,q} = (\mathbb{R}^{p+q}, g)$ where $g_x = \langle\cdot|\cdot\rangle_{p,q}$ for all $x \in \mathbb{R}^{p+q}$.*
*For $q = 0$, $\mathbb{E}^p = \mathbb{R}^{p,0}$ is called the **Euclidean space**.*
*For $q = 1$, $\mathbb{M}^n = \mathbb{R}^{n-1,1}$ is called the **Minkowski space**.*

Another elementary way of producing pseudo-Riemannian manifolds is through products. If (M,g) and (M,g') are pseudo-Riemannian manifolds, then we can define the **product pseudo-Riemannian manifold** $(M \times M', g \oplus g')$ where the metric is defined by:

$$(g \oplus g')_{(x,x')}((v,v'),(w,w')) = g_x(v,w) + g'_{x'}(v',w')$$

Note that if (M,g) and (M',g') are Riemannian, then so is the product.

This construction generalizes to the product of a finite number of pseudo-Riemannian manifolds.

8.2.2 Local expression of a pseudo-Riemannian metric

Let (M,g) be a pseudo-Riemannian manifold of dimension d. Given local coordinates $(x^1,\ldots,x^d)$, for $u = \sum_{i=1}^d u^i \partial_i$ and $v = \sum_{i=1}^d v^i \partial_i$, we find $g_x(u,v) = \sum_{1\le i,j\le d} g_{i,j}(x) u^i v^j$, where $g_{i,j}(x) = g_x(\partial_i, \partial_j)$. We write:

$$g = \sum_{1\le i,j\le d} g_{i,j}(x)\, dx^i dx^j$$

Here we use the notation $dx^i dx^j = \frac{1}{2}\left(dx^i \otimes dx^j + dx^j \otimes dx^i\right)$.

It is quite frequent to see the notation ds^2 for a pseudo-Riemannian metric, especially in coordinates:

$$ds^2 = \sum_{1\leq i,j\leq d} g_{i,j}(x)dx^i dx^j$$

Given another coordinate system $(y^1,\dots,y^d)$, if we write $g'_{i,j}$ the metric in these coordinates, i.e. $g'_{i,j}(x) = g_x\left(\frac{\partial}{\partial y^i}, \frac{\partial}{\partial y^j}\right)$, the formula for the coordinate change is:

$$g'_{i,j} = \sum_{1\leq k,l\leq d} \frac{\partial x^k}{\partial y^i}\frac{\partial x^l}{\partial y^j} g_{k,l}$$

8.2.3 Isometric maps

Definition 8.2.4. *Let (M,g) and (M',g') be pseudo-Riemannian manifolds. A smooth map $f : M \to M'$ is an* ***isometric immersion*** *if $f^*g' = g$.*
It is called an ***isometry*** *if it is also a diffeomorphism.*
A ***local isometry*** *is a local diffeomorphism which is an isometric immersion.*

Note that an isometric immersion is indeed an immersion, since $\ker d_x f \subset \ker g_x$. In particular, its existence implies that $\dim M' \geq \dim M$.

Definition 8.2.5. *Let (M,g) be a pseudo-Riemannian manifold. The* ***isometry group*** *of (M,g) is:*

$$\mathrm{Isom}(M,g) = \{f \in \mathrm{Diff}(M) \,|\, f^*g = g\}$$

The isometry group is a subgroup of $\mathrm{Diff}(M)$, and we will discuss its topology later. It is important however to understand that even though most of the examples that we will work on have many isometries, a typical pseudo-Riemannian manifold (i.e. a generic metric for an appropriate topology on the set of metrics) has no non trivial isometry. Indeed, the equation $f^*g = g$ where the map f is the unknown is an overdetermined partial differential equation.

Example: Given $\gamma \in \mathrm{O}(p,q)$ and $v \in \mathbb{R}^{p+q}$, the affine map $x \mapsto \gamma x + v$ is an isometry of $\mathbb{R}^{p,q}$. This means that the group $\mathrm{O}(p,q) \ltimes \mathbb{R}^{p+q}$ of affine transformations whose linear part is in $\mathrm{O}(p,q)$ is a subgroup of $\mathrm{Isom}(\mathbb{R}^{p,q})$.

Proposition 8.2.6. $\mathrm{Isom}(\mathbb{R}^{p,q}) = \mathrm{O}(p,q) \ltimes \mathbb{R}^{p+q}$

Proof. Let $f \in \mathrm{Isom}(\mathbb{R}^{p,q})$. This means that for all $x,u,v \in \mathbb{R}^{p+q}$, we have:

$$\langle d_x f(u) | d_x f(v)\rangle_{p,q} = \langle u|v\rangle_{p,q} \tag{8.1}$$

Let us differentiate this expression with respect to x at some vector w.

$$\left\langle d_x^2 f(u,w) \middle| d_x f(v) \right\rangle_{p,q} + \left\langle d_x f(u) \middle| d_x^2 f(v,w) \right\rangle_{p,q} = 0 \tag{8.2}$$

The same formula remains true when switching u and w.

$$\left\langle d_x^2 f(w,u) \middle| d_x f(v) \right\rangle_{p,q} + \left\langle d_x f(w) \middle| d_x^2 f(v,u) \right\rangle_{p,q} = 0 \tag{8.3}$$

Since f is smooth, $d^2 f_x$ is symmetric, and subtracting (8.3) from (8.2) yields:

$$\left\langle d_x f(u) \middle| d_x^2 f(w,u) \right\rangle_{p,q} = \left\langle d_x f(w) \middle| d_x^2 f(v,u) \right\rangle_{p,q} \tag{8.4}$$

Now a cyclic permutation of u, v, w in (8.2) gives:

$$\left\langle d_x^2 f(v,u) \middle| d_x f(w) \right\rangle_{p,q} + \left\langle d_x f(v) \middle| d_x^2 f(w,u) \right\rangle_{p,q} = 0 \tag{8.5}$$

Combining (8.4) and (8.5), we find:

$$\left\langle d_x f(u) \middle| d_x^2 f(v,w) \right\rangle_{p,q} = 0$$

Since f is a diffeomorphism, we find that $d_x^2 f = 0$, i.e. f is affine. Now (8.1) shows that the linear part of f is in $\mathrm{O}(p,q)$. □

For pseudo-Riemannian manifolds of the same dimension, another interesting type of isometric immersions are those that are covering maps.

Definition 8.2.7. *Let (M,g) and $(\widetilde{M},\widetilde{g})$ be pseudo-Riemannian manifolds. A map $f : \widetilde{M} \to M$ is called a **pseudo-Riemannian covering** if it is an isometric immersion and a covering map.*

8.3 Volume and angles

8.3.1 Pseudo-Riemannian volume

If g is a Riemannian metric, the **Riemannian volume** is the unique Borel measure Vol^g on M such that, for any chart (U,φ) and continuous function f with support in U,

$$\int_U f \, d\,\mathrm{Vol}^g = \int_{\varphi(U)} f \circ \varphi^{-1} \sqrt{\det(g \circ \varphi^{-1})} \, d\lambda$$

where λ denotes the Lebesgue measure on $\mathbb{R}^d$ et $g(x)$ est la matrix $(g_{i,j}(x))_{1 \le i,j \le d}$.

If moreover M is oriented, the **Riemannian volume form** is the volume form $\mathrm{vol}^g \in \Gamma(\Lambda^d T^*M)$ defined in oriented coordinates by:

$$\mathrm{vol}_x^g = \sqrt{\det(g(x))} \, dx^1 \wedge \cdots \wedge dx^d$$

The volume and the volume form are also defined for pseudo-Riemannian metrics by considering the absolute value of the determinant.

8.3.2 Conformal metrics

If (M, g) is a pseudo-Riemannian manifold and $\phi \in \mathcal{C}^\infty(M,]0, +\infty[)$, then ϕg is also a pseudo-Riemannian metric on M, with the same signature as g.

Definition 8.3.1. *Let M be a manifold. Two pseudo-Riemannian metrics g and g' are called* ***conformal*** *if there is $\phi \in \mathcal{C}^\infty(M,]0, +\infty[)$ such that $g' = \phi g$.*
The ***conformal class*** *of a pseudo-Riemannian metric is the set $[g] \subset \Gamma\left(S^2 T^* M\right)$ of pseudo-Riemannian metrics that are conformal to g.*

Note that two conformal metrics have the same signature. Their pseudo-Riemannian volumes are related by $\mathrm{dvol}_{g'} = \phi^{\frac{\dim M}{2}} \mathrm{dvol}_g$.

In order to understand the geometric meaning of conformal Riemannian metrics, we have to define a notion of angles in Riemannian geometry.

Definition 8.3.2. *Let (M, g) be a Riemannian manifold. Let $x \in M$, and $v, w \in T_x M \setminus \{0\}$. The* ***Riemannian angle*** *between v and w is the angle $\sphericalangle_x(v, w) \in [0, \pi]$ defined by:*

$$\cos \sphericalangle_x(v, w) = \frac{\langle v | w \rangle_x}{\|v\|_x \|w\|_x}$$

One can easily prove that two Riemannian metrics on a given manifold are conformal if and only if they define the same Riemannian angles.

For a pseudo-Riemannian manifold (M, g) of signature (p, q) with $pq \neq 0$, the situation is different. Here the conformal class is characterized by the isotropic cone: given another pseudo-Riemannian metric h on M, we find:

$$h \in [g] \quad \Longleftrightarrow \quad \forall x \in M \ \{v \in T_x M \,|\, g_x(v, v) = 0\} = \{v \in T_x M \,|\, h_x(v, v) = 0\}$$

8.4 Examples of pseudo-Riemannian manifolds

8.4.1 Pseudo-Riemannian quotients

Just as examples of covering maps can be constructed from group actions, examples of pseudo-Riemannian coverings can be constructed from isometric actions.

Theorem 8.4.1. *Let $(\widetilde{M}, \widetilde{g})$ be a pseudo-Riemannian manifold, and $\Gamma \subset \mathrm{Isom}(\widetilde{M}, \widetilde{g})$ a subgroup such that the action $\Gamma \curvearrowright \widetilde{M}$ is free and properly discontinuous. There is a unique pseudo-Riemannian metric g on $M = \widetilde{M}/\Gamma$ for which the projection $\widetilde{M} \to M$ is a pseudo-Riemannian covering.*

Proof. Let $\pi : \widetilde{M} \to M$ be the projection. Let $x \in M$, and consider a local inverse f of π. Set $g_x = (f^* \widetilde{g})_x$. Since two local inverses differ by an element of Γ, the metric does not depend on the choice of f, and defines a smooth Riemannian metric on M. $\square$

Examples:

- For any $\lambda > 0$, we can define the **circle** $\mathbb{E}^1/\lambda\mathbb{Z}$ as the quotient of $\mathbb{E}^1$ by the group generated by the translation $x \mapsto x + \lambda$.
- The **flat torus** $\mathbb{E}^2/\mathbb{Z}^2$. More generally we can define $\mathbb{E}^n/\Lambda$ where Λ is the subgroup of $\mathrm{Isom}(\mathbb{E}^n)$ generated by n linearly independent translations.
- The **Clifton-Pohl torus**, which is the quotient of $(\mathbb{R}^2\setminus\{0\}, \frac{2dxdy}{x^2+y^2})$ by the map $(x,y) \mapsto (2x, 2y)$. It is a Lorentzian manifold often used to point out the differences between Riemannian and Lorentzian geometries.

8.4.2 Pseudo-Riemannian submanifolds

Definition 8.4.2. *Let (M,g) be a pseudo-Riemannian manifold. An immersed submanifold $N \subset M$ is called a* ***pseudo-Riemannian submanifold*** *if there are integers p', q' with $p' + q' = \dim N$ such that the restriction of g to $T_xN \times T_xN$ is non degenerate and has signature (p', q') for all $x \in N$.*
It is a ***Riemannian submanifold*** *if $q' = 0$.*
If $N \subset M$ is a pseudo-Riemannian submanifold, then the pseudo-Riemannian metric defined on N by restriction of g to tangent spaces is called the ***induced metric****, or* ***restricted metric****, or the* ***first fundamental form****.*

Note that every submanifold of a Riemannian manifold is a Riemannian submanifold. In particular, every submanifold of $\mathbb{R}^d$ inherits a Riemannian metric in this way. It is the case for spheres $\mathbb{S}^n \subset \mathbb{R}^{n+1}$. The induced metric g_{sph} is called the **round metric**, or standard metric on $\mathbb{S}^n$.

If $N \subset M$ is a pseudo-Riemannian embedded submanifold of (M,g) and $f \in \mathrm{Isom}(M,g)$ preserves M, then the restriction of f to N is an isometry. However N can have many more isometries.

Applied to the sphere $\mathbb{S}^n \subset \mathbb{R}^{n+1}$, we find that $\mathrm{O}(n+1) \subset \mathrm{Isom}(\mathbb{S}^n)$. We will see later that this is an equality.

Submanifolds of a pseudo-Riemannian manifold of arbitrary signature are not always pseudo-Riemannian submanifolds, since the restriction of the metric can be degenerate. If (M,g) is a pseudo-Riemannian manifold of signature (p,q), then a pseudo-Riemannian submanifold of (M,g) can have any signature (p',q') with $p' \leq p$ and $q' \leq q$.

We will now see an example of a Riemannian submanifold of the Minkowski space $\mathbb{M}^{n+1}$. Consider one sheet of the one-sheeted hyperboloid:

$$\mathcal{H}^n = \left\{(x_1, \dots, x_{n+1}) \in \mathbb{R}^{n+1} \,\middle|\, \langle x|x\rangle_{n,1} = -1 \text{ \& } x_{n+1} > 0\right\}$$

For $x \in \mathcal{H}^n$, the tangent space is:

$$T_x\mathcal{H}^n = \left\{v \in \mathbb{R}^{n+1} \,\middle|\, \langle x|v\rangle_{n,1} = 0\right\}$$

Since $\langle x|x\rangle_{n,1} = -1$, its orthogonal complement is definite positive, i.e. $\mathcal{H}^n$ is a Riemannian submanifold of $\mathbb{M}^{n+1}$. This Riemannian manifold $\mathbb{H}^n = (\mathcal{H}^n, g_{\mathcal{H}^n})$ is called the **real hyperbolic space**.

By the same considerations as for the sphere, we find that $\mathrm{O}_+(n,1) \subset \mathbb{H}^n$, where $\mathrm{O}_+(n,1)$ is the subgroup of $\mathrm{O}(n,1)$ preserving each sheet of the two-sheeted hyperboloid $\mathcal{H}^n$ (it has index two in $\mathrm{O}(n,1)$).

8.4.3 Riemannian manifolds of dimension 1

Theorem 8.4.3. *Let (M,g) be a connected Riemannian manifold of dimension 1. Then (M,g) is isometric to an interval of $\mathbb{E}^1$ or to a circle $\mathbb{E}^1/\lambda\mathbb{Z}$.*

Proof. Consider two isometric maps $\varphi : I \to M$ and $\psi : J \to M$. Now consider the set

$$X = \left\{t \in I \cap J \,\middle|\, \varphi(t) = \psi(t) \text{ \& } \dot\varphi(t) = \dot\psi(t)\right\}$$

It is a closed subset of $I \cap J$. Now let $t_0 \in X$. Since φ is an immersion and $\dim M = 1$, it is a local diffeomorphism. Considering a local inverse φ^{-1} near $\varphi(t_0)$, we see that $f = \varphi^{-1} \circ \psi$, which is defined on an open interval K containing t_0, satisfies $|\dot f(t)| = 1$ for all $t \in K$. The fact that $t_0 \in X$ implies that $f(t_0) = t_0$ and $\dot f(t_0) = 1$. It follows that $f = \mathrm{Id}_K$, i.e. $\varphi = \psi$ on K. This argument shows that the set X is open in $I \cap J$.

Let $x_0 \in M$, and $v_0 \in T_{x_0}M$ such that $g_{x_0}(v_0, v_0) = 1$ (there are exactly two such vectors). Consider the set E of pairs (I, φ) where $I \subset \mathbb{R}$ is an open interval containing 0 and $\varphi : I \to M$ is an isometric map such that $\varphi(0) = x_0$ and $\dot\varphi(0) = v_0$.

By the above discussion, if $(I,\varphi), (J,\psi) \in E$ then φ and ψ coincide on $I \cap J$. This allows us to define a maximal element of E: set $I_M = \bigcup_{(I,\varphi)\in E} I$, and define $\varphi_M : \mathcal{I} \to M$ by $\varphi_M(t) = \varphi(t)$ if $(I,\varphi) \in E$ and $t \in I$.

First, let us check that E is not empty. For this, start with a curve $\gamma : J \to M$ such that $\gamma(0) = x_0$ and $\dot\gamma(0) = v_0$. Up to shrinking J, we can assume that γ is an immersion. Then the function $\lambda : I \to \mathbb{R}$ defined by $\lambda(t) = \int_0^t \sqrt{g_{\gamma(t)}(\dot\gamma(t), \dot\gamma(t))}$ is increasing, so it is a diffeomorphism onto its image $I \subset \mathbb{R}$. Now set $\varphi = \gamma \circ \lambda^{-1}$, so that $(I, \varphi) \in E$.

Since E is not empty, we find that $(I_M, \varphi_M) \in E$. Now notice that since φ_M is an immersion, its image $\varphi_M(I_M)$ is open in M. Its complement is the union of such open sets obtained by starting at a different base point. It follows that $\varphi_M(I_M)$ is closed. Since M is assumed to be connected, we see that φ_M is onto.

If φ_M is injective, then it is an isometry between an interval of $\mathcal{E}^1$ to (M,g). Now assume that φ_M is not injective. We wish to show that it is periodic. Up to changing x_0 and v_0, we can assume that there is $T > 0$ such that $\varphi_M(T) = x_0$, and that T is the smallest positive real number with this property. First assume that $\dot{\varphi}_M(T) = v_0$ (we will see that it is always the case). Then $(I_M - T, \varphi_M(\cdot + T)) \in E$, and it follows from the maximality of I_M that I_M is stable by translation by T, therefore $I_M = \mathbb{R}$, and that $\varphi_M(\cdot + T) = \varphi_M$. It follows that there is a map $\varphi : \mathbb{E}^1/T\mathbb{Z} \to M$ such that $\varphi_M = \varphi \circ \pi$ where $\pi : \mathbb{E}^1 \to \mathbb{E}^1/T\mathbb{Z}$ is the canonical projection. Since T was chosen to be minimal, the map φ is injective, and it is an isometry between the circle $\mathbb{E}^1/T\mathbb{Z}$ and (M,g).

Now let us see why we must have $\dot{\varphi}_M(T) = v_0$. If not, then $\dot{\varphi}_M(T) = -v_0$. The same type of argument as above shows that $I_M = \mathbb{R}$ and that $\varphi_M(T-t) = \varphi_M(t)$ for all $t \in \mathbb{R}$. This implies that $\dot{\varphi}_M\left(\frac{T}{2}\right) = -\dot{\varphi}_M\left(\frac{T}{2}\right)$, i.e. $\dot{\varphi}_M\left(\frac{T}{2}\right) = 0$. This is a contradiction with the fact that φ_M is isometric. $\square$

This implies that the isometries of a submanifold are not all restrictions of isometries, since one dimensional submanifolds have isometries but need not be preserved by any isometry of the ambient space.

8.4.4 Conformal models of the real hyperbolic space

The Poincaré **ball model** of the real hyperbolic space $\mathbb{H}^n$ is $(\mathbb{B}^n, g_{hyp})$ where

$$\mathbb{B}^n = \{x \in \mathbb{R}^n \,|\, \|x\| < 1\}$$

and

$$g_{hyp} = \frac{4}{\left(1 - \|x\|^2\right)^2}\left(dx_1^2 + \cdots + dx_n^2\right)$$

It is a pseudo-Riemannian manifold conformal to the unit ball in the Euclidean space. Its volume element is $\mathrm{dvol}_{hyp} = \frac{2^n}{\left(1-\|x\|^2\right)^n} dx_1 \cdots dx_n$.

Consider the following map f from the hyperboloid model to the ball model of $\mathbb{H}^n$: set $p = (0,\ldots,0,-1) \in \mathbb{M}^{n+1}$, and embed $\mathbb{R}^n$ into $\mathbb{M}^{n+1}$ as the hyperplane $x_{n+1} = 0$. For $x \in \mathcal{H}^n$ we let $f(x)$ be the intersection of the line from x to p with $\mathbb{R}^n$.

Precisely, we have:

$$f(x_1,\ldots,x_{n+1}) = \left(\frac{x_1}{1 + x_{n+1}}, \ldots, \frac{x_n}{1 + x_{n+1}}\right)$$

The map f is an isometry.

The Lobachevsky **upper half-space model** of the real hyperbolic space $\mathbb{H}^n = \mathbb{H}^n_{\mathbb{R}}$ is $(\mathbb{R}^n_+, ds^2_{hyp})$ where

$$\mathbb{R}^n_+ = \{x \in \mathbb{R}^n \,|\, x_n > 0\}$$

and

$$ds^2_{hyp} = \frac{dx_1^2 + \cdots + dx_n^2}{x_n^2}$$

It is conformal to a half Euclidean space. The volume element is $\mathrm{dvol}_{hyp} = \frac{dx_1 \cdots dx_n}{x_n^n}$.

8.4.5 Invariant metrics on Lie groups

Let G be a Lie group with Lie algebra $\mathfrak{g}$. A pseudo-Riemannian g on G is **left invariant** if $\{L_x \,|\, x \in G\} \subset \mathrm{Isom}(G,g)$, i.e.

$$\forall x \in G \; L_x^* g = g$$

It is **right invariant** if $\{R_x \,|\, x \in G\} \subset \mathrm{Isom}(G,g)$, i.e.

$$\forall x \in G \; R_x^* g = g$$

It is **bi-invariant** if it is both left and right invariant.

Any bilinear symmetric non degenerate form $b : \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$ defines a unique left invariant pseudo-Riemannian metric g on G such that $g_e = b$.

The metric g is bi-invariant if and only if b is Ad-invariant, i.e. $\{\mathrm{Ad}(x) \,|\, x \in G\} \subset \mathrm{O}(b)$.

8.4.6 The space of ellipsoids

For $n \geq 2$, we consider the set

$$\mathcal{E}_n = \left\{ x \in \mathcal{M}_n(\mathbb{R}) \,\middle|\, {}^t x = x,\, x > 0,\, \det x = 1 \right\}$$

It is a submanifold of $\mathcal{M}_n(\mathbb{R})$, and it can be identified with the set of volume 1 ellipsoids of $\mathbb{R}^n$ centred at 0 (by identifying $x \in \mathcal{E}_n$ with $\left\{ v \in \mathbb{R}^n \,\middle|\, {}^t x v x \leq 1 \right\}$).

The tangent spaces are easily described, as $T_{1_n}\mathcal{E}_n = \left\{ X \in \mathcal{M}_n(\mathbb{R}) \,\middle|\, {}^t X = X,\, \mathrm{Tr}\, X = 0 \right\}$, and more generally $T_x\mathcal{E}_n = \left\{ X \in \mathcal{M}_n(\mathbb{R}) \,\middle|\, {}^t X = X,\, \mathrm{Tr}(x^{-1}X) = 0 \right\}$.

Consider the Riemannian metric on $\mathcal{E}_n$ defined as

$$\langle X | Y \rangle_x = \mathrm{Tr}(x^{-1} X x^{-1} Y)$$

To prove that it is a Riemannian metric (the hard part is the positive definiteness), it is convenient to use the existence of a square root of x, i.e. an element $\sqrt{x} \in \mathcal{E}_n$ such that $\sqrt{x}^2 = x$.

For $g \in \mathrm{SL}(n,\mathbb{R})$ the map $x \mapsto {}^t g x g$ is an isometry of $\mathcal{E}_n$. Since -1_n acts as the identity, this gives an embedding of $\mathrm{PSL}(n,\mathbb{R})$ into $\mathrm{Isom}(\mathcal{E}_n)$. We will see later that this map is surjective.

8.5 Raising and lowering indices

8.5.1 The musical isomorphisms

A non degenerate symmetric bilinear form $\langle\cdot|\cdot\rangle$ on a finite dimensional real vector space V induces isomorphisms:

$$\flat : \left\{ \begin{array}{ccc} V & \to & V^* \\ v & \mapsto & v^\flat = \langle v|\cdot\rangle \end{array} \right. \text{ and } \sharp = \flat^{-1} : \left\{ \begin{array}{ccc} V^* & \to & V \\ \lambda & \mapsto & \lambda_\sharp \end{array} \right.$$

A pseudo-Riemannian metric g on a manifold M therefore defines isomorphisms between TM and T^*M. We can apply these isomorphisms to tensor powers, and find isomorphisms between $\mathcal{T}^{r,s}(M)$ and $\mathcal{T}^{r',s'}(M)$ whenever $r+s=r'+s'$.

In particular we get an isomorphism $\flat : \mathcal{X}(M) = \mathcal{T}^{0,1}(M) \to \Omega^1(M) = \mathcal{T}^{1,0}(M)$ from vector fields to 1-forms. For $X \in \mathcal{X}(M)$, if we write $X = \sum_{i=1}^d X^i \partial_i$ in coordinates, then

$$X^\flat = \sum_{i=1}^d \left(\sum_{j=1}^d g_{i,j} X^j \right) dx^i$$

Similarly, the inverse $\sharp : \Omega^1(M) \to \mathcal{X}(M)$ writes for $\omega = \sum_{i=1}^d \omega_i dx^i$ as

$$\omega_\sharp = \sum_{i=1}^d \left(\sum_{j=1}^d g^{i,j} \omega_j \right) \partial_j$$

Einstein's convention makes these formulae very concise:

$$X_i^\flat = g_{i,j} X^j \text{ and } \omega_\sharp^i = g^{i,j} \omega_j$$

When writing tensors in coordinates, the lower indices correspond to the covariant part and the upper indices to the contravariant part. Because of this convention for notations, applying an isomorphism $\mathcal{T}^{r,s}(M) \to \mathcal{T}^{r+1,s-1}(M)$ is called **lowering an index**, and applying an isomorphism $\mathcal{T}^{r,s}(M) \to \mathcal{T}^{r-1,s+1}(M)$ is called **raising an index** (note that this terminology is consistent with the musical notations $\flat$ and $\sharp$).

Note that there are several isomorphisms $\mathcal{T}^{r,s}(M) \to \mathcal{T}^{r+1,s-1}(M)$, depending on which index we lower (i.e. on which factor of $TM^{\otimes q}$ we apply $\flat$). If $s=1$, there is no possible confusion. For example, if we have $T \in \mathcal{T}^{3,1}(M)$, then there is only one index to lower, and we find $T^\flat \in \mathcal{T}^{4,0}(M)$ defined locally by

$$T^\flat_{i,j,k,l} = \sum_{a=1}^d g_{l,a} T^a_{i,j,k}$$

8.5.2 Contractions of tensors

Given a finite dimension real vector space V, there is a unique linear map $V^* \otimes V \to \mathbb{R}$ sending $\lambda \otimes v$ to $\lambda(v)$. The identification of $V^* \otimes V$ with $\mathrm{End}(V)$ identifies this map with the trace in $\mathrm{End}(V)$.

This map also yields maps $(V^*)^{\otimes r} \otimes V^{\otimes s} \to (V^*)^{\otimes r-1} \otimes V^{\otimes s-1}$ (where we have to choose the factors of $(V^*)^{\otimes r}$ and $V^{\otimes s}$ to which we apply the map $V^* \otimes V \to \mathbb{R}$).

On a manifold M, this defines maps $\mathcal{T}^{r,s}(M) \to \mathcal{T}^{r-1,s-1}(M)$ called the **contraction** of a tensor. For $T \in \mathcal{T}^{3,1}(M)$, its contraction $U \in \mathcal{T}^{2,0}(M)$ is given by:

$$U_{i,j} = \sum_{a=1}^{d} T^{a}_{a,i,j}$$

On a pseudo-Riemannian manifold (M, g), by combining this with the musical isomorphisms, we are able to contract covariant (and contravariant) tensors. For $R \in \mathcal{T}^{4,0}(M)$, its contraction $S \in \mathcal{T}^{2,0}(M)$ is defined by

$$S_{i,j} = \sum_{1 \le a,b \le d} g^{a,b} R_{a,i,j,b}$$

For $V \in \mathcal{T}^{2,0}(M)$, its contraction is a function $W \in \mathcal{C}^\infty(M) = \mathcal{T}^{0,0}(M)$ defined by

$$W = \sum_{1 \le i,j \le d} g^{i,j} R_{i,j}$$

Chapter 9

The Levi-Civita connection

9.1 The fundamental theorem of pseudo-Riemannian geometry

Theorem 9.1.1. *Let (M, g) be a pseudo-Riemannian manifold. There is a unique connection ∇ on TM with the following two properties:*

1. *∇ is torsion-free: $\forall X, Y \in \mathcal{X}(M)\ \nabla Y(X) - \nabla X(Y) = [X, Y]$*

2. *g is parallel for ∇:*

$$\forall X, Y, Z \in \mathcal{X}(M) \quad X \cdot g(Y, Z) = g(\nabla Y(X), Z) + g(Y, \nabla Z(X))$$

Remarks.

- *This connection $\nabla = \nabla^g$ is called the **Levi-Civita connection** of g.*

- *Condition 2. is equivalent to $\nabla g = 0$ (where ∇ also denotes the induced connection on $T^*M \otimes T^*M$).*

Proof. Let us start with uniqueness. If ∇ satisfies *1.* and *2.*, we find, for $X, Y, Z \in \mathcal{X}(M)$:

$$\begin{aligned}
X \cdot g(Y, Z) &= g(\nabla Y(X), Z) + g(Y, \nabla Z(X)) \\
Y \cdot g(Z, X) &= g(\nabla Z(Y), X) + g(Z, \nabla X(Y)) \\
Z \cdot g(X, Y) &= g(\nabla X(Z), Y) + g(X, \nabla Y(Z))
\end{aligned}$$

By adding the first two lines and subtracting the third, then simplifying because ∇ is torsion-free, we find:

$$\begin{aligned}
2g(\nabla Y(X), Z) = {} & X \cdot g(Y, Z) + Y \cdot g(Z, X) + Z \cdot g(X, Y) \\
& + g([X, Y], Z) - g([X, Z], Y) - g([Y, Z], X)
\end{aligned}$$

Since g is non-degenerate, this formula defines $\nabla Y(X)$, hence the uniqueness.

For the existence, we use this formula to define ∇. For this we first have to check that the formula is tensorial with respect to Z, so that it defines $\nabla Y(X) \in \mathcal{X}(M)$, then that it is tensorial in X, so that it defines $\nabla Y \in \mathcal{T}^{1,1}(M)$. Finally we can check that it satisfies the Leibniz rule. □

Example: For the pseudo-Euclidean space $\mathbb{R}^{p,q}$, the Levi-Civita connection is the trivial connection on $T\mathbb{R}^{p+q} = \mathbb{R}^{p+q} \times \mathbb{R}^{p+q}$.

9.2 Parallel transport in pseudo-Riemannian manifolds

Proposition 9.2.1. *Let (M,g) be a pseudo-Riemannian manifold, and $c : I \to M$ be a smooth curve. If $X, Y : I \to TM$ are vector fields along c, then:*

$$\frac{d}{dt}g(X,Y) = g\left(\frac{D}{dt}X, Y\right) + g\left(X, \frac{D}{dt}Y\right)$$

Proof. The definition of the intrinsic derivative $\frac{D}{dt}g$ induced on the tensor bundle $T^*M \otimes T^*M$ leads directly to:

$$\frac{d}{dt}g(X,Y) = \left(\frac{D}{dt}g\right)(X,Y) + g\left(\frac{D}{dt}X, Y\right) + g\left(X, \frac{D}{dt}Y\right)$$

Since g is a tensor defined on all of M, we have that $\frac{D}{dt}g = \nabla g(\dot{c}) = 0$. The result follows. □

Proposition 9.2.2. *Let (M,g) be a pseudo-Riemannian manifold and $c : I \to M$ a piecewise smooth curve. For every $t_0, t_1 \in I$, the parallel transport for the Levi-Civita connection*

$$\|_{t_0}^{t_1} : (T_{c(t_0)}M, g_{c(t_0)}) \to (T_{c(t_1)}M, g_{c(t_1)})$$

is isometric.

Remark. *Consequently, the holonomy group* Hol_x *is a subgroup of* $\mathrm{O}(g_x) \approx \mathrm{O}(p,q)$.

Proof. If $X, Y : I \to TM$ are vector fields along M, we get:

$$\frac{d}{dt}g_{c(t)}(X(t), Y(t)) = g_{c(t)}\left(\frac{D}{dt}X(t), Y(t)\right) + g_{c(t)}\left(X(t), \frac{D}{dt}Y(t)\right)$$

If X and Y are parallel, it follows that $g(X,Y)$ is constant. □

9.3 The Christoffel symbols

Let (M,g) be a pseudo-Riemannian manifold, and consider local coordinates $(x^1,\dots,x^d)$. The connection form of the Levi-Civita connection is usually denoted by Γ. Its components $\Gamma_{i,j}^k$ are called the **Christoffel symbols.**

$$g_{ij} = g(\partial_i, \partial_j) \ ; \ \ g = \sum g_{i,j} dx^i dx^j \ ; \ \ \nabla\partial j(\partial_i) = \sum \Gamma_{i,j}^k \partial_k$$

Since the Levi-Civita connection is torsion-free, we have:

$$\Gamma_{j,i}^k = \Gamma_{i,j}^k$$

The formula used to prove its existence yields:

$$2g(\nabla\partial j(\partial_i), \partial_k) = \partial_i g_{j,k} + \partial_j g_{k,i} - \partial_k g_{i,j}$$

To obtain the Christoffel symbols, we must consider the inverse matrix $(g^{i,j})_{1\le i,j\le d}$ of $(g_{i,j})_{1\le i,j\le d}$.

$$\Gamma_{i,j}^k = \frac{1}{2}\sum_{l=1}^{d} g^{l,k}(\partial_i g_{j,l} + \partial_j g_{i,l} - \partial_l g_{i,j})$$

9.4 Differential operators on Riemannian manifolds

Definition 9.4.1. *Let (M,g) be a Riemannian manifold, and let $f \in \mathcal{C}^\infty(M)$. The **gradient** of f is the vector field $\vec{\nabla} f \in \mathcal{X}(M)$ defined by:*

$$\forall x \in M \ \forall v \in T_xM \quad d_x f(v) = g_x(\vec{\nabla} f(x), v)$$

Remark. *With the notation of the musical isomorphisms, we have $\vec{\nabla} f = df_\sharp$.*

Note that the notation $\vec{\nabla}$ has nothing to do with the Levi-Civita connection. In coordinates, we find:

$$\vec{\nabla} f = \sum_{1\le i,j\le d} g^{i,j}\partial_i f \partial_j$$

Definition 9.4.2. *Let (M,g) be a Riemannian manifold, and let $X \in \mathcal{X}(M)$. The **divergence** of X is the function $\operatorname{div} X \in \mathcal{C}^\infty(M)$ such that for all $x \in M$, $\operatorname{div} X(x)$ is the trace of the map $v \mapsto \nabla X(v)$, where ∇ is the Levi-Civita connection.*

In local coordinates, for $X = X^i\partial_i$ (using Einstein's convention), we find $\operatorname{div} X = \partial_i X^i + \Gamma_{i,j}^j X^i$. Note that $\operatorname{div} X$ is the contraction of the tensor $\nabla X \in \mathcal{T}^{1,1}(M)$.

Definition 9.4.3. *Let (M,g) be a Riemannian manifold, and let $f \in \mathcal{C}^\infty(M)$. The **Laplacian** of f is the function $\Delta f \in \mathcal{C}^\infty(M)$ defined by $\Delta f = \operatorname{div}\vec{\nabla} f$.*

We have already seen that a connection ∇ on TM allows us to define a Hessian $\operatorname{Hess} f$. If ∇ is the Levi-Civita connection of a Riemannian manifold, one can check that $\Delta f = \operatorname{Tr}_g(\operatorname{Hess} f)$.

9.5 Examples of Levi-Civita connections

9.5.1 The Levi-Civita connection of a submanifold

If (M, g) is a pseudo-Riemannian manifold and $N \subset M$ is a pseudo-Riemannian submanifold, then recall that we can consider the restriction of the Levi-Civita connection ∇ to N, which is a connection on $TM|_N$. This allows us to define $\nabla_x X(v)$ for $x \in N$, $v \in T_x N$ and $X \in \Gamma(TM|_N)$. In other terms, the vector field X is only defined on N but is not necessarily tangent to N. However, requiring X to be tangent to N does not ensure that $\nabla_x X(v)$ is.

The tangent bundle TN is a vector subbundle of $TM|_N$. Since N is a pseudo-Riemannian submanifold, according to Proposition 8.1.10 the vector bundle $TM|_N$ splits as a direct sum $TM|_N = TN \oplus TN^\perp$. The vector bundle $\nu N = TN^\perp$ is called the **normal bundle** of N. For $x \in N$ and $v \in T_x M$, we write $v = v^\top + v^\perp$ its decomposition according to this direct sum.

Lemma 9.5.1. *Let M be a manifold and $N \subset M$ an immersed submanifold. For all $x \in N$ and $v \in T_x N$, there is a vector field $X \in \mathcal{X}(M)$ and a neighbourhood $U \subset N$ of x such that $X(x) = v$ and $X(y) \in T_y N$ for all $y \in U$.*

Proof. Use the linearisation of immersions and a plateau function on M. □

Proposition 9.5.2. *Let (M, g) be a pseudo-Riemannian manifold, and $N \subset M$ a pseudo-Riemannian submanifold, with induced metric $\overline{g}$. For all $X \in \mathcal{X}(N)$, $x \in N$ and $v \in T_x N$, we have $\overline{\nabla}_x X(v) = (\nabla_x X(v))^\top$ where $\overline{\nabla}$ is the Levi-Civita connection of $(N, \overline{g})$ and ∇ is the Levi-Civita connection of (M, g).*

Proof. First check, let us check that the formula $\overline{\nabla}_x X(v) = (\nabla_x X(v))^\top$ defines a connection $\overline{\nabla}$ on TN. It is a map from $\mathcal{X}(N)$ to $\Omega^1(N)$ (because the map $v \mapsto v^\top$ is a vector bundle morphism).

Let $f \in \mathcal{C}^\infty(N)$, $X \in \mathcal{X}(N)$, $x \in N$ and $v \in T_x N$. Since ∇ is a connection we have:

$$\nabla_x (fX)(v) = d_x f(v) X(x) + f(x) \nabla_x X(v)$$

Since $X(x) \in T_x N$, we have $X(x)^\top = X(x)$ and projecting on TN yields:

$$\overline{\nabla}_x (fX)(v) = d_x f(v) X(x) + f(x) \overline{\nabla}_x X(v)$$

This shows that $\overline{\nabla}$ is a connection on TN. We now wish to compute its torsion. For this, we first consider vector fields $X, Y \in \mathcal{X}(M)$ whose restrictions $\overline{X}, \overline{Y}$ to N are tangent to N. Since ∇ is torsion free, we find:

$$\nabla Y(X) - \nabla X(Y) = [X, Y]$$

Now evaluating this at some point $x \in N$ and projecting on $T_x N$, the left hand side is $\overline{\nabla Y}(\overline{X}) - \overline{\nabla X}(\overline{Y})$, and the right hand side is $[\overline{X}, \overline{Y}]$, so we find:

$$\overline{\nabla Y}(\overline{X}) - \overline{\nabla X}(\overline{Y}) = [\overline{X}, \overline{Y}]$$

Hence $\overline{T}(\overline{X},\overline{Y}) = 0$ where $\overline{T}$ is the torsion of $\overline{\nabla}$. Using the tensoriality of $\overline{T}$ and Lemma 9.5.1, we find that $\overline{\nabla}$ is torsion free.

Now let $X, Y, Z \in \mathcal{X}(M)$ be such that their restrictions $\overline{X},\overline{Y},\overline{Z}$ to N are tangent to N. Since g is parallel for ∇, we have:

$$X \cdot g(Y,Z) = g(\nabla Y(X), Z) + g(Y, \nabla Z(X))$$

When restricting to N, the left hand side becomes $\overline{X} \cdot \overline{g}(\overline{Y},\overline{Z})$. Let us compute the first term of the right hand side:

$$\begin{aligned} g\left(\nabla Y(X), \overline{Z}\right) &= \overline{g}\left(\overline{\nabla Y}(\overline{X}), \overline{Z}\right) + \underbrace{g\left(\left(\nabla \overline{Y}(\overline{X})\right)^{\perp}, \overline{Z}\right)}_{=0} \\ &= \overline{g}\left(\overline{\nabla Y}(\overline{X}), \overline{Z}\right) \end{aligned}$$

In the end, we find:

$$\overline{X} \cdot \overline{g}(\overline{Y},\overline{Z}) = \overline{g}\left(\overline{\nabla Y}(\overline{X}), \overline{Z}\right) + \overline{g}\left(\overline{Y}, \overline{\nabla Z}(\overline{X})\right)$$

This shows that $\overline{\nabla}\overline{g}(\overline{X})(\overline{Y},\overline{Z}) = 0$. Once again by using the tensoriality of $\overline{\nabla}\overline{g}$ and Lemma 9.5.1, we find that $\overline{\nabla}\overline{g} = 0$, so $\overline{\nabla}$ is the Levi-Civita connection of $\overline{g}$. □

9.5.2 Conformal metrics

Consider a pseudo-Riemannian manifold (M, g) and a conformal metric $g' = e^{2\sigma} g$ with $\sigma \in \mathcal{C}^\infty(M)$. Then the Levi-Civita connections ∇, ∇' of g, g' respectively are related by:

$$\nabla'_x X(v) = \nabla_x X(v) + d_x\sigma(v) X(x) + d_x\sigma(X(x)) v - g_x(X(x), v)\vec{\nabla}\sigma(x)$$

Where the gradient $\vec{\nabla}\sigma$ is considered for the metric g.

9.5.3 The space of ellipsoids

For $n \geq 2$, we consider the set

$$\mathcal{E}_n = \left\{ x \in \mathcal{M}_n(\mathbb{R}) \,\middle|\, {}^t x = x,\, x > 0,\, \det x = 1 \right\}$$

Recall that the tangent spaces are given by $T_x\mathcal{E}_n = \left\{ X \in \mathcal{M}_n(\mathbb{R}) \,\middle|\, {}^t X = X,\, \mathrm{Tr}(x^{-1}X) = 0 \right\}$, and the Riemannian metric is defined as:

$$\langle X | Y \rangle_x = \mathrm{Tr}(x^{-1} X x^{-1} Y)$$

The Levi-Civita connection ∇ of $\mathcal{E}_n$ is given by:

$$\nabla_x X(v) = d_x X(v) - \frac{v x^{-1} X(x) + X(x) x^{-1} v}{2}$$

To prove this, one should first check that it is well defined (i.e. that $\nabla_x X(v) \in T_x\mathcal{E}_n$), that it is a connection, that the torsion vanishes, and that the metric is parallel.

9.6 Pseudo-Riemannian geodesics

9.6.1 The geodesic flow

Definition 9.6.1. *Let (M,g) be a pseudo-Riemannian manifold. A* ***geodesic*** *of (M,g) is a geodesic of its Levi-Civita connection ∇, i.e. a smooth curve $c: I \to M$ such that $\frac{D}{dt}\dot{c} = 0$.*

Example: Since the Levi-Civita connection of $\mathbb{R}^{p,q}$ is the trivial connection, geodesics of $\mathbb{R}^{p,q}$ are affinely parametered straight lines.

Even though it is not possible to solve explicitly the geodesic equation, it does have a first integral.

Proposition 9.6.2. *Let (M,g) be a pseudo-Riemannian manifold, and $c: I \to M$ a geodesic. Then $g(\dot{c},\dot{c})$ is constant.*

Proof. It is a straightforward consequence of Proposition 9.2.2 and the fact that $\frac{D}{dt}\dot{c} = 0$. □

Let us fix a notation: for $v \in TM$, we let $I_v \subset \mathbb{R}$ be the maximal interval on which the geodesic c_v is defined.

Definition 9.6.3. *Let (M,g) be a pseudo-Riemannian manifold, and set*

$$U = \bigcup_{v \in TM} I_v \times \{v\} \subset \mathbb{R} \times TM$$

The ***geodesic flow*** *of (M,g) is the map*

$$\Phi : \left\{ \begin{array}{ccc} U & \to & TM \\ (t,v) & \mapsto & \dot{c}_v(t) \end{array} \right.$$

Write $\Phi(t,v) = \Phi^t(v)$. It is a local flow: if $t, t+s \in I_v$ then $s \in I_{\Phi^t(v)}$ and $\Phi^s(\Phi^t(v)) = \Phi^{t+s}(v)$.

The corresponding vector field $\mathcal{Z}^g = \frac{d}{dt}\Big|_{t=0} \Phi^t \in \mathcal{X}(TM)$ is called the **geodesic spray**.

If the Levi-Civita connection ∇ is complete, then $U = \mathbb{R} \times TM$, and $(\Phi^t)_{t \in \mathbb{R}}$ is a one parameter subgroup of $\mathrm{Diff}(TM)$.

If g is Riemannian, we can consider the **unit tangent bundle**

$$T^1 M = \{v \in TM \,|\, \|v\| = 1\}$$

The geodesic flow Φ_t preserves T^1M (Proposition 9.6.2), so we get a dynamical system $(T^1M, (\Phi_t))$. Note that if M is compact, then so is T^1M.

9.6.2 Normal coordinates and the injectivity radius

Definition 9.6.4. *Let (M,g) be a Riemannian manifold. Consider $x \in M$, open sets $U \subset T_xM$ and $V \subset M$ such that $\exp_x|_U : U \to V$ is a diffeomorphism, and an orthonormal basis $(e_1,\dots,e_d)$ of T_xM. The coordinates $(x^1,\dots,x^d)$ on V defined by $x^i = g_x\left((\exp_x|_U)^{-1},\cdot\right)$ are called **normal coordinates**.*

Normal coordinates $(x^1,\dots,x^d)$ correspond to the chart $(x^1,\dots,x^d) \mapsto \exp_x(x^1 e_1 + \dots + x^d e_d)$.

Proposition 9.6.5. *Let (M,g) be a Riemannian manifold. Consider $x \in M$ and normal coordinates $(x^1,\dots,x^d)$ around x. Then*

$$\forall i,j,k \in \{1,\dots,d\} \quad g_{i,j}(x) = \delta_{i,j} \ \ ; \ \ \Gamma^k_{i,j}(x) = \partial_k g_{i,j}(x) = 0$$

Remark. *This means that a Riemannian metric is "Euclidean up to order one". This makes it hopeless to find invariants of Riemannian metrics that only involve first order derivatives.*

Proof. By definition of normal coordinates, $(\partial_1(x),\dots,\partial_d(x))$ is an orthonormal basis of T_xM, so $g_{i,j}(x) = g_x(\partial_i(x),\partial_j(x)) = \delta_{i,j}$.

Now fix some $v = (v^1,\dots,v^d) \in \mathbb{R}^d$. The curve c defined by $c(t) = \exp_x(tv^1\partial_1(x) + \dots + tv^d\partial_d(x))$ is a geodesic. Recall the geodesic equation in coordinates:

$$\forall k \in \{1,\dots,d\} \quad \ddot{c}^k(t) + \sum_{1\le i,j\le d} \Gamma^k_{i,j}(c(t))\dot{c}^i(t)\dot{c}^j(t) = 0$$

Here $c^i(t) = tv^i$, so the equation simplifies:

$$\forall k \in \{1,\dots,d\} \qquad \sum_{1\le i,j\le d} \Gamma^k_{i,j}(c(t))v^i v^j = 0$$

In particular we have $\sum_{1\le i,j\le d}\Gamma^k_{i,j}(x)v^iv^j = 0$. This being true for all $v \in \mathbb{R}^d$, we find $\Gamma^k_{i,j}(x) = 0$.

Now we also have

$$\forall i,j,k \in \{1,\dots,d\} \quad \partial_i g_{j,k}(x) + \partial_j g_{i,k}(x) - \partial_k g_{i,j}(x) = 2\sum_{l=1}^{d} g_{k,l}(x)\Gamma^l_{i,j}(x) = 0$$

By permuting the indices we also get

$$\forall i,j,k \in \{1,\dots,d\} \quad \partial_k g_{i,j}(x) + \partial_i g_{k,j}(x) - \partial_j g_{k,i}(x) = 0$$

By adding these last two equalities and using the symmetry of g, we find $\partial_i g_{j,k}(x) = 0$. ☐

Definition 9.6.6. *Let (M,g) be a Riemannian manifold, and let $x \in M$. The* ***injectivity radius*** *of (M,g) at x is $\mathrm{inj}_x = \sup E \in \,]0,+\infty]$ where $E \subset \,]0,+\infty[$ is the set of positive real numbers r such that $\exp_x$ is diffeomorphic from $B_{T_xM}(0,r)$ onto its image.*

The ***injectivity radius*** *of (M,g) is $\mathrm{inj}M = \inf_{x\in M} \mathrm{inj}_x \in [0,+\infty]$.*

Remark. *The theory of ODEs shows that $x \mapsto \mathrm{inj}_x$ is lower semi-continuous. Therefore if M is compact then $\mathrm{inj}M > 0$.*

9.6.3 Isometries and geodesics

Proposition 9.6.7. *Let (N,h) and (M,g) be pseudo-Riemannian manifolds. If $\varphi : N \to M$ is an isometric immersion, and $c : I \to N$ is a smooth curve such that $\varphi \circ c$ is a geodesic, then c is a geodesic.*

Proof. Since being a geodesic is a local condition, we can work in coordinates. In other words, we can assume that N is an open set of $\mathbb{R}^n$, M is an open set of $\mathbb{R}^d$, and φ is the map $\varphi(x) = (x,0)$. Write $\Gamma_{i,j}^k$ the Christoffel symbols of g, and $c(t) = (x^1(t),\dots,x^n(t))$. Then $\varphi \circ c(t) = (x^1(t),\dots,x^n(t),0,\dots,0)$ and the fact that $\varphi \circ c$ is a geodesic gives

$$\forall k \in \{1,\dots,n\} \quad \ddot{x}^k(t) + \sum_{1\le i,j\le n} \Gamma_{i,j}^k(c(t),0)\dot{x}^i(t)\dot{x}^j(t) = 0$$

Now let $\overline{\Gamma}_{i,j}^k$ be the Christoffel symbols of h. Since φ is an isometry, we find that $h_{i,j}(x) = g_{i,j}(x,0)$ for all $1 \le i,j \le k$. It follows that $\overline{\Gamma}_{i,j}^k(x) = \Gamma_{i,j}^k(x,0)$, and the geodesic equation for $\varphi \circ c$ yields the geodesic equation for c. □

This is far from being an equivalence. It can be seen in the proof: the geodesic equation for $\varphi \circ c$ has $d-n$ more constraints that we did not use:

$$\forall k \in \{n+1,\dots,d\} \quad \sum_{1\le i,j\le n} \Gamma_{i,j}^k(c(t),0)\dot{x}^i(t)\dot{x}^j(t) = 0$$

To find some explicit examples, consider φ to be the inclusion of a submanifold in $\mathbb{E}^d$. Most submanifolds do not contain any straight line, and geodesics always exist.

Corollary 9.6.8. *Let (N,h) and (M,g) be pseudo-Riemannian manifolds. Let $\varphi : N \to M$ be an isometry. If $x \in M$ and $v \in T_xM$ are such that $\exp_x(v)$ is well defined, then:*

$$\varphi(\exp_x(v)) = \exp_{\varphi(x)}(d_x\varphi(v))$$

Proposition 9.6.9. *Let (N,h) and (M,g) be pseudo-Riemannian manifolds, with N connected. Let $\varphi,\psi : N \to M$ be isometries. If there is some $x \in N$ such that $\varphi(x) = \psi(x)$ and $d_x\varphi = d_x\psi$, then $\varphi = \psi$.*

Proof. Consider the set $X = \{x \in N \,\big|\, \varphi(x) = \psi(x) \;\&\; d_x\varphi = d_x\psi\}$. It is open because of Corollary 9.6.8 and closed because φ and ψ are smooth. □

Examples: We can use Proposition 9.6.9 to find the isometry groups of the sphere and the hyperbolic space. Indeed, in both cases, we have found a subgroup G of $\mathrm{Isom}(M,g)$ such that for all $x, y \in M$ and any linear isometry $L : T_xM \to T_xM$, there is $\varphi \in G$ such that $\varphi(x) = y$ and $d_x\varphi = L$ ($G = \mathrm{O}(n+1)$ for $\mathbb{S}^n$ and $G = \mathrm{O}_+(n,1)$ for $\mathbb{H}^n$). It follows that $G = \mathrm{Isom}(M,g)$.

Proposition 9.6.9 says that an isometry φ can be recovered from the image $\varphi(x_0)$ of a given point x_0 and the differential $d_{x_0}\varphi$. It is actually possible to use this to define charts on the isometry group, and prove that it is a Lie group.

The topology that we consider on $\mathrm{Isom}(M,g)$ is the compact-open topology: a basis is given by $\{\varphi \in \mathrm{Isom}(M,g) \,|\, \varphi(K) \subset U\}$ where $K \subset M$ is compact and $U \subset M$ is open.

Theorem 9.6.10 (Myers-Steenrod)**.** *Let (M,g) be a pseudo-Riemannian manifold. The group* $\mathrm{Isom}(M,g)$ *has a unique Lie group structure for the compact-open topology such that the action on M is smooth.*

9.6.4 Examples of geodesics

We already mentioned that geodesics of $\mathbb{R}^{p,q}$ are affinely parameterised straight lines.

A straightforward consequence of Proposition 9.5.2 is that for any pseudo-Riemannian submanifold $N \subset \mathbb{R}^{p,q}$ and smooth curve $c : I \to N$, we have:

$$\frac{D}{dt}\dot{c} = \ddot{c}^{\top}$$

It follows that c is a geodesic if and only if:

$$\forall t \in I \;\; \ddot{c}(t) \in T_{c(t)}N^{\perp}$$

For $x \in \mathbb{S}^d \subset \mathbb{E}^d$ and $v \in T_x\mathbb{S}^d = x^{\perp}$ with $\|v\| = 1$, the geodesic is:

$$c_v(t) = \cos(t)x + \sin(t)v$$

For $x \in \mathcal{H}^d \subset \mathbb{M}^{d+1}$ and $v \in T_x\mathcal{H}^d = x^{\perp}$ with $\|v\| = 1$, the geodesic is:

$$c_v(t) = \cosh(t)x + \sinh(t)v$$

In both cases, we find that images of geodesics are intersections with linear planes in the ambiant vector space.

We can use the isometries between the various models of the hyperbolic space to see that geodesics in the ball and hyperboloid models of $\mathbb{H}^d$ are circle arcs perpendicular to the boundary.

Finally, if $p : \widetilde{M} \to M$ is a Riemannian covering, then geodesics of M are compositions of geodesics of $\widetilde{M}$ with p.

In the space of ellipsoids $\mathcal{E}_n$, geodesics through 1_n are exactly the curves $t \mapsto \exp(tX)$ where X is a traceless symmetric matrix.

Chapter 10

Riemannian manifolds as metric spaces

10.1 The Riemannian distance

10.1.1 Lengths of curves

We now consider a **connected Riemannian manifold** (M,g).

Recall that we use the convention that piece-wise smooth paths $c:[a,b]\to M$ are continuous.

Definition 10.1.1. *Let (M,g) be a Riemannian manifold, and $c:[a,b]\to M$ be a piece-wise smooth curve. The **length** of c is*

$$L(c)=\int_a^b \|\dot{c}(t)\|\,dt$$

Definition 10.1.2. *Let (M,g) be a Riemannian manifold, and $c:[a,b]\to M$ be a piece-wise smooth curve. We say that c has **constant speed** if $\|\dot{c}\|$ is constant.*

Remark. *Geodesics have constant speed.*

We can use the lengths of curves to define a distance.

Definition 10.1.3. *Let (M,g) be a connected Riemannian manifold. The **Riemannian distance** of (M,g) is $d:M\times M\to[0,+\infty[$ given by*

$$d(x,y)=\inf\{L(c)\,\big|\,c:[a,b]\to M\ \text{ piece-wise smooth},\ c(a)=x,\ c(b)=y\}$$

Remark. *Since M is connected, it is also path connected, so $d(x,y)\in[0,+\infty[$ is well defined (understand: it is finite).*

Staring at this definition won't get you very far. The space of curves joining two given points is infinite dimensional in its nature, so minimising a functional on this space is by no means easy.

Before we attempt general methods and subtle definitions, let us work out the case of the Euclidean space.

Theorem 10.1.4. *The Riemannian distance of the Euclidean space $\mathbb{E}^n$ is equal to the Euclidean distance.*

Remark. *This can be summarized by the fact that the shortest path joining two points is the straight line.*

Proof. Let d be the Riemannian distance on $\mathbb{E}^n$, and $x, y \in \mathbb{E}^n$ such that $x \neq y$. Consider the straight line

$$\gamma : \left\{ \begin{array}{ccc} [0,1] & \to & \mathbb{E}^n \\ t & \mapsto & x + t(y-x) \end{array} \right.$$

joining x and y. Since $L(\gamma) = \|x-y\|$, we get $d(x,y) \leq \|x-y\|$.

Now let $c : [0,1] \to \mathbb{E}^n$ be a piecewise smooth curve such that $c(0) = x$ and $c(1) = y$. Decompose $c(t) = a(t) + b(t)$ where $a(t)$ is on the line joining x and y, and $b(t)$ is orthogonal to it (i.e. $a(t) = x + \left\langle c(t) - x \middle| \frac{y-x}{\|y-x\|} \right\rangle \frac{y-x}{\|y-x\|}$ and $b(t) = c(t) - a(t)$).

Since $\dot{a}(t)$ and $\dot{b}(t)$ are orthogonal, we have that $\|\dot{c}(t)\| \geq \|\dot{a}(t)\|$, hence $L(c) \geq L(a)$.

Now write $a(t) = x + \lambda(t)(y-x)$, so that

$$L(a) = \int_0^1 |\dot{\lambda}(t)| \left\|y - x\right\| dt \geq \left| \int_0^1 \dot{\lambda}(t) dt \right| \left\|x - y\right\| = \left\|y - x\right\|$$

It follows that $L(c) \geq L(a) = \left\|y - x\right\|$, hence $d(x,y) \geq \left\|y - x\right\|$. □

We will keep this result in mind when we prove that the Riemannian distance is always a distance. The idea is that locally, one can compare the Riemannian metric to a Euclidean metric in charts.

Lemma 10.1.5. *Let (M,g) be a Riemannian manifold, and d^g the Riemannian distance.*

1. *If h is another Riemannian metric on M and $g \geq h$, then $d^g \geq d^h$ where d^h is the Riemannian distance of (M,h).*

2. *If $U \subset M$ is open, then $d^U(x,y) \geq d^g(x,y)$ for all $x, y \in U$, where d^U is the Riemannian distance of $(U, g|_U)$.*

Remark. *By $g \geq h$ for Riemannian metrics, we mean that $g_x(v,v) \geq h_x(v,v)$ for all $x \in M$ and $v \in T_xM$.*

Proof.

1. If $c : I \to M$ is a piece-wise smooth curve, then we let $L^g(c)$ (resp. $L^h(c)$) be its length with respect to g (resp. h). Since $g_{c(t)}(\dot{c}(t), \dot{c}(t)) \geq h_{c(t)}(\dot{c}(t), \dot{c}(t))$ for all $t \in I$, we find that $L^g(c) \leq L^h(c)$, therefore $L^g(c) \geq d^h(x, y)$ if c joins x and y, and finally $d^g(x, y) \geq d^h(x, y)$.

2. Given $x, y \in U$, a curve in U joining x and y is also a curve in M joining x and y, i.e. $d^U(x, y)$ is the infimum of a subset contained in the one defining $d^g(x, y)$, therefore $d^U(x, y) \geq d^g(x, y)$.

□

We can use this principle to prove that the Riemannian distance is a distance.

Theorem 10.1.6. *Let (M, g) be a connected Riemannian manifold. The Riemannian distance d is a distance on M that defines the manifold topology.*

Proof. One easily checks that d is well defined, non negative, symmetric, and that it satisfies the triangle inequality (this is why we work with piece-wise smooth curves: they are stable under concatenation).

Let $x, y \in M$ be such that $x \neq y$. Consider local coordinates $(x^1, \dots, x^d)$ on an open domain $U \subset M$ such that $g_{i,j}(x) = \delta_{i,j}$ (e.g. normal coordinates) and $y \notin U$. To simplify notations, assume that $U \subset \mathbb{R}^d$.

Since the functions $g_{i,j}$ are continuous, we can shrink U and assume the existence of $a > 0$ such that:

$$\forall z \in U \ \forall v \in T_z U \quad \frac{1}{a^2} \|v\|_{eucl}^2 \leq g_z(v, v) \leq a^2 \|v\|_{eucl}^2$$

Let $\varepsilon > 0$ be such that $B_{eucl}(x, \varepsilon) \subset U$. For $z \in B_{eucl}(x, \varepsilon)$, consider a piece-wise smooth path $c : [0, 1] \to M$ joining x and z. If $c([0, 1]) \subset U$, then the first point of Lemma 10.1.5 implies that $L(c) \geq \frac{\|z-x\|_{eucl}}{a}$. If the path c leaves U, consider t_∂ the smallest parameter such that $c(t_\partial) \in \partial B_{eucl}(x, \varepsilon)$, then the restriction of c to $[0, t_\partial]$ is a path contained in U joining x to $c(t_\partial)$, and it is shorter than c, hence $L(c) \geq \frac{\varepsilon}{a} \geq \frac{\|z-x\|_{eucl}}{a}$.

This shows all for $z \in B_{eucl}(x, \varepsilon)$ satisfy $d(x, z) \geq \frac{\|z-x\|_{eucl}}{a}$.

Since any continuous curve from x to y must cross $\partial B_{eucl}(x, \varepsilon)$, we also find that $d(x, y) \geq \frac{\varepsilon}{a} > 0$, therefore $d(x, y) \neq 0$, and d is a distance.

The fact that $d(x, z) \geq \frac{\|z-x\|_{eucl}}{a}$ for all $z \in B_{eucl}(x, \varepsilon)$ shows that $B_d(x, r) \subset B_{eucl}(x, ar)$ for all $r \leq a\varepsilon$.

The first point of Lemma 10.1.5 and the fact that $g \leq a g_{eucl}$ imply that $B_{eucl}(x, r) \subset B_d(x, ar)$ for all $r \leq \varepsilon$. It follows that d defines the manifold topology. □

10.1.2 Minimising curves

The computation of the Riemannian distance for the Euclidean space is based on the fact that we can find an explicit formula for the shortest path between two points. In other terms, the infimum defining the Riemannian distance is a minimum.

This will not always be the case. Consider $\mathbb{R}^2 \setminus \{0\}$ with the Euclidean metric. Considering paths that take arbitrarily small detours around the origin, we see that the Riemannian distance is still equal to the Euclidean distance, however given two opposite points, there is no shortest curve in $\mathbb{R}^2 \setminus \{0\}$. This problem will be avoided by working locally.

Definition 10.1.7. *Let (M,g) be a connected Riemannian manifold, and d the Riemannian distance.*
*A piece-wise smooth curve $c : [a,b] \to M$ is called **minimising** if $L(c) = d(c(a), c(b))$.*

It will be practical to consider minimising curves defined on infinite intervals, so we need a definition that does not involve the endpoints. Notice that the notion of minimising curve is stable under restrictions.

Lemma 10.1.8. *Let (M,g) be a connected Riemannian manifold, d the Riemannian distance, and $c : [a,b] \to M$ a minimising curve. For all $[a',b'] \subset [a,b]$, the restriction $c|_{[a',b']}$ is minimising.*

Proof. We already have $d(c(a'), c(b')) \le L\left(c|_{[a',b']}\right)$.

Consider a piece-wise smooth curve $\gamma : [a',b'] \to M$ such that $\gamma(a') = c(a')$ and $\gamma(b') = c(b')$. Let $\widetilde{\gamma} : [a,b] \to M$ be the piece-wise smooth curve defined by $\widetilde{\gamma}(t) = c(t)$ if $t \in [a,a']$ or $t \in [b',b]$ and $\widetilde{\gamma}(t) = \gamma(t)$ for $t \in [a',b']$. Since $\widetilde{\gamma}$ is a piece-wise smooth curve joining $c(a)$ and $c(b)$, we have that $L(\widetilde{\gamma}) \ge d(c(a), c(b)) = L(c)$.

By writing out the integral that defines the length, we find that $L(\gamma) \ge L\left(c|_{[a',b']}\right)$. Therefore $c|_{[a',b']}$ is minimising. □

If $I \subset \mathbb{R}$ is any interval and $c : I \to M$ is a curve, where M is a manifold, we say that c is **piece-wise smooth** if its restriction to any compact interval is piece-wise smooth.

Definition 10.1.9. *Let (M,g) be a Riemannian manifold, and d the Riemannian distance.*
*If $I \subset \mathbb{R}$ is any interval, then a piece-wise smooth curve $c : I \to M$ is called **minimising** if*

$$\forall s,t \in I \;\; L(c|_{[s,t]}) = d(c(s), c(t))$$

*It is **locally minimising** if*

$$\forall t \in I \;\; \exists \varepsilon > 0 \;\; c|_{[t-\varepsilon, t+\varepsilon]} \text{ is minimising}$$

The two definitions of minimising curves coincide for a compact interval. Minimising curves are locally minimising, but we will see that the converse is not true.

10.1.3 Riemannian spherical coordinates

We now wish to find out some more about the relationship between geodesics and the Riemannian distance. In the Euclidean space, we used an orthogonal projection to find the Riemannian distance, which is somehow related to Cartesian coordinates. However, these coordinates are not well defined in Riemannian geometry. We can however define some spherical coordinates, and they will be very useful.

Theorem 10.1.10 (Gauß Lemma)**.**
Let (M,g) be a Riemannian manifold, and $x \in M$. If $\exp_x$ is defined at $v \in T_xM$, then:

$$\forall w \in T_xM \quad \left\langle d_v \exp_x(v) \middle| d_x \exp_x(w) \right\rangle_{\exp_x(v)} = \langle v|w\rangle_x$$

Proof. Given $\varepsilon > 0$ small enough, consider:

$$f : \left\{ \begin{array}{ccc}]-\varepsilon, 1+\varepsilon[\times]-\varepsilon, \varepsilon[& \to & M \\ (t,s) & \mapsto & \exp_x(tv + stw) \end{array} \right.$$

We find that $\frac{\partial f}{\partial t}(1,0) = d_v \exp_x(v)$ and $\frac{\partial f}{\partial s}(1,0) = d_v \exp_x(w)$, so we wish to compute $\left\langle \frac{\partial f}{\partial t} \middle| \frac{\partial f}{\partial s} \right\rangle$ at $(t,s) = (1,0)$.

Note that for s fixed, $f(\cdot, s)$ is a geodesic, so $\frac{D}{\partial t}\frac{\partial f}{\partial t} = 0$.

$$\begin{aligned} \frac{\partial}{\partial t}\left\langle \frac{\partial f}{\partial t} \middle| \frac{\partial f}{\partial s} \right\rangle &= \underbrace{\left\langle \frac{D}{\partial t}\frac{\partial f}{\partial t} \middle| \frac{\partial f}{\partial s} \right\rangle}_{=0} + \left\langle \frac{\partial f}{\partial t} \middle| \frac{D}{\partial t}\frac{\partial f}{\partial s} \right\rangle \\ &= \left\langle \frac{\partial f}{\partial t} \middle| \frac{D}{\partial s}\frac{\partial f}{\partial t} \right\rangle \\ &= \frac{1}{2}\frac{\partial}{\partial s}\left\langle \frac{\partial f}{\partial t} \middle| \frac{\partial f}{\partial t} \right\rangle \end{aligned}$$

Since $f(\cdot, s)$ is a geodesic, it has constant speed, i.e. $\left\langle \frac{\partial f}{\partial t} \middle| \frac{\partial f}{\partial t} \right\rangle$ does not depend on t. Since $\frac{\partial f}{\partial t}(0,s) = v + sw$, we find $\left\langle \frac{\partial f}{\partial t} \middle| \frac{\partial f}{\partial t} \right\rangle = \|v + sw\|^2$, and:

$$\frac{\partial}{\partial t}\left\langle \frac{\partial f}{\partial t} \middle| \frac{\partial f}{\partial s} \right\rangle = \frac{1}{2}\frac{\partial}{\partial s}\left\langle \frac{\partial f}{\partial t} \middle| \frac{\partial f}{\partial t} \right\rangle = \langle v|w\rangle + s\|w\|^2$$

Integrating yields:

$$\left\langle \frac{\partial f}{\partial t} \middle| \frac{\partial f}{\partial s} \right\rangle (1,s) = \left\langle \frac{\partial f}{\partial t} \middle| \frac{\partial f}{\partial s} \right\rangle (0,s) + \langle v|w\rangle + s\,\|w\|^2$$

Since $\frac{\partial f}{\partial s}(0,s) = 0$, we finally get:

$$\left\langle \frac{\partial f}{\partial t} \middle| \frac{\partial f}{\partial s} \right\rangle (1,0) = \langle v|w\rangle$$

□

This has a nice interpretation in terms of spherical coordinates.

Proposition 10.1.11. *Let (M,g) be a Riemannian manifold, and $x \in M$. Define*

$$\Phi : \left\{ \begin{array}{ccc}]0,\mathrm{inj}_x[\times T_x^1 M & \to & M \\ (r,v) & \mapsto & \exp_x(rv) \end{array} \right.$$

There is a smooth family of Riemannian metrics $(h(r))_{r\in]0,\mathrm{inj}_x[}$ on the sphere $T_x^1 M$ such that $(\Phi^* g)_{r,v} = dr^2 + h(r)_v$.

Remark. *By smooth family of Riemannian metrics, we mean that each $h(r)$ is a Riemannian metric on $T_x^1 M$, and the map $(r,v) \mapsto h(r)_v$ is a smooth from $]0,\mathrm{inj}_x[\times T_x^1 M$ to the total space of the vector bundle $S^2 T^*(T_x^1 M)$. In human language, this means that the expressions in coordinates are smooth functions in (r,v).*

Proof. Since $T_{(r,v)}\left( ]0,\mathrm{inj}_x[\times T_x^1 M \right) = \mathbb{R} \times T_v T_x^1 M$, any Riemannian metric on $]0,\mathrm{inj}_x[\times T_x^1 M$ can be written as

$$\alpha(r,v) dr^2 + dr \otimes \omega(r)_v + h(r)_v$$

Where $\omega(r)$ is a smooth family of 1-forms on $T_x^1 M$ and $h(r)$ is a smooth family of Riemannian metrics.

First, we have that $\alpha(r,v) = \left\| d_{(r,v)}\Phi(1,0) \right\|^2 = \|\dot{c}_v(r)\|^2$ where c_v is the geodesic satisfying $\dot{c}_v(0) = v$. It follows that $\alpha(r,v) = \|v\|^2 = 1$.

Since any $w \in T_v T_x^1 M$ satisfies $\langle v|w\rangle_x = 0$, the Gauss Lemma yields:

$$\omega(r)_v(w) = \left\langle d_{(r,v)}\Phi(0,w) \middle| d_{(r,v)}\Phi(1,0) \right\rangle = \left\langle d_{rv}\exp_x(rw) \middle| d_{rv}\exp_x(rv) \right\rangle = 0$$

□

In the following examples, the metric $h(r)$ can be computed easily.

$\mathbb{E}^2$	$ds^2 = dr^2 + r^2 d\theta^2$	$(r > 0)$
$\mathbb{S}^2$	$ds^2_{sph} = dr^2 + \sin^2 r d\theta^2$	$(0 < r < \pi)$
$\mathbb{H}^2$	$ds^2_{hyp} = dr^2 + \sinh^2 r d\theta^2$	$(r > 0)$

10.1.4 Shortest paths and geodesics

Theorem 10.1.12. *Let (M,g) be a Riemannian manifold, and $c : I \to M$ a piece-wise smooth curve. Then c is a locally minimising if and only if c is a reparametrization of a geodesic.*

It is a consequence of the following more precise statement.

Proposition 10.1.13. *Let (M,g) be a Riemannian manifold. Let $x \in M$, $v \in T_x^1 M$, and let c be the geodesic with initial velocity v. If $t \in]0, \mathrm{inj}_x[$, then:*

1. *The curve $c|_{[0,t]}$ is minimising.*
2. *Any minimising curve joining x and $\exp_x(tv)$ is a reparametrization of $c|_{[0,t]}$.*
3. *$B_M(x,t) = \exp_x\left(B_{T_xM}(0,t)\right)$.*

Proof. Let $\gamma : [a,b] \to M$ be a piece-wise smooth curve such that $\gamma(a) = x$ et $\gamma(b) = \exp_x(tv)$. Let b' be the first time at which γ exits $\exp_x\left(B_{T_xM}(0,t)\right)$, i.e. $b' = \inf\left\{s \in [a,b] \,\middle|\, c(s) \notin \exp_x\left(B_{T_xM}(0,t)\right)\right\} \in]a,b]$.

For $s \in [a,b']$, we set $r(s) = \left\|\exp_x^{-1}(c(s))\right\|$. Using the Riemannian spherical coordinates of Proposition 10.1.11, we can write $c(s) = \exp_x(r(s)u(s))$ for $s \in [a,b']$ where $u(s) \in T_x^1 M$. We find:

$$\left\|\dot\gamma\right\|^2 = \dot r^2 + h(r)_u(\dot u, \dot u)$$

It follows that $\left\|\dot\gamma(s)\right\| \geq |\dot r(s)|$ for all $s \in [a,b']$. We can use this estimate the length of γ:

$$\begin{aligned} L(\gamma) &= \int_a^b \left\|\dot\gamma(s)\right\| ds \\ &\geq \int_a^{b'} \left\|\dot\gamma(s)\right\| ds \\ &\geq \int_a^{b'} |\dot r(s)| ds \\ &\geq \left|\int_a^{b'} \dot r(s) ds\right| = |r(b') - r(a)| = t \end{aligned}$$

This shows that $L(\gamma) \geq L\left(c|_{[0,t]}\right)$, i.e. the geodesic $c|_{[0,t]}$ is minimising. If γ is also minimising, then every inequality that we used is an equality. In particular, we have that $\int_{b'}^b \left\|\dot\gamma(s)\right\| ds = 0$, which shows that $b' = b$, and $h(r)_u(\dot u, \dot u) = 0$, i.e. u is constant, however $\gamma(b) = \exp_x(v)$ implies that $u(s) = v$ for all s, hence $\gamma(s) = \exp_x(r(s)v) = c(r(s))$, and γ is a reparametrization of

$c|_{[0,t]}$.

Since $c|_{[0,t]}$ is minimising, we find that $d(x, \exp_x(tv)) = t$. This being true for all $t \in]0, \mathrm{inj}_x[$ and $v \in T^1_x M$, we find $\exp_x\left(B_{T_xM}(0,t)\right) \subset B_M(x,t)$. Since we have shown that every piece-wise smooth curve starting from x and leaving $\exp_x\left(B_{T_xM}(0,t)\right)$ has length at least t, we also find that $B_M(x,t) \subset \exp_x\left(B_{T_xM}(0,t)\right)$.

□

Proposition 10.1.14. *Let (M,g) be a Riemannian manifold. Every $x_0 \in M$ has a neighbourhood U such that:*

1. *Any $x,y \in U$ are joined by a unique unit-speed minimising geodesic $c_{x,y}$.*
2. *There is $\varepsilon > 0$ such that $c_{x,y}$ is defined on $]-\varepsilon, \varepsilon[$ for all $x,y \in U$, and the map $\left\{ \begin{array}{ccc} U \times U \times]-\varepsilon,\varepsilon[& \to & M \\ (x,y,t) & \mapsto & c_{x,y}(t) \end{array} \right.$ is smooth.*
3. *If $x,y,z \in U$ and $d(x,y) + d(y,z) = d(x,z)$, then $y = c_{x,z}(d(x,y))$.*
4. *If $c : I \to U$ satifies $d(c(t),c(s)) = |t-s|$ for all $t,s \in I$, then c is a unit speed geodesic.*

Proof. Using the lower semi-continuity of the injectivity radius, we can find an open set $V \subset M$ containing x_0 such that for all $x \in V$, we have $V \subset B(x, \mathrm{inj}_x)$.

As seen for connections, we can find an open set $U \subset V$ containing x_0, and a smooth map $\varphi : U \times U \to TM$ such that $\varphi(x,y) \in T_xM$, $\varphi(x,x) = 0$ and $\exp_x(\varphi(x,y)) = y$ for all $x,y \in U$. Now $c_{x,y}(t) = \exp_x(t\varphi(x,y))$ is a smooth function of (x,y,t).

Proposition 10.1.13 implies the fist point because $U \subset V$, and the remark above implies the second point.

Now let $x,y,z \in U$ and $d(x,y) + d(y,z) = d(x,z)$. The concatenation of $c_{x,y}$ and $c_{y,z}$ is a minimising curve from x to z, so it must be a geodesic, hence $y = c_{x,z}(d(x,y))$.

Finally, if a curve $c : I \to U$ satifies $d(c(t),c(s)) = |t-s|$ for all $t,s \in I$, the previous point shows that $c(t) = c_{c(a),c(b)}(t-a)$ for $t \in [a,b] \subset I$. □

Lemma 10.1.15. *Let (M,g) be a Riemannian manifold, and let $x \in M$. For $v,w \in T_xM$, we have:*

$$d(\exp_x(tv), \exp_x(tw)) = t\|v-w\|_x + o(t)$$

Proof. Consider a neighbourhood $U \subset M$ given by Proposition 10.1.14, and the smooth map $\varphi : U \times U \to TM$ used in its proof.

Let $W = \exp_x^{-1}(U)$, and consider the function $F : W \times W \to \mathbb{R}$ defined by $F(u,v) = d(\exp_x(u), \exp_x(v))^2$. Then $F(u,v) = \left\|\varphi(\exp_x(u), \exp_x(v))\right\|^2_{\exp_x(u)}$, which shows that F is smooth.

Using the fact that $F(u,v) = F(v,u)$, $F(0,v) = \|v\|_x^2$ and $F(u,u) = 0$, we can recover the first and second order differentials of F at $(0,0)$. We find $d_{(0,0)}F = 0$ and:

$$d^2_{(0,0)}F((u,v),(z,w)) = 2g_x(u-v,z-w)$$

This leads to:

$$d(\exp_x(tv),\exp_x(tw))^2 = t^2\,\|v-w\|_x^2 + o(t^2)$$

□

10.1.5 Recovering a Riemannian metric from the distance

Proposition 10.1.16. *Let M be a manifold, and g, g' Riemannian metrics on M. If the Riemannian distances d_g and $d_{g'}$ are equal, then $g = g'$.*

Proof. At first we only consider the metric g. Let $x \in M$, and define $f : M \to \mathbb{R}$ by $f(y) = \frac{1}{2}d_g(x,y)^2$. By Proposition 10.1.13, we find that $f(\exp_x(v)) = \frac{1}{2}\|v\|^2$ for $v \in T_xM$ small enough. It follows that f is smooth in a neighbourhood of x, that $d_xf = 0$ and that $d_x^2 f = g_x$ (where $d_x^2 f$ is the Hessian of a function at a critical point).

The same being true for g', we find that $g_x = g'_x$. □

Proposition 10.1.17. *Let (M,g) be a Riemannian manifold, and d the Riemannian distance. If $f : M \to M$ is an isometry of the metric space (M,d), then it is an isometry of the Riemannian manifold (M,g).*

Proof. If f is smooth, then the Riemannian distance of f^*g is equal to the Riemannian distance of g, and Proposition 10.1.16 implies that $f \in \mathrm{Isom}(M,g)$. So it remains to show that f is smooth.

Let $x \in M$. For all $v \in T_xM$, the image of the geodesic $t \mapsto \exp_x(tv)$ under f is a geodesic because of the last point in Proposition 10.1.14, so there is $L(v) \in T_{f(x)}M$ such that $f(\exp_x(tv)) = \exp_{f(x)}(tL(v))$ for t small enough. The map $L : T_xM \to T_{f(x)}M$ fixes 0, and it satisfies:

$$\begin{aligned}\|L(u)-L(v)\|_{f(x)} = \lim_{t\to 0}\frac{d(f(\exp_x(tu)),f(\exp_x(tv)))}{t} &= \lim_{t\to 0}\frac{d(\exp_x(tu),\exp_x(tv))}{t}\\ &= \|u-v\|_x\end{aligned}$$

It follows that L is a linear isometry. Since $f(\exp_x(v)) = \exp_{f(x)}(L(v))$ for v small enough and $\exp_x$ is a local diffeomorphism at x, we find that f is smooth around x. □

This implies that $\mathrm{Isom}(M,g)$ is closed in $\mathrm{Homeo}(M)$. This is also true for pseudo-Riemannian manifolds, but considerably more difficult.

If M is compact, then so is $\mathrm{Isom}(M,g)$ (by Ascoli's Theorem). Any compact Lie group acting on a manifold preserves a Riemannian metric.

Proposition 10.1.18. *Let $G \curvearrowright M$ be a smooth action of a compact Lie group on a manifold. There is a Riemannian metric on M for which G acts isometrically.*

Proof. We use a left-invariant volume form ω on G and any Riemannian metric h on M to define:

$$H_x(u,v) = \int_G h_{gx}(d_x g(u), d_x g(v))\, d\omega(g)$$

Then H is a Riemannian metric on M, invariant under the action of G. □

10.1.6 Closed geodesics

Corollary 10.1.19. *Let (M,g) be a compact Riemannian manifold. Any non trivial free homotopy class contains a closed geodesic.*

Remark. *The statement is false for non compact manifolds.*

Proof. Since M is compact, we know that $r = \mathrm{inj} M > 0$.

Let $\mathcal{C} \subset \mathcal{C}^0([0,1],M)$ be a non trivial free homotopy class. It is a closed subset of $\mathcal{C}^0([0,1],M)$. Let $\mathcal{C}' \subset \mathcal{C}$ be the subset of piecewise $\mathcal{C}^1$ paths.

For $c \in \mathcal{C}'$, we can find a piecewise geodesic $\gamma \in \mathcal{C}'$, with at most $\frac{L(c)}{r}$ pieces, such that $L(\gamma) \leq L(c)$. Set:

$$L = \inf\{L(c) \,\big|\, c \in \mathcal{C}'\}$$

Let us prove that $L > 0$. If not, we could find a sequence of paths (c_k) in $\mathcal{C}'$ such that $L(c_k) \to 0$. Since M is compact, up to a considering a subsequence this means that c_k converges to a constant path, which must be in $\mathcal{C}$ because it is closed. This contradicts the non triviality of $\mathcal{C}$.

Consider a sequence $c_k \in \mathcal{C}'$ such that $L(c_k) \to L$. Using the above remark, we can assume that c_k is piecewise geodesic with at most $\frac{L+1}{r}$ pieces.

Using Ascoli's Theorem, we can assume that c_k converges in $\mathcal{C}^0([0,1],M)$ to some path $c \in \mathcal{C}$. We find that c is a geodesic. □

10.2 Geodesics and calculus of variations

10.2.1 Energy and the variational approach to geodesics

The fact that minimising the length is invariant under a change of parameter is a major technical issue. It means that if we find an equation describing minimising curves, then this equation must have an infinite dimensional space of solutions (a physicist might say that the group of diffeomorphisms of the interval act as gauge transformations, and infinite dimensional gauge is something that one should stay away from).

A first clue towards finding a way around this problem is to consider only curves with constant speed. Indeed, any regular curve (i.e. with non vanishing velocity) can be reparametrized so that it has constant speed, and this reparametrization is unique.

The appropriate solution consists in finding a functional on curves that is minimised by length-minimising curves with constant speed. This functional is the energy.

Definition 10.2.1. *Let (M,g) be a Riemannian manifold, and $c : [a,b] \to M$ be a piece-wise smooth curve. The **energy** of c is*

$$E(c) = \frac{1}{2}\int_a^b \|\dot{c}(t)\|^2\,dt$$

Remark. *The energy still makes sense in pseudo-Riemannian manifolds, but the length does not (artificially defining it with an absolute value before the square root can be useful but only if we restrict the study to subspaces of curves).*

There is a simple inequality between the length and the energy of a curve, and the equality case is only achieved by curves with constant speed.

Lemma 10.2.2. *Let (M,g) be a Riemannian manifold, and $c : [a,b] \to M$ be a piece-wise smooth curve. Then $L(c)^2 \le 2(b-a)E(c)$, and equality holds if and only if c has constant speed.*

Proof. It is a consequence of the Cauchy-Bunyakovsky-Schwarz inequality, and its equality case. □

We can define energy-minimising curves in a similar fashion as for the length.

Definition 10.2.3. *Let (M,g) be a connected Riemannian manifold, and d the Riemannian distance.*
*A piece-wise smooth curve $c : [a,b] \to M$ is called **energy-minimising** if any other piece-wise smooth curve $\gamma : [a,b] \to M$ such that $\gamma(a) = c(a)$ and $\gamma(b) = c(b)$ satisfies $E(\gamma) \ge E(c)$.*

Note that considering curves defined on the same interval is vital in this definition, since the energy changes when rescaling to a different interval.

We still have the same properties for restrictions.

Lemma 10.2.4. *Let (M,g) be a connected Riemannian manifold, d the Riemannian distance, and $c : [a,b] \to M$ an energy-minimising curve. For all $[a',b'] \subset [a,b]$, the restriction $c|_{[a',b']}$ is energy-minimising.*

Proof. The proof of Lemma 10.1.8 can be carried out mutatis mutandis. □

This also allows for a notion of energy-minimising curves defined on an arbitrary interval.

Definition 10.2.5. *Let (M,g) be a Riemannian manifold, and d the Riemannian distance.*
If $I \subset \mathbb{R}$ is any interval, then a piece-wise smooth curve $c : I \to M$ is called ***energy-minimising*** *if for all $s,t \in I$, the restriction $c|_{[s,t]}$ is energy-minimising.*
It is ***locally energy-minimising*** *if*

$$\forall t \in I \ \ \exists \varepsilon > 0 \ \ c|_{[t-\varepsilon,t+\varepsilon]} \text{ is energy-minimising}$$

Proposition 10.2.6. *Let (M,g) be a Riemannian manifold, and let $c : I \to M$ be a piece-wise smooth curve. Then c is energy-minimising if and only if it is minimising and has constant speed.*
Similarly, it is locally energy-minimising if and only if it is locally minimising and has constant speed.

Proof. Note that the global statement implies the local one, since constant speed is a local property. Without loss of generality we can assume that I is a compact interval $[a,b]$.

First assume that c is minimising and has constant speed. If $\gamma : [a,b] \to M$ is a piece-wise smooth curve such that $\gamma(a) = c(a)$ and $\gamma(b)$, then Lemma 10.2.2 gives $E(\gamma) \geq \frac{L(\gamma)^2}{2(b-a)}$ Since c is minimising, it follows that $E(\gamma) \geq \frac{L(c)^2}{2(b-a)}$. However c has constant speed, so Lemma 10.2.2 implies that $\frac{L(c)^2}{2(b-a)} = E(c)$, hence $E(\gamma) \geq E(c)$, and c is energy-minimising.

Now assume that c is energy-minimising. Let $\widetilde{c} : [a,b] \to M$ be the constant speed reparametrization of c. By Lemma 10.2.2, we get $E(c) \geq \frac{L(c)^2}{2(b-a)} = \frac{L(\widetilde{c})^2}{2(b-a)} = E(\widetilde{c})$. Since c is energy-minimising, these are equalities, i.e. $E(c) = \frac{L(c)^2}{2(b-a)}$, and according to Lemma 10.2.2 c has constant speed.

Let $\gamma : [a,b] \to M$ be a piece-wise smooth curve such that $\gamma(a) = c(a)$ and $\gamma(b) = c(b)$, and let $\widetilde{\gamma}$ be the constant speed reparametrisation of γ. Lemma 10.2.2 and the invariance of the length under reparametrisations each applied twice give:

$$L(\gamma) = L(\widetilde{\gamma}) = \sqrt{2(b-a)E(\widetilde{g})} \geq \sqrt{2(b-a)E(c)} = L(c)$$

It follows that c is minimising. $\square$

10.2.2 The first variation formula

Since we are looking for minimisers of the energy functional, a possible approach is to try to find its critical points, then compute the second order derivative at these points and try to evaluate its sign. For now we will just focus on the first derivative.

It is possible to formalize this in terms of functions on infinite dimensional manifolds of paths, but we will not go down this path.

Instead, we will work with variations of curves.

Definition 10.2.7. *Let (M,g) be a Riemannian manifold, and $c:[a,b]\to M$ a smooth curve.*
*A **variation** of c is a smooth map $f:[a,b]\times\,]-\varepsilon,\varepsilon[\,\to M$ for some $\varepsilon>0$ such that, if $c_s: I\to M$ is the curve defined by $c_s(t)=f(t,s)$, then $c_0=c$.*
*It has **fixed endpoints** if $c_s(a)=c(a)$ and $c_s(b)=c(b)$ for all $s\in\,]-\varepsilon,\varepsilon[$.*
*The **variation field** of f the the vector field J along c defined by $J(t)=\frac{\partial}{\partial s}\big|_{s=0}c_s(t)\in T_{c(t)}M$.*

Remarks.

- *We consider smooth curves in order to avoid complicated definitions of piece-wise smooth functions of two variables.*
- *If a variation f has fixed endpoints, then the variation field J satisfies $J(a)=0$ and $J(b)=0$.*
- *One can show that any vector field along c is the variation field of a variation of c, and that the variation can be chosen with fixed endpoints if the vector field vanishes at the endpoints of c. However, it will not be necessary for our applications.*

Theorem 10.2.8 (First variation formula for the energy)**.**
Let (M,g) be a Riemannian manifold, $c:[a,b]\to M$ a smooth curve, $f:[a,b]\times\,]-\varepsilon,\varepsilon[\,\to M$ a variation of c, and J its variation field. We have:

$$\frac{d}{ds}\bigg|_{s=0}E(c_s)=-\int_a^b g_{c(t)}\Big(J(t),\frac{D}{dt}\dot c(t)\Big)dt+g_{c(b)}(\dot c(b),J(b))-g_{c(a)}(\dot c(a),J(a))$$

Remark. *If f has fixed endpoints, then the formula simplifies to:*

$$\frac{d}{ds}\bigg|_{s=0}E(c_s)=-\int_a^b g_{c(t)}\Big(J(t),\frac{D}{dt}\dot c(t)\Big)dt$$

Proof. Since we are considering smooth functions and integrating on a compact interval, we can differentiate before integrating:

$$\frac{d}{ds}E(c_s)=\frac{1}{2}\int_a^b\frac{\partial}{\partial s}g_{c_s(t)}(\dot c_s(t),\dot c_s(t))\,dt$$

We can compute the integrand:

$$\begin{aligned}\frac{\partial}{\partial s}g_{c_s(t)}(\dot c_s(t),\dot c_s(t))&=\frac{\partial}{\partial s}g\left(\frac{\partial f}{\partial t},\frac{\partial f}{\partial t}\right)\\&=2g\left(\frac{\partial f}{\partial t},\frac{D}{\partial s}\frac{\partial f}{\partial t}\right)\\&=2g\left(\frac{\partial f}{\partial t},\frac{D}{\partial t}\frac{\partial f}{\partial s}\right)\\&=2\frac{\partial}{\partial t}g\left(\frac{\partial f}{\partial t},\frac{\partial f}{\partial s}\right)-2g\left(\frac{D}{\partial t}\frac{\partial f}{\partial t},\frac{\partial f}{\partial s}\right)\end{aligned}$$

At $s = 0$, this simplifies as:

$$\left.\frac{\partial}{\partial s}\right|_{s=0} g_{c_s(t)}\left(\dot{c}_s(t), \dot{c}_s(t)\right) = 2\frac{d}{dt}g\left(\dot{c}, J\right) - 2g\left(\frac{D}{dt}\dot{c}, J\right)$$

Integration yields the desired formula. □

We little effort, one can show that critical points of the energy must satisfy $\frac{D}{dt}\dot{c} = 0$, i.e. be geodesics.

A similar formula can be obtained for the variation of the length, but it is only practical if we assume the curve c to have constant speed.

Theorem 10.2.9 (First variation formula for the length)**.**
Let (M, g) be a Riemannian manifold, $c : [a, b] \to M$ a smooth curve with constant speed, $f : [a, b] \times]-\varepsilon, \varepsilon[\to M$ a variation of c with fixed endpoints, and J its variation field. We have:

$$\left.\frac{d}{ds}\right|_{s=0} L(c_s) = -\sqrt{\frac{b-a}{L(c)}} \int_a^b g_{c(t)}\left(J(t), \frac{D}{dt}\dot{c}(t)\right) dt$$

Proof. We proceed in the same way as we did for Theorem 10.2.8, and find that:

$$\frac{d}{ds}L(c_s) = \int_a^b \frac{\partial}{\partial s}\left[g\left(\frac{\partial f}{\partial t}, \frac{\partial f}{\partial t}\right)^{\frac{1}{2}}\right] dt$$

Since c has constant speed, we get $g(\dot{c}, \dot{c}) = \frac{L(c)}{b-a}$. Using the computations made in the proof of Theorem 10.2.8, we find:

$$\left.\frac{\partial}{\partial s}\right|_{s=0}\left[g\left(\frac{\partial f}{\partial t}, \frac{\partial f}{\partial t}\right)^{\frac{1}{2}}\right] = \sqrt{\frac{b-a}{L(c)}}\left(\frac{d}{dt}g\left(\dot{c}, J\right) - g\left(\frac{D}{dt}\dot{c}, J\right)\right)$$

Integration once again yields the desired formula. □

10.3 Completeness of Riemannian manifolds

10.3.1 The Hopf-Rinow Theorem

Definition 10.3.1. *A Riemannian manifold (M, g) is called* ***geodesically complete*** *if any geodesic $c : I \to M$ extends to $\mathbb{R}$.*
It is called ***geodesically connected*** *if any pair of points in M is linked by a minimising geodesic.*

Examples: $\mathbb{E}^n$ is geodesically complete, but not $\mathbb{R}^n \setminus \{0\}$.

Geodesic completeness and connectedness are linked.

Lemma 10.3.2. *Let (M,g) be a connected Riemannian manifold. Let $x \in M$ be such that $\exp_x$ is defined on all T_xM. For all $y \in M$, there is a minimising geodesic from x to y.*

Proof. Consider $\varepsilon \in]0, \mathrm{inj}_x[$, and denote by $S(x,\varepsilon)$ the sphere of radius ε around x, i.e.

$$S(x,\varepsilon) = \{z \in M \,|\, d(x,z) = \varepsilon\}$$

According to Proposition 10.1.13, we have:

$$S(x,\varepsilon) = \exp_x\Big(S_{T_xM}(0,\varepsilon)\Big)$$

It follows that $S(x,\varepsilon)$ is compact, and we can find $z_\varepsilon \in S(x,\varepsilon)$ such that:

$$\forall z \in S(x,\varepsilon) \quad d(y,z) \leq d(y,z_\varepsilon)$$

Let γ be a piece-wise smooth curve such that $\gamma(0) = x$ and $\gamma(1) = y$. Let t be the first time at which γ exits $S(x,\varepsilon)$, i.e.

$$t = \min\{s \in [0,1] \Big| \gamma(s) \in S(x,\varepsilon)\}$$

We find:

$$\begin{aligned} L(\gamma) &\geq L\Big(\gamma|_{[0,t]}\Big) + L\Big(\gamma|_{[t,1]}\Big) \\ &\geq d(x,\gamma(t)) + d(y,\gamma(t)) \\ &\geq \varepsilon + d(y,z_\varepsilon) \end{aligned}$$

Since the right term does not depend on γ, we find:

$$d(x,y) \leq \varepsilon + d(y,z_\varepsilon)$$

Since $d(x,z_\varepsilon) = \varepsilon$, the triangle inequality yields$\varepsilon + d(y,z_\varepsilon) \geq d(x,y)$, hence $d(x,y) = \varepsilon + d(y,z_\varepsilon)$.

Let $v \in T^1_xM$ be such that $\exp_x(\varepsilon v) = z_\varepsilon$, and set:

$$I = \{t \in [0, d(x,y)] \Big| d(y, \exp_x(tv)) + t = d(x,y)\}$$

The set $I \subset [0, d(x,y)]$ is closed, contains 0, and is open according to the previous discussion. Consequently $d(x,y) \in I$, i.e. $\exp_x(d(x,y)v) = y$. □

Theorem 10.3.3 (Hopf-Rinow)**.** *Let (M,g) be a connected Riemannian manifold, ∇ its Levi-Civita connection and d the Riemannian distance. The following assertions are equivalent.*

1. *(M,g) is geodesically complete.*

2. *$\forall x \in M$ $\exp_x$ is defined on T_xM.*

3. *$\exists x \in M$ $\exp_x$ is defined on T_xM.*

4. *$\forall x \in M$ $\forall R > 0$ $\overline{B}(x,r)$ is compact.*

5. *(M,d) is complete.*

Furthermore, if there conditions are satisfied then (M,g) is geodesically connected.

Remark. *We will say that (M,g) is **complete** if it satisfies these conditions.*

Proof. First note that *(1)⇒(2)* is a matter of definitions, and that *2.⇒(3)* is just specification.
3.⇒4. Lemma 10.3.2 implies that $\overline{B}(x,R) \subset \exp_x\left(\overline{B}_{T_xM}(0,R)\right)$, hence the compactness.
4.⇒5. is a general fact for metric spaces (Cauchy sequences are bounded).
5.⇒1. Let $c :]a,b[\to M$ be a geodesic. Without loss of generality, we can assume that $\|\dot{c}\| = 1$ and $0 \in]a,b[$.
If $b < +\infty$, consider a sequence (t_k) such that $t_k \to b$.
Since c is 1-Lispchitz, the sequence $(c(t_k))$ is Cauchy, therefore converges to some $y \in M$.
For $\varepsilon > 0$ small enough, there is a geodesic γ from $c(b-\varepsilon)$ to y, which can be used to extend c. □

Corollary 10.3.4. *Let (M,g) be a complete Riemannian manifold, and d the Riemannian distance. If the metric space (M,d) is bounded, then M is compact.*

Proof. Let $x \in M$ and $R > 0$ be such that $M = \overline{B}(x,R)$. The Hopf-Rinow Theorem implies that $\overline{B}(x,R)$ is compact, and so is M. □

Corollary 10.3.5. *Let (M,g) be a connected Riemannian manifold. If M is compact, then (M,g) is geodesically complete and geodesically connected.*

This is not true for general pseudo-Riemannian manifolds: the Clifton-Pohl torus is a compact Lorentzian manifold yet is not geodesically complete.

Proposition 10.3.6. *Let (M,g) be a Riemannian manifold. There exists $\sigma \in \mathcal{C}^\infty(M)$ such that the conformal metric $e^\sigma g$ is complete.*

10.3.2 Riemannian coverings

Proposition 10.3.7. *Let (M,g) and (N,h) be connected Riemannian manifolds, and $f : M \to N$ a smooth map.*

1. *If f is a local isometry and (M,g) is complete, then f is a Riemannian covering and (N,h) is complete.*

2. *If (N,h) is complete and f is a Riemannian covering, then (M,g) is complete.*

Proof.

1. Let us start by showing that f is onto, using the connectedness of N. Since f is a local diffeormphism, $f(M)$ is open in N. Let us show that it is also closed.
   Let $x \in M$ and $v \in T_{f(x)}N$. Since f is a local diffeomorphism, set $w = (d_x f)^{-1}(w) \in T_xM$. We then have that $c_v(t) = f(c_w(t))$. Since c_w is defined on $\mathbb{R}$, it is also the case for c_v, and the Riemannian manifold $\left(f(M), h|_{f(M)}\right)$ is complete. Since the Riemannian distance of (N,h), when restricted to $f(M)$, is smaller than the Riemannian distance of $f(M)$ (Lemma 10.1.5). It follows that a sequence (y_k) in $f(M)$ converging to $y \in N$ is Cauchy for the Riemannian distance of $f(M)$. It is therefore convergent in $f(M)$, and since the topologies on $f(M)$ are the same, we find that $y \in f(M)$, i.e. $f(M)$ is closed in N.
   This shows not only that $f(M) = N$, but also that (N,h) is complete.

   Let $y \in N$, and $\varepsilon \in]0, \frac{1}{2}\mathrm{inj}_y N[$. For $x \in f^{-1}(\{y\})$, we set $U_x = \exp_x(B_{T_xM}(0,\varepsilon))$ (it is well defined because (M,g) is complete).

   If $x \neq x'$, let us prove by contradiction that $U_x \cap U_{x'} = \emptyset$. Consider $z \in U_x \cap U_{x'}$, and let $v, v' \in B_{T_xM}(0,\varepsilon)$ be such that $z = \exp_x(v) = \exp_x(v')$. The geodesics $f \circ c_v$ and $f \circ c_{v'}$ join $f(z)$ to y and $d(f(z),y) < \mathrm{inj}_y$, so they must be equal. Consequently $\dot{c}_v(1) = \dot{c}_{v'}(1)$, and $c_v = c_{v'}$, hence $x = c_v(0) = c_{v'}(0) = x'$.

   The restruction $f|_{U_x}$ is injective because $f \circ \exp_x = \exp_y \circ d_x f$ and $\exp_y$ is injective on $B_{T_yN}(0,\varepsilon)$.

   We finally have to show that $f^{-1}(B(y,\varepsilon)) = \bigcup_{f(x)=y} U_x$.

   For $z \in f^{-1}(B(y,\varepsilon))$, we can write $y = \exp_{f(z)}(w)$ where $\|w\|_{f(z)} < \varepsilon$. Let $\widehat{w} = (d_z f)^{-1}(w) \in T_zM$. Since $f(c_{\widehat{w}}(t)) = \exp_{f(z)}(tw)$, if we set $x = c_{\widehat{w}}(1) \in f^{-1}(\{y\})$, we find $z \in U_x$.

   SInce f is 1-lispchitz, we also have $f(U_x) \subset B(y,\varepsilon)$ for any $x \in f^{-1}(\{y\})$.

2. Let $c : I \to M$ be a geodesic. Then $f \circ c$ is a geodesic of N, and extends to $\mathbb{R}$. Every lift of a geodesic of N is a geodesic of M.

□

Proposition 10.3.8. *Let (M,g) and (N,h) be Riemannian manifolds. If (M,g) is complete and $f : M \to N$ is a smooth map such that $f^*h \geq g$, then f is a covering map.*

Proof. Since $f^*h \geq g$, we see that f^*g is a Riemannian metric on M. Now the inequality $f^*h \geq g$ integrates to an inequality on the Riemannian distances:

$$\forall x,y \in M \quad d_{f^*h}(x,y) \geq d_g(x,y)$$

This means that for $x \in M$ and $r > 0$, we have an inclusion of closed balls:

$$\overline{B}_{f^*h}(x,r) \subset \overline{B}_g(x,r)$$

The Hopf-Rinow Theorem assures that $\overline{B}_g(x,r)$ is compact, and so is $\overline{B}_{f^*h}(x,r)$. According to the Hopf-Rinow Theorem, the Riemannian manifold (M,f^*h) is complete. Since f is an isometry from (M,f^*h) to (N,h), we can apply Proposition 10.3.7 and find that f is a covering map. □

10.3.3 Completeness and vector fields

The classical property of finite time explosion for ODEs generalizes to complete Riemannian manifolds.

Proposition 10.3.9. *Let (M,g) be a complete Riemannian manifold, and let $X \in \mathcal{X}(M)$. For $x \in M$, let $I \subset \mathbb{R}$ be maximal the interval on which the flow line $\left(\varphi_X^t(x)\right)_{t\in I}$ is defined. If $t_0 = \sup I$ is finite, then*

$$\limsup_{t\to t_0} \left\| X\left(\varphi_X^t(x)\right)\right\|_{\varphi_X^t(x)} = +\infty$$

Proof. Assume the contrary. Then for any sequence $t_k \in I$ with $t_k \to t_0$, we find that $\varphi_X^{t_k}(x)$ is a Cauchy sequence for the Riemannian distance, so it must converge to some $y \in M$ because (M,g) is complete. Then the flow line starting at y extends the flow line of x on a larger interval, hence the contradiction. □

Corollary 10.3.10. *Let (M,g) be a complete Riemannian manifold. If $X \in \mathcal{X}(M)$ is bounded for the metric g, then it is complete.*

Chapter 11

Pseudo-Riemannian curvature

11.1 The various notions of curvature

11.1.1 Symmetries and contractions of the curvature tensor

Definition 11.1.1. *Let (M,g) be a pseudo-Riemannian manifold, and ∇ the Levi-Civita connection. The curvature field of ∇ is called the* ***Riemann tensor*** *of type (3,1) of (M,g). It is denoted by $R \in \Omega^2(\mathrm{End}(TM))$.*

Let us list its symmetries.

Proposition 11.1.2. *Let (M,g) be a pseudo-Riemannian manifold, ∇ the Levi-Civita connection, and R its type (3,1) Riemann tensor.*
For all $x \in M$ and $u,v,w,z \in T_xM$, we have the following symmetries:

1. $R_x(u,v)w = -R_x(v,u)w$.

2. $R_x(u,v)w + R_x(v,w)u + R_x(w,u)v = 0$.

3. $g_x(R_x(u,v)w,z) = -g_x(R_x(u,v)z,w)$.

4. $g_x(R_x(u,v)w,z) = g_x(R_x(w,z)u,v)$.

Remark. *Property 3. can be written as $R_x(u,v) \in \mathfrak{so}(g_x)$.*

Proof. Property *1.* is a consequence of the skew-symmetry of the curvature, i.e. $R \in \Omega^2(\mathrm{End}(TM))$.

Property *2.* is the first Bianchi identity.

In order to prove property *3.* we consider a smooth function $f : \mathbb{R}^2 \to M$ such that $f(0) = x$, $\frac{\partial f}{\partial t}(0) = u$ and $\frac{\partial f}{\partial s}(0) = v$. Also consider sections $\sigma, \tau \in$

$\Gamma(f^*TM)$ such that $\sigma(0) = w$ and $\tau(0) = z$. First we compute:

$$\begin{aligned}
g\left(R\left(\frac{\partial f}{\partial t}, \frac{\partial f}{\partial s}\right)\sigma, \tau\right) &= g\left(\left[\frac{D}{\partial t}, \frac{D}{\partial s}\right]\sigma, \tau\right) \\
&= g\left(\frac{D}{\partial t}\frac{D}{\partial s}\sigma, \tau\right) - g\left(\frac{D}{\partial s}\frac{D}{\partial t}\sigma, \tau\right) \\
&= -g\left(\frac{D}{\partial s}\sigma, \frac{D}{\partial t}\tau\right) - \frac{\partial}{\partial t} g\left(\frac{D}{\partial s}\sigma, \tau\right) \\
&\quad + g\left(\frac{D}{\partial t}\sigma, \frac{D}{\partial s}\tau\right) + \frac{\partial}{\partial s} g\left(\frac{D}{\partial t}\sigma, \tau\right)
\end{aligned}$$

When symmetrizing in σ and τ, half of the terms disappear and the rest yields:

$$\begin{aligned}
g\left(R\left(\frac{\partial f}{\partial t}, \frac{\partial f}{\partial s}\right)\sigma, \tau\right) + g\left(R\left(\frac{\partial f}{\partial t}, \frac{\partial f}{\partial s}\right)\tau, \sigma\right) &= \frac{\partial}{\partial s}\left[g\left(\frac{D}{\partial t}\sigma, \tau\right) + g\left(\sigma, \frac{D}{\partial t}\tau\right)\right] \\
&\quad - \frac{\partial}{\partial t}\left[g\left(\frac{D}{\partial s}\sigma, \tau\right) + g\left(\sigma, \frac{D}{\partial s}\tau\right)\right] \\
&= \frac{\partial^2}{\partial s \partial t} g(\sigma, \tau) - \frac{\partial^2}{\partial t \partial s} g(\sigma, \tau) \\
&= 0
\end{aligned}$$

Evaluating at 0 yields property *3.*

In order to prove property *4.*, let us use normal coordinates $(x^1, \ldots, x^d)$ around x. We then have that $g_{i,j}(x) = g^{i,j}(x) = \delta_{i,j}$ and $\partial_k g_{i,j}(x) = 0$ (also $\Gamma^k_{i,j}(x) = 0$), hence $R^l_{i,j,k}(x) = \partial_i \Gamma^l_{j,k}(x) - \partial_j \Gamma^l_{i,k}(x)$. Derivatives at x simplify a lot:

$$\begin{aligned}
\partial_i \Gamma^l_{j,k}(x) &= \partial_i \left(\frac{1}{2}\sum_{m=1}^{d} g^{l,m}(\partial_j g_{k,m} + \partial_k g_{j,m} - \partial_m g_{j,j})\right)(x) \\
&= \frac{1}{2}(\partial^2_{i,j} g_{k,l}(x) + \partial^2_{i,k} g_{j,l}(x) - \partial^2_{i,l} g_{j,k}(x))
\end{aligned}$$

We now get:

$$\begin{aligned}
g_x(R_x(\partial_i, \partial_j)\partial_k, \partial_l) &= R^l_{i,j,k}(x) \\
&= \frac{1}{2}\left(\partial^2_{i,k} g_{j,l}(x) - \partial^2_{i,l} g_{j,k}(x) - \partial^2_{j,k} g_{i,l}(x) + \partial^2_{j,l} g_{i,k}(x)\right)
\end{aligned}$$

This formula remains invariant when switching (i, j) and (k, l), which is the desired symmetry (and all other symmetries can be retrieved through this formula). $\square$

These symmetries are all encoded in the type (4,0) tensor obtained by lowering an index of the type (3,1) curvature tensor.

Definition 11.1.3. *Let (M,g) be a pseudo-Riemannian manifold, ∇ the Levi-Civita connection, and R its type (3,1) Riemann tensor.*
The ***type (4,0) Riemann tensor*** *is the tensor $R \in \Gamma\left(T^*M^{\otimes 4}\right)$ defined by:*

$$\forall x \in M \ \forall u,v,w,z \in T_xM \quad R_x(u,v,w,z) = g_x(R_x(u,v)w,z)$$

We use the same letter for the type (3,1) and the type (4,0) tensor as there is very little chance of it inducing a confusion.

Proposition 11.1.4. *Let (M,g) be a pseudo-Riemannian manifold. The type (4,0) Riemann tensor R satisfies the following symmetries, for $x \in M$ and $u,v,w,z \in T_xM$:*

$$\begin{aligned} R_x(u,v,w,z) &= -R_x(v,u,w,z) \\ &= -R_x(u,v,z,w) \\ &= R(w,z,u,v) \end{aligned}$$

Remark. *These symmetries are summarized by $R \in \Gamma\left(S^2(\Lambda^2 T^*M)\right)$.*

Proof. These are just concise versions of Proposition 11.1.2. □

In local coordinates, we write $R_{i,j,k,l} = R(\partial_i, \partial_j, \partial_k, \partial_l)$, and find:

$$R_{i,j,k,l} = \sum_{m=1}^{n} g_{m,l} R_{i,j,k}^{m}$$

Definition 11.1.5. *Let (M,g) be a pseudo-Riemannian manifold. For $x \in M$ and a vector plane $P \subset T_xM$ non degenerate for g_x, if (v,w) is a vector basis of P, the* ***sectional curvature*** *$K(P)$ is defined by:*

$$K(P) = \frac{R_x(v,w,w,v)}{g_x(v,v)g_x(w,w) - g_x(v,w)^2}$$

Remarks.

- *It does not depend on the choice of a vector basis (v,w) of P.*
- *The fact that P is non degenerate for g_x is equivalent to the non vanishing of the denominator.*
- *If g is Riemannian, and (v,w) is an orthonormal basis of P, then $K(P) = R_x(v,w,w,v)$.*

- *If g is Riemannian, then planes are always non degenerate, so the sectional curvature can be defined as a function $K : \mathcal{G}_2(TM) \to \mathbb{R}$ where $\mathcal{G}_2(\mathbb{R})$ is the (total space of the) fibre bundle above M whose fibre over $x \in M$ is the Grassmannian $\mathcal{G}_2(T_xM)$. The function K is smooth. It follows that for a compact M, the sectional curvature is bounded.*

One can show that the sectional curvature determines the Riemann tensor R (it is a consequence of the symmetries of the Riemann tensor).

Definition 11.1.6. *Let (M,g) be a pseudo-Riemannian manifold, and R its type (3,1) Riemann tensor. The* ***Ricci curvature*** *of (M,g) is the type (2,0) tensor $\mathrm{Ric} \in \Gamma\left((T^*M)^{\otimes 2}\right)$ defined for $x \in M$ and $v, w \in T_xM$ by:*

$$\mathrm{Ric}_x(v,w) = \mathrm{Tr}\,(z \mapsto R_x(z,v)w)$$

Proposition 11.1.7. *Let (M,g) be a pseudo-Riemannian manifold. The Ricci curvature Ric is symmetric, i.e. $\forall x \in M\ \forall v,w \in T_xM \quad \mathrm{Ric}_x(v,w) = \mathrm{Ric}_x(w,v)$.*

Remark. *This is summarized by $\mathrm{Ric} \in \Gamma(S^2T^*M)$.*

Proof. Let $(e_1,\dots,e_d)$ be an orthonormal basis of T_xM. Then we have:

$$\mathrm{Ric}_x(v,w) = \sum_{i=1}^{d} R_x(e_i,v,w,e_i)$$

The symmetries of the type (4,0) tensor imply that Ric_x is symmetric. □

In local coordinates, we write $R_{i,j} = \mathrm{Ric}(\partial_i,\partial_j)$, and we find:

$$R_{i,j} = \sum_{k=1}^{d} R^k_{k,i,j} = \sum_{1\le k,l\le d} g^{k,l} R_{k,i,j,l}$$

Note that the Ricci curvature is of the same type as the metric, so it makes sense to compare them.

Definition 11.1.8. *An* ***Einstein manifold*** *is a pseudo-Riemannian manifold (M,g) for which there is $\lambda \in \mathbb{R}$ such that $\mathrm{Ric} = \lambda g$.*

Remarks.

- *If (M,g) has constant sectional curvature equal to κ, then $\mathrm{Ric} = (n-1)\kappa g$.*
- *This implies that $\nabla\mathrm{Ric} = 0$, and it is almost an equivalence (in other terms, an Einstein manifold should be interpreted as a pseudo-Riemannian manifold with constant Ricci curvature).*

Definition 11.1.9. *Let (M,g) be a pseudo-Riemannian manifold, and* $\mathrm{Ric} \in \Gamma(S^2T^*M)$ *its Ricci curvature. The **scalar curvature** of (M,g) is the function* $R = \mathrm{Scal} \in \mathcal{C}^\infty(M)$ *defined by:*

$$\forall x \in M \quad R(x) = \mathrm{Scal}(x) = \mathrm{Tr}_{g_x}(\mathrm{Ric}_x)$$

where Tr_{g_x} *is the trace of a quadratic form with reference g_x.*

Remarks.

- *If $(e_i)_{1\leq i\leq d}$ is a g_x-orthonormal frame of T_xM, then $R(x) = \sum_{i=1}^d \mathrm{Ric}_x(e_i,e_i)$.*
- *It can also be defined as $R(x) = \mathrm{Tr}(f_x)$ where $f_x \in \mathrm{End}(T_xM)$ is the g_x-self adjoint operator such that $\mathrm{Ric}_x(v,w) = g_x(v, f_x(w))$ for all $v,w \in T_xM$.*
- *In local coordinates, $R = \sum_{1\leq i,j\leq d} g^{i,j}R_{i,j}$.*

Einstein's equation: in the theory of General Relativity, a spacetime is represented by a 4-dimensional Lorentzian manifold (M,g) (Special Relativity corresponds to the Minkowski space $\mathbb{M}^4$). The physics of a spacetime are encoded in a type (2,0) tensor $T \in \Gamma(S^2(T^*M))$, called the stress-energy tensor, and Einstein's equation is an equation on the Lorentzian metric g:

$$\mathrm{Ric} - \frac{1}{2}R + \Lambda g = T$$

where Λ is called the cosmological constant.

Proposition 11.1.10. *Let (M,g) be a pseudo-Riemannian manifold. For $\lambda > 0$, the following tensors and functions associated to the metrics g and $\lambda^2 g$ are related in the following way:*

1. $\nabla^{\lambda^2 g} = \nabla^g$
2. $\mathrm{dvol}^{\lambda^2 g} = \lambda^n \,\mathrm{dvol}^g$
3. *If g is Riemannian, then $d_{\lambda^2 g} = \lambda d_g$.*
4. *$R^{\lambda^2 g} = R^g$ (where R is the type (3,1) Riemann tensor).*
5. $K^{\lambda^2 g} = \frac{1}{\lambda^2}K^g$
6. $\mathrm{Ric}^{\lambda^2 g} = \mathrm{Ric}^g$
7. $\mathrm{Scal}^{\lambda^2 g} = \frac{1}{\lambda^2}\mathrm{Scal}^g$.

Moreover, isometries preserve R, K, Ric and Scal.

Definition 11.1.11. *We say that a Riemannian manifold (M,g) has* ***pinched sectional curvature*** *if there are $\alpha,\beta \in \mathbb{R}$ such that $\alpha \le K(P) \le \beta$ for every plane $P \in \mathcal{G}_2(TM)$.*
We say that it has ***negative pinched curvature*** *if there are $b < a < 0$ such that $-b^2 \le K(P) \le -a^2$ for every plane $P \in \mathcal{G}_2(TM)$.*

Remark. *These definitions would make sense for pseudo-Riemannian manifolds, but they are totally irrelevant.*

The formulae for different types of curvature look quite intimidating when it comes to computing them in examples. For this, we can try to cheat and use the abundance of isometries in the three main examples.

If (M,g) is Riemannian, and $\mathrm{Isom}(M,g) \curvearrowright M$ is transitive, then (M,g) has pinched sectional curvature, and Scal is constant.

If moreover $\mathrm{Isom}(M,g) \curvearrowright \mathcal{G}_2(TM)$ is transitive, then (M,g) has constant sectional curvature.

Consequence: the Riemannian manifolds $\mathbb{E}^n$, $(\mathbb{S}^n, \lambda^2 g_{sph})$ and $(\mathbb{H}^n, \lambda^2 g_{hyp})$ have constant sectional curvature (we will see that their values are respectively $0, \frac{1}{\lambda^2}$ and $-\frac{1}{\lambda^2}$).

11.1.2 Jacobi fields as variation fields

Consider a Riemannian manifold (M,g) and a geodesic $c : I \to M$.

Reminder: A Jacobi field along c is $J : I \to TM$ such that $\forall t \in I$ $J(t) \in T_{c(t)}M$ and $\frac{D}{dt}\frac{D}{dt}J + R(J,\dot c)\dot c = 0$.

A Jacobi field J is determined by $J(t_0)$ and $\frac{D}{dt}J(t_0)$ for a given $t_0 \in I$.

Proposition 11.1.12. *Let (M,g) be a Riemannian manifold, and $c : I \to M$ a geodesic. If J is a Jacobi field along c, and if*

$$\exists t_0 \in I \quad J(t_0) \perp \dot c(t_0) \text{ and } \frac{D}{dt}J(t_0) \perp \dot c(t_0)$$

then

$$\forall t \in I \quad J(t) \perp \dot c(t) \text{ and } \frac{D}{dt}J(t) \perp \dot c(t)$$

Proof. Since c is a geodesic, we have that:

$$\frac{d}{dt} g\left(\frac{D}{dt}J, \dot c\right) = g\left(\frac{D}{dt}\frac{D}{dt}J, \dot c\right)$$

Using the fact that J is a Jacobi field, it follows that:

$$\frac{d}{dt} g\left(\frac{D}{dt}J, \dot c\right) = -R(J, \dot c, \dot c, \dot c) = 0$$

In other words, $g(\frac{D}{dt}J, \dot{c})$ is constant. If it vanishes a t_0, then it vanishes everywhere.
We now have $\frac{d}{dt}g(J, \dot{c}) = g(\frac{D}{dt}J, \dot{c}) = 0$, so $g(J, \dot{c})$ is also constant, and vanishes on I. □

Lemma 11.1.13. *Let (M, g) be a Riemannian manifold, and $c : I \to M$ a geodesic. Consider a function $f \in \mathcal{C}^\infty(I)$, and set $J = f\dot{c}$. Then J is a Jacobi field if and only if $\ddot{f} = 0$.*

Proof. Since $\frac{D}{dt}\dot{c} = 0$, we find $\frac{D}{dt}\frac{D}{dt}J = \frac{D}{dt}(\dot{f}\dot{c}) = \ddot{f}\dot{c}$. Now $R(J, \dot{c})\dot{c} = fR(\dot{c}, \dot{c})\dot{c} = 0$ because of skew-symmetry. □

Definition 11.1.14. *Let (M, g) be a Riemannian manifold, and $c : I \to M$ a geodesic. A Jacobi field J along c is called **orthogonal** if*

$$\forall t \in I \quad J(t) \perp \dot{c}(t) \text{ and } \frac{D}{dt}J(t) \perp \dot{c}(t)$$

*It is called **tangent** if there are $a, b \in \mathbb{R}$ such that $J(t) = (at + b)\dot{c}(t)$ for all $t \in I$.*

Proposition 11.1.15. *Let (M, g) be a Riemannian manifold, $c : I \to M$ a geodesic and J a Jacobi field along c. Then g is tangent if and only if there is $t_0 \in I$ such that $J(t_0)$ and $\frac{D}{dt}J(t_0)$ are proportional to $\dot{c}(t_0)$.*

Proof. If J is tangent, then the computation made in Lemma 11.1.13 shows that J and $\frac{D}{dt}J$ are proportional to $\cdot J$ everywhere.
If $\frac{D}{dt}J(t_0) = a\dot{c}(t_0)$ and $J(t_0) = (at_0 + b)\dot{c}(t_0)$, then the $t \mapsto (at + b)\dot{c}(t)$ is a Jacobi field according to Lemma 11.1.13, and has the same initial condition at t_0 as J, so it must be equal to J, therefore J is tangent. □

Proposition 11.1.16. *Let (M, g) be a Riemannian manifold, and $c : I \to M$ a geodesic. If J is a Jacobi field along c, then J decomposes uniquely as $J = J^T + J^\perp$ where J^T is a tangent Jacobi field along c and $J^\perp$ is an orthogonal Jacobi field along c.*

Proof. The uniqueness comes from the fact that tangent and orthogonal Jacobi fields form vector spaces whose intersection in null.
For the existence, fix $t_0 \in I$, and decompose $\frac{D}{dt}J(t_0) = a\dot{c}(t_0) + u$ and $J(t_0) = (at_0 + b)\dot{c}(t_0) + v$ where $a, b \in \mathbb{R}$ and $u, v \in \dot{c}(t_0)^\perp$. Let $J^T(t) = (at + b)\dot{c}(t)$, and let $J^\perp$ be the Jacobi field along c such that $J^\perp(t_0) = u$ and $\frac{D}{dt}J^\perp(t_0) = v$. Then $J^T + J^\perp$ is a Jacobi field along c with the same initial data at t_0 as J, therefore $J = J^T + J^\perp$.
By construction J^T is tangent, and $J^\perp$ is orthogonal thanks to Proposition 11.1.12. □

Definition 11.1.17. *Let (M, g) be a Riemannian manifold, and $c : I \to M$ a smooth curve.*

*A **variation** of c is a smooth map $f : I \times]-\varepsilon,\varepsilon[\to M$ for some $\varepsilon > 0$ such that, if $c_s : I \to M$ is the curve defined by $c_s(t) = f(t,s)$, then $c_0 = c$.*
*It is a **geodesic variation** if all the curves c_s are geodesics.*
*The **variation field** of f the the vector field J along c defined by $J(t) = \frac{\partial}{\partial s}\big|_{s=0} c_s(t) \in T_{c(t)}M$.*

Proposition 11.1.18. *Let (M,g) be a Riemannian manifold, $c : I \to M$ a geodesic, and J a vector field along c.*
Then J is a Jacobi field if and only if for every $t_0 \in I$, there is an open interval $I_0 \subset I$ with $t_0 \in I$ such that $J|_{I_0}$ is the variation field of a geodesic variation of $c|_{I_0}$.

Remark. *If (M,g) is geodesically complete, then the local asumption can be removed: J is a Jacobi field if and only if it is the variation field of a geodesic variation of c.*

Proof. The fact that the variation field of a geodesic variation is a Jacobi field was proved in section **??**.

If J is a Jacobi field and $t_0 \in I$, first consider a geodesic $\gamma : [-\varepsilon,\varepsilon] \to M$ such that $\gamma(0) = c(t_0)$ and $\dot\gamma(0) = J(t_0)$.

Let $X, Y :]-\varepsilon,\varepsilon[\to TM$ be the parallel vector fields along γ such that $X(0) = \dot c(t_0)$ and $Y(0) = \frac{D}{dt}J(t_0)$.

Let $I_0 \subset I$ be an open interval such that $t_0 \in I$ and the geodesic c_s with initial condition $c_s(t_0) = \gamma(s)$ and $\dot c_s(t_0) = X(s) + sY(s)$ is defined on I_0 (note that $c_0 = c$).
On pose $c_s(t) = \exp_{\gamma(s)}(tX(s) + stY(s))$.

Then $f : I_0 \times]-\varepsilon,\varepsilon[\to M$ defined by $f(t,s) = c_s(t)$ is a geodesic variation of c. It follows that $J_1 = \frac{\partial}{\partial s}\big|_{s=0} c_s$ is a Jacobi field.

Since $c_s(t_0) = \gamma(s)$, we have that $J_1(t_0) = \dot\gamma(0) = J(0)$. We also have that $\frac{D}{\partial t}\frac{\partial f}{\partial s} = \frac{D}{\partial s}\frac{\partial f}{\partial t}$ and $\frac{\partial f}{\partial t}(t_0,s) = \dot c_s(t_0) = X(s) + sY(s)$, so $\frac{D}{\partial s}\frac{\partial f}{\partial t}(t_0,0) = Y(0)$ (because X and Y are parallel along γ). This shows that $\frac{D}{dt}J_1(t_0) = \frac{D}{dt}J(t_0)$, so $J_1 = J$, and J is the variation field of a geodesic variation on I_0. □

11.1.3 The second variation formula

Jacobi fields also arise naturally from the variational study of geodesics.

Definition 11.1.19. *Let (M,g) be a Riemannian manifold, R its type $(3,1)$ curvature tensor and $c : [a,b] \to M$ a geodesic. The bilinear form*

$$I : \begin{cases} \Gamma(c^*TM) \times \Gamma(c^*TM) & \to & \mathbb{R} \\ (X,Y) & \mapsto & -\int_a^b g\left(X, \frac{D}{dt}\frac{D}{dt}Y + R(Y,\dot c)\dot c\right) dt \end{cases}$$

*is called the **index form** of c.*

Theorem 11.1.20. *Let (M,g) be a Riemannian manifold, $c:[a,b]\to M$ a geodesic, $f:[a,b]\times\,]-\varepsilon,\varepsilon[\,\to M$ a variation of c, and J its variation field. We have:*

$$\left.\frac{d^2}{ds^2}\right|_{s=0} E(c_s) = I(J,J)$$

11.1.4 Jacobi fields and sectional curvature

Let (M,g) be a Riemannian manifold, $x\in M$, $P\subset T_xM$ a plane, and (v,w) vector basis of T_xM.

Let $c=c_v$ be the geodesic with initial velocity v, and J a Jacobi field along c such that $J(0)=w$.

$$K(P)=\frac{g_x(R_x(v,w)w,v)}{g_x(v,v)g_x(w,w)-g_x(v,w)^2}=-\frac{g_x\left(\frac{D}{dt}\frac{D}{dt}J(0),J(0)\right)}{g_x(\dot c(0),\dot c(0))g_x(J(0),J(0))-g_x(\dot c(0),J(0))^2}$$

In particular, if $g_x(v,v)=1$, $g_x(v,w)=0$, and J is orthogonal along c, we find:

$$K\left(\mathbb{R}\dot c(0)\oplus\mathbb{R}J(0)\right)=-\frac{\left\langle\frac{D}{dt}\frac{D}{dt}J(0)\middle|J(0)\right\rangle}{\|J(0)\|^2}$$

If we know the explicit expressions of geodesics of a given Riemannian manifold, then we can compute Jacobi fields using geodesic variations, and this formula gives the sectional curvature.

11.2 Curvature and topology

11.2.1 Riemannian manifolds with constant sectional curvature

Theorem 11.2.1. *For $n\geq 2$, the sectional curvature of $(\mathbb{S}^n,g_{sph})$ is $+1$, that of $(\mathbb{H}^n,g_{hyp})$ is -1.*

Proof. For $x\in\mathbb{S}^n\subset\mathbb{E}^{n+1}$ and $v\in T_x\mathbb{S}^n=x^\perp$ such that $\|v\|=1$, the geodesic c_v is given by:

$$c(t)=\cos t x+\sin t v$$

If $w\in T_x\mathbb{S}^n$ is such that $\|w\|=1$ and $\langle v|w\rangle=0$, then:

$$f(s,t)=\cos t x+\sin t(\cos s v+\sin s w)$$

is a geodesic variation of c. Therefore $J(t)=\sin t w$ is a Jacobi field along c.

Now $\frac{D}{dt}\frac{D}{dt}J=\ddot J=-J$, so for t such that $\sin t\neq 0$, we have $K(\mathbb{R}\dot c(t)\oplus\mathbb{R}J(t))=-\frac{\langle\ddot J(t)|J(t)\rangle}{\langle J(t)|J(t)\rangle}=1$.

Since we have already seen that the sectional curvature is constant, we find that it is 1 everywhere.

For the hyperbolic space $\mathbb{H}^n$, consider the hyperboloïd model $\mathcal{H}^n \subset \mathbb{M}^{n+1}$. The geodesic c_v is now given by:

$$c(t) = \cosh t x + \sinh t v$$

The geodesic variation is:

$$f(s,t) = \cosh t x + \sinh t(\cos s v + \sin s w)$$

The associated Jacobi field is $J(t) = \sinh t w$, it satisfies $\frac{D}{dt}\frac{D}{dt}J = \ddot{J} = J$, and the sectional curvature is $K = -1$. □

Theorem 11.2.2. *Let (M,g) be a Riemannian manifold of dimension n with constant sectional curvature equal to $\kappa \in \mathbb{R}$. Then every $x \in M$ has a neighbourhood U isometric to an open set of:*

$$\begin{cases} \mathbb{E}^n & \text{if } \kappa = 0 \\ (\mathbb{S}^n, \frac{1}{\kappa} g_{sph}) & \text{if } \kappa > 0 \\ (\mathbb{H}^n, -\frac{1}{\kappa} g_{hyp}) & \text{if } \kappa < 0 \end{cases}$$

Moreover, if (M,g) is complete and simply connected, then it is globally isometric to this model space.

Lemma 11.2.3. *Let (M,g) be a Riemannian manifold with constant sectional curvature equal to $\kappa \in \mathbb{R}$. For all $x \in M$, if (u,v) is an orthonormal basis of a plane $P \subset T_xM$, then $R_x(u,v)v = \kappa u$.*

Proof. Let us start with showing that $R_x(u,v)v \in \mathbb{R}.v \oplus \mathbb{R}.w$. If $\dim M = 2$, it is automatic. If $\dim M \geq 3$, we can consider $w \in T_xM$ such that $\|w\|_x = 1$ and $g_x(u,w) = g_x(v,w) = 0$.
Notice that since (u,v) and (v,w) are both orthonormal, we have:

$$R_x(u,v,v,u) = R_x(w,v,v,w) = \kappa$$

Expressing the sectional curvature of the plane generated by v and $u+w$ yields $R(u+w,v,v,u+w) = 2\kappa$.
The symmetries of the (4,0) Riemann tensor show that $R_x(w,v,v,u) = R_x(u,v,v,w)$, and multi-linearity leads to:

$$\underbrace{R_x(u+w,v,v,u+w)}_{=2\kappa} = \underbrace{R_x(u,v,v,u)}_{=\kappa} + 2R_x(u,v,v,w) + \underbrace{R_x(w,v,v,w)}_{=\kappa}$$

It follows that $R_x(u,v,v,w) = 0 = g_x(R(u,v)v,w)$. This being true for any unitary vector w orthogonal to u and w, we find that $R_x(u,v)v = \lambda u + \mu v$ for some $\lambda, \mu \in \mathbb{R}$.
Now $R_x(u,v,v,v) = 0$ yields $\mu = 0$, and $R_x(u,v,v,u) = \kappa$ yields $\lambda = \kappa$, hence the result. □

Lemma 11.2.4. *Let (M,g) be a Riemannian manifold with constant sectional curvature equal to $\kappa \in \mathbb{R}$. For $x \in M$, and $u,v \in T_xM$ such that $g_x(u,v) = 0$ and $g_x(v,v) = 1$, we let U be the parallel vector field along c_v such that $U(0) = u$, and J the Jacobi field along c_v such that $J(0) = 0$ and $\frac{D}{dt}J(0) = u$.*
For $t \in I_v$, we have $J(t) = f(t)U(t)$, where

$$f(t) = \begin{cases} t & \text{if } \kappa = 0 \\ \frac{1}{\sqrt{\kappa}} \sin\left(\sqrt{\kappa} t\right) & \text{if } \kappa > 0 \\ \frac{1}{\sqrt{-\kappa}} \sinh\left(\sqrt{-\kappa} t\right) & \text{if } \kappa < 0 \end{cases}$$

Proof. Notice that the function f satisfies (and is determined by) $f(0) = 0$, $\dot f(0) = 1$, and $\ddot f = -\kappa f$.

Set $J_1(t) = f(t)U(t)$. Since U is parallel along c_v, we find $\frac{D}{dt}J_1 = \dot f U$, and $\frac{D}{dt}\frac{D}{dt}J_1 = \ddot f U = -\kappa J_1$.

According to Lemma 11.2.3, we have $R(J_1, \dot c_v)\dot c_v = \kappa J_1$. So J_1 is a Jacobi field along c_v. But $J_1(0) = 0$ and $\frac{D}{dt}J_1(0) = u$, therefore $J_1 = J$. □

Proof of Theorem 11.2.2. Multiplying g by some well chosen $\lambda > 0$ if necessary, we can assume that $\kappa = 0$, 1 or -1.
Fix some $x \in M$.

Flat case: Let us show that $\exp_x : (T_xM, g_x) \to (M,g)$ is a local isometry.

Let $v \in T_xM$ be such that $\exp_x(v)$ is well defined. Set $c(t) = \exp_x(tv)$.

For $w \in T_xM$, we have $d_v \exp_x(w) = J(1)$, where J is the Jacobi field along c satisfying $J(0) = 0$ and $\frac{D}{dt}J(0) = w$. Write $w = \lambda v + u$ where $g_x(v,u) = 0$.

Lemma 11.2.4 shows that $J(1) = U(1)$ where U is the parallel vector field along c such that $U(0) = w$.

It follows that $d_v \exp_x(w) = J(1) + \lambda \dot c(1)$ is the value at 1 of a parallel vector field along c. Since the parallel transport is isometric, it follows that $d_v \exp_x$ is isometric, i.e. $\exp_x$ is a local isometry.

In the complete case, the map $\exp_x$ is local isometry from $\mathbb{E}^n$ to M, hence a Riemannian covering by Proposition 10.3.7, and an isometry if M is simply connected.

Negative curvature case: Now Lemma 11.2.4 gives $\left\| d_{tv} \exp_x(tw) \right\| = \sinh t$ when $g_x(v,w) = 0$.

Consider some $\widetilde{x} \in \mathbb{H}^n$. We know that $\exp_{\widetilde{x}} = T_{\widetilde{x}}\mathbb{H}^n \to \mathbb{H}^n$ is a diffeomorphism.

Fix a linear isometry $f : T_{\widetilde{x}}\mathbb{H}^n \to T_xM$. Just as in the flat case, Lemma 11.2.4 shows that $\exp_{\widetilde{x}}^* g_{hyp} = (\exp_x \circ f)^* g$.

The map $F = \exp_x \circ f \circ \exp_{\widetilde{x}}^{-1}$ is an isometry (local or global depending on the hypothesis) with $\mathbb{H}^n$.

Positive curvature case: Now Lemma 11.2.4 gives $\left\|d_{tv}\exp_x(tw)\right\| = \sin t$ when $g_x(v,w) = 0$.

The construction of a local isometry works exactly as in the negative curvature case.

In the complete and simply connected case, we have to be more careful. Using the inverse of the exponential map, we build a local isometry $F : \mathbb{S}^n \setminus \{-\widetilde{x}\} \to M$.

Fix some $\widetilde{y} \in \mathbb{S}^n \setminus \{\widetilde{x}, -\widetilde{x}\}$. Consider $y = F(\widetilde{y})$, and the same construction as for F gives a local isometry $G : \mathbb{S}^n \setminus \{-\widetilde{y}\}$ such that $G(\widetilde{y}) = y$ and $d_{\widetilde{y}}G = d_{\widetilde{y}}F$.

Since $\mathbb{S}^n \setminus \{\pm\widetilde{x}, \pm\widetilde{y}\}$ is connected, It follows from Proposition 9.6.9 that $F = G$ on this subset, and we have built a local isometry $\mathbb{S}^n \to M$, which is an isometry because M is simply connected. □

Consequence: If a Riemannian manifold (M,g) has constant sectional curvature equal to κ, then (M,g) is isometric to a quotient $\mathbb{X}^n_\kappa/\Gamma$ where $\Gamma \subset \mathrm{Isom}(\mathbb{X}^n_\kappa)$ is isomorphic to $\pi_1(M)$.

All surfaces posses a Riemannian metric with constant sectional curvature. This is not true for higher dimensional manifolds (e.g. $\mathbb{S}^2 \times \mathbb{S}^1$).

Theorem 11.2.5 (Poincaré-Koebe Uniformisation Theorem)**.**
Let (M,g) be a two-dimensional Riemannian manifolds. There is a constant curvature metric g' conformal to g. It is unique up to homothety.

Case $\kappa = 0$: A Riemannian manifold (M,g) with constant sectional curvature equal to 0 is called **flat**.

Theorem 11.2.6 (Bieberbach Theorem)**.**
If $\mathbb{E}^n/\Gamma$ is compact and orientable, then $\Gamma \cap \mathbb{R}^n$ is the group generated by n linearly independent translations, and has finite index in G.

Flat compact surfaces are the torus $\mathbb{T}^2 = \mathbb{E}^2/\mathbb{Z}^2$ and the Klein bottle $\mathbb{E}^2/\Gamma$ where Γ is the group generated by $(x,y) \mapsto (x+1,y)$ and $(x,y) \mapsto (1-x,y+1)$.

Case $\kappa = 1$: A Riemannian manifold with constant sectional curvature equal to $+1$ is called **spherical** (note that it is not the negation of aspherical…).

A complete spherical manifold is compact.

Compact spherical surfaces are the sphere $\mathbb{S}^2$ and the projective plane $\mathbb{RP}^2$.

Case $\kappa = -1$**:** A Riemannian manifold with constant sectional curvature equal to -1 is called **hyperbolic**.

Hyperbolic manifolds are plentiful (e.g. all the remaining compact surfaces), their study is still an active area of research.

11.2.2 The topology of non positively curved Riemannian manifolds

Given $\kappa \in \mathbb{R}$, we define the function $f_\kappa : \mathbb{R} \to \mathbb{R}$ by:

$$f_\kappa(t) = \begin{cases} t & \text{if } \kappa = 0 \\ \frac{1}{\sqrt{\kappa}} \sin\left(\sqrt{\kappa}t\right) & \text{if } \kappa > 0 \\ \frac{1}{\sqrt{-\kappa}} \sinh\left(\sqrt{-\kappa}t\right) & \text{if } \kappa < 0 \end{cases}$$

It is the solution of the ODE $\ddot{y} + \kappa y = 0$ with initial conditions $f_\kappa(0) = 0$ and $\dot{f}_\kappa(0) = 1$.

Recall that if J is a Jacobi field along a unit speed geodesic in a Riemannian manifold with constant sectional curvature equal to κ, and $J(0) = 0$, then $\|J(t)\| = \left\|\frac{D}{dt}J(0)\right\| f_\kappa(t)$ for $t \geq 0$ (and $t < \frac{\pi}{\sqrt{\kappa}}$ when $\kappa > 0$).

Lemma 11.2.7. *Let (M,g) be a Riemannian manifold with sectional curvature bounded from above by $\kappa_0 \in \mathbb{R}$.*
If $c : I \to M$ is a unit speed geodesic, and $J : I \to TM$ is an orthogonal Jacobi field along c such that $J(0) = 0$, then $\|J(t)\| \geq \left\|\frac{D}{dt}J(0)\right\| f_{\kappa_0}(t)$ for $t \in I$ (and $t < \frac{\pi}{\sqrt{\kappa_0}}$ when $\kappa_0 > 0$).

Proof. Assume that $\left\|\frac{D}{dt}J(0)\right\| = 1$ (which is always possible unless $\frac{D}{dt}J(0) = 0$, in which case $J = 0$ and the result is straightforward).

Set $u(t) = \|J(t)\|$. If $u(t) \neq 0$, we find:

$$\dot{u}(t) = \frac{\left\langle J(t) \middle| \frac{D}{dt}J(t)\right\rangle}{\|J(t)\|}$$

Let us differentiate once more:

$$\begin{aligned}
\ddot{u}(t) &= \frac{\left\langle \frac{D}{dt}J \middle| \frac{D}{dt}J \right\rangle + \left\langle J \middle| \frac{D}{dt}\frac{D}{dt}J \right\rangle}{\|J\|} - \frac{\left\langle J \middle| \frac{D}{dt}J \right\rangle^2}{\|J\|^3} \\
&= \frac{\left\| \frac{D}{dt}J \right\|^2 \|J\|^2 - \left\langle J \middle| \frac{D}{dt}J \right\rangle^2}{\|J\|^3} - \frac{\langle R(J,\dot{c})\dot{c}|J\rangle}{\|J\|} \\
&\geq -\frac{\langle R(J,\dot{c})\dot{c}|J\rangle}{\|J\|} \\
&\geq -\kappa_0 \|J\|
\end{aligned}$$

Hence $\ddot{u} \geq -\kappa_0 u$.

In order to justify this formula, let us prove that $u(t) \neq 0$ when $t > 0$ is small enough. Set $v = u^2 = g(J,J)$. Then $\dot{v} = 2\left\langle J \middle| \frac{D}{dt}J \right\rangle$, hence $\ddot{v} = 2\left\| \frac{D}{dt}J \right\|^2 + 2\left\langle J \middle| \frac{D}{dt}\frac{D}{dt}J \right\rangle$. We find $\ddot{v}(0) = 1$, hence the result.

We wish to compare u and f_{κ_0}. We have:

$$\frac{d}{dt}\left(\frac{u}{f_{\kappa_0}}\right) = \frac{\dot{u} f_{\kappa_0} - u \dot{f}_{\kappa_0}}{f_{\kappa_0}^2}$$

Since $u(0) = f_{\kappa_0}(0)$ and $\dot{u}(0) = \dot{f}_{\kappa_0}(0)$, we find $(\dot{u} f_{\kappa_0} - u \dot{f}_{\kappa_0})(0) = 0$.

$$\frac{d}{dt}(\dot{u} f_{\kappa_0} - u \dot{f}_{\kappa_0}) = f_{\kappa_0}(\ddot{u} + \kappa_0 u) \geq 0$$

It follows that $\frac{u}{f_{\kappa_0}}$ is non-decreasing, hence $u \geq k_{\kappa_0}$, which is the desired result. □

Definition 11.2.8. *Let (M,g) be a Riemannian manifold. Two points $x,y \in M$ are called **conjugate** if there is a geodesic $c : [0,1] \to M$ such that $c(0) = x$ and $c(1) = y$, and a non trivial Jacobi field J along c such that $J(0) = 0$ and $J(1) = 0$.*

Proposition 11.2.9. *Let (M,g) be a Riemannian manifold with non-positive sectional curvature. Then M has no pairs of conjugate points.*

Proof. If c is a geodesic and J a Jacobi field along c such that $J(0) = 0$ and $J(1) = 0$, Lemma 11.2.7 yields $\frac{D}{dt}J(0) = 0$, hence $J = 0$. □

Theorem 11.2.10 (Cartan-Hadamard Theorem)**.**
Let (M,g) be a connected complete Riemannian manifold. If the sectional curvature is non-positive, then for all $x \in M$, the map $\exp_x : T_xM \to M$ is a covering map.
In particular, its universal cover is diffeomorphic to $\mathbb{R}^d$ (where $d = \dim M$).

Proof. By completeness, $\exp_x$ is defined on all T_xM. Recall (Proposition **??**) that for $v, w \in T_xM$, the differential $d_v \exp_x(w)$ is equal to $J(1)$ where J is the Jacobi field along the geodesic c_v satisfying $J(0) = 0$ and $\frac{D}{dt}J(0) = w$.

According to Lemma 11.2.7, we have $\left\|d_v \exp_x(w)\right\|_{\exp_x(v)} \geq \|w\|_x$. This means that $\exp_x^* g \geq g_x$. It follows from Proposition 10.3.8 that $\exp_x$ is a covering map. □

Consequence: if M is simply connected and $\kappa \leq 0$, then two points are linked by a unique geodesic.

11.2.3 The topology of positively curved Riemannian manifolds

Theorem 11.2.11 (Myers).
Let (M,g) be a complete Riemannian manifold of dimension d. If there is $r > 0$ such that:

$$\mathrm{Ric} \geq \frac{d-1}{r^2} g$$

Then $\mathrm{diam} M \leq \pi r$. *In particular, M is compact and $\pi_1(M)$ is finite.*

Remarks.

- *Under these conditions,* $\mathrm{diam} M = \pi r$ *if and only if* (M,g) *is isometric to* $\mathbb{S}^n$ *(Cheng).*
- *If* $\kappa \geq \frac{1}{r^2}$, *then* $\mathrm{Ric} \geq \frac{d-1}{r^2} g$.

This result can appear weaker than the Cartan-Hadamard Theorem, since it does not determine the topology of M. The reason for this is that positively curved simply connected Riemannian manifolds can have different topologies (e.g. $\mathbb{S}^n$ and $\mathbb{CP}^n$). In even dimension, the Synge Theorem asserts that an oriented complete Riemmannian manifold with positive sectional curvature is simply connected (this is not true in odd dimensions, as show the Lens spaces, quotients of $\mathbb{S}^3$ by finite cyclic groups).

To obtain that M is covered by $\mathbb{S}^n$, we need to add a condition on the curvature. The story starts in 1926 with a conjecture of Hopf stating that a simply connected Riemannian manifold with sectional curvature close enough to 1 should be homeomorphic to a sphere. This was first proved in 1951 by Rauch: if the sectional curvature κ of a complete simply connected Riemannian manifold (M,g) satifies $\frac{3}{4} \leq \kappa \leq 1$, then M is homeomorphic to a sphere. The optimal constant was found in 1961, a result of Berger (heavily relying on the work of Klingenberg) states that if the sectional curvature κ satisfies $\frac{1}{4} < \kappa \leq 1$, then M is homeomorphic to a sphere. Berger also showed that if it satisfies $\frac{1}{4} \leq \kappa \leq 1$ but M is not homeomorphic to a sphere, then (M,g) is isometric to a standard Riemannian metric on a projective space $\mathbb{CP}^n$, $\mathbb{HP}^n$ or $\mathbb{OP}^2$.

The question of differentiability in the Sphere Theorem stayed open for many years after the work of Berger. A first version with non optimal pinching constants was obtained by Gromoll and Calabi in 1966. The final version was proved in Brendle and Schoen in 2007: if (M,g) is a complete simply connected Riemannian manifold with sectional curvature κ satisfying $\frac{1}{4} < \kappa \leq 1$, then M is diffeomorphic to a sphere. Note that there are examples of manifolds that are homeomorphic to a sphere $\mathbb{S}^n$ but not diffeomorphic to $\mathbb{S}^n$ (e.g. for $n = 7$), known as exotic spheres. Gromoll and Meyer exhibited in 1974 an exotic sphere with a positively curved Riemannian metric.

The main tool used by Brendle and Schoen is the Ricci flow, which is famous for being used by Perelman in his proof of the Poincaré conjecture.

11.3 The geometry of non positively curved Riemannian manifolds

Definition 11.3.1. *A **Cartan-Hadamard manifold** is a simply connected complete Riemannian manifold of non positive sectional curvature.*

We have seen that if (M,g) is a Cartan-Hadamard manifold and $x \in M$, then $\exp_x$ is a diffeomorphism. For $x,p,q \in M$, we can define the angle $\sphericalangle_x(p,q)$ to be $\sphericalangle(u,v)$ where $p = \exp_x(u)$ and $q = \exp_x(v)$.

Lemma 11.3.2. *Let (M,g) be a Cartan-Hadamard manifold, $x \in M$ and $u,v \in T_xM$. We have:*

$$d(\exp_x(u), \exp_x(v)) \geq \|u - v\|_x$$

Proof. We saw in the proof of the Cartan-Hadamard Theorem that $\left\|d_w \exp_x(z)\right\| \geq \|z\|_x$ for all $w,z \in T_xM$. Now consider the curve $\gamma(t) = \exp_x(u + t(v-u))$ for $t \in [0,1]$. We find:

$$\left\|\dot\gamma(t)\right\|_{\gamma(t)} = \left\|d_{u+t(v-u)} \exp_x(v-u)\right\|_{\gamma(t)} \geq \|v-u\|_x$$

This yields $d(\exp_x(u), \exp_x(v) \geq L(\gamma) \geq \|v-u\|_x$. □

Lemma 11.3.3. *Let (M,g) be a Cartan-Hadamard manifold, and let $x,y,z \in M$. Set $a = d(x,z)$, $b = d(y,z)$, $c = d(x,y)$ and $\gamma = \sphericalangle_z(x,y)$. We have the following relation:*

$$c^2 \geq a^2 + b^2 - 2ab\cos\gamma$$

Proof. Consider $u,v \in T_zM$ such that $x = \exp_z(u)$ and $y = \exp_z(v)$.

$$\begin{aligned} c^2 &= d(\exp_z(u), \exp_z(v))^2 \\ &\geq \|u-v\|_x^2 = \|u\|_x^2 + \|v\|_x^2 - 2\langle u|v\rangle_x \\ &\geq a^2 + b^2 - 2ab\cos(\gamma) \end{aligned}$$

□

Proposition 11.3.4. *Let (M,g) be a Cartan-Hadamard manifold, and let $S \subset M$ be a non empty bounded subset. There is a unique closed ball of minimal radius containing S.*

Proof. Set $E = \left\{(x,r) \in M \times \mathbb{R}^+ \,\middle|\, S \subset \overline{B}(x,r)\right\}$ and $R = \inf\{r \,|\, \exists x \in M \ (x,r) \in E\}$.

Consider a sequence $(x_k, r_k) \in E$ such that $r_k \to R$. Let us show that (x_k) is a Cauchy sequence.

Let $\varepsilon > 0$, and let $k_0 > 0$ be such that: $\forall k \geq k_0 \ r_k^2 \leq R^2 + \varepsilon$. For $k, l \geq k_0$ and $p \in S$, we let q be the middle point of the geodesic segment joining x_k and x_l, hence

$$\sphericalangle_q(p, x_k) + \sphericalangle_q(p, x_l) = \pi$$

Up to exchanging k and l, we can assume that

$$\cos \sphericalangle_q(p, x_k) \leq 0$$

We now get

$$\begin{aligned} R^2 + \varepsilon &\geq r_k^2 \\ &\geq d(x_k, p)^2 \\ &\geq d(x_k, q)^2 + d(p,q)^2 = \frac{d(x_k, x_l)^2}{4} + d(p,q)^2 \end{aligned}$$

Since S is not included in $B(q, R-\varepsilon)$ (by definition of R), we can choose $p \in S$ that satisfies $d(p,q)^2 \geq R^2 - \varepsilon$. Therefore

$$R^2 + \varepsilon \geq \frac{d(x_k, x_l)^2}{4} + R^2 - \varepsilon$$

It follows that (x_k) is a Cauchy sequence.

Existence: Consider any sequence (x_k, r_k) such as above, and let $x = \lim x_k \in M$. Then $S \subset \cap_{k \geq 0} \overline{B}(x_k, r_k) \subset \overline{B}(x, R)$.

Uniqueness: If $S \subset B(x,R)$ and $S \subset B(y,R)$, then consider the sequence (x_k, r_k) in E such that $r_k = R$, $x_{2k} = x$ and $x_{2k+1} = y$. We find that (x_k) converges in M, so $x = y$.

□

Corollary 11.3.5. *Let (M,g) be a Cartan-Hadamard manifold, and let K be a compact group that acts continously on M by isometries. Then K fixes a point in M.*

Remark. *By a continuous action, we mean that the map $K \times M \to M$ is continuous.*

Proof. Let $x \in M$. The orbit $K.x$ is compact, hence bounded and Proposition 11.3.4 says there is a unique closed ball of minimal radius $\overline{B}(x_0, R)$ containing $K.x$. For $\varphi \in K$, we find that $B(\varphi(x_0), R)$ contains $\varphi(K.x) = K.x$, so $\varphi(x_0) = x_0$ by uniqueness. □

11.3.1 The boundary at infinity of a Cartan-Hadamard manifold

Definition 11.3.6. *Let (M,g) be a complete Riemannian manifold. Two unit speed geodesics $c_1, c_2 : \mathbb{R} \to M$ are called* ***positively asymptotic*** *if there is M such that $d(c_1(t), c_2(t)) \leq M$ for all $t \geq 0$.*

Lemma 11.3.7. *Let (M,g) be a Cartan-Hadamard manifold. For any unit speed geodesic $c : \mathbb{R} \to M$ and any $x \in M$, there is a unique $v \in T^1_x M$ such that c_v is positively asymptotic to c.*

Proof. For $t \geq 0$, we let $v_t \in T^1_x M$ be such that the geodesic c_{v_t} goes through $c(t)$. Let us show that $t \mapsto v_t$ is Cauchy.

Set $d_0 = d(x, c(0))$. Lemma 11.3.3 yields

$$d(c(t),c(s))^2 \geq d(c(t),x)^2 + d(c(s),x)^2 - 2d(c(t),x)d(c(s),x)\cos\sphericalangle(v_t,v_s)$$

The triangle inequality gives

$$t - d_0 \leq d(c(t),x) \leq t + d_0$$

$$s - d_0 \leq d(c(s),x) \leq s + d_0$$

For t,s large enough, we find

$$\begin{aligned}\cos\sphericalangle(v_t,v_s) &\geq \frac{(t-d_0)^2 + (s-d_0)^2 - (t-s)^2}{2(t+d_0)(s+d_0)} \\ &\geq \frac{(t-d_0)(s-d_0)}{(t+d_0)(s+d_0)} \end{aligned} \tag{11.1}$$

We can set $v = \lim_{t\to+\infty} v_t \in T^1_x M$. For $t,s \geq 0$, we have

$$d(c_v(t),c(t)) \leq d(c_v(t),c_{v_s}(t)) + d(c_{v_s}(t),c(t))$$

For $s > t$, Lemma 11.3.2 yields $d(c_{v_s}(t'),c(t)) \leq d(c_{v_s}(z),c(0))$ where $t' = t + s' - s$ and $z = s' - s$, with $c_{v_s}(s') = c(s)$.

The triangle inequality gives $s - d_0 \leq s' \leq s + d_0$, and we find

$$\begin{aligned} d(c_{v_s}(t),c(t)) &\leq d(c_{v_s}(t),c_{v_s}(t')) + d(c_{v_s}(t'),c(t)) \\ &\leq d_0 + d(c_{v_s}(z),c(0)) \\ &\leq d_0 + d(c_{v_s}(z),x) + d_0 \\ &\leq 3d_0 \end{aligned}$$

It follows that $d(c_v(t),c(t)) \leq d(c_v(t),c_{v_s}(t)) + 3d_0$, and $s \to +\infty$ leads to $d(c_v(t),c(t)) \leq 3d_0$ for all $t \geq 0$.

The uniqueness is also a consequence of Lemma 11.3.3, as for all $v' \in T_x^1 M$, we have

$$d(c_v(t), c_{v'}(t))^2 \geq 2t^2 (1 - \cos \sphericalangle(v, v'))$$

□

We now let $\mathcal{G}(M)$ be the set of unit speed geodesics in M, and $\partial_\infty M = \mathcal{G}(M)/\sim$ where $c_1 \sim c_2$ if c_1 and c_2 are positively asymptotic.

According to Lemma 11.3.7, for all $x \in M$ the map

$$\varphi_x : \left\{ \begin{array}{ccc} T_x^1 M & \to & \partial_\infty M \\ v & \mapsto & [c_v] \end{array} \right.$$

is a bijection.

Lemma 11.3.8. *Let (M, g) be a Cartan-Hadamard manifold. For $x, y \in M$, the map $\varphi_y^{-1} \circ \varphi_x : T_x^1 M \to T_y^1 M$ is a homeomorphism.*

Proof. We only have to prove that $\varphi_y^{-1} \circ \varphi_x : T_x^1 M \to T_y^1 M$ is continuous.

For this purpose we consider $u_k \to u \in T_x^1 M$. For $t \geq 0$, we let $v_k(t) \in T_y^1 M$ be such that $c_{v_k(t)}$ goes through $c_{u_k}(t)$ and $v(t) \in T_y^1 M$ be such that $c_{v(t)}$ goes through $c_u(t)$.

Now set $v_k = \varphi_y^{-1} \circ \varphi_x(u_k)$ and $v = \varphi_y^{-1} \circ \varphi_x(u)$. Applying Lemma 11.3.3 to the triangle with vertices y, $c_{u_k}(t)$ and $c_u(t)$, we find

$$\cos \sphericalangle(v_k(t), v(t)) \geq \left(\frac{t - d(x,y)}{t + d(x,y)} \right)^2 - \frac{1}{2} \left(\frac{d(c_{u_k}(t), c_u(t))}{t - d(x,y)} \right)^2$$

The uniformity in (11.1) in the proof of Lemma 11.3.7 shows that $v_k(t) \to v_k$ and $v(t) \to v$ as $t \to +\infty$.

So for all $\varepsilon > 0$, we can find $t > 0$ such that

$$\cos \sphericalangle(v_k, v) \geq \cos \sphericalangle(v_k(t), v(t)) - \varepsilon$$

and

$$\left(\frac{t - d(x,y)}{t + d(x,y)} \right)^2 \geq 1 - \varepsilon$$

it follows that

$$\cos \sphericalangle(v_k, v) \geq 1 - 2\varepsilon - \frac{1}{2} \left(\frac{d(c_{u_k}(t), c_u(t))}{t - d(x,y)} \right)^2$$

But $c_{u_k}(t) \to c_u(t)$ because $\exp_x$ is continuous, which leads to $v_k \to v$. □

Theorem 11.3.9. *Let (M, g) be a Cartan-Hadamard manifold. There is a unique topology on $\overline{M} = M \cup \partial_\infty M$ such that:*

- *For all* $x \in M$*, the map* $\varphi_x : T^1_x M \to \partial_\infty M$ *is a homeomorphism.*
- M *is open and dense in* $\overline{M}$*, and the induced topology is the manifold topology.*
- *For any unit speed geodesic* $c : \mathbb{R} \to M$*, we have* $\lim_{t\to+\infty} c(t) = [c]$.
- $\overline{M}$ *is compact.*

Note that by uniqueness, the group $\mathrm{Isom}(M)$ acts by homeomorphisms on $\overline{M}$, hence on $\partial_\infty M$.

Chapter 12

Riemannian submanifolds

Let (M, g) be a Riemannian manifold, and $N \subset M$ a submanifold (immersed or embedded). Recall that the **first fundamental form** of N, also called the **induced metric** is the Riemannian metric $\overline{g}$ on N defined as the restriction of g to TN.

We will use a bar to denote everything that relates to N: the Levi-Civita connection of N is $\overline{\nabla}$, the curvature tensor is $\overline{R}$, the Riemannian distance is $\overline{d}$, etc...

For $x \in N$ and $v \in T_xM$, we will write $v = v^\top + v^\perp$ where $v^\top \in T_xN$ and $v^\perp \in (T_xN)^\perp$.

Recall that the Levi-Civita connection ∇ of (M, g) restricts to N (i.e. $\nabla_x X(v)$ is well defined for $X \in \Gamma(TM|_N)$ and $v \in T_xN$), and that the Levi-Civita connection $\overline{\nabla}$ of $(N, \overline{g})$ satisfies $\overline{\nabla}_x X(v) = (\nabla_x X(v))^\top$ for all $X \in \mathcal{X}(N)$, $x \in N$ and $v \in T_xN$.

12.1 The second fundamental form

Definition 12.1.1. *Let (M, g) be a Riemannian manifold, and $N \subset M$ a submanifold. The **normal bundle** of N is the vector sub-bundle νN of $TM|_N$ defined by $\nu_x N = (T_xN)^\perp \subset T_xM$.*

The orthogonal decomposition induces an isomorphism of vector bundles $TN \oplus \nu N = TM|_N$.

Example: the normal bundle of $\mathbb{S}^n \subset \mathbb{E}^{n+1}$ is a trivialisable line bundle ($\nu_x \mathbb{S}^n = \mathbb{R}.x$).

Lemma 12.1.2. *Let (M, g) be a Riemannian manifold, and $N \subset M$ a submanifold. The map $A : \mathcal{X}(N \times \mathcal{X}(N) \to \Gamma(\nu N)$ defined by:*

$$\forall X, Y \in \mathcal{X}(N) \quad A(X, Y) = (\nabla Y(X))^\perp$$

is tensorial and symmetric.

Proof. Tensoriality in X comes from the definition of a connection. For $f \in \mathcal{C}^\infty(N)$, we have $A(X, fY) = fA(X,Y) + (X\cdot f)Y^\perp = fA(X,Y)$ since $Y^\perp = 0$. Now that we know that it is tensorial, in order to prove the symmetry we can consider the case where X, Y are vector fields on M whose restriction to N is tangent, thanks to Lemma 9.5.1.

$$\begin{aligned} A(X,Y) - A(Y,X) &= \nabla Y(X)^\perp - \nabla X(Y)^\perp \\ &= [X,Y]^\perp \\ &= 0 \end{aligned}$$

□

Definition 12.1.3. *Let (M,g) be a Riemannian manifold, and $N \subset M$ a submanifold. The* ***second fundamental form*** *of N is $\vec{\mathrm{II}} \in \Gamma\left(S^2(T^*N) \otimes \nu N\right)$ defined by:*

$$\forall X, Y \in \mathcal{X}(N) \quad \vec{\mathrm{II}}(X,Y) = (\nabla Y(X))^\perp$$

Note that for $X, Y \in \mathcal{X}(N)$, we find $\nabla Y(X) = \overline{\nabla} Y(X) + \vec{\mathrm{II}}(X,Y)$ (this is known as the Gauß formula).

The second fundamental form has values in the normal bundle. Given a normal vector $n \in \nu_x N$, the map $(u,v) \mapsto g_x\left(n, \vec{\mathrm{II}}_x(u,v)\right)$ is a symmetric bilinear form T_xN, so it can be represented by a self adjoint operator of T_xN.

Definition 12.1.4. *Let (M,g) be a Riemannian manifold, and $N \subset M$ a submanifold. The* ***shape operator*** *of N is $S \in \Gamma\left((\nu N)^* \otimes \mathrm{End}(TN)\right)$ defined by:*

$$\forall x \in N \ \forall n \in \nu_x N \ \forall u, v \in T_xN \quad \overline{g}_x(S_x(n)u, v) = -g_x\left(n, \vec{\mathrm{II}}_x(u,v)\right)$$

Remark. *The shape operator is also called the* ***Weingarten operator****.*

Proposition 12.1.5 (Weingarten formula)**.** *Let (M,g) be a Riemannian manifold, and $N \subset M$ a submanifold. For all $n \in \Gamma(\nu N)$, we have*

$$S(n) = (\nabla n)^\top$$

Proof. Consider $X, Y \in \mathcal{X}(N)$. Since the formula is local, we can assume that X, Y, n extend to vector fields on M. Note that we only assume that $g(n,Y) = 0$ on N, but this is enough to find that $X \cdot g(n,Y) = 0$ on N.

$$\begin{aligned} \overline{g}\left((\nabla n(X))^\top, Y\right) &= g(\nabla n(X), Y) \\ &= X \cdot g(n,Y) - g(n, \nabla Y(X)) \\ &= 0 - g\left(n, \vec{\mathrm{II}}(X,Y) + \overline{\nabla} Y(X)\right) \\ &= -\overline{g}(S(n)X, Y) \end{aligned}$$

□

Theorem 12.1.6 (Gauß equation). *Let (M,g) be a Riemannian manifold with Levi-Civita connection ∇ and curvature tensor R, and let $N \subset M$ be a submanifold. Let $\overline{g}$ be the restricted metric on N, and $\overline{R}$ the curvature tensor of $\overline{g}$. All $x \in N$ and $u,v \in T_xN$ satisfy*

$$\overline{R}_x(u,v,v,u) = R_x(u,v,v,u) - g_x\left(\vec{\mathrm{II}}_x(u,v),\vec{\mathrm{II}}_x(u,v)\right) + g_x\left(\vec{\mathrm{II}}_x(u,u),\vec{\mathrm{II}}_x(v,v)\right)$$

Proof. Recall that according to Lemma 9.5.1, we can consider that $u = X(x)$ and $v = Y(x)$ where $X,Y \in \mathcal{X}(M)$ are vector fields that restrict to vector fields of N.

First, we consider two other tangent vectors $w,z \in T_xN$, and use the Weingarten formula to obtain:

$$\begin{aligned} g_x\left(\nabla_x\left(\vec{\mathrm{II}}(X,Y)\right)(w),z\right) &= -\overline{g}_x\left(S_x\left(\vec{\mathrm{II}}(X,Y)\right)(w),z\right) \\ &= g_x\left(\vec{\mathrm{II}}_x(u,v),\vec{\mathrm{II}}_x(w,z)\right) \end{aligned} \tag{12.1}$$

By using the decomposition $\nabla = \overline{\nabla} + \vec{\mathrm{II}}$ for vector fields on N, we find:

$$R_x(u,v,v,u) - \overline{R}_x(u,v,v,u) = g_x\left(\nabla_x\left(\vec{\mathrm{II}}(Y,Y)\right)(u),u\right) - g_x\left(\nabla_x\left(\vec{\mathrm{II}}(Y,X)\right)(v),u\right)$$

Using (12.1), we find the desired formula. □

12.2 Hypersurfaces

Consider a hypersurface $N \subset M$, and $n \in \Gamma(\nu N)$ unitary (it is always possible to find such a field locally, and there are exactly two choices).

We can consider the **scalar second fundamental form** defined by $\mathrm{II} = \left\langle n \middle| \vec{\mathrm{II}} \right\rangle$.

The Eigenvalues of the shape operator $S(n)$ are called the **principal curvatures** of N.

The **Gauß curvature** is $K = \det S(n)$. It is the product of the principal curvatures.

Theorem 12.1.6 in the case $M = \mathbb{R}^3$ says that the Gauß curvature of N is equal to its sectional curvature for the induced metric. This result implies the *Theorema Egregium* of Gauß: the Gauß curvature is invariant under isometries.

Proposition 12.2.1. *Let N be an immersed hypersurface of a Riemannian manifold (M,g), with unitary normal field n. Let $x \in N$, $v \in T_xN$, and consider a smooth curve $c :]-\varepsilon,\varepsilon[\to N$ such that $c(0) = x \in N$ and $\dot{c}(0) = v$. The scalar second fundamental form satisfies:*

$$\mathrm{II}_x(v,v) = g_x\left(\frac{D}{dt}\dot{c}(0), n(x)\right)$$

Proof. Differentiating the fact that $\dot{c}(t)$ and $n(c(t))$ are orthogonal for all t, we find that:

$$\begin{aligned} g\left(\frac{D}{dt}\dot{c},n\right) &= -g\left(\dot{c},\frac{D}{dt}n\right) \\ &= -g\left(\dot{c},\nabla n(\dot{c})\right) \\ &= -g\left(\dot{c},S(n)\dot{c}\right) \\ &= g\left(n,\vec{\mathrm{II}}(\dot{c},\dot{c})\right) \\ &= \mathrm{II}(\dot{c},\dot{c}) \end{aligned}$$

□

Hypersurfaces of $\mathbb{E}^d$. If (M,g) is the Euclidean space $\mathbb{E}^d$, then the shape operator is simply dn, i.e. it is already tangent to N. Indeed, the normal field n can be seen as a map $n : N \to \mathbb{S}^{d-1}$, and its differential $d_x n$ at $x \in N$ is a map from T_xN to $T_{n(x)}\mathbb{S}^{d-1} = n(x)^\perp = T_xN$.

To compute the second fundamental form, we start with a smooth curve $c :]-\varepsilon,\varepsilon[\to N$ such that $c(0) = x \in N$ and $\dot{c}(0) = v$. Proposition 12.2.1 yields

$$\mathrm{II}_x(v,v) = \langle \ddot{c}(0) | n(x) \rangle$$

i.e. $\mathrm{II}_x(v,v)$ is the curvature of the curve obtained by intersecting N with a plane spanned by the normal direction to N and v.

The Gauß curvature of a surface in $\mathbb{R}^3$: it is the product of the principal curvatures λ_1,λ_2, which are the extrema of the curvature of curves drawn on N.

Ruled surfaces: a ruled surface is a surface that is obtained as a union of straight lines.
A ruled surface has non positive Gauß curvature $\kappa \le 0$, it vanishes if and only if it stays on one side of the tangent plane. The main examples are cones, cylinders, and the one-sheeted hyperboloids.

Definition 12.2.2. *Let $N \subset M$ be an immersed submanifold of a Riemannian manifold (M,g). The **mean curvature** of N is $H \in \Gamma(\nu N)$ given by $H(x) = \mathrm{Tr}\,\mathrm{II}_x$.*
*A submanifold is called **minimal** if $H = 0$.*

Remark. *It is the trace of a quadratic form, i.e. $H(x) = \sum_{i=1}^n \mathrm{II}_x(v_i,v_i)$ where (v_i) is an orthonormal basis of T_xN.*

If N is a hypersurface, we locally choose a unit normal field $\vec{n} \in \Gamma(\nu N)$, and consider the scalar mean curvature $H(x) = \mathrm{Tr}\,\mathrm{II}_x$.

Theorem 12.2.3. *If N is compact, and $X \in \mathcal{X}(M)$ is a complete vector field with flow (φ^t), then:*

$$\left.\frac{d}{dt}\right|_{t=0} \mathrm{Vol}\Big(\varphi^t(N)\Big) = -\int_N g(H,X)\,\mathrm{dvol}_{\overline{g}}$$

12.3 Totally geodesic submanifolds

Proposition 12.3.1. *Let (M,g) be a Riemannian manifold and $N \subset M$ an immersed submanifold. The following are equivalent:*

1. $\forall x \in M \quad \vec{\mathrm{II}}_x = 0.$
2. *Any geodesic of N is a geodesic of M.*
3. *For all $x \in N$, there are neighbourhoods $V \subset N$ of x and $U \subset T_xN$ of 0 such that $\exp_x(U) = V$ (where $\exp_x$ is the exponential in (M,g)).*
4. *TN is stable under ∇, i.e.*

$$\forall X \in \mathcal{X}(N)\ \forall x \in N\ \forall v \in T_xN \quad \nabla_x X(v) \in T_xN$$

A submanifold satisfying these properties is called **totally geodesic.**

Proof. First note that 1. $\Longleftrightarrow$ 4. comes from the definition of $\vec{\mathrm{II}}$.
2. $\Rightarrow$ 3. is a consequence of the local surjectivity of the exponential map.
3. $\Rightarrow$ 2. is a consequence of the uniqueness of geodesics.
2. $\Longleftrightarrow$ 4. comes from the geodesic equation on a submanifold (see the discussion following Proposition 9.6.7). □

Exercise: The totally geodesic submanifolds of $\mathbb{E}^n$ are open subsets of affine subspaces.

Lemma 12.3.2. *Let (M,g) be a Riemannian manifold, and let $N \subset M$ be a totally geodesic submanifold. For all $x \in N$ and $u,v,w \in T_xN$, we have $R_x(u,v)w \in T_xN$ where R is the Riemann tensor of (M,g).*

Proof. We let $\overline{R}$ be the Riemann tensor of N, so that $\overline{R}_x(u,v)w \in T_xN$. The Gauß equation (Theorem 12.1.6) associated to the fact that $\vec{\mathrm{II}} = 0$ assures that for $z \in T_xN^\perp$, we have $0 = \left\langle \overline{R}_x(u,v)w \middle| z \right\rangle_x = \langle R_x(u,v)w|z\rangle$, hence $R_x(u,v)w \in T_xN$ (and furthermore $R_x(u,v)w = \overline{R}_x(u,v)w$). □