Contents

Chapter 13

Globally and locally symmetric spaces

13.1 Globally symmetric spaces

Definition 13.1.1. *A **Riemannian symmetric space** is a connected Riemannian manifold $(X, \langle\cdot|\cdot\rangle)$ such that for all $x \in X$, there is an isometry $s_x \in \mathrm{Isom}(X)$ such that $s_x(x) = x$ and $d_x s_x = -\mathrm{Id}_{T_xX}$.*

Note that such an isometry s_x is unique because of Proposition **??**.

Lemma 13.1.2. *Let X be a Riemannian symmetric space. Then X is complete, and if $c : \mathbb{R} \to X$ is a geodesic, we have $s_{c(t)}(c(s)) = c(2t - s)$ for all $t \in \mathbb{R}$.*

Proof. First notice that if $c : I \to \mathbb{R}$ is a geodesic and $x = c(0)$, then $s_x \circ c$ is also a geodesic, with velocity vector $-\dot{c}(0)$, hence $-I = I$ and $s_x \circ c(t) = c(-t)$. Up to a translation of the parameter, this proves the second point under the assumption of completeness.

Let $c : [a, b] \to X$ be a geodesic with $a, b \in \mathbb{R}$. Set $z = c(b)$. Now $\gamma : [b, 2b - a]$ defined by $\gamma(t) = s_z(c(2b - t))$ is a geodesic such that $\gamma(b) = c(b)$ and $\dot{\gamma}(b) = \dot{c}(b)$, so it extends c to $[a, 2b - a]$. Repeating this argument shows that c is extendable to $\mathbb{R}$, i.e. X is complete. $\square$

Proposition 13.1.3. *If X is a Riemannian symmetric space, then X is homogeneous, i.e. the isometry group $\mathrm{Isom}(X)$ acts transitively on X.*

Proof. Let $x, y \in X$. Since X is complete by Lemma 13.1.2, the Hopf-Rinow Theorem **??** provides a geodesic $c : \mathbb{R} \to X$ such that $c(0) = x$ and $c(1) = y$. By Lemma 13.1.2, we find that $y = s_z(x)$ where $z = c\left(\frac{1}{2}\right)$. $\square$

Notation: If X is a symmetric space, we consider $G = \mathrm{Isom}_\circ(X)$ the identity component of the isometry group. Recall that it is a Lie group and that the action on X is smooth (Myers-Steenrod Theorem). We fix some $x_0 \in M$, and set $K = \mathrm{Stab}_G(x_0)$. It is a compact Lie subgroup of G.

For $g \in G$, we have $s_{g(x)} = g \circ s_x \circ g^{-1}$.

Lemma 13.1.4. *Let $\mathbb{X}$ be a Riemannian symmetric space. The map*

$$\begin{cases} \mathbb{X} & \to & \mathrm{Isom}(\mathbb{X}) \\ x & \mapsto & s_x \end{cases}$$

is smooth.

Proof. Set $\overline{G} = \mathrm{Isom}(\mathbb{X})$ and let $o \in \mathbb{X}$. Since $\overline{G} \curvearrowright \mathbb{X}$ is transitive, the orbit map $\varphi_o : \overline{G} \to \mathbb{X}$ is a submersion.

Since $s_{g(o)} = g \circ s_o \circ g^{-1}$, the map $x \mapsto s_x$ lifts through the submersion φ_o to the map $g \mapsto g \circ s_o \circ g^{-1}$ which is smooth, so $x \mapsto s_x$ is smooth. □

Lemma 13.1.5. *Let X a Riemannian symmetric space, and $G = \mathrm{Isom}_\circ(X)$. For all $x, y \in X$, we have that $s_x \circ s_y \in G$. The action of G on X is transitive.*

Proof. Let $c : \mathbb{R} \to X$ be a geodesic such that $c(0) = x$ and $c(1) = y$. Then $s_x \circ s_{c(t)}$ is a continuous path in $\mathrm{Isom}(X)$ that links Id and $s_x \circ s_y$.

By letting $z = c\left(\frac{3}{2}\right)$, we find $y = s_z \circ s_y(x)$, hence the transitivity. □

Examples : $\mathbb{E}^n$, $\mathbb{S}^n$ and $\mathbb{H}^n$ are symmetric spaces.

13.2 Locally symmetric spaces

Motivated by Proposition 13.1.3, we can try to define symmetries in arbitrary Riemannian manifolds.

Definition 13.2.1. *Let (M, g) be a Riemannian manifold, and let $x \in M$. The **geodesic symmetry** through x is the map $s_x = \exp_x \circ \left(-\exp_x^{-1}\right)$ defined on $B(x, \mathrm{inj}_x)$.*

In general, it is not a local isometry, but it is in some cases.

Definition 13.2.2. *A **Riemannian locally symmetric space** is a Riemannian manifold (M, g) such that for all $x \in M$, the geodesic symmetry s_x is isometric on a neighbourhood of x.*

Theorem 13.2.3 (Cartan-Ambrose-Hicks)**.** *Let (M, g) be a Riemannian manifold. Then (M, g) is a locally symmetric space if and only if $\nabla R = 0$.*
If (M, g) is a complete and simply connected Riemannian locally symmetric space, then it is a symmetric space.

The condition $\nabla R = 0$ will be used in the following way:

Lemma 13.2.4. *Let (M,g) be a Riemannian manifold, and $T \in \mathcal{T}^{p,0}(M)$ be a covariant tensor. The following are equivalent:*

1. $\nabla T = 0$
2. *For any smooth curve $I \to M$ and parallel vector fields $X_1,\dots,X_p \in \Gamma(c^*TM)$ along c, $T(X_1,\dots,X_p)$ is constant.*

Lemma 13.2.5. *Let (M,g) be a Riemannian manifold with curvature tensor R. If $\nabla R = 0$, then for all $x \in M$ and $v \in T_xM$ with $\|v\|_x < \mathrm{inj}_x$, the differential $d_{\exp_x(v)}s_x : T_{\exp_x(v)}M \to T_{\exp_x(-v)}M$, is equal to the parallel transport along the geodesic $t \mapsto \exp_x(tv)$.*

Proof. Write $y = \exp_x(v)$, and let $u \in T_yM$. From the definition of s_x and the chain-rule we find that $d_ys_x(u) = -d_{-v}\exp_x(w)$ where $w = (d_v\exp_x)^{-1}(u) \in T_xM$.

Let $c : I \to M$ be the geodesic defined by $\dot c(0) = v$, and J the Jacobi field along c such that $J(0) = 0$ and $\frac{D}{dt}J(0) = w$. The formula for the differential of the exponential map (Proposition **??**) yields $w = J(1)$ and $d_ys_x(w) = J(-1)$.

Consider a parallel orthonormal frame $(\varepsilon_1,\dots,\varepsilon_d)$ along c, and decompose $J(t) = \sum_{i=1}^d J^i(t)\varepsilon_i(t)$. Since we have chosen a parallel frame, we have that $\frac{D}{dt}\frac{D}{dt}J(t) = \sum_{i=1}^d \ddot J^i(t)\varepsilon_i(t)$. Using the fact that it is an orthonormal frame, we find:

$$\begin{aligned} R_{c(t)}(J(t),\dot c(t))\dot c(t) &= \sum_{j=1}^d J^j(t)R_{c(t)}(\varepsilon_j(t),\dot c(t))\dot c(t) \\ &= \sum_{1\le i,j\le d} J^j(t)R_{c(t)}(\varepsilon_j(t),\dot c(t),\dot c(t),\varepsilon_i(t))\varepsilon_i(t) \end{aligned}$$

Let $\alpha_{i,j} = R_x(\varepsilon_j(0),v,v,\varepsilon_i(0)) \in \mathbb{R}$. According to Lemma 13.2.4, we have that:

$$\forall t \in I \quad R_{c(t)}(J(t),\dot c(t))\dot c(t) = \sum_{1\le i,j\le d} \alpha_{i,j}J^j(t)\varepsilon_i(t)$$

The Jacobi field equation writes as:

$$\forall i \ \ddot J^i + \sum_{j=1}^d \alpha_{i,j}J^j = 0$$

It is a linear differential equation with constant coefficients. It follows that the vector field $\widetilde J$ along c defined by $\widetilde J(t) = -\sum_{i=1}^d J^i(-t)\varepsilon_i(t)$ is also a Jacobi field.

Since $\widetilde J(0) = 0 = J(0)$ and $\frac{D}{dt}\widetilde J(0) = \sum_{i=1}^d \dot J^i(0)\varepsilon_i(0) = \frac{D}{dt}J(0)$, we find that

$\widetilde{J} = J$.

Finally we get:

$$\begin{aligned} d_y s_x(u) &= J(-1) \\ &= \widetilde{J}(-1) \\ &= -\sum_{i=1}^{d} J^i(1)\varepsilon_i(-1) \\ &= -\|_1^{-1} J(1) \\ &= -\|_1^{-1} u \end{aligned}$$

□

Since the parallel transport is isometric (proposition **??**), it follows that s_x is a local isometry.

Lemma 13.2.5 admits the following generalisation:

Lemma 13.2.6. *Let (M,g) be a complete Riemannian manifold, and assume that $\nabla R = 0$. For all $x, y \in M$ and all linear isometry $\varphi : T_xM \to T_yM$ which preserves the Riemann tensor (i.e. $R_y(\varphi(u),\varphi(v))\varphi(w) = \varphi(R_x(u,v)w)$ for all $u, v, w \in T_xM$), there is a local isometry $f : B(x, \mathrm{inj}_x) \to M$ such that $f(x) = y$ and $d_xf = \varphi$.*

If we drop the completeness hypothesis, we can still build f on $B(x,r)$ where $r \leq \mathrm{inj}_x$ and $\exp_y$ is defined on $B_{T_yM}(0,r)$.

Lemma 13.2.7. *Let (M,g) be a complete Riemannian manifold, and assume that $\nabla R = 0$. For all $x, y \in M$ and smooth curve $c : [0,1] \to M$ such that $c(0) = x$ and $c(1) = y$, there is a neighbourhood $U \subset M$ of $c([0,1])$ and a local isometry $f : U \to M$ such that $f(x) = x$ and $d_xf = -\mathrm{Id}$.*

Proof. Considering the open cover $c([0,1]) \subset \bigcup_{t\in[0,1]} B(c(t), \mathrm{inj}_{c(t)})$, we can consider a finite sequence $0 = t_0 < t_1 < \cdots < t_N = 1$ such that $c([t_i, t_{i+1}]) \subset U_i$ where $x_i = c(t_i)$ and $U_i = B(x_i, \mathrm{inj}_{x_i})$.

Our goal is to inductively construct a connected open set $V_i \subset M$ containing $c([0, t_{i+1}])$ and a local isometry $f_i : V_i \to M$ such that $f_i(x) = x$ et $d_xf_i = -\mathrm{Id}$.

The open set $V_0 = U_0$ and the local isometry f_0 are given by Lemma 13.2.5.

Assume that we have V_i and $f_i : V_i \to M$ as described above. Since $x_{i+1} \in V_i$, Lemma 13.2.6 guarantees the existence of a local isometry $\widetilde{f} : U_{i+1} \to M$ such that $\widetilde{f}(x_{i+1}) = y_i$ and $d_{x_{i+1}}\widetilde{f} = d_{x_{i+1}} f_i$.

Let W be the connected component of x_{i+1} in $U_{i+1} \cap V_i$. The restrictions of f_i and $\widetilde{f}$ to W are equal.

We now denote by V_{i+1} the connected component of x_{i+1} in $V_i \cup U_{i+1}$. We can define $f_{i+1} : V_{i+1} \to M$ that extends both f_i and $\widetilde{f}$, since they are equal on the intersection of their domains, and satisfy all the requirements.

Finally $U = V_{N-1}$ and $f = f_{N-1}$ answer the initial problem. □

Proof of Theorem 13.2.3. Assume that (M,g) is locally symmetric. Near $x \in M$, we have $s_x^*(\nabla R) = \nabla R$. Evaluating this at x, since $s_x(x) = x$ and $d_x s_x = -\mathrm{Id}$, we find $(s_x^*(\nabla R))_x = -(\nabla R)_x$ (because ∇R is a type (4,1) tensor). Hence $(\nabla R)_x = 0$.

If $\nabla R = 0$, then Lemma 13.2.5 shows that (M,g) is locally symmetric (because parallel transport is isometric).

We now assume that (M,g) is locally symmetric, complete and simply connected. Let $x \in M$. We wish to construct s_x.

Lemma 13.2.7 assures that for every $y \in M$, we can find a connected open set $U_y \subset M$ containing x and y, and a local isometry $f_y : U_y \to M$ such that $f_y(x) = x$ et $d_x f_y = -\mathrm{Id}$. Let us show that $f_y(y)$ does not depend on the choice of the curve to which we apply 13.2.7.

Consider two curves $c_0, c_1 : [0,1] \to M$ such that $c_0(0) = c_1(0) = x$ and $c_0(1) = c_1(1) = y$. Since M is simply connected, we can consider a smooth homotopy $H : [0,1]^2 \to M$ such that $H(0,\cdot) = c_0$, $H(1,\cdot) = c_1$, $H(s,0) = x$ and $H(s,1) = y$ for all $s \in [0,1]$.

Let c_s be the curve $c_s = H(s,\cdot)$ for $s \in [0,1]$, also U_s the connected open set and $f_s : U_s \to M$ the local isometry obtained by applying Lemma 13.2.7 to c_s.

Let us show that the map $\left\{ \begin{array}{ccc} [0,1] & \to & M \\ s & \mapsto & f_s(y) \end{array} \right.$ is locally constant. Note that in order to show that $f_s(y) = f_{s'}(y)$, we only need to check that x and y are in the same connected component of $U_s \cap U_{s'}$.

For $s \in [0,1]$, we set $r = \min\left\{\mathrm{inj}_{c_s(t)} \,\middle|\, t \in [0,1]\right\}$. Let $\eta > 0$ be such that:

$$|s - s'| < \eta \Rightarrow \forall t \in [0,1] \; c_{s'}(t) \in B(c_s(t), r)$$

We find that $c_{s'}(t) \in U_s$ for $s \in]s-\eta, s+\eta[$ and $t \in [0,1]$. Hence $U_s \cap U_{s'} \supset c_{s'}([0,1])$, and y is in the connected component of $U_s \cap U_{s'}$ which contains x. Hence $f_{s'}(y) = f_s(y)$.

Since $[0,1]$ is connected, we get $f_0(y) = f_1(y)$.

We now define s_x in the following way: for $y \in M$, we choose a smooth path c_y from x to y and we set $s_x(y) = f_y(y)$ where f_y is given by Lemma 13.2.7.

The map s_x is isometric: for $z \in B(y, \mathrm{inj}_y)$, if we choose the concatenation of a path from x to y and the minimising geodesic from y to z in order to construct $s_x(z)$, we find that $s_x(z) = f_y(z)$, and f_y is isometric on a neighbourhood of y.

Since (M,g) is complete, the local isometry $s_x : M \to M$ is a Riemannian covering, hence a diffeomorphism because M is simply connected.

□

Corollary 13.2.8. *Let $\mathbb{X}$ be a symmetric space. Then $\nabla R = 0$.*

Proposition 13.2.9. *Let $\mathbb{X}$ be a symmetric space. If $c : \mathbb{R} \to \mathbb{X}$ is a geodesic, then for all $t,s \in \mathbb{R}$ the differential at $c(t)$ of $s_{c(s)}$ is equal to the opposite of the parallel transport along c.*

Proof. The same computations as in Lemma 13.2.5 can be carried out. □

13.3 The symmetric space of ellipsoids

For $n \geq 2$, we let $\mathcal{E}_n = \left\{x \in \mathcal{M}_n(\mathbb{R}) \,\middle|\, {}^t x = x,\, x > 0,\, \det x = 1\right\}$. Let $x_0 = 1_n \in \mathcal{E}_n$, and $\mathfrak{p} = T_{x_0}\mathcal{E}_n$. Note that

$$\mathfrak{p} = \left\{X \in \mathcal{M}_n(\mathbb{R}) \,\middle|\, {}^t X = X,\, \operatorname{Tr} X = 0\right\}$$

The map $\left\{\begin{array}{ccc} \mathfrak{p} & \to & \mathcal{E}_n \\ X & \mapsto & \exp(X) \end{array}\right.$ is a diffeomorphism, and we let Log be its inverse. The map $x \mapsto \sqrt{x} = e^{\frac{1}{2}\operatorname{Log} x}$ is a diffeomorphism of $\mathcal{E}_n$.

Consider the action $\mathrm{SL}(n,\mathbb{R}) \curvearrowright \mathcal{E}_n$ defined by $g.x = gx^t g$. Then $\mathrm{Stab}_{\mathrm{SL}(n,\mathbb{R})}(x_0) = \mathrm{SO}(n,\mathbb{R})$. This action is transitive, so we can identify $\mathcal{E}_n$ with $\mathrm{SL}(n,\mathbb{R})/\mathrm{SO}(n,\mathbb{R})$.

Endow $\mathfrak{p}$ with the inner product $\langle X|Y\rangle_{x_0} = \operatorname{Tr}(XY) = \sum_{i,j} X_{i,j} Y_{i,j}$.
It is invariant under the action of $\mathrm{SO}(n,\mathbb{R})$.

For $X \in T_x\mathcal{E}_n$, we set $\|X\|_x = \left\|\sqrt{x}^{-1} X \sqrt{x}^{-1}\right\|_{x_0}$. The polarized form is:

$$\langle X|Y\rangle_x = \operatorname{Tr}(Xx^{-1}Yx^{-1})$$

It is a Riemannian metric on $\mathcal{E}_n$, and $\mathrm{SL}(n,\mathbb{R})$ acts isometrically.

The map $s_{x_0} : \left\{\begin{array}{ccc} \mathcal{E}_n & \to & \mathcal{E}_n \\ x & \mapsto & x^{-1} \end{array}\right.$ is an isometry. It fixes x_0 and satisfies $d_{x_0}s_{x_0} = -\mathrm{Id}$. Therefore $\mathcal{E}_n$ is a symmetric space.

$G = \mathrm{Isom}_\circ(\mathcal{E}_n) = \mathrm{PSL}(n,\mathbb{R}) = \mathrm{SL}(n,\mathbb{R})/\{\pm 1_n\}$ and $K = \mathrm{PSO}(n,\mathbb{R})$.

Exercise: Show that $\mathcal{E}_2$ is isometric to $\mathbb{H}^2$.

13.4 The algebraic structure of symmetric spaces

13.4.1 Symmetric spaces and involutions of Lie groups

Proposition 13.4.1. *Let X be a Riemannian symmetric space, $G = \mathrm{Isom}_\circ(X)$, $x_0 \in M$ and $K = \mathrm{Stab}_G(x_0)$. Let $H = \left\{g \in G \,\middle|\, s_{x_0} g = g s_{x_0}\right\}$ and $H_\circ$ its identity component. Then:*

$$H_\circ \subset K \subset H$$

Conversely, if G is a connected Lie group, $\sigma : G \to G$ an involutive Lie group automorphism, and K a compact subgroup of G such that $H_\circ \subset K \subset H$, where $H = \{g \in G \,|\, \sigma(g) = g\}$, then any G-invariant Riemannian metric on G/K is symmetric.

Remarks.

- *If K is compact, then G-invariant Riemannian metrics on G/K exist.*
- *The double inclusion $H_\circ \subset K \subset H$ should be interpreted as the fact that K as the same Lie algebra as H.*

Proof. Let X be a Riemannian symmetric space, $G = \mathrm{Isom}_\circ(X)$, $x_0 \in M$ and $K = \mathrm{Stab}_G(x_0)$. Let $H = \left\{g \in G \,\middle|\, s_{x_0} g = g s_{x_0}\right\}$ and $H_\circ$ its identity component.

If (γ_t) is a one-parameter subgroup of H, then $s_{x_0}(\gamma_t(x_0)) = \gamma_t(x_0)$ for all t. Since x_0 is an isolated fixed point of s_{x_0}, it follows that $\gamma_t \in K$, hence $H_\circ \subset K$.

If $g \in K$, then $h = s_{x_0} g s_{x_0}$ is an isometry of X satisfying $h(x_0) = x_0 = g(x_0)$ and $d_{x_0}h = d_{x_0}g$, hence $h = g$. Therefore $K \subset H$.

We now consider a connected Lie group G, an involutive Lie group automorphism $\sigma : G \to G$, and K a compact subgroup of G such that $H_\circ \subset K \subset H$, where $H = \{g \in G \,|\, \sigma(g) = g\}$. Let $\mathfrak{k}$ be the Lie algebra of K.

Remarquons que l'algèbre de Lie de G^σ est $\{X \in \mathfrak{g} \,|\, \theta(X) = X\}$. Comme $G^\sigma_\circ \subset K \subset G^\sigma$, l'algèbre de Lie de G^σ est celle de $\mathfrak{k}$, et on trouve :

$$\mathfrak{k} = \{X \in \mathfrak{g} \,|\, \theta(X) = X\}$$

Let $\pi : G \to G/K$ be the projection, and let $o = \pi(e)$. The map $s : \left\{\begin{array}{ccc} G/K & \to & G/K \\ \pi(g) & \mapsto & \pi(\sigma(g)) \end{array}\right.$ is well defined because $K \subset H$. We have $s(o) = o$.

Let us show that $d_o s = -\mathrm{Id}$. Since $s \circ \pi = \pi \circ \sigma$, we have that $d_o s \circ d_e \pi = d_e \pi \circ d_e \sigma$.

Since $(d_e\sigma)^2 = \mathrm{Id}$, and the Lie algebra of H, which is equal to $\mathfrak{k}$, is the eigenspace of $d_e\sigma$ for the eigenvalue 1, the eigenspace of $d_e\sigma$ for the eigenvalue -1 is supplementary to $\ker d_e\pi = \mathfrak{k}$, hence $d_o s = -\mathrm{Id}$.

For $g \in G$, we let $m_g : G/K \to G/K$ be the multiplication by g. We find:

$$\begin{aligned}
m_g \circ s \circ \pi &= m_g \circ \pi \circ \sigma \\
&= \pi \circ L_g \circ \sigma \\
&= \pi \circ \sigma \circ L_{\sigma(g)} \\
&= s \circ \pi \circ L_{\sigma(g)} \\
&= s \circ m_{\sigma(g)} \circ \pi
\end{aligned}$$

It follows that $m_g \circ s = s \circ m_{\sigma(g)}$. We now consider a G-invariant Riemannian metric Ω on G/K. Then $s^*\Omega$ is also G-invariant:

$$\begin{aligned}
m_g^*(s^*\Omega) &= (s \circ m_g)^*\Omega \\
&= (m_{\sigma(g)} \circ s)^*\Omega \\
&= s^*(m_{\sigma(g)}^*\Omega) \\
&= s^*\Omega
\end{aligned}$$

Moreover, $(\sigma^*\Omega)_o = \Omega_o$, and a G-invariant metric is characterized by its value at o, hence $s^*\Omega = \Omega$, and $(G/K, \Omega)$ is a Riemannian symmetric space. $\square$

13.4.2 The Cartan involution

Definition 13.4.2. *Let X be a Riemannian symmetric space, $G = \mathrm{Isom}_\circ(X)$, $\mathfrak{g}$ its Lie algebra, $x_0 \in M$ and $K = \mathrm{Stab}_G(x_0)$. The* ***Cartan involution*** *relatively to x_0 is the map $\theta = d_{\mathrm{Id}}\sigma : \mathfrak{g} \to \mathfrak{g}$ where $\sigma : G \to G$ is defined by $\sigma(g) = s_{x_0} \circ g \circ s_{x_0}$.*

Exemple : Pour $\mathcal{E}_n$, et $x_0 = 1_n$, on a $s_{x_0}(x) = x^{-1}$. Pour $[g] \in \mathrm{PSL}(n, \mathbb{R})$, notons $\alpha([g]) \in G$, i.e. $\alpha([g])(x) = gx^tg$. On trouve :

$$s_{x_0} \circ \alpha([g]) \circ s_{x_0}(x) = (gx^{-1}\,{}^tg)^{-1} = {}^tg^{-1}x^t({}^tg^{-1})$$

Donc $\sigma([g]) = [{}^tg^{-1}]$, et $\theta(X) = -{}^tX$ pour $X \in \mathfrak{sl}(n, \mathbb{R})$.

Exemple : Pour $\mathcal{E}_n$, on a $K = G^\sigma$.

Proposition 13.4.3. *Let $\mathbb{X}$ be a Riemannian symmetric space, $G = \mathrm{Isom}_\circ(\mathbb{X})$, $\mathfrak{g}$ its Lie algebra, $o \in \mathbb{X}$ and $K = G_o$. Let θ be the Cartan involution relatively to o, and B the Killing form of $\mathfrak{g}$. Set $\mathfrak{k} = \{X \in \mathfrak{g} \mid \theta(X) = X\}$ and $\mathfrak{p} = \{X \in \mathfrak{g} \mid \theta(X) = -X\}$.*

1. *$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$, and this decomposition is $\mathrm{Ad}(K)$-invariant.*
2. *$[\mathfrak{k}, \mathfrak{k}] \subset \mathfrak{k}$, $[\mathfrak{k}, \mathfrak{p}] \subset \mathfrak{p}$ et $[\mathfrak{p}, \mathfrak{p}] \subset \mathfrak{k}$.*
3. *$\mathfrak{k}$ is the Lie algebra of K and $d_e\varphi_o : \mathfrak{p} \to T_o\mathbb{X}$ is an isomorphism.*
4. *$B|_{\mathfrak{k} \times \mathfrak{p}} = 0$.*

5. *$B|_{\mathfrak{k}\times\mathfrak{k}}$ is negative definite.*

Remark. *The decomposition $\mathfrak{g} = \mathfrak{p} \oplus \mathfrak{k}$ is called the* ***Cartan decomposition*** *of $\mathfrak{g}$.*

Proof.

1. The decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ is a consequence of $\theta^2 = \mathrm{Id}$.
   If $\sigma : G \to G$ denotes the conjugacy by s_o, then $i_g \circ \sigma = \sigma \circ i_{\sigma(g)}$ for all $g \in G$. For $g \in K$, we have $\sigma(g) = g$, hence $i_g \circ \sigma = \sigma \circ i_g$. Differentiating at Id yields $\mathrm{Ad}(g) \circ \theta = \theta \circ \mathrm{Ad}(g)$, hence the $\mathrm{Ad}(g)$-invariance of the decomposition $\mathfrak{k} \oplus \mathfrak{p}$.

2. All three inclusions are a consequence of the fact that θ is a Lie algebra morphism.

3. We have seen in the proof of Proposition 13.4.1 that $\mathfrak{k}$ is the Lie algebra of K.
   The map $d_e\varphi_o$ is surjective because the action of G is transitive. Its kernel is the Lie algebra of K, hence supplementary to $\mathfrak{p}$, therefore $d_e\varphi_o : \mathfrak{p} \to T_oX$ is an isomorphism.

4. If $X \in \mathfrak{k}$ and $Y \in \mathfrak{p}$, then the matrix of $\mathrm{ad}(X) \circ \mathrm{ad}(Y)$ in a basis adapted to the decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ has vanishing diagonal blocs, hence $\mathrm{Tr}(\mathrm{ad}(X) \circ \mathrm{ad}(Y)) = 0$, i.e. $B(X,Y) = 0$.

5. Let $\langle\langle \cdot | \cdot \rangle\rangle$ be an $\mathrm{Ad}(K)$-invariant inner product on $\mathfrak{g}$. Let $(e_1, \ldots, e_n)$ be an orthonormal basis of $\mathfrak{g}$.
   For $X \in \mathfrak{k}$, we find:
$$\begin{aligned} B(X,X) &= \mathrm{Tr}(\mathrm{ad}(X) \circ \mathrm{ad}(X)) \\ &= \sum_{i=1}^{n} \langle\langle \mathrm{ad}(X) \circ \mathrm{ad}(X) e_i | e_i \rangle\rangle \\ &= -\sum_{i=1}^{n} \langle\langle \mathrm{ad}(X) e_i | \mathrm{ad}(X) e_i \rangle\rangle \\ &= -\sum_{i=1}^{n} \| [X, e_i] \|^2 \end{aligned}$$
   We get $B(X,X) \le 0$. Moreover, if $B(X,X) = 0$, then $X \in \mathfrak{z}(\mathfrak{g})$. It follows that for all $t \in \mathbb{R}$ and $Y \in \mathfrak{g}$, $\exp(tX)$ commutes with $\exp(Y)$.
   Since G is connected, we find that $\exp(tX) \in Z(G)$.
   The G-action on M is transitive, and for all $g \in G$ we have:
$$\exp(tX)(g(o)) = g(\exp(tX)(o)) = g(o)$$
   It follows that $\exp(tX) = \mathrm{Id}$, hence $X = 0$.

□

Exemple : Pour $\mathcal{E}_n$, on a $\mathfrak{k} = \mathfrak{so}(n,\mathbb{R})$ et $\mathfrak{p} = \left\{X \in \mathfrak{sl}(n,\mathbb{R}) \middle| {}^tX = X\right\}$. La forme de Killing de $\mathfrak{g} = \mathfrak{sl}(n,\mathbb{R})$ est :

$$B(X,Y) = 4n\,\mathrm{Tr}(XY)$$

Chapter 14

The geometry of symmetric spaces

14.1 Geodesics in symmetric spaces

Definition 14.1.1. *Let (M,g) be a Riemannian manifold. An isometry $\varphi \in \mathrm{Isom}(M,g)$ is a* ***transvection*** *if there are a non constant geodesic $c : \mathbb{R} \to M$ and $t_0 \in \mathbb{R}$ such that for all $t \in \mathbb{R}$, we have $\varphi(c(t)) = c(t+t_0)$ and $d_{c(t)}\varphi = \|_t^{t+t_0}$.*

Lemma 14.1.2. *Let $\mathbb{X}$ be a symmetric space, and let $c : \mathbb{R} \to M$ be a non constant geodesic. For $t \in \mathbb{R}$, consider $g_t = s_{c(\frac{t}{2})} \circ s_{c(0)}$. Then g_t is a transvection, and $t \mapsto g_t$ is a one-parameter subgroup of G.*

Proof. Lemma 13.1.2 shows that $g_t(c(s)) = c(t+s)$, and Proposition 13.2.9 shows that $d_{c(s)}g_t = \|_s^{s+t}$, so g_t is a transvection.

The isometries g_{t+s} and $g_t \circ g_s$ both send $o = c(0)$ to $c(t+s)$, and their differential at o is the parallel transport along c (Proposition 13.2.9). It follows that they are equal, i.e. $t \mapsto g_t$ is a one parameter subgroup of G (it is smooth thanks to Lemma 13.1.4). □

Proposition 14.1.3. *Let $\mathbb{X}$ be a Riemannian symmetric space, $G = \mathrm{Isom}_\circ(\mathbb{X})$, $\mathfrak{g}$ its Lie algebra, $o \in \mathbb{X}$ and $\mathfrak{g} = \mathfrak{p} \oplus \mathfrak{k}$ the associated Cartan decomposition. For all $X \in \mathfrak{p}$ and $g \in G$, the isometry $g \exp_G(X) g^{-1}$ is a transvection of $\mathbb{X}$. Moreover, any transvection of $\mathbb{X}$ has this form.*

Proof. Since the conjugate of a transvection by an isometry is a transvection, so we only have to show that $\exp_G(X)$ is a transvection for $X \in \mathfrak{p} \setminus \{0\}$. Set $v = d_e\varphi_o(X)$, and $c = c_v$ (it is a non constant geodesic because of Proposition 13.4.3). For $t \in \mathbb{R}$, we let $g_t = s_{c(\frac{t}{2})} \circ s_o$.

According to Lemma 14.1.2, $t \mapsto g_t$ is a one-parameter subgroup of G, so we can consider $X' \in \mathfrak{g}$ such that $g_t = \exp_G(tX)$. First, we wish to show that X' is in $\mathfrak{p}$. For this we compute $\sigma\left(\exp_G(tX')\right)$

$$\begin{aligned}
\sigma\left(\exp_G(tX')\right) &= s_o \circ s_{c(\frac{t}{2})} \circ s_o \circ s_o \\
&= s_o \circ s_{c(\frac{t}{2})} \\
&= \left(s_{c(\frac{t}{2})} \circ s_o\right)^{-1} \\
&= g_t^{-1} \\
&= g_{-t} \\
&= \exp_G(-tX')
\end{aligned}$$

The derivative at $t = 0$ yields $\theta(X') = -X'$, i.e. $X' \in \mathfrak{p}$. Since $g_t(o) = c(t)$, we find:

$$\begin{aligned}
v &= \left.\frac{d}{dt}\right|_{t=0} g_t(o) \\
&= \left.\frac{d}{dt}\right|_{t=0} \varphi_o\left(\exp_G(tX')\right) \\
&= d_e\varphi_o(X')
\end{aligned}$$

It follows from Proposition 13.4.3 that $X' = X$. Hence $\exp_G(X) = g_1$ is a transvection.

Reciprocally, let $g \in \mathrm{Isom}(\mathbb{X})$ be a transvection along a geodesic c. Let $t_0 \in \mathbb{R}$ be given by the definition of a transvection. Since G acts transitively on $\mathbb{X}$, we can assume that $c(0) = o$.

Consider $g_t = s_{c(\frac{t}{2})} \circ s_o$ as in Lemma 14.1.2. Then g and g_{t_0} have the same one-jet at o, so they are equal.

Let $X \in \mathfrak{g}$ be such that $g_t = \exp_G(tX)$ for all $t \in \mathbb{R}$. The computation above shows that $X \in \mathfrak{p}$. We find that $g = \exp_G(t_0X)$. □

Corollary 14.1.4. *Let $\mathbb{X}$ be a Riemannian symmetric space, $G = \mathrm{Isom}_\circ(\mathbb{X})$, $\mathfrak{g}$ its Lie algebra, $o \in \mathbb{X}$ and $\mathfrak{g} = \mathfrak{p} \oplus \mathfrak{k}$ the associated Cartan decomposition. The geodesics going through o are exactly the curves $t \mapsto \exp_G(tX)o$ for $X \in \mathfrak{p}$.*

Proof. Let $v \in T_o\mathbb{X}$, and set $c = c_v$. According to Proposition 13.4.3, there is $X \in \mathfrak{p}$ such that $v = d_e\varphi_o(X)$. Following the proof of Proposition 14.1.3, we find:

$$\begin{aligned}
\exp_G(tX).o &= s_{c(\frac{t}{2})} \circ s_o(o) \\
&= s_{c(\frac{t}{2})}(o) \\
&= c(t)
\end{aligned}$$

□

14.2 The Levi-Civita connection of a symmetric space

In order to relate the Levi-Civita connection of a symmetric space with computations in the Lie algebra of its isometry group, we can can start by seeing that a smooth action $G \curvearrowright M$ induces a Lie algebra anti-morphism (i.e. reversing the bracket) from the Lie algebra of G to the Lie algebra $\mathcal{X}(M)$ of vector fields on M.

Definition 14.2.1. *Let G be a Lie group with Lie algebra $\mathfrak{g}$, and M a manifold. Consider a smooth action $G \curvearrowright M$. For $X \in \mathfrak{g}$, the **fundamental vector field** associated to X is $\overline{X} \in \mathcal{X}(M)$ defined by:*

$$\overline{X}(x) = d_e\varphi_x(X)$$

Lemma 14.2.2. *Let $G \curvearrowright M$ be a smooth action of a Lie group. For $X, Y \in \mathfrak{g}$, we have $[\overline{X}, \overline{Y}] = \overline{[Y, X]}$.*

Remark. *This result can be interpreted by seeing* $\mathrm{Diff}(M)$ *as an infinite dimensional Lie group, whose Lie algebra is $\mathcal{X}(M)$, but the bracket is the opposite of the usual Lie bracket of vector fields.*

Proof. First notice that the flow φ^t of $\overline{X}$ is given by:

$$\varphi^t(x) = \exp_G(tX).x$$

Indeed, $(t, x) \mapsto \exp_G(tX).x$ is a flow and we have:

$$\begin{aligned} \left.\frac{d}{dt}\right|_{t=0} \exp_G(tX).x &= \left.\frac{d}{dt}\right|_{t=0} \varphi_x\Big(\exp_G(tX)\Big) \\ &= d_e\varphi_x(X) \\ &= \overline{X}(x) \end{aligned}$$

So the flow ψ^t of $\overline{Y}$ is also given by:

$$\psi^t(x) = \exp_G(tY).x$$

We can compute $[\overline{X}, \overline{Y}]$ by looking at the commutators of the flows:

$$\begin{aligned} [\overline{X}, \overline{Y}](x) &= \left.\frac{d}{dt}\right|_{t=0^+} \psi_{\sqrt{t}} \circ \varphi_{\sqrt{t}} \circ \psi_{-\sqrt{t}} \circ \varphi_{-\sqrt{t}}(x) \\ &= \left.\frac{d}{dt}\right|_{t=0^+} (\exp(\sqrt{t}Y)\exp(\sqrt{t}X)\exp(-\sqrt{t}Y)\exp(-\sqrt{t}X)).x \end{aligned}$$

But we also have:

$$\left.\frac{d}{dt}\right|_{t=0^+} \exp(\sqrt{t}Y)\exp(\sqrt{t}X)\exp(-\sqrt{t}Y)\exp(-\sqrt{t}X) = [Y, X]$$

It follows that $[\overline{X}, \overline{Y}] = \overline{[Y, X]}$. □

Definition 14.2.3. *Let (M,g) be a Riemannian manifold. A **Killing field** is $X \in \mathcal{X}(M)$ whose flow preserves g, i.e.*

$$g_{\varphi_X^t(x)}\left(d_x\varphi_X^t(u), d_x\varphi_X^t(v)\right) = g_x(u,v)$$

whenever $\varphi_X^t(x)$ is defined.

Remark. *This condition is equivalent to $\mathcal{L}_X g = 0$, where $\mathcal{L}_X$ is the Lie derivative.*

If a Lie group G acts smoothly on a Riemannian manifold (M,g) by isometries, i.e. the maps $x \mapsto g.x$ are isometries, then the fundamental vector fields are Killing fields.

Lemma 14.2.4. *Let (M,g) a Riemmanian manifold with Levi-Civita connection ∇. Let $X \in \mathcal{X}(M)$ be a Killing field. For all $V, W \in \mathcal{X}(M)$, we have:*

$$[X, \nabla W(V)] = \nabla W([X,V]) + \nabla[X,W](V)$$

Proof. For $\varphi \in \mathrm{Isom}(M,g)$, we have $\varphi^*\nabla = \nabla$ (because $\varphi^*\nabla$ is the Levi-Civita connection of $\varphi^* g = g$). This means $\varphi^*(\nabla W(V)) = \nabla\varphi^* W(\varphi^* V)$.

We can apply this to the flow φ^t of X, and find:

$$(\varphi^t)^*(\nabla W(V)) = \nabla(\varphi^t)^* W\left((\varphi^t)^* V\right)$$

The derivative of the left hand side at $t = 0$ is $[X, \nabla W(V)]$ by definition of the Lie bracket. The right hand side derivates to $\nabla W([X,V]) + \nabla[X,W](V)$. □

Theorem 14.2.5. *Let $\mathbb{X}$ be a Riemannian symmetric space, $G = \mathrm{Isom}_\circ(\mathbb{X})$, $\mathfrak{g}$ its Lie algebra, $o \in \mathbb{X}$ and $\mathfrak{g} = \mathfrak{p} \oplus \mathfrak{k}$ the associated Cartan decomposition. Consider $X \in \mathfrak{p}$ and $v = d_e\varphi_o(X)$. For all $V \in \mathcal{X}(M)$, we have $\nabla_o V(v) = [\overline{X}, V](o)$.*

Proof. Let φ^t be the flow of $\overline{X}$, i.e. $\varphi^t(x) = \exp_G(tX).x$. The relationship between a connection and its parallel transport yields

$$\nabla_o V(v) = \left.\frac{d}{dt}\right|_{t=0} \Big\|_{\varphi^t(o)}^{o} V\left(\varphi^t(o)\right)$$

Here $\|_{\varphi^t(o)}^{o}$ is the parallel transport along the flow line $t \mapsto \varphi^t(o) = \exp_G(tX)$. According to Proposition 14.1.3, it is equal to the differential of the transvection φ^t, hence:

$$\nabla_o V(v) = \left.\frac{d}{dt}\right|_{t=0} \left(d_o\varphi^t\right)^{-1}\left(V\left(\varphi^t(o)\right)\right)$$

We recognize $\left(\varphi^t\right)^* V(o)$, and find $\nabla_o V(v) = [\overline{X}, V](o)$. □

14.3 The curvature of a symmetric space

Theorem 14.3.1. *Let $\mathbb{X}$ be a Riemannian symmetric space, $G = \mathrm{Isom}_\circ(\mathbb{X})$, $\mathfrak{g}$ its Lie algebra, $o \in \mathbb{X}$ and $\mathfrak{g} = \mathfrak{p} \oplus \mathfrak{k}$ the associated Cartan decomposition. For $X, Y, Z \in \mathfrak{p}$, we set $u = d_e\varphi_o(X), v = d_e\varphi_o(Y)$ and $w = d_e\varphi_o(Z) \in T_o\mathbb{X}$. We have:*

$$R_o(u,v)w = -d_e\varphi_o\left([[X,Y],Z]\right)$$

Remark. *Note that $[X,Y] \in \mathfrak{k}$, hence $[[X,Y],Z] \in \mathfrak{p}$ (Proposition 13.4.3). In particular, we find $R_o(u,v)w = 0 \iff [[X,Y],Z] = 0$.*

Proof. We use the fundamental vector fields $\overline{X}, \overline{Y}, \overline{Z}$ to compute the curvature.

$$\begin{aligned} R_o(u,v)w &= R(\overline{X},\overline{Y})\overline{Z}(x_0) \\ &= \nabla_o\left(\nabla\overline{Z}(\overline{Y})\right)(u) - \nabla_o\left(\nabla\overline{Z}(\overline{X})\right)(v) - \nabla_o\overline{Z}\left([\overline{X},\overline{Y}](o)\right) \end{aligned} \tag{14.1}$$

According to Lemma 14.2.2, we find:

$$\begin{aligned} [\overline{X},\overline{Y}](o) &= -\overline{[X,Y]}(o) \\ &= -d_e\varphi_o([X,Y]) \end{aligned}$$

But Proposition 13.4.3 shows that $[X,Y] \in \mathfrak{k} = \ker d_e\varphi_o$, hence $[\overline{X},\overline{Y}](o) = 0$, and (14.1) simplifies:

$$R_o(u,v)w = \nabla_o\left(\nabla\overline{Z}(\overline{Y})\right)(u) - \nabla_o\left(\nabla\overline{Z}(\overline{X})\right)(v) \tag{14.2}$$

Theorem 14.2.5 yields

$$\nabla_o\left(\nabla\overline{Z}(\overline{Y})\right)(u) = \left[\overline{X}, \nabla\overline{Z}(\overline{Y})\right](o)$$

Since $\overline{X}$ is a Killing field, we can apply Lemma 14.2.4.

$$\left[\overline{X}, \nabla\overline{Z}(\overline{Y})\right] = \nabla\overline{Z}\left([\overline{X},\overline{Y}]\right) + \nabla[\overline{X},\overline{Z}](\overline{Y})$$

Evaluating at o, we have seen that $[\overline{X},\overline{Y}](o) = 0$, so all that remains is:

$$\left[\overline{X}, \nabla\overline{Z}(\overline{Y})\right](o) = \nabla_o[\overline{X},\overline{Z}](v)$$

Applying Theorem 14.2.5 once again, we find:

$$\left[\overline{X}, \nabla\overline{Z}(\overline{Y})\right](o) = [\overline{Y},[\overline{X},\overline{Z}]](o)$$

Two applications of Lemma 14.2.2 yield:

$$\left[\overline{X}, \nabla\overline{Z}(\overline{Y})\right](o) = \overline{[Y,[X,Z]]}(o)$$

We have shown:

$$\nabla_o\left(\nabla\overline{Z}(\overline{Y})\right)(u) = \overline{[Y,[X,Z]]}(o) \tag{14.3}$$

Injecting (14.3) back into (14.2), we find:

$$R_o(u,v)w = \overline{[Y,[X,Z]]}(o) - \overline{[X,[Y,Z]]}(o)$$

The Jacobi identity yields the desired formula. □

Example: For $\mathcal{E}_n$, we can compute the sectional curvature of the plane generated by $X, Y \in \mathfrak{p}$:

$$R(X,Y,Y,X) = -\mathrm{Tr}([[X,Y],Y]X) = \mathrm{Tr}([X,Y]^2)$$

We find that the sectional curvature is 0 if $[X,Y] = 0$, and negative otherwise.

14.4 Totally geodesic submanifolds of symmetric spaces

Recall that a submanifold N of a Riemannian manifold $(M,g)M$ is called **totally geodesic** if any geodesic of N is a geodesic of M.

Now we start with a point $x \in M$ and a vector subspace $V \subset T_xM$, and wonder whether there is a totally geodesic submanifold $N \subset M$ such that $x \in N$ and $T_xN = V$. Actually there is not much choice for N, as it should be an open subset of $\exp_x(V)$. But in general, $\exp_x(V)$ is not totally geodesic.

We have already seen that a necessary condition is for V to be stable under the Riemann tensor: if $u,v,w \in V$, then $R_x(u,v)w \in V$. When (M,g) is a symmetric space, we will show that it is also a sufficient condition. In the general case, it cannot be sufficient as one should at least impose some stability by the covariant derivatives of the Riemann tensor.

This stability under the Riemann tensor has a nice interpretation in Lie algebraic terms for a symmetric space.

Definition 14.4.1. *Let $\mathfrak{g}$ be a Lie algebra. A vector subspace $\mathfrak{v} \subset \mathfrak{g}$ is called a **Lie triple system** if it satisfies:*

$$\forall X,Y,Z \in \mathfrak{v} \quad [[X,Y],Z] \in \mathfrak{v}$$

Remark. *This can be summarized as $[\mathfrak{v},[\mathfrak{v},\mathfrak{v}]] \subset \mathfrak{v}$.*

Lemma 14.4.2. *Let $\mathfrak{g}$ be a Lie algebra. If $\mathfrak{v} \subset \mathfrak{g}$ is a Lie triple system, then $[\mathfrak{v},\mathfrak{v}]$ and $\mathfrak{v} + [\mathfrak{v},\mathfrak{v}]$ are Lie subalgebras of $\mathfrak{g}$.*

Proof. For $X, Y, Z, W \in \mathfrak{v}$, the Jacobi identity yields:

$$[[X,Y],[Z,W]] = -[Z,\underbrace{[W,[X,Y]]}_{\in \mathfrak{v}} - [X,\underbrace{[[X,Y],Z]}_{\in \mathfrak{v}} \in [\mathfrak{v},\mathfrak{v}]$$

For $X, Y \in \mathfrak{v} + [\mathfrak{v},\mathfrak{v}]$, there are three cases to deal with:

- If $X \in \mathfrak{v}$ and $Y \in \mathfrak{v}$ then $[X,Y] \in [\mathfrak{v},\mathfrak{v}] \subset \mathfrak{v} + [\mathfrak{v},\mathfrak{v}]$.
- If $X \in \mathfrak{v}$ and $Y \in [\mathfrak{v},\mathfrak{v}]$ then $[X,Y] \in \mathfrak{v} \subset \mathfrak{v} + [\mathfrak{v},\mathfrak{v}]$ by definition of a Lie triple system.
- If $X \in [\mathfrak{v},\mathfrak{v}]$ and $Y \in [\mathfrak{v},\mathfrak{v}]$ then $[X,Y] \in [\mathfrak{v},\mathfrak{v}] \subset \mathfrak{v} + [\mathfrak{v},\mathfrak{v}]$ because $[\mathfrak{v},\mathfrak{v}]$ is a Lie subalgebra.

These three cases show that $\mathfrak{v} + [\mathfrak{v},\mathfrak{v}]$ is a Lie subalgebra of $\mathfrak{g}$. □

Theorem 14.4.3. *Let $\mathbb{X}$ be a Riemannian symmetric space, $G = \mathrm{Isom}_\circ(\mathbb{X})$, $\mathfrak{g}$ its Lie algebra, $o \in \mathbb{X}$ and $\mathfrak{g} = \mathfrak{p} \oplus \mathfrak{k}$ the associated Cartan decomposition. Let $V \subset T_o\mathbb{X}$ be a vector subspace. The following are equivalent:*

1. *V is stable under the Riemann tensor R_o.*
2. *$(d_e\varphi_o)^{-1}(V) \cap \mathfrak{p}$ is a Lie triple system in $\mathfrak{g}$.*
3. *There is a totally geodesic submanifold N of $\mathbb{X}$ such that $x \in N$ and $T_xN = V$.*

Moreover, if these conditions are met, then $\exp_o(V)$ is an immersed totally geodesic submanifold of $\mathbb{X}$, and it is a symmetric space.

Proof. Note that *1.* $\Longleftrightarrow$ *2.* is a straightforward consequence of Theorem 14.3.1.
3.$\Rightarrow$*1.* is a general fact for Riemannian manifolds (Lemma **??**). We will now prove *2.* implies *3.* and the last statement.
Let $\mathfrak{v} = (d_e\varphi_o)^{-1}(V) \cap \mathfrak{p}$, and assume that it is a Lie triple system. According to Lemma 14.4.2, $\mathfrak{h} = \mathfrak{v} + [\mathfrak{v},\mathfrak{v}]$ is a Lie subalgebra of $\mathfrak{g}$, so we can consider the connected Lie subgroup $H \subset G$ such that $T_eH = \mathfrak{h}$.

Note that $H.o$ is an immersed submanifold on M (it is an orbit of a smooth action), and that $T_oH.o = d_e\varphi_o\mathfrak{h} = V$ (because $[\mathfrak{v},\mathfrak{v}] \subset \mathfrak{k} = \ker d_e\varphi_o$).

Let us show that $\exp_o(V) \subset H.o$. Let $v \in V$, and consider $X \in \mathfrak{v}$ such that $d_e\varphi_o(X) = v$. Corollary 14.1.4 yields $\exp_o(v) = \exp_G(X).o$. Since $X \in \mathfrak{v} \subset \mathfrak{h}$, we find $\exp_G(X) \in H$ and $\exp_o(v) \in H.o$. So we have proved $\exp_o(V) \subset H.o$.

This means that any geodesic starting from o and tangent to V lies in $H.o$. Given $x \in H.o$, we write $x = g.o$ for some $g \in H$, and we have $T_xH.o = d_og(V)$. For $v \in V$, by setting $w = d_og(v)$, we find $c_w(t) = g.c_v(t) \in g(H.o) = H.o$. This shows that $H.o$ is totally geodesic, i.e. we have proven *3.*

Note that $H.o$ is complete. Indeed, for $v \in V$ the geodesic c_v of $H.o$ is defined on $\mathbb{R}$. It follows from the Hopf-Rinow Theorem that $H.o \subset \exp_o(V)$, hence $H.o = \exp_o(V)$. So $\exp_o(V)$ is an immersed totally geodesic submanifold of $\mathbb{X}$.

Now note that $\exp_o(V)$ is stable under s_o. This implies that $H.o$ is also stable under $s_{g.o} = gs_og^{-1}$ for $g \in H$, so $H.o$ is a symmetric space. □

Definition 14.4.4. *Let $\mathbb{X}$ be a symmetric space. A **flat** of $\mathbb{X}$ is a complete and connected totally geodesic submanifold $F \subset M$ which is flat.*
*The **rank** of $\mathbb{X}$ is the maximal dimension of a flat of $\mathbb{X}$.*

Remark. *The rank is sometimes called the geometric rank.*

Proposition 14.4.5. *Let $\mathbb{X}$ be a Riemannian symmetric space, $G = \mathrm{Isom}_\circ(\mathbb{X})$, $\mathfrak{g}$ its Lie algebra, $o \in \mathbb{X}$ and $\mathfrak{g} = \mathfrak{p} \oplus \mathfrak{k}$ the associated Cartan decomposition. Let $V \subset T_o\mathbb{X}$ be a vector subspace. There is a flat $F \subset \mathbb{X}$ such that $T_oF = V$ if and only if $(d_e\varphi_o)^{-1}(V) \cap \mathfrak{p}$ is an abelian subalgebra of $\mathfrak{g}$.*

Remark. *One only need to ask of $(d_e\varphi_o)^{-1}(V) \cap \mathfrak{p}$ to be a Lie subalgebra, since $[\mathfrak{p},\mathfrak{p}] \subset \mathfrak{k}$ shows that a vector subspace of $\mathfrak{p}$ must be abelian in order to be a Lie subalgebra of $\mathfrak{g}$.*

Proof. Let $\mathfrak{v} = (d_e\varphi_o)^{-1}(V) \cap \mathfrak{p}$. If $\mathfrak{v}$ is an abelian subalgebra of $\mathfrak{g}$, then it is a Lie triple system and Theorem 14.4.3 shows that there is a totally geodesic submanifold $F \subset X$ such that $T_oF = V$, and that F is a symmetric space. The curvature of F can be computed thanks to Theorem 14.3.1, and it vanishes.

Now assume that there is a flat $F \subset \mathbb{X}$ such that $T_oF = V$. Theorem 14.3.1 yields $[[X,Y],Z] = 0$ for all $X,Y,Z \in \mathfrak{v}$. Let B be the Killing form of $\mathfrak{g}$. Since B is ad-invariant, we find for $X,Y \in \mathfrak{v}$:

$$\begin{aligned} B([X,Y],[X,Y]) &= B(\mathrm{ad}(X)Y,[X,Y]) \\ &= -B(Y,\mathrm{ad}(X)[X,Y]) \\ &= B(Y,[[X,Y],X]) \end{aligned}$$

Since $[[X,Y],X] = 0$, we find that $B([X,Y],[X,Y]) = 0$. But we showed in Proposition 13.4.3 that B is negative on $\mathfrak{k}$, and $[X,Y] \in [\mathfrak{p},\mathfrak{p}] \subset \mathfrak{k}$. It follows that $[X,Y] = 0$, i.e. $\mathfrak{v}$ is an abelian subalgebra of $\mathfrak{g}$. □

Examples: The rank of $\mathbb{E}^n$ is n, the rank of $\mathbb{S}^n$ and $\mathbb{H}^n$ is 1. For $\mathcal{E}_n$, a flat corresponds to matrices that commute, so they are diagonalisable in the same basis. It follows that the rank of $\mathcal{E}_n$ is $n-1$.

14.5 The sectional curvature of symmetric spaces

In order to relate the geometry of a symmetric space $\mathbb{X}$ and the Lie algebra $\mathfrak{g}$ of its isometry group, we first notice that $\mathfrak{p}$ in the Cartan decomposition has two quadratic forms: the Riemannian metric of $\mathbb{X}$ and the Killing form of $\mathfrak{g}$. To understand their relationship, we must first ask if they have the same signature.

Definition 14.5.1. *Let $\mathbb{X}$ be a symmetric space, $G = \mathrm{Isom}(\mathbb{X})$, $\mathfrak{g}$ its Lie algebra, and let $o \in \mathbb{X}$ with corresponding Cartan decomposition $\mathfrak{g} = \mathfrak{p} \oplus \mathfrak{k}$. We say that $\mathbb{X}$ is:*

- *Of **Euclidean type** if $\mathfrak{p}$ is an abelian ideal of $\mathfrak{g}$.*
- *Of **compact type** if $B|_{\mathfrak{p}\times\mathfrak{p}}$ is negative definite.*
- *Of **non compact type** if $B|_{\mathfrak{p}\times\mathfrak{p}}$ is positive definite.*

Remark. *By using the homogeneity, we can see that this definition does not depend on the choice of $o \in \mathbb{X}$.*

The Euclidean type can be interpreted in terms of curvature.

Proposition 14.5.2. *Let $\mathbb{X}$ be a symmetric space. Then $\mathbb{X}$ is of Euclidean type if and only if $\mathbb{X}$ is flat.*

Proof. According to Proposition 14.4.5 applied to $T_o\mathbb{X}$, we see that $\mathbb{X}$ is flat if and only if $\mathfrak{p}$ is abelian. If $\mathfrak{p}$ is abelian, then $[\mathfrak{p}, \mathfrak{k}] \subset \mathfrak{p}$ shows that it is an ideal, which completes the proof. □

Consequently, a simply connected symmetric space of Euclidean type is isometric to the Euclidean space $\mathbb{E}^n$, hence the terminology.

Not every symmetric space is of one of these types, but the classification of symmetric spaces reduces to these three types. Note that the product of symmetric spaces is always a symmetric space, so a classification of symmetric spaces requires the understanding of which symmetric spaces can split into a product.

Definition 14.5.3. *Let $\mathbb{X}$ be a symmetric space, $G = \mathrm{Isom}(\mathbb{X})$, $\mathfrak{g}$ its Lie algebra, and let $o \in \mathbb{X}$ with corresponding Cartan decomposition $\mathfrak{g} = \mathfrak{p} \oplus \mathfrak{k}$. We say that $\mathbb{X}$ is **irreducible** if the only vector subspaces $\mathfrak{v} \subset \mathfrak{p}$ satisfying $[\mathfrak{k}, \mathfrak{v}] \subset \mathfrak{v}$ are $\{0\}$ and $\mathfrak{p}$.*

Remarks.

- *By using the homogeneity, we can see that this definition does not depend on the choice of $o \in \mathbb{X}$.*
- *This is equivalent to stating that the action of the identity component of K on $T_o\mathbb{K}$ is irreducible.*

Irreducible symmetric spaces must be of one of the three types described above.

Proposition 14.5.4. *Every irreducible symmetric space is either of Euclidean, compact or non compact type.*

Before we can prove Proposition 14.5.4, we need to introduce some notations.

Definition 14.5.5. *Let $\mathbb{X}$ be a symmetric space, $G = \mathrm{Isom}_\circ(\mathbb{X})$, $\mathfrak{g}$ its Lie algebra and B its Killing form, $o \in \mathbb{X}$ and $\mathfrak{g} = \mathfrak{p} \oplus \mathfrak{k}$ the associated Cartan decomposition. The **Riemannian form** is the inner product $\langle \cdot | \cdot \rangle$ on $\mathfrak{p}$ defined by:*

$$\forall X, Y \in \mathfrak{p} \quad \langle X | Y \rangle = \langle d_e\varphi_o(X) | d_e\varphi_o(Y) \rangle_o$$

*The **Killing operator** is the self-adjoint (for $\langle \cdot | \cdot \rangle$) operator $b \in \mathrm{End}(\mathfrak{p})$ defined by:*

$$\forall X, Y \in \mathfrak{p} \quad B(X, Y) = \langle b(X) | Y \rangle$$

Lemma 14.5.6. *Let $\mathbb{X}$ be a symmetric space, $G = \mathrm{Isom}_\circ(\mathbb{X})$, $\mathfrak{g}$ its Lie algebra, $o \in \mathbb{X}$ and $\mathfrak{g} = \mathfrak{p} \oplus \mathfrak{k}$ the associated Cartan decomposition. The Killing operator b commutes with* $\mathrm{ad}(X)$ *for all $X \in \mathfrak{k}$.*

Proof. Since $\exp_G(tX) \in \mathrm{Isom}(\mathbb{X})$, we have:

$$\forall v, w \in T_o\mathbb{X} \quad \Big\langle d_o \exp_G(tX)(v) \Big| d_o \exp_G(tX)(w) \Big\rangle_{\exp_G(tX).o} = \langle v | w \rangle_o$$

The derivative at $t = 0$ pulled back by φ_o yields:

$$\forall Y, Z \in \mathfrak{p} \quad \langle \mathrm{ad}(X)Y | Z \rangle + \langle Y | \mathrm{ad}(X)Z \rangle = 0 \tag{14.4}$$

The ad-invariance of the Killing form now translates to b as:

$$\forall Y, Z \in \mathfrak{p} \quad \underbrace{\langle b \circ \mathrm{ad}(X)Y | Z \rangle}_{=B(\mathrm{ad}(X)Y, Z)} + \underbrace{\langle b(Y) | \mathrm{ad}(X)Z \rangle}_{=B(Y, \mathrm{ad}(X)Z)} = 0 \tag{14.5}$$

Putting (14.4) and (14.5) together, we find for all $Y, Z \in \mathfrak{p}$:

$$\begin{aligned} \langle b \circ \mathrm{ad}(X)(Y) | Z \rangle &= -\langle b(Y) | \mathrm{ad}(X)(Z) \rangle \\ &= \langle \mathrm{ad}(X) \circ b(Y) | Z \rangle \end{aligned}$$

Since $\langle \cdot | \cdot \rangle$ is positive definite, it follows that $b \circ \mathrm{ad}(X) = \mathrm{ad}(X) \circ b$. □

Proof of Proposition 14.5.4. Since b is self-adjoint, there is an orthonormal basis $X_1, \ldots, X_n$ of $\mathfrak{p}$ such that $b(X_i) = \lambda_i X_i$ for $\lambda_i \in \mathbb{R}$. We set:

$$\begin{aligned} \mathfrak{v}_0 &= \mathrm{Vect}\{X_i \,|\, \lambda_i = 0\} \\ \mathfrak{v}_- &= \mathrm{Vect}\{X_i \,|\, \lambda_i < 0\} \\ \mathfrak{v}_+ &= \mathrm{Vect}\{X_i \,|\, \lambda_i > 0\} \end{aligned}$$

We have a direct sum $\mathfrak{p} = \mathfrak{v}_0 \oplus \mathfrak{v}_- \oplus \mathfrak{v}_+$, this decomposition is orthogonal for both $\langle \cdot | \cdot \rangle$ and $B|_{\mathfrak{p}\times\mathfrak{p}}$. It is also $\mathrm{ad}(\mathfrak{k})$-invariant because of Lemma 14.5.6.

If $\mathbb{X}$ is irreducile, then one of the three K-invariant spaces $\mathfrak{v}_0$, $\mathfrak{v}_-$, $\mathfrak{v}_+$ is equal to $\mathfrak{p}$. If $\varphi = \mathfrak{v}_0$, then $\mathbb{X}$ is of Euclidean type. If $\varphi = \mathfrak{v}_+$, then $\mathbb{X}$ is of compact type. If $\varphi = \mathfrak{v}_-$, then $\mathbb{X}$ is of non compact type.

□

Theorem 14.5.7. *Let $\mathbb{X}$ be an irreducible symmetric space of compact or non compact type, $G = \mathrm{Isom}_\circ(\mathbb{X})$, $\mathfrak{g}$ its Lie algebra and B its Killing form, $o \in \mathbb{X}$ and $\mathfrak{g} = \mathfrak{p} \oplus \mathfrak{k}$ the associated Cartan decomposition.*
There is $\alpha \in \mathbb{R} \setminus \{0\}$ such that $B(X,Y) = \alpha \langle X|Y \rangle$ for all $X, Y \in \mathfrak{p}$.
If (u,v) is an orthonormal basis of a plane $P \subset T_o\mathbb{X}$, and $X, Y \in \mathfrak{p}$ are such that $d_e\varphi_o(X) = u$ and $d_e\varphi_o(Y) = v$, then:

$$K(P) = \frac{1}{\alpha} B([X,Y],[X,Y])$$

If $\mathbb{X}$ is of compact (resp. non compact) type, then it has non negative (resp. non positive) sectional curvature.

Proof. The Killing operator b is diagonalisable, and $\mathrm{ad}(X)$ commutes with b so it must preserve its eigenspaces for each $X \in \mathfrak{k}$, so there is $\alpha \in \mathbb{R}$ such that $b = \alpha \,\mathrm{Id}$. Therefore $B(X,Y) = \alpha \langle X|Y \rangle$ for all $X, Y \in \mathfrak{p}$.

By looking at the sign of B, we see that $\alpha < 0$ (resp. $\alpha > 0$) when $\mathbb{X}$ is of compact (resp. non compact) type.

$$\begin{aligned} K(P) &= R_o(u,v,v,u) \\ &= \langle R_o(u,v)v|u \rangle_o \\ &= \langle -[[X,Y],Y]|X \rangle \\ &= \frac{1}{\alpha} B([Y,[X,Y]],X) \\ &= \frac{1}{\alpha} B([X,Y],[X,Y]) \end{aligned}$$

The sign of the sectional curvature comes from the fact that B is negative definite on $\mathfrak{k}$. □

Chapter 15

Classification of symmetric spaces

15.1 Decomposition into irreducible symmetric spaces

Recall that the universal cover of a symmetric space is still a symmetric space, and that the product of symmetric spaces is also a symmetric space. Up to these manipulations, the classification of symmetric spaces reduces to the irreducible ones.

Theorem 15.1.1. *Every simply connected symmetric space is isometric to a product $\mathbb{X}_1 \times \cdots \times \mathbb{X}_k$ where each $\mathbb{X}_i$ is an irreducible symmetric space.*

Lemma 15.1.2. *Let $\mathbb{X}$ be a symmetric space, $G = \mathrm{Isom}_\circ(\mathbb{X})$, $\mathfrak{g}$ its Lie algebra and B its Killing form, $o \in \mathbb{X}$ and $\mathfrak{g} = \mathfrak{p} \oplus \mathfrak{k}$ the associated Cartan decomposition. The vector space $\mathfrak{p}$ admits a decomposition*

$$\mathfrak{p} = \mathfrak{v}_0 \oplus \mathfrak{v}_1 \oplus \cdots \oplus \mathfrak{v}_{k_+} \oplus \mathfrak{v}_{-1} \oplus \cdots \oplus \mathfrak{v}_{k_-}$$

such that:

- $[\mathfrak{k}, \mathfrak{v}_i] \subset \mathfrak{v}_i$ *and* $\mathfrak{v}_i$ *is irreducible for* $i \neq 0$.
- *If $i \neq j$ then $\mathfrak{v}_i$ et $\mathfrak{v}_j$ are orthogonal for both the Riemannian form $\langle \cdot | \cdot \rangle$ and the Killing form B.*
- $\mathfrak{v}_1 \oplus \cdots \oplus \mathfrak{v}_{k_+}$ *is equal to the sum $\mathfrak{v}_+$ of eigenspaces attached to positive eigenvalues of the Killing operator b and $\mathfrak{v}_{-1} \oplus \cdots \oplus \mathfrak{v}_{k_-}$ is equal to the sum $\mathfrak{v}_-$ of eigenspaces attached to negative eigenvalues of b.*
- $\mathfrak{v}_0 = \ker b$.

Proof. Since $\mathrm{ad}(X)$ preserves the Riemannian form on $\mathfrak{p}$ for all $X \in \mathfrak{k}$, the orthogonal of an $\mathrm{ad}(\mathfrak{k})$-invariant subspace is also invariant, and we can find a decomposition of $\mathfrak{p}$ into irreducible subspaces. Each of them must be included in an eigenspace of b. □

Definition 15.1.3. *Let $\mathbb{X}$ be a symmetric space, $G = \mathrm{Isom}_\circ(\mathbb{X})$, $\mathfrak{g}$ its Lie algebra, $o \in \mathbb{X}$ and $\mathfrak{g} = \mathfrak{p} \oplus \mathfrak{k}$ the associated Cartan decomposition.*
The decomposition $\mathfrak{p} = \bigoplus_{k_- \leq i \leq k_+} \mathfrak{v}_i$ given by Lemma 15.1.2 is called the ***decomposition into irreducible factors of*** $\mathfrak{p}$.

Remark. *The space $\mathfrak{v}_0$ may not be irreducible.*

Lemma 15.1.4. *Let $\mathbb{X}$ be an irreducible symmetric space of compact or non compact type, $G = \mathrm{Isom}_\circ(\mathbb{X})$, $\mathfrak{g}$ its Lie algebra and B its Killing form, $o \in \mathbb{X}$ and $\mathfrak{g} = \mathfrak{p} \oplus \mathfrak{k}$ the associated Cartan decomposition.*
The kernel of the Killing operator $\mathfrak{v}_0 = \ker b \subset \mathfrak{p}$ is equal to $\ker B$. It is an abelian ideal of $\mathfrak{g}$, and $[\mathfrak{v}_0, \mathfrak{p}] = \{0\}$.

Proof. The definition of $\mathfrak{v}_0$ yields $B(X,Y) = 0$ for $X \in \mathfrak{v}_0$ and $Y \in \mathfrak{p}$. Since $\mathfrak{k}$ is B-orthogonal to $\mathfrak{p}$, we also have $B(X,Y) = 0$ for $X \in \mathfrak{v}_0$ and $Y \in \mathfrak{k}$, hence $\mathfrak{v}_0 \subset \ker B$.

Since B is negative definite on $\mathfrak{k}$, we find that $\ker B$ is included in the orthogonal for B of $\mathfrak{k}$, i.e. in $\mathfrak{p}$. It follows that $\ker B \subset \ker b = \mathfrak{v}_0$.

This implies that $\mathfrak{v}_0$ is an ideal of $\mathfrak{g}$. Moreover, for $X \in \mathfrak{v}_0$ and $Y \in \mathfrak{p}$, we have $[X,Y] \in \mathfrak{k} \cap \mathfrak{v}_0 = \{0\}$, hence $[\mathfrak{v}_0, \mathfrak{p}] = \{0\}$, and $\mathfrak{v}_0$ is abelian. □

Lemma 15.1.5. *Let $\mathbb{X}$ be a symmetric space, $G = \mathrm{Isom}_\circ(\mathbb{X})$, $\mathfrak{g}$ its Lie algebra, $o \in \mathbb{X}$, $\mathfrak{g} = \mathfrak{p} \oplus \mathfrak{k}$ the associated Cartan decomposition and $\mathfrak{p} = \bigoplus_{k_- \leq i \leq k_+} \mathfrak{v}_i$ the decomposition into irreducible factors.*
For $i \neq 0$, write $\mathfrak{l}_i = [\mathfrak{v}_i, \mathfrak{v}_i]$ and $\mathfrak{h}_i = \mathfrak{v}_i \oplus \mathfrak{l}_i$.

1. *$\mathfrak{v}_i$ is a Lie triple system of $\mathfrak{g}$.*

2. *$\mathfrak{l}_i$ is an ideal of $\mathfrak{k}$.*

3. *$\mathfrak{h}_i$ is an ideal of $\mathfrak{g}$.*

4. *If $i \neq j$ are both different from 0, then $[\mathfrak{h}_i, \mathfrak{h}_j] = \{0\}$.*

5. *If $i \neq j$ are both different from 0, the ideals $\mathfrak{l}_i$ and $\mathfrak{l}_j$ of $\mathfrak{k}$ are orthogonal for B.*

Proof. 1. Since $[\mathfrak{v}_i, \mathfrak{v}_i] \subset [\mathfrak{p}, \mathfrak{p}] \subset \mathfrak{k}$, we have $[[\mathfrak{v}_i, \mathfrak{v}_i], \mathfrak{v}_i] \subset [\mathfrak{k}, \mathfrak{v}_i] \subset \mathfrak{v}_i$, so $\mathfrak{v}_i$ is a Lie triple system.

2. For $X \in \mathfrak{k}$ and $Y, Z \in \mathfrak{v}_i$, we find:

$$[X,[Y,Z]] = -[\underbrace{Y}_{\in \mathfrak{v}_i}, \underbrace{[Z,X]}_{\in [\mathfrak{v}_i, \mathfrak{k}] \subset \mathfrak{v}_i}] - [\underbrace{Z}_{\in \mathfrak{v}_i}, \underbrace{[X,Y]}_{\in [\mathfrak{k}, \mathfrak{v}_i] \subset \mathfrak{v}_i}] \in [\mathfrak{v}_i, \mathfrak{v}_i] = \mathfrak{l}_i$$

3. Let us start by proving that $[\mathfrak{v}_i, \mathfrak{v}_j] = \{0\}$. Let $X \in \mathfrak{v}_j$ and $Y \in \mathfrak{v}_i$. Since $\mathfrak{v}_i$ and $\mathfrak{v}_j$ are B-orthogonal, we find:
$$B([X,Y],[X,Y]) = B(X, \underbrace{[Y,[X,Y]]}_{\in[\mathfrak{v}_i,\mathfrak{k}]\subset\mathfrak{v}_i}) = 0$$
Since B is negative definite on $\mathfrak{k}$, we find $[X,Y] = 0$, and $[\mathfrak{v}_j, \mathfrak{v}_i] = \{0\}$.
We already know that $[\mathfrak{k}, \mathfrak{l}_i] \subset \mathfrak{l}_i$ and $[\mathfrak{k}, \mathfrak{v}_i] \subset \mathfrak{v}_i$, so we find that $[\mathfrak{k}, \mathfrak{h}_i] \subset \mathfrak{h}_i$.
The fact that $[\mathfrak{v}_i, \mathfrak{v}_j] = \{0\}$ for $j \neq i$ shows that $[\mathfrak{p}, \mathfrak{v}_i] \subset [\mathfrak{v}_i, \mathfrak{v}_i] = \mathfrak{l}_i$. Finally, for $X \in \mathfrak{p}$ and $Y, Z \in \mathfrak{v}_i$, we have:
$$[X,[Y,Z]] = -[\underbrace{Y}_{\in\mathfrak{v}_i}, \underbrace{[Z,X]}_{\in[\mathfrak{v}_i,\mathfrak{p}]\subset\mathfrak{l}_i}] - [\underbrace{Z}_{\in\mathfrak{v}_i}, \underbrace{[X,Y]}_{\in[\mathfrak{p},\mathfrak{v}_i]\subset\mathfrak{l}_i}] \in [\mathfrak{v}_i, \mathfrak{l}_i] \subset \mathfrak{v}_i$$

4. Since $\mathfrak{h}_i$ and $\mathfrak{h}_j$ are both ideals, we have $[\mathfrak{h}_i, \mathfrak{h}_j] \subset \mathfrak{h}_i \cap \mathfrak{h}_j = \{0\}$.

5. For $X_i, Y_i \in \mathfrak{v}_i$, and $X_j, Y_j \in \mathfrak{v}_j$, we find:
$$B([X_i, Y_i],[X_j, Y_j]) = B(X_i, \underbrace{[Y_i,[X_j, Y_j]]}_{=0} = 0$$

$\square$

Thanks to Theorem 14.4.3 and Lemma 15.1.5, we know that the $\mathbb{X}_i = \exp_o(\mathfrak{v}_i)$ are totally geodesic subspaces of $\mathbb{X}$, and symmetric spaces.

Lemma 15.1.6. *Let $\mathbb{X}$ be a symmetric space, $G = \mathrm{Isom}_\circ(\mathbb{X})$, $\mathfrak{g}$ its Lie algebra, $o \in \mathbb{X}$, $\mathfrak{g} = \mathfrak{p} \oplus \mathfrak{k}$ the associated Cartan decomposition and $\mathfrak{p} = \bigoplus_{k_- \leq i \leq k_+} \mathfrak{v}_i$ the decomposition into irreducible factors.*
For $i \neq 0$, write $\mathfrak{l}_i = [\mathfrak{v}_i, \mathfrak{v}_i]$ and $\mathfrak{h}_i = \mathfrak{v}_i \oplus \mathfrak{l}_i$. Denote by $\mathfrak{l}_0$ the B-orthogonal of $\bigoplus_{i\neq 0} \mathfrak{l}_i$ in $\mathfrak{k}$, and write $\mathfrak{h}_0 = \mathfrak{l}_0 \oplus \mathfrak{v}_0$.

1. *$\mathfrak{h}_0$ is an ideal of $\mathfrak{g}$.*

2. *For $i \neq 0$ we have $[\mathfrak{h}_i, \mathfrak{h}_0] = \{0\}$.*

3. *The decomposition $\mathfrak{g} = \bigoplus_{k_- \leq i \leq k_+} \mathfrak{h}_i$ is B-orthogonal.*

Proof. Let us start by showing that for $i \neq 0$, we have $[\mathfrak{h}_i, \mathfrak{v}_0] = \{0\}$. It follows from $[\mathfrak{v}_i, \mathfrak{v}_0] \subset [\mathfrak{p}, \mathfrak{v}_0] = \{0\}$ and:
$$[\mathfrak{l}_i, \mathfrak{v}_0] = [[\mathfrak{v}_i, \mathfrak{v}_i], \mathfrak{v}_0] \subset [\underbrace{[\mathfrak{v}_i, \mathfrak{v}_0]}_{=\{0\}}, \mathfrak{v}_i] + [\underbrace{[\mathfrak{v}_0, \mathfrak{v}_i]}_{=\{0\}}, \mathfrak{v}_i] = \{0\}$$

1. Thanks to Lemma 15.1.4 we know that $[\mathfrak{g},\mathfrak{v}_0]\subset\mathfrak{v}_0$, so it only remains to show that $[\mathfrak{g},\mathfrak{l}_0]\subset\mathfrak{h}_0$. Let $X\in\mathfrak{k}$, $Y\in\mathfrak{l}_0$ and $Z\in\mathfrak{l}_i$ for some $i\neq 0$.

$$B([X,Y],Z)=B(Y,\underbrace{[Z,X]}_{\in[\mathfrak{l}_i,\mathfrak{k}]\subset\mathfrak{l}_i})=0$$

   It follows that $[\mathfrak{k},\mathfrak{l}_0]\subset\mathfrak{l}_0$. Now let $X\in\mathfrak{p}$, $Y\in\mathfrak{l}_0$ and $Z\in\mathfrak{v}_i$ for some $i\neq 0$.

$$B([X,Y],Z)=B(Y,\underbrace{[Z,X]}_{\in[\mathfrak{v}_i,\mathfrak{p}]\subset\mathfrak{l}_i})=0$$

   It follows that $[\mathfrak{p},\mathfrak{k}_0]$ is orthogonal to $\mathfrak{v}_i$. Since it is included in $\mathfrak{p}$, we find $[\mathfrak{p},\mathfrak{l}_0]\subset\mathfrak{v}_0$.

2. Since $\mathfrak{h}_i$ and $\mathfrak{h}_0$ are both ideals of $\mathfrak{g}$ we have $[\mathfrak{h}_i,\mathfrak{h}_0]\subset\mathfrak{h}_i\cap\mathfrak{h}_0=\{0\}$.

3. Since the decompositions $\mathfrak{p}=\bigoplus_{k_-\leq i\leq k_+}\mathfrak{v}_i$ and $\mathfrak{k}=\bigoplus_{k_-\leq i\leq k_+}\mathfrak{l}_i$ are B-orthogonal, and so is $\mathfrak{g}=\mathfrak{p}\oplus\mathfrak{k}$, the same goes for $\mathfrak{g}=\bigoplus_{k_-\leq i\leq k_+}\mathfrak{h}_i$.

□

Lemma 15.1.7. *Let $\mathbb{X}$ be a symmetric space, $G=\mathrm{Isom}_\circ(\mathbb{X})$, $\mathfrak{g}$ its Lie algebra, $o\in\mathbb{X}$, $\mathfrak{g}=\mathfrak{p}\oplus\mathfrak{k}$ the associated Cartan decomposition and $\mathfrak{p}=\bigoplus_{k_-\leq i\leq k_+}\mathfrak{v}_i$ the decomposition into irreducible factors.*
For $i\neq 0$, write $\mathfrak{l}_i=[\mathfrak{v}_i,\mathfrak{v}_i]$ and $\mathfrak{h}_i=\mathfrak{v}_i\oplus\mathfrak{l}_i$. Denote by $\mathfrak{l}_0$ the B-orthogonal of $\bigoplus_{i\neq 0}\mathfrak{l}_i$ in $\mathfrak{k}$, and write $\mathfrak{h}_0=\mathfrak{l}_0\oplus\mathfrak{v}_0$.
Let H_0 be the connected Lie subgroup of G with Lie algebra $\mathfrak{h}_0$. The symmetric space $\mathbb{X}_0=H_0.o$ is of Euclidean type.

Proof. This is a consequence of Proposition 14.4.5 and Lemma 15.1.4. □

Lemma 15.1.8. *Let $\mathbb{X}$ be a symmetric space, $G=\mathrm{Isom}_\circ(\mathbb{X})$, $\mathfrak{g}$ its Lie algebra, $o\in\mathbb{X}$, $\mathfrak{g}=\mathfrak{p}\oplus\mathfrak{k}$ the associated Cartan decomposition and $\mathfrak{p}=\bigoplus_{k_-\leq i\leq k_+}\mathfrak{v}_i$ the decomposition into irreducible factors.*
For $i\neq 0$, write $\mathfrak{l}_i=[\mathfrak{v}_i,\mathfrak{v}_i]$ and $\mathfrak{h}_i=\mathfrak{v}_i\oplus\mathfrak{l}_i$. Denote by $\mathfrak{l}_0$ the B-orthogonal of $\bigoplus_{i\neq 0}\mathfrak{l}_i$ in $\mathfrak{k}$, and write $\mathfrak{h}_0=\mathfrak{l}_0\oplus\mathfrak{v}_0$.
Let H_i be the connected Lie subgroup of G with Lie algebra $\mathfrak{h}_i$. If $i<0$ (resp. $i>0$), the symmetric space $\mathbb{X}_i=H_i.o$ is of compact (resp. non compact) type and irreducible.

Proof. Since $\mathfrak{h}_i$ is an ideal of $\mathfrak{g}$, its Killing form is the restriction of the Killing form of $\mathfrak{g}$. It is non degenerate, so according to Cartan's criterion $\mathfrak{h}_i$ is semi-simple.

Let $\mathfrak{g}_i$ be the Lie algebra of the isometry group of $\mathbb{X}_i$, $G_i = \mathrm{Isom}_\circ(\mathbb{X}_i)$, and $\mathfrak{g}_i = \mathfrak{p}_i \oplus \mathfrak{k}_i$ the Cartan decomposition associated to o. Since H_i acts isometrically on $\mathbb{X}_i$, we have a Lie group morphism $f_i : H_i \to G_i$. The kernel of f_i acts trivially on $\mathbb{X}_i$ so it fixes o, hence $\ker f_i \subset K$. It follows that $d_e f_i$ is injective on $\mathfrak{v}_i$. Hence $d_e f_i(\mathfrak{v}_i) = \mathfrak{p}_i$ by equality of dimensions. We also have $d_e f_i(\mathfrak{l}_i) \subset \mathfrak{k}_i$.

Let us show that $\mathbb{X}_i$ is irreducible. If $\mathfrak{v} \subset \mathfrak{p}_i$ is $\mathrm{ad}(\mathfrak{k}_i)$-invariant, then $\mathfrak{v}' = (d_e f_i)^{-1}(\mathfrak{v}) \cap \mathfrak{v}_i$ must be $\mathrm{ad}(\mathfrak{l}_i)$-invariant. Indeed, given $X \in \mathfrak{v}'$ and $Y \in \mathfrak{l}_i$, we have $[X,Y] \in [\mathfrak{v}_i, \mathfrak{l}_i] \subset \mathfrak{v}_i$ and since $d_e f_i$ is a Lie algebra morphism we find:

$$d_e f_i([X,Y]) = [\underbrace{d_e f_i(X)}_{\in \mathfrak{v}}, \underbrace{d_e f_i(Y)}_{\in \mathfrak{k}_i}] \in \mathfrak{v}$$

Since $[\mathfrak{l}_j, \mathfrak{v}_i] = \{0\}$ for $j \neq i$, it follows that $\mathfrak{v}'$ is $\mathrm{ad}(\mathfrak{k})$-invariant, so $\mathfrak{v}' = \{0\}$ or $\mathfrak{v}' = \mathfrak{v}_i$, hence $\mathfrak{v} = \{0\}$ or $\mathfrak{v} = \mathfrak{p}_i$, and $\mathbb{X}_i$ is irreducible.

The same computation as in Theorem 14.5.7 shows that $\mathbb{X}_i$ has non negative (resp. non positive) sectional curvature when $i < 0$ (resp. $i > 0$), so $\mathbb{X}_i$ must have compact (resp. non compact) type. □

Lemma 15.1.9. *Let G be a Lie group with Lie algebra $\mathfrak{g}$. Let $H_1, H_2 \subset G$ be connected Lie subgroups with respective Lie algebras $\mathfrak{h}_1, \mathfrak{h}_2 \subset \mathfrak{g}$.*
If $[\mathfrak{h}_1, \mathfrak{h}_2] = \{0\}$, then any elements $g_1 \in H_1$ and $g_2 \in H_2$ commute:

$$g_1 g_2 = g_2 g_1$$

Proof. Since H_1 is connected, we find $\mathrm{Ad}(g_1)|_{\mathfrak{h}_2} = \mathrm{Id}_{\mathfrak{h}_2}$. By considering the one-parameter subgroup associated to $X_2 \in \mathfrak{h}_2$, we find that g_1 commutes with $\exp_G(X_2)$. Since H_2 is connected, g_1 commutes with any element of H_2. □

Proof of Theorem 15.1.1. We consider the decomposition into irreducible factors $\mathfrak{p} = \bigoplus_{k_- \leq i \leq k_+} \mathfrak{v}_i$. For $i \neq 0$, write $\mathfrak{l}_i = [\mathfrak{v}_i, \mathfrak{v}_i]$ and $\mathfrak{h}_i = \mathfrak{v}_i \oplus \mathfrak{l}_i$. Denote by $\mathfrak{l}_0$ the B-orthogonal of $\bigoplus_{i \neq 0} \mathfrak{l}_i$ in $\mathfrak{k}$, and write $\mathfrak{h}_0 = \mathfrak{l}_0 \oplus \mathfrak{v}_0$.
Let H_i be the connected Lie subgroup of G with Lie algebra $\mathfrak{h}_i$, and recall that the symmetric space $\mathbb{X}_i = H_i.o$ is a totally geodesic submanifold of $\mathbb{X}$. Consider $L_i = H_i \cap K_i$, so that $\mathbb{X}_i$ can be identified with $L_i \backslash H_i$.

We will now show that $\mathbb{X}$ is isometric to the product $\mathbb{X}_{k_-} \times \cdots \times \mathbb{X}_{k_+}$. This implies that $\mathbb{X}_0$ is simply connected, so it is isometric to the Euclidean space $\mathbb{E}^k$, and therefore is irreducible. So the proof of Theorem 15.1.1 will be complete thanks to Lemma 15.1.7 and 15.1.8.

We start by considering the map:

$$\overline{\varphi} : \begin{cases} H_{k_-} \times \cdots \times H_{k_+} & \to & \mathbb{X} \\ (g_{k_-}, \ldots, g_{k_+}) & \mapsto & g_{k_-} \cdots g_{k_+} o \end{cases}$$

It is smooth, and Lemma 15.1.9 shows that for all $(\ell_{k_-},\dots,\ell_{k_+}) \in L_{k_-}\times\cdots\times L_{k_+}$ we have $\varphi(\ell_{k_-}g_{k_-},\dots,\ell_{k_+}g_{k_+}) = \varphi(g_{k_-},\dots,g_{k_+})$. Thus the map:

$$\varphi : \begin{cases} \mathbb{X}_{k_-}\times\cdots\times\mathbb{X}_{k_+} & \to & \mathbb{X} \\ (g_{k_-}o,\dots,g_{k_+}o) & \mapsto & g_{k_-}\cdots g_{k_+}o \end{cases}$$

is well defined and smooth.

Let us prove that it is isometric (then the completeness of $\mathbb{X}_{k_-}\times\cdots\times\mathbb{X}_{k_+}$, the simple connectedness of $\mathbb{X}$ and a count of dimensions will imply that φ is an isometry).

We start by computing the differential of $\overline{\varphi}$. Since the elements in different groups H_i commute, we can easily compute the partial derivatives:

$$\begin{aligned} d_{(g_{k_-},\dots,g_{k_+})}\overline{\varphi}(0,\dots,0,X_i,0,\dots,0) &= d_o\left(g_{k_-}\cdots\widehat{g_i}\cdots g_{k_+}\right)\circ d_{g_i}\varphi_o(X_i) \\ &= d_o\left(g_{k_-}\cdots g_{k_+}\right)\circ(d_og_i)^{-1}\left[d_{g_i}\varphi_o(X_i)\right] \end{aligned}$$

This leads to:

$$d_{(g_{k_-}o,\dots,g_{k_+}o)}\varphi(0,\dots,0,u_i,0,\dots,0) = d_o\left(g_{k_-}\cdots g_{k_+}\right)\circ(d_og_i)^{-1}(u_i)$$

Since the spaces $T_o\mathbb{X}_i$ are pairwise orthogonal, we find that φ is isometric. □

15.2 Symmetric spaces without Euclidean factors

Definition 15.2.1. *A symmetric space $\mathbb{X}$ has* ***no Euclidean factor*** *if none of the factors in the decomposition of its universal cover $\widetilde{\mathbb{X}}$ given by Theorem 15.1.1 is of Euclidean type.*

Lemma 15.2.2. *Let $\mathbb{X}$ be a symmetric space, $G = \mathrm{Isom}_\circ(\mathbb{X})$, $\mathfrak{g}$ its Lie algebra, B its Killing form, $o \in \mathbb{X}$ and $\mathfrak{g} = \mathfrak{p}\oplus\mathfrak{k}$ the associated Cartan decomposition.*
For $X \in \mathfrak{p}\setminus\{0\}$, the following are equivalent:

1. *$X \in \ker B$.*
2. *Any plane $P \subset T_o\mathbb{X}$ containing $d_e\varphi_o(X)$ has vanishing sectional curvature.*

Proof. First assume that $X \in \ker B$, and consider a plane $P \subset T_o\mathbb{X}$ spanned by $d_e\varphi_o(X)$ and $d_e\varphi_o(Y)$ for some $Y \in \mathfrak{p}$. According to Lemma 15.1.4, we have $[X,Y] = 0$, so Theorem 14.3.1 shows that the curvature of P is 0.

Now assume that any plane $P \subset T_o\mathbb{X}$ containing $d_e\varphi_o(X)$ has vanishing sectional curvature. Then according to Proposition 14.4.5, we find that $[X,Y] = 0$ for all $Y \in \mathfrak{p}$. Let $\mathfrak{c} = \{X' \in \mathfrak{p} \mid \forall Y \in \mathfrak{p}\ [X',Y] = 0\}$. Let us see that $\mathfrak{c}$

is stable under the Killing operator b. For $X' \in \mathfrak{c}$, $Y \in \mathfrak{p}$ and $Z \in \mathfrak{k}$ we can compute:

$$\begin{aligned} B(Z,[Y,b(X')]) &= B([Z,Y],b(X')) \\ &= \langle b([Z,Y]) \big| b(X') \rangle \\ &= \langle [Z,b(Y)] \big| b(X') \rangle \\ &= B([Z,b(Y)],X') \\ &= B(Z,[b(Y),X']) \\ &= 0 \end{aligned}$$

Since B is negative definite on $\mathfrak{k}$, we find $[Y,b(X')] = 0$, i.e. $b(X') \in \mathfrak{c}$. It follows that $\mathfrak{c}$ is the direct sum of its intersections with eigenspaces of b. Let $X' \in \mathfrak{c}$ be such that $b(X') = \lambda X'$ for some $\lambda \neq 0$. For $Z \in \mathfrak{k}$ and $Y \in \mathfrak{p}$, we find:

$$B(Y,[X',Z]) = B([Y,X'],Z) = 0$$

It follows that $[X',Z] \in \ker b$. However the eigenspaces of b are invariant under $\mathrm{ad}(Z)$, hence $[X',Z] = 0$, and $X' \in \mathfrak{z}(\mathfrak{g}) \subset \ker B$, therefore $X' = 0$. It follows that $\mathfrak{c} \subset \ker B$, and $X \in \ker B$. □

Proposition 15.2.3. *Let $\mathbb{X}$ be a symmetric space, $G = \mathrm{Isom}_\circ(\mathbb{X})$ and $\mathfrak{g}$ its Lie algebra. Then $\mathbb{X}$ has no Euclidean factor if and only if $\mathfrak{g}$ is semi-simple.*

Proof. Cartan's criterion states that $\mathfrak{g}$ is semi-simple if and only if $\ker B = \{0\}$. But Lemma 15.2.2 shows that the condition $\ker B = \{0\}$ remains invariant under coverings, so we can assume that $\mathbb{X}$ is simply connected. If $\mathbb{X}$ has a Euclidean factor $\mathbb{X}_0$, then any element $X \in \mathfrak{v}_0$ is in $\ker B$, so $\mathfrak{g}$ is not semi-simple. If $\mathbb{X}$ has no Euclidean factor, then B is non degenerate on $\mathfrak{p}$, so it is non degenerate on $\mathfrak{g}$ and $\mathfrak{g}$ is semi-simple. □

Proposition 15.2.4. *A symmetric space with no Euclidean factor is of compact (resp. non compact) type if and only if it has non negative (resp. non positive) sectional curvature.*

For a symmetric space $\mathbb{X}$ of compact or non compact type, we find that all the irreducible factors must be of the same type.

Proposition 15.2.5. *Let $\mathbb{X}$ be a simply connected symmetric space. If $\mathbb{X}$ is of compact (resp. non compact) type, there are irreducible symmetric spaces $\mathbb{X}_1, \ldots, \mathbb{X}_k$ of compact (resp. non compact) type such that $\mathbb{X}$ is isometric to $\mathbb{X}_1 \times \cdots \times \mathbb{X}_k$.*

Proof. If $\mathbb{X}$ is of compact (resp. non compact) type, then $\mathfrak{p} = \mathfrak{v}_-$ (resp. $\mathfrak{p} = \mathfrak{v}_+$) in Lemma 15.1.2. The rest of the proof of Theorem 15.1.1 can be carried out in the same way. □

Geometrically, the three types of symmetric spaces are determined by the sign of the curvature.

Proposition 15.2.6. *Let $\mathbb{X}$ be a symmetric space. If $\mathbb{X}$ is of compact (resp. non compact) type, then the sectional curvature of $\mathbb{X}$ is non negative (resp. non positive).*

Proof. This is a consequence of the fact that the universal cover of $\mathbb{X}$ has the same property, so we can apply Theorem 14.5.7, Proposition 15.2.5 and the fact that a product of Riemannian manifolds with non negative (resp. non positive) sectional curvature also has non negative (resp. non positive) sectional curvature). □

Lemma 15.2.7. *Let $\mathbb{X}$ be a symmetric space. The universal cover $\widetilde{\mathbb{X}}$ is of Euclidean (resp. compact, non compact) type if and only if $\mathbb{X}$ is of Euclidean (resp. compact, non compact) type.*

Proof. The Euclidean case is a consequence of Proposition 14.5.2. □

Proposition 15.2.8. *If $\mathbb{X}_1,\dots,\mathbb{X}_k$ are symmetric spaces and are all of Euclidean (resp. compact, non compact) type, then $\mathbb{X}_1 \times \cdots \times \mathbb{X}_k$ is of Euclidean (resp. compact, non compact) type.*

Proof. □

Theorem 15.2.9. *Any simply connected symmetric space $\mathbb{X}$ is isometric to a product $\mathbb{X}_0 \times \mathbb{X}_- \times \mathbb{X}_+$ where:*

- *$\mathbb{X}_0$ is a symmetric space of Euclidean type.*
- *$\mathbb{X}_-$ is a symmetric space of compact type.*
- *$\mathbb{X}_+$ is a symmetric space of non compact type.*

15.3 Symmetric spaces of compact type

Proposition 15.3.1. *Let $\mathbb{X}$ be a simply connected symmetric space of compact type. There are irreducible symmetric spaces $\mathbb{X}_1,\dots,\mathbb{X}_k$ of compact type such that $\mathbb{X}$ is isometric to $\mathbb{X}_1 \times \cdots \times \mathbb{X}_k$.*

Proof. If $\mathbb{X}$ is of compact type, then $\mathfrak{p} = \mathfrak{v}_-$ in Lemma 15.1.2. The rest of the proof of Theorem 15.1.1 can be carried out in the same way. □

Proposition 15.3.2. *A symmetric space of compact type has non negative sectional curvature.*

Proof. Non negative sectional curvature for a symmetric space $\mathbb{X}$ or for its universal cover $\widetilde{\mathbb{X}}$ are equivalent, so we can apply Theorem 14.5.7, Proposition 15.2.5 and the fact that a product of Riemannian manifolds with non negative sectional curvature also has non negative sectional curvature. □

Proposition 15.3.3. *Let $\mathbb{X}$ be a symmetric space, $G = \mathrm{Isom}_\circ(\mathbb{X})$ and $\mathfrak{g}$ its Lie algebra. If $\mathbb{X}$ is of compact type, then G is compact, and $\mathfrak{g}$ is semi-simple.*

Remark. *Since G acts transitively on $\mathbb{X}$, it is also compact.*

Proof. The Killing form of $\mathfrak{g}$ is negative definite, so $\mathfrak{g}$ is semi-simple.

The opposite of the Killing form of $\mathfrak{g}$ induces a bi-invariant Riemannian metric on G, and computations show that its sectional curvature must be positive. The Myers Theorem implies that G is compact. □

Definition 15.3.4. *A real Lie algebra $\mathfrak{g}$ is called* ***compact*** *if its Killing form is negative definite.*

If $\mathfrak{g}$ is compact, then it is semi-simple, and so is $\mathfrak{g} \otimes \mathbb{C}$.

Theorem 15.3.5. *The map $\mathfrak{g} \mapsto \mathfrak{g} \otimes \mathbb{C}$ is a bijection from the set of compact Lie algebras (up to isomorphism) to the set of complex semi-simple Lie algebras (up to isomorphism).*

Compact Lie algebras	Complex simple Lie algebras
$\mathfrak{so}(n)$	$\mathfrak{so}(n, \mathbb{C})$
$\mathfrak{su}(n)$	$\mathfrak{sl}(n, \mathbb{C})$
$\mathfrak{sp}(n)$	$\mathfrak{sp}(n, \mathbb{C})$

Note that even if $\mathbb{X}$ is irreducible, the group G may not be simple. The main example is a compact Lie group itself. If $\mathfrak{h}$ is a compact Lie algebra, there is a unique (up to isomorphism) Lie group H with trivial centre and Lie algebra isomorphic to $\mathfrak{h}$. We can endow H with a bi-invariant Riemannian metric whose value at $\mathfrak{h}$ is the opposite of the Killing form. Then H is a symmetric space, and $G = H \times H$ acts on the left and on the right on H.

Classical examples include spheres $\mathbb{S}^n = SO(n+1)/SO(n)$ and projective spaces $\mathbb{CP}^n = SU(n+1)/U(n)$.

15.4 Symmetric spaces of non compact type

We will admit the following result.

Proposition 15.4.1. *A symmetric space of non compact type is simply connected.*

A symmetric space of non compact type is therefore a Cartan-Hadamard manifold. According to the Cartan-Hadamard Theorem, it is diffeomorphic to the Euclidean space.

Proposition 15.4.2. *Let $\mathbb{X}$ be a symmetric space of non compact type. There are irreducible symmetric spaces $\mathbb{X}_1, \ldots, \mathbb{X}_k$ of non compact type such that $\mathbb{X}$ is isometric to $\mathbb{X}_1 \times \cdots \times \mathbb{X}_k$.*

Proof. If $\mathbb{X}$ is of non compact type, then $\mathfrak{p} = \mathfrak{v}_+$ in Lemma 15.1.2. The rest of the proof of Theorem 15.1.1 can be carried out in the same way. □

Proposition 15.4.3. *A symmetric space of non compact type has non positive sectional curvature.*

Proof. Simply apply Theorem 14.5.7, Proposition 15.2.5 and the fact that a product of Riemannian manifolds with non positive sectional curvature also has non positive sectional curvature. □

Proposition 15.4.4. *Let $\mathbb{X}$ be a symmetric space of non compact type, and $G = \mathrm{Isom}_\circ(\mathbb{X})$. Any compact subgroup $K \subset G$ fixes a point of $\mathbb{X}$.*

Proof. This was already shown to be true for any Cartan-Hadamard manifold. □

Proposition 15.4.5. *Let $\mathbb{X}$ be a symmetric space of non compact type, $G = \mathrm{Isom}_\circ(\mathbb{X})$ and $o \in \mathbb{X}$. The stabiliser $K = G_o$ is a maximal compact subgroup of G, i.e. any compact subgroup $L \subset G$ containing K is equal to K.*

Definition 15.4.6. *A semi-simple Lie algebra has **no compact factor** if it has no compact ideal.*

Theorem 15.4.7. *The map sending $\mathbb{X}$ to the Lie algebra $\mathfrak{g}$ of its isometry group is a bijection from the set of symmetric spaces of non compact type (up to multi-homothety) to the set of non compact semi-simple real Lie algebras (up to isomorphism).*

A multi-homothety is a map that is homothetic on each irreducible factor, but possibly with different constants .

Starting with a semi-simple Lie algebra with no compact factor $\mathfrak{g}$, we can choose a Lie group G whose Lie algebra is $\mathfrak{g}$, and with trivial centre $Z(G) = \{1\}$. We then consider the symmetric space $\mathbb{X} = G/K$ where $K \subset G$ is a maximal compact subgroup.

Chapter 16

Real semi-simple Lie algebras

16.1 The algebraic structure of real semi-simple Lie algebras

16.1.1 Restricted roots

Definition 16.1.1. *Let $\mathfrak{g}$ be a finite dimensional real semi-simple Lie algebra with Killing form B. A* ***Cartan involution*** *of $\mathfrak{g}$ is a Lie algebra automorphism θ of $\mathfrak{g}$ such that $\theta^2 = \mathrm{Id}$ and such that the bilinear form $\langle\cdot|\cdot\rangle_\theta$ on $\mathfrak{g}$ defined by*

$$\forall X, Y \in \mathfrak{g} \quad \langle X|Y\rangle_\theta = -B(X, \theta(Y))$$

is positive definite.
Given a Cartan involution θ, we write $\mathfrak{p} = \{X \in \mathfrak{g} \,|\, \theta(X) = -X\}$ and $\mathfrak{k} = \{X \in \mathfrak{g} \,|\, \theta(X) = X\}$. The decomposition $\mathfrak{g} = \mathfrak{p} \oplus \mathfrak{g}$ is called the ***Cartan decomposition****.*

Note that it is symmetric because θ is a Lie algebra morphism. The same computations as in Proposition 13.4.3 show that $[\mathfrak{p}, \mathfrak{p}] \subset \mathfrak{k}$, $[\mathfrak{k}, \mathfrak{k}]^{\subset}\mathfrak{k}$ and $[\mathfrak{k}, \mathfrak{p}] \subset \mathfrak{p}$.

Proposition 16.1.2. *Let $\mathbb{X}$ be a symmetric space of non compact type, $G = \mathrm{Isom}_\circ(\mathbb{X})$, and $\mathfrak{g}$ its Lie algebra. For all $o \in \mathbb{X}$, the Cartan involution of $\mathbb{X}$ associated to o is a Cartan involution of $\mathfrak{g}$.*

Lemma 16.1.3. *Let $\mathfrak{g}$ be a finite dimensional real semi-simple Lie algebra, θ a Cartan involution of $\mathfrak{g}$ and $\mathfrak{g} = \mathfrak{p} \oplus \mathfrak{k}$ the associated Cartan decomposition.*
For all $X \in \mathfrak{p}$, the map $\mathrm{ad}(X) : \mathfrak{g} \to \mathfrak{g}$ is self-adjoint for the inner product $\langle\cdot|\cdot\rangle_\theta$.

Proof. Recall that θ is a Lie algebra automorphism of $\mathfrak{g}$, so $\mathrm{ad}(\theta(X)) \circ \theta = \theta \circ \mathrm{ad}(X)$, hence $\mathrm{ad}(X) \circ \theta = -\theta \circ \mathrm{ad}(X)$ since $X \in \mathfrak{p}$.

For $Y, Z \in \mathfrak{g}$, we find:

$$\begin{aligned}\langle \mathrm{ad}(X)Y|Z\rangle_\theta &= -B(\mathrm{ad}(X)Y, \theta(Z))\\ &= B(Y, \mathrm{ad}(X)\circ\theta(Z))\\ &= -B(Y, \theta(\mathrm{ad}(X)Z))\\ &= B(Y, \mathrm{ad}(X)Z)\end{aligned}$$

□

Consequently, the map $\mathrm{ad}(X)$ is diagonalisable for all $X \in \mathfrak{p}$.

Definition 16.1.4. *Let $\mathfrak{g}$ be a finite dimensional real semi-simple Lie algebra, θ a Cartan involution of $\mathfrak{g}$ and $\mathfrak{g} = \mathfrak{p} \oplus \mathfrak{k}$ the associated Cartan decomposition. A **Cartan subspace** of $\mathfrak{g}$ is a maximal abelian subalgebra of $\mathfrak{p}$.*

Definition 16.1.5. *Let $\mathbb{X}$ be a symmetric space. A **maximal flat** of $\mathbb{X}$ is a flat $F \subset \mathbb{X}$ that is maximal for the inclusion.*

Proposition 16.1.6. *Let $\mathbb{X}$ be a symmetric space of non compact type, $G =$ $\mathrm{Isom}_\circ(\mathbb{X})$, $\mathfrak{g}$ its Lie algebra, B its Killing form, $o \in \mathbb{X}$, and $\mathfrak{g} = \mathfrak{p} \oplus \mathfrak{k}$ the Cartan decomposition.*
Let $\mathfrak{a} \subset \mathfrak{p}$ be a vector subspace. Then $\mathfrak{a}$ is Cartan subspace if and only if $d_e\varphi_o(\mathfrak{a}) \subset T_o\mathbb{X}$ is the tangent space of a maximal flat.

Proof. This is a consequence of Proposition 14.4.5. □

For $\alpha \in \mathfrak{a}^*$, we write:

$$\mathfrak{g}_\alpha = \{Y \in \mathfrak{g} \,|\, \forall X \in \mathfrak{a}\ [X, Y] = \alpha(X)Y\}$$

Definition 16.1.7. *Let $\mathfrak{g}$ be a finite dimensional real semi-simple Lie algebra, θ a Cartan involution of $\mathfrak{g}$ and $\mathfrak{g} = \mathfrak{p} \oplus \mathfrak{k}$ the associated Cartan decomposition. Let $\mathfrak{a} \subset \mathfrak{p}$ be a Cartan subspace. A **restricted root** is $\alpha \in \mathfrak{a}^* \setminus \{0\}$ such that $\mathfrak{g}_\alpha \neq \{0\}$.*

We will denote by $\Sigma \subset \mathfrak{a}^*$ the set of restricted roots. We have a decomposition

$$\mathfrak{g} = \mathfrak{g}_0 \oplus \bigoplus_{\alpha\in\Sigma} \mathfrak{g}_\alpha$$

Moreover, this decomposition is orthogonal for the Cartan form.

Theorem 16.1.8. *Let $\mathfrak{g}$ be a finite dimensional real semi-simple Lie algebra, θ a Cartan involution of $\mathfrak{g}$ and $\mathfrak{g} = \mathfrak{p} \oplus \mathfrak{k}$ the associated Cartan decomposition. Let $\mathfrak{a} \subset \mathfrak{p}$ be a Cartan subspace, and $\Sigma \subset \mathfrak{a}^*$ the set of restricted roots.*

1. *Σ is a root system of $\mathfrak{a}^*$.*

2. $\forall \alpha, \beta \in \mathfrak{a}^*$ $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subset \mathfrak{g}_{\alpha+\beta}$.

3. $\forall \alpha \in \mathfrak{a}^*$ $\theta(\mathfrak{g}_\alpha) = \mathfrak{g}_{-\alpha}$

4. $\mathfrak{g}_0 \cap \mathfrak{p} = \mathfrak{a}$ *and* $\mathfrak{g}_0 = \mathfrak{a} \oplus (\mathfrak{g}_0 \cap \mathfrak{k})$.

5. *If* $\alpha + \beta \neq 0$, *then* $\mathfrak{g}_\alpha$ *and* $\mathfrak{g}_\beta$ *are orthogonal for* B.

Contrary to the complex case, the root system Σ is not always reduced.
For $\alpha \in \Sigma$, the root space $\mathfrak{g}_\alpha$ does not decompose as the sum of its intersections with $\mathfrak{k}$ and $\mathfrak{p}$. The root space decomposition and the Cartan decomposition are related in a more complicated way.

Lemma 16.1.9. *Let* $\mathfrak{g}$ *be a finite dimensional real semi-simple Lie algebra,* θ *a Cartan involution of* $\mathfrak{g}$ *and* $\mathfrak{g} = \mathfrak{p} \oplus \mathfrak{k}$ *the associated Cartan decomposition.*
Let $\mathfrak{a} \subset \mathfrak{p}$ *be a Cartan subspace, and* $\Sigma \subset \mathfrak{a}^*$ *the set of restricted roots. For* $X \in \mathfrak{a}$ *and* $Y \in \mathfrak{g}_\alpha$, *we write* $Y = Y_\mathfrak{k} + Y_\mathfrak{p} \in \mathfrak{k} \oplus \mathfrak{p}$ *the Cartan decomposition of* Y. *The Lie bracket* $[X, Y_\alpha]$ *decomposes as:*

$$[X, Y_\mathfrak{k}] = \alpha(X) Y_\mathfrak{p} \text{ and } [X, Y_\mathfrak{p}] = \alpha(X) Y_\mathfrak{k}$$

Proof. We have $[X, Y_\mathfrak{k}] + [X, Y_\mathfrak{p}] = \alpha(X)Y_\mathfrak{p} + \alpha(X)Y_\mathfrak{k}$. Since $[X, Y_\mathfrak{k}] \in [\mathfrak{p}, \mathfrak{k}] \subset \mathfrak{p}$ and $[X, Y_\mathfrak{p}] \in [\mathfrak{p}, \mathfrak{p}] \subset \mathfrak{k}$, we can identify the factors. □

16.1.2 Some examples

The hyperbolic space $\mathbb{H}^n$

To understand the description of the Lie algebra $\mathfrak{g} = \mathfrak{so}(n,1)$, we will use a decomposition in blocks of size n and 1. For $A \in \mathfrak{gl}(n,\mathbb{R})$, $u, v \in \mathbb{R}^n$ and $\lambda \in \mathbb{R}$ we find:

$$\begin{pmatrix} A & {}^t v \\ u & \lambda \end{pmatrix} \in \mathfrak{so}(n,1) \iff A \in \mathfrak{so}(n),\ v = u,\ \lambda = 0$$

For the symmetric space $\mathbb{H}^n$, by fixing the point $o = (0, \ldots, 0, 1)$ in the hyperboloid model, we find that the Cartan involution of is simply $\theta(X) = -{}^t X$.

$$\mathfrak{k} = \left\{ \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix} \middle| A \in \mathfrak{so}(n) \right\} \ ; \ \mathfrak{p} = \left\{ \begin{pmatrix} 0 & {}^t u \\ u & 0 \end{pmatrix} \middle| u \in \mathbb{R}^n \right\}$$

The bracket of two elements in $\mathfrak{p}$ can be computed explicitly.

$$\left[\begin{pmatrix} 0 & {}^t u \\ u & 0 \end{pmatrix}, \begin{pmatrix} 0 & {}^t v \\ v & 0 \end{pmatrix} \right] = \begin{pmatrix} {}^t u v - {}^t v u & 0 \\ 0 & 0 \end{pmatrix}$$

This bracket can only vanish if $u = 0$ or $v = 0$. It follows that a maximal abelian subalgebra of $\mathfrak{p}$ has dimension 1. Set $\mathfrak{a} = \mathbb{R}.X$ where

$$H = \begin{pmatrix} 0 & {}^t h \\ h & 0 \end{pmatrix} ;\ h = (0,\ldots,0,1)$$

The bracket between H and an arbitrary element of $\mathfrak{g}$ can be computed explicitly.

$$\left[H, \begin{pmatrix} A & {}^t v \\ v & 0 \end{pmatrix}\right] = \begin{pmatrix} {}^t hv - {}^t vh & -A^t h \\ hA & 0 \end{pmatrix}$$

The line hA is simply the last line of A, and for $v = (v_1,\ldots,v_n) \in \mathbb{R}^n$, we have:

$${}^t hv - {}^t vh = \begin{pmatrix} & & & -v_1 \\ & 0 & & \vdots \\ & & & -v_{n-1} \\ v_1 & \cdots & v_{n-1} & 0 \end{pmatrix}$$

The centraliser of $\mathfrak{a}$ can be found easily:

$$\mathfrak{g}_0 \cap \mathfrak{k} = \left\{ \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix} \middle| A \in \mathfrak{so}(n-1) \right\}$$

Here we identify $\mathfrak{so}(n-1)$ with the top left block diagonal embedding in $\mathfrak{so}(n)$. The roots are $\pm\alpha$ where $\alpha(H) = 1$, and the root spaces are:

$$\mathfrak{g}_\alpha = \left\{ \begin{pmatrix} 0 & -{}^t u & {}^t u \\ u & 0 & 0 \\ u & 0 & 0 \end{pmatrix} \middle| u \in \mathbb{R}^{n-1} \right\}$$

$$\mathfrak{g}_{-\alpha} = \left\{ \begin{pmatrix} 0 & {}^t u & {}^t u \\ -u & 0 & 0 \\ u & 0 & 0 \end{pmatrix} \middle| u \in \mathbb{R}^{n-1} \right\}$$

The space of ellipsoids $\mathcal{E}_n$

The decomposition that we find for $\mathfrak{g} = \mathfrak{sl}(n,\mathbb{R})$ is exactly the same as for $\mathfrak{sl}(n,\mathbb{C})$. Indeed, we can choose $\mathfrak{a}$ to be the space of diagonal traceless matrices. It happens to be a Cartan subalgebra, i.e. $\mathfrak{g}_0 = \mathfrak{a}$. A real semi-simple Lie algebra with this property is called a **real split** Lie algebra. There is a one to one correspondence between real split Lie algebras and complex semi-simple Lie algebras.

Real split Lie algebras	Complex Lie algebras
$\mathfrak{sl}(n,\mathbb{R})$	$\mathfrak{sl}(n,\mathbb{C})$
$\mathfrak{so}(n,n+1)$	$\mathfrak{so}(2n+1,\mathbb{C})$
$\mathfrak{sp}(2n,\mathbb{R})$	$\mathfrak{sp}(2n,\mathbb{C})$
$\mathfrak{so}(n,n)$	$\mathfrak{so}(2n,\mathbb{C})$

16.1.3 Restricted roots and geometry

Lemma 16.1.10. *Let $\mathfrak{g}$ be a finite dimensional real semi-simple Lie algebra, θ a Cartan involution of $\mathfrak{g}$ and $\mathfrak{g} = \mathfrak{p} \oplus \mathfrak{k}$ the associated Cartan decomposition.*
Let $\mathfrak{a} \subset \mathfrak{p}$ be a Cartan subspace, and $\Sigma \subset \mathfrak{a}^*$ the set of restricted roots.
For $\alpha \in \Sigma$, consider $Y_\alpha \in \mathfrak{g}_\alpha$ such that $\langle Y_\alpha | Y_\alpha \rangle_\theta = 1$. Then $X_\alpha = [\theta(Y_\alpha), Y_\alpha] \in \mathfrak{a}$ satisfies:

$$\forall X \in \mathfrak{a} \quad \langle X_\alpha | X \rangle_\theta = \alpha(X)$$

Proof. It follows from Theorem 16.1.8 that $[\theta(Y_\alpha), Y_\alpha] \in \mathfrak{g}_0$. By using the Cartan decomposition of Y_α we also find $[\theta(Z), Z] \in \mathfrak{p}$ for any $Z \in \mathfrak{g}$, so $[\theta(Y_\alpha), Y_\alpha] \in \mathfrak{g}_0 \cap \mathfrak{p} = \mathfrak{a}$.

$$\begin{aligned}
\langle X_\alpha | X \rangle_\theta &= -B([\theta(Y_\alpha), Y_\alpha], X) \\
&= -B(\theta(Y_\alpha), [Y_\alpha, X]) \\
&= B(\theta(Y_\alpha), \alpha(X) Y_\alpha \\
&= \alpha(X) \langle Y_\alpha | Y_\alpha \rangle_\theta \\
&= \alpha(X)
\end{aligned}$$

□

Definition 16.1.11. *Let $\mathbb{X}$ be a symmetric space of non compact type, $G =$ $\mathrm{Isom}_\circ(\mathbb{X})$, $\mathfrak{g}$ its Lie algebra, B its Killing form, $o \in \mathbb{X}$, and $\mathfrak{g} = \mathfrak{p} \oplus \mathfrak{k}$ the Cartan decomposition. We say that $\mathbb{X}$ is* ***normalized*** *if the Killing form and the Riemannian form are equal on $\mathfrak{p}$.*

Proposition 16.1.12. *Let $\mathbb{X}$ be a symmetric space of non compact type, $G =$ $\mathrm{Isom}_\circ(\mathbb{X})$, $\mathfrak{g}$ its Lie algebra, B its Killing form, $o \in \mathbb{X}$, $\theta : \mathfrak{g} \to \mathfrak{g}$ the associated Cartan involution and $\mathfrak{g} = \mathfrak{p} \oplus \mathfrak{k}$ the Cartan decomposition.*
Let $\mathfrak{a} \subset \mathfrak{p}$ be a Cartan subspace, and $\Sigma \subset \mathfrak{a}^*$ the set of restricted roots.
For $\alpha \in \Sigma$, consider $Y_\alpha \in \mathfrak{g}_\alpha$ such that $\langle Y_\alpha | Y_\alpha \rangle_\theta = 1$, and $X_\alpha = [\theta(Y_\alpha), Y_\alpha]$. There is a totally geodesic surface $S \subset \mathbb{X}$ containing o such that $T_o S$ is spanned by $d_e \varphi_o(X_\alpha)$ and $d_e \varphi_o(Y_\alpha)$.
If $\mathbb{X}$ is normalized, then the sectional curvature of S is $-\|\alpha\|^2$.

Proof. Use $B(Y, \theta(Y)) = 1$ and $B(Y, Y) = 0$. □

16.1.4 Regular elements and Weyl chambers

Proposition 16.1.13. *Let $\mathfrak{g}$ be a real semi-simple Lie algebra, $\theta : \mathfrak{g} \to \mathfrak{g}$ a Cartan involution and $\mathfrak{g} = \mathfrak{p} \oplus \mathfrak{k}$ the Cartan decomposition.*
For $X \in \mathfrak{p}$, the following assertions are equivalent.

1. *$\mathfrak{z}(X) \cap \mathfrak{p}$ is abelian.*
2. *X belongs to a unique Cartan subspace.*
3. *For any Cartan subspace $\mathfrak{a} \subset \mathfrak{p}$ containing X, we have $\forall \alpha \in \Sigma\ \alpha(X) \neq 0$.*
4. *There is a Cartan subspace $\mathfrak{a} \subset \mathfrak{p}$ containing X, such that $\forall \alpha \in \Sigma\ \alpha(X) \neq 0$.*

Definition 16.1.14. *Such an element $X \in \mathfrak{p}$ is called **regular**.*

Proof. *1.⇒2.*If $\mathfrak{z}(X) \cap \mathfrak{p}$ is abelian, then it is a Cartan subspace. If $\mathfrak{a} \subset \mathfrak{p}$ is a Cartan subspace containing X, then $\mathfrak{a} \subset \mathfrak{z}(X)$, hence $\mathfrak{a} = \mathfrak{z}(X) \cap \mathfrak{p}$ because of maximality.

(2)⇒(1) Let $\mathfrak{a} \subset \mathfrak{p}$ be the Cartan subspace containing X. For $Y \in \mathfrak{z}(X) \cap \mathfrak{p}$, the abelian subalgebra $\mathbb{R}.X + \mathbb{R}.Y$ is contained in Cartan subalgebra, so $Y \in \mathfrak{a}$, and $\mathfrak{z}(X) \cap \mathfrak{p} \subset \mathfrak{a}$ is abelian.

(3)⇒(4) is just specification.

(1)⇒(3) Assume that $\mathfrak{z}(X) \cap \mathfrak{p}$ is abelian, and consider a Cartan subspace $\mathfrak{a} \subset \mathfrak{p}$ that contains X.
If $\alpha(X) = 0$ for some $\alpha \in \Sigma$, then Lemma 16.1.9 shows that $Y_{\mathfrak{p}} \in \mathfrak{z}(X) \cap \mathfrak{p} = \mathfrak{a}$ for all $Y \in \mathfrak{g}_\alpha$. Hence $[X', Y_{\mathfrak{p}}] = \alpha(X')Y_{\mathfrak{k}} = 0$ for all $X' \in \mathfrak{a}$, and $\mathfrak{g}_\alpha \subset \mathfrak{p}$, which leads to $\alpha = 0$.

(4)⇒(1) For $Y \in \mathfrak{z}(X) \cap \mathfrak{p}$, we write $Y = Y_0 + \sum_{\alpha \in \Sigma} Y_\alpha$ its decomposition in $\mathfrak{g} = \mathfrak{g}_0 \oplus \bigoplus_{\alpha \in \Sigma} \mathfrak{g}_\alpha$. Since $0 = [X, Y] = \sum_{\alpha \in \Sigma} \alpha(X) Y_\alpha$, we get $Y_\alpha = 0$, hence $Y \in \mathfrak{g}_0 \cap \mathfrak{p} = \mathfrak{a}$. □

Note that a consequence of the fourth point is that every Cartan subspace contains regular elements (because $\bigcup_{\alpha \in \Sigma} \ker \alpha$ has empty interior in $\mathfrak{a}$).

Now consider a symmetric space of non compact type $\mathbb{X}$. The stabiliser K of a point $o \in \mathbb{X}$ acts on the set of Cartan subspaces of $\mathfrak{p}$ (an element $g \in K$ acts on a Cartan subspace $\mathfrak{a} \subset \mathfrak{p}$ by $g.\mathfrak{a} = \mathrm{Ad}(g)\mathfrak{a}$).

Proposition 16.1.15. *Let $\mathbb{X}$ be a symmetric space of non compact type, $G = \mathrm{Isom}_\circ(\mathbb{X})$, $\mathfrak{g}$ its Lie algebra, $o \in \mathbb{X}$, $K = G_o$ its stabiliser and $\mathfrak{g} = \mathfrak{p} \oplus \mathfrak{k}$ the Cartan*

decomposition.
The action of K on the set of Cartan subspaces of $\mathfrak{p}$ is transitive.

Proof. Let $\mathfrak{a}, \mathfrak{b} \subset \mathfrak{p}$ be Cartan subspaces. Consider regular elements $X \in \mathfrak{a}$ and $Y \in \mathfrak{b}$, and the function

$$f : \left\{ \begin{array}{ccc} K & \to & \mathbb{R} \\ g & \mapsto & B(\mathrm{Ad}(g)X, Y) \end{array} \right.$$

Since K is compact, f reaches its maximum at some $g_0 \in K$. Up to replacing X with $\mathrm{Ad}(g_0)X$ and $\mathfrak{a}$ with $\mathrm{Ad}(g_0)\mathfrak{a}$, we can assume that $g_0 = \mathrm{Id}$.

For all $Z \in \mathfrak{k}$, we have $\frac{d}{dt}\big|_{t=0} f(\exp_G(tZ)) = 0$. This derivative can be computed:

$$\left.\frac{d}{dt}\right|_{t=0} f(\exp_G(tZ)) = B(\mathrm{ad}(Z)X, Y) = B(Z, [X, Y])$$

Since $[X, Y] \in \mathfrak{k}$ and B is negative definite on $\mathfrak{k}$, it follows that $[X, Y] = 0$. The elements X and Y being regular, we find $\mathfrak{a} = \mathfrak{b}$. □

Definition 16.1.16. *Let $\mathfrak{g}$ be a real semi-simple Lie algebra, $\theta : \mathfrak{g} \to \mathfrak{g}$ a Cartan involution and $\mathfrak{g} = \mathfrak{p} \oplus \mathfrak{k}$ the Cartan decomposition.*
Let $\mathfrak{a} \subset \mathfrak{p}$ be a Cartan subspace, and Σ the set of restricted roots.
A ***Weyl chamber*** *of $\mathfrak{a}$ is a connected component of $\mathfrak{a} \setminus \bigcup_{\alpha \in \Sigma} \ker \alpha$.*

Proposition 16.1.17. *Let $\mathbb{X}$ be a symmetric space of non compact type, $G = \mathrm{Isom}_\circ(\mathbb{X})$, $\mathfrak{g}$ its Lie algebra, $o \in \mathbb{X}$, $K = G_o$ its stabiliser and $\mathfrak{g} = \mathfrak{p} \oplus \mathfrak{k}$ the Cartan decomposition.*
The group K acts transitively on the set of pairs $(\mathfrak{a}^+, \mathfrak{a})$ where $\mathfrak{a} \subset \mathfrak{p}$ is a Cartan subspace and $\mathfrak{a}^+ \subset \mathfrak{a}$ is a Weyl chamber.

Proof. Consider $X \in \mathfrak{a}^+ \subset \mathfrak{a}$, $Y \in \mathfrak{b}^+ \subset \mathfrak{b}$, and the function f introduced in the proof of Proposition 16.1.15.

$$f : \left\{ \begin{array}{ccc} K & \to & \mathbb{R} \\ g & \mapsto & B(\mathrm{Ad}(g)X, Y) \end{array} \right.$$

We can still assume that f reaches its maximum at e (hence $\mathfrak{a} = \mathfrak{b}$). For $Z \in \mathfrak{k}$, we have $\frac{d^2}{dt^2}\big|_{t=0} f(\exp_G(tZ)) \leq 0$. Since $\mathrm{Ad}(\exp_G(tZ)) = \exp_{\mathrm{GL}(\mathfrak{g})}(t\,\mathrm{ad}(Z))$, we find:

$$\begin{aligned} \left.\frac{d^2}{dt^2}\right|_{t=0} f(\exp_G(tZ)) &= B(\mathrm{ad}(Z)^2 X, Y) \\ &= B([Z,[Z,X]], Y) \\ &= -B([Z,X],[Z,Y]) \\ &= -B([X,Z],[Y,Z]) \end{aligned}$$

We are left with:

$$B([X,Z],[Y,Z]) \geq 0$$

For $\alpha \in \Sigma$, choose $Y \in \mathfrak{g}_\alpha \setminus \{0\}$ and decompose $Y = Y_\mathfrak{k} + Y_\mathfrak{p} \in \mathfrak{k} \oplus \mathfrak{p}$. We can apply the previous inequality with $Z = Y_\mathfrak{k}$.

Since $[X, Y_\mathfrak{k}] = \alpha(X) Y \mathfrak{p}$, we find:

$$\alpha(X)\alpha(Y)B(Y_\mathfrak{p}, Y_\mathfrak{p}) \geq 0$$

Note that $Y_\mathfrak{p} \neq 0$. Since B is positive definite on $\mathfrak{p}$, we find that $\alpha(X)\alpha(Y) \geq 0$. It follows that X and Y are in the same Weyl chamber.

□

Proposition 16.1.18 (Polar decomposition). *Let $\mathbb{X}$ be a symmetric space of non compact type, $G = \mathrm{Isom}_\circ(\mathbb{X})$, $\mathfrak{g}$ its Lie algebra, $o \in \mathbb{X}$, $K = G_o$ its stabiliser and $\mathfrak{g} = \mathfrak{p} \oplus \mathfrak{k}$ the Cartan decomposition.*
For all $g \in G$, there is a unique pair $(k, X) \in K \times \mathfrak{p}$ such that $g = k\exp_G(X)$.

Proof. Let $X \in \mathfrak{p}$ be such that $g(o) = \exp_G(X).o$ (i.e. $X = (d_e\varphi_o^{-1}(v)$ where $g(o) = \exp_o(v)$). Now $k = g\exp_G(-X) \in K$ satisfies $g = k\exp_G(X)$.

The uniqueness comes from the fact that $\exp_o$ is a diffeomorphism. □

Proposition 16.1.19 (KAK decomosition). *Let $\mathbb{X}$ be a symmetric space of non compact type, $G = \mathrm{Isom}_\circ(\mathbb{X})$, $\mathfrak{g}$ its Lie algebra, $o \in \mathbb{X}$, $K = G_o$ its stabiliser and $\mathfrak{g} = \mathfrak{p} \oplus \mathfrak{k}$ the Cartan decomposition.*
Let $\mathfrak{a} \subset \mathfrak{p}$ be a Cartan subspace and $\mathfrak{a}^+ \subset \mathfrak{a}$ a Weyl chamber.

For all $g \in G$, there are $k, k' \in K$ and $X \in \overline{\mathfrak{a}^+}$ such that :

$$g = k\exp_G(X)k'$$

Proof. Following Proposition 16.1.18, we write $g = l\exp_G(Y)$ with $Y \in \mathfrak{p}$ and $l \in K$.

Note that any element of $\mathfrak{a}$ is in the closure of a Weyl chamber, so according to Proposition 16.1.17 there is $k' \in K$ such that $X = \mathrm{Ad}(k')Y \in \overline{\mathfrak{a}^+}$. Set $k = lk'^{-1} \in K$, so that we find:

$$\begin{aligned} k\exp_G(X)k' &= lk'^{-1}\exp_G(X)k' \\ &= lk'\exp_G(\mathrm{Ad}(k'^{-1})X) \\ &= l\exp_G(Y) \\ &= g \end{aligned}$$

□

16.2 Compactifications of symmetric spaces

16.2.1 The visual boundary of a symmetric space of non compact type

Proposition 16.2.1. *Let* $\mathbb{X}$ *be a symmetric space of non compact type,* $G = \mathrm{Isom}_\circ(\mathbb{X})$*,* $o \in \mathbb{X}$ *and* $K = G_o$ *its stabiliser.*
The map $\psi_o : T^1_o\mathbb{X} \to \partial_\infty\mathbb{X}$ *is* K*-equivariant.*

Proof. For $g \in K$ and $v \in T^1_{x_0}\mathbb{X}$, we find $g(c_v(t)) = c_{g.v}(t)$ where $g.v = d_{x_0}g(v)$. □

Proposition 16.2.2. *Let* $\mathbb{X}$ *be a symmetric space of non compact type,* $G = \mathrm{Isom}_\circ(\mathbb{X})$*,* $o \in \mathbb{X}$ *and* $K = G_o$ *its stabiliser.*
The following assertions are equivalent.

1. $G \curvearrowright \partial_\infty\mathbb{X}$ *is transitive.*
2. $K \curvearrowright T^1_o\mathbb{X}$ *is transitive.*
3. $\mathbb{X}$ *has rank* 1.
4. *The sectional curvature of* $\mathbb{X}$ *is negative.*

Proof. *(1)* $\iff$ *(2)* is a consequence of Proposition 16.2.1.

(3) $\iff$ *(4)* is a consequence of the formula for sectional curvature (Proposition **??**).

(3)$\Rightarrow$*(2)* is a consequence of Proposition 16.1.17.

(2)$\Rightarrow$*(3)* is a consequence of Proposition **??**.

□

Theorem 16.2.3. *Any rank* 1 *symmetric space of non compact type is homothetic to one of the following:*

- *The real hyperbolic space* $\mathbb{H}^n$ *(and* $\mathfrak{g} = \mathfrak{so}(n,1)$*).*
- *The complex hyperbolic space* $\mathbb{H}^n_{\mathbb{C}}$ *(and* $\mathfrak{g} = \mathfrak{su}(n,1)$*).*
- *The quaternionic hyperbolic space* $\mathbb{H}^n_{\mathbb{H}}$ *(and* $\mathfrak{g} = \mathfrak{sp}(n,1)$*).*
- *The octonionic hyperbolic plane* $\mathbb{H}^2_{\mathbb{O}}$ *(and* $\mathfrak{g} = \mathfrak{f}_4^{-20}$*).*

In particular, a rank 1 symmetric space of non compact type is irreducible (this is the first part of the proof, and a consequence of the more general fact that the rank is additive under products).

Proposition 16.2.4. *Let $\mathbb{X}$ be a symmetric space of non compact type, $G = \mathrm{Isom}_\circ(\mathbb{X})$, $o \in \mathbb{X}$ and $K = G_o$ its stabiliser.*
For all $\xi \in \partial_\infty \mathbb{X}$, we have $G.\xi = K.\xi$, i.e.

$$\forall g \in G\, \exists k \in K \quad g\xi = k\xi$$

More over, the stabiliser $G_\xi \subset G$ acts transitively on $\mathbb{X}$.

Proof. Let γ be the unit speed geodesic such that $\gamma(0) = o$ and $\gamma \in \xi$. Let γ_t be the 1-parameter group of transvections along γ (i.e. $\dot\gamma(0) = d_e\varphi_o(X)$ and $\gamma_t = \exp_G(tX)$ for some $X \in \mathfrak{p}$).

Let $g \in G$. For $t > 0$, we let σ_t be the unit speed geodesic such that $\sigma_t(0) = x_0$ and passing through $g\gamma(t) = g\gamma_t(x_0)$. Set $\xi_t = [\sigma_t]$.

Denote by q_t the transvection along σ_t such that $q_t(x_0) = g\gamma_t(x_0)$. Note that $q_t.\xi_t = \xi_t$.

Set $k_t = q_t^{-1} g\gamma_t$. We have $k_t(x_0) = x_0$, i.e. $k_t \in K$. Hence:

$$\sphericalangle_{x_0}(k_t\xi, \xi_t) = \sphericalangle_{q_t(x_0)}(g\xi, q_t\xi_t) = \sphericalangle_{g\gamma(t)}(x_0, g(x_0)) \to 0$$

We also find:

$$\sphericalangle_{x_0}(\xi_t, g\xi) = \sphericalangle_{x_0}(g\gamma(t), g\xi) \le \pi - \sphericalangle_{g\gamma(t)}(g\xi, x_0) = \sphericalangle_{g\gamma(t)}(x_0, g(x_0)) \to 0$$

It follows that $\sphericalangle_{x_0}(k_t\xi, g\xi) \to 0$, hence $k_t\xi \to g\xi$. Therefore $g\xi \in \overline{K.\xi} = K.\xi$ (because K is compact).

The transitivity of $G_\xi \curvearrowright \mathbb{X} = G/K$ is a rewriting of the transitivity of $K \curvearrowright G.\xi = G_\xi \backslash G$:

$$\begin{aligned}\forall g \in G\, \exists k \in K \quad gk \in G_\xi &\iff \forall g \in G\, \exists k \in K\, \exists p \in G_\xi \quad gk = p\\ &\iff \forall g \in G\, \exists p \in G_\xi\, \exists k \in K \quad gk = p\\ &\iff \forall g \in G\, \exists p \in G_\xi\, \exists k \in K \quad pg = k\end{aligned}$$

Therefore $G_\xi \curvearrowright K\backslash G = M$ is transitive. □

16.2.2 The Furstenberg boundary

Definition 16.2.5. *Let $\mathbb{X}$ be a symmetric space of non compact type, $G = \mathrm{Isom}_\circ(\mathbb{X})$, and $\mathfrak{g}$ its Lie algebra.*
*An **asymptotic Weyl chamber** is $\psi_o(d_e\varphi_o(\mathfrak{a}^+)) \subset \partial_\infty\mathbb{X}$ where $\mathfrak{a}^+ \subset \mathfrak{p}$ is a Weyl chamber and $\mathfrak{g} = \mathfrak{p} \oplus \mathfrak{k}$ is the Cartan decomposition associated to some $o \in \mathbb{X}$.*
*The **Furstenberg boundary** $\partial_F\mathbb{X}$ is the set of asymptotic Weyl chambers.*

An asymptotic Weyl chamber can be described geometrically: it is the boundary at infinity of a maximal flat.

Proposition 16.2.6. *Let $\mathbb{X}$ be a symmetric space of non compact type. The group $G = \mathrm{Isom}_\circ(\mathbb{X})$ acts transitively on the Furstenberg boundary $\partial_F\mathbb{X}$.*

Definition 16.2.7. *Let $\mathbb{X}$ be a symmetric space of non compact type. A point $\xi \in \partial_\infty\mathbb{X}$ is called* ***regular*** *if there are $o \in \mathbb{X}$ and a regular vector $v \in T_o^1\mathbb{X}$ such that $\xi = \psi_o(v)$.*

Proposition 16.2.8. *Let $\mathbb{X}$ be a symmetric space of non compact type, and $G =$ $\mathrm{Isom}_\circ(\mathbb{X})$. If $\xi \in \partial_\infty\mathbb{X}$ is regular, then ξ belongs to a unique asymptotic Weyl chamber C, and $\{g \in G \,|\, gC = C\} = G_\xi$.*

If $\xi \in \partial_\infty M$ is regular, we fix $o \in \mathbb{X}$ and the Weyl chamber $\mathfrak{a}^+ \subset \mathfrak{a} \subset \mathfrak{p}$ such that $\xi = \lim_{t\to+\infty} \exp_G(tX)o$ with $X \in \mathfrak{a}^+$. We set:

$$A_\xi = \exp_G(\mathfrak{a})$$

$$\mathfrak{n}_\xi = \bigoplus_{\alpha(X)>0} \mathfrak{g}_\alpha$$

$$N_\xi = \exp_G(\mathfrak{n}_\xi)$$

Theorem 16.2.9 (Iwasawa decomposition). *N_ξ is a subgroup of G, and the multiplication $K \times A_\xi \times N_\xi \to G$ is a diffeomorphism.*

16.3 Lattices in semi-simple Lie groups

If (M,g) is a complete locally symmetric space, then its universal cover is a symmetric space $\mathbb{X}$. So the study of locally symmetric spaces is related to the study of discrete subgroups of the isometry groups of symmetric spaces. For the Euclidean type, these groups are, up to finite index, abelian (Bieberbach's Theorem). For the compact type, they must be finite. The non compact type leads to a very rich theory.

Definition 16.3.1. *Let G be a Lie group. A* ***lattice*** *of G is a discrete subgroup $\Gamma \subset G$ such that $\Gamma\backslash G$ has finite volume. We say that Γ is* ***uniform*** *(or* ***cocompact****) if $\Gamma\backslash G$ est compact.*

Theorem 16.3.2 (Borel, Harish-Chandra). *Every semi-simple Lie group possesses a lattice.*

If $G = \mathrm{Isom}_\circ(\mathbb{X})$ where $\mathbb{X}$ is a symmetric space of non compact type, then a lattice $\Gamma \leq G$ is the fundamental group of a complete locally symmetric space $\Gamma\backslash\mathbb{X}$ if Γ is also required to be torsion-free (i.e. non trivial elements have infinite order).

Examples:

1. $\mathrm{SL}(n,\mathbb{Z})$ is a non uniform lattice in $\mathrm{SL}(n,\mathbb{R})$.

2. If $P \subset \mathbb{H}^2$ is a regular right angled polygon with $4g$ sides (it exists for any $g \geq 2$) labelled $A_1, B_1, A_1^{-1}, B_1^{-1}, A_2, \ldots B_g^{-1}$, consider the isometry a_i (resp. b_i) sending A_i to A_i^{-1} (resp. B_i to B_i^{-1}) and reversing the orientation of the edges. The subgroup Γ of $\mathrm{PSL}(2,\mathbb{R}) = \mathrm{Isom}_\circ(\mathbb{H}^2)$ generated by $a_1, b_1, \ldots, a_g, b_g$ is a torsion-free uniform lattice, and the quotient $\Gamma \backslash \mathbb{H}^2$ is a compact orientable surface of genus g.

Consider two torsion-free lattices $\Gamma_1, \Gamma_2 \leq G$, and the associated locally symmetric spaces $M_i = \Gamma_i \backslash \mathbb{X}$, $i = 1, 2$. If Γ_1 and Γ_2 are conjugate, i.e. if $\Gamma_2 = g\Gamma_1 g^{-1}$ for some $g \in G$, then the isometry g of $\mathbb{X}$ induces an isometry from M_1 to M_2. Reciprocally, an isometry $\varphi : M_1 \to M_2$ induces an isometry $g \in \mathrm{Isom}(\mathbb{X})$ such that $\Gamma_2 = g\Gamma_1 g^{-1}$.

Theorem 16.3.3 (Mostow rigidity)**.** *Let $\mathbb{X}$ be an irreducible symmetric space of non compact type which is not homothetic to $\mathbb{H}^2$. If $\Gamma_1, \Gamma_2 \subset G = \mathrm{Isom}_\circ(\mathbb{X})$ are lattices, and $\theta : \Gamma_1 \to \Gamma_2$ is a group isomorphism, there is $g \in G$ such that:*

$$\forall \gamma \in \Gamma_1 \quad \theta(\gamma) = g\gamma g^{-1}$$

For $\mathbb{H}^2$, the situation is very different. If S is a closed orientable surface of genus $g \geq 2$ and $\Gamma = \pi_1(S)$, then the space of representations $\rho : \Gamma \to \mathrm{PSL}(2,\mathbb{R})$ such that $\rho(\Gamma)$ is a lattice, up to conjugation in $\mathrm{PGL}(2,\mathbb{R}) = \mathrm{Isom}(\mathbb{H}^2)$, is homeomorphic (for a suitable topology) to $\mathbb{R}^{6g-6}$. This space is called the **Teichmüller space** of S.