Groups and geometry

Mid-term exam

Exercise 1

The goal of this exercise is to show that the exponential map $\exp : \mathfrak{gl}(n,\mathbb{C}) \to \mathrm{GL}(n,\mathbb{C})$ is surjective.

Let $A \in \mathrm{GL}(n,\mathbb{C})$. For $P = \sum_{k=0}^{d} a_x X^k \in \mathbb{C}[X]$, we write $P(A) = \sum_{k=0} a_k A^k \in \mathcal{M}_n(\mathbb{C})$. Let:

$$\mathbb{C}[A] = \{P(A) \,|\, P \in \mathbb{C}[X]\} \subset \mathcal{M}_n(\mathbb{C})$$

We also consider $\mathbb{C}[A]^\times = \mathbb{C}[A] \cap \mathrm{GL}(n,C)$.

1. Prove that $A^{-1} \in \mathbb{C}[A]$, then show that $\mathbb{C}[A]^\times$ is a subgroup of $\mathrm{GL}(n,\mathbb{C})$.

   Solution: Consider $P_A(X) = \det(X1_n - A)$ the characteristic polynomial of A. According to the Cayley-Hamilton Theorem, we have $P_A(A) = 0$. Write $P_A = \sum_{k=0}^{n} a_k X^k$. Note that $a_0 = (-1)^n \det A \neq 0$, so we can consider the polynomial $Q_A = -\sum_{k=0}^{n-1} \frac{a_{k+1}}{a_0} X^k$ and write $P_A(X) = a_0 - a_0 X Q_A(X)$. Now $P_A(A) = 0$ leads to $A^{-1} = Q_A(A) \in \mathbb{C}[A]$.

   Let us now show that $\mathbb{C}[A]^\times$ is a subgroup of $\mathrm{GL}(n,\mathbb{C})$. It contains $1_n = 1(A)$. Given $B = P(A), C = Q(A) \in \mathbb{C}[A]^\times$, we have $BC = PQ(A) \in \mathbb{C}[A]$, and $BC \in \mathrm{GL}(n,\mathbb{C})$, so $BC \in \mathbb{C}[A]^\times$.
   Let $B = P(A) \in \mathbb{C}[A]^\times$. Applying the previous result to B, there is a polynomial $Q_B \in \mathbb{C}[X]$ such that $B^{-1} = Q_B(B)$, so $B^{-1} = Q_B(B) = Q_B \circ P(A) \in \mathbb{C}[A]$, hence $B^{-1} \in \mathbb{C}[A]^\times$ and this concludes the proof that $\mathbb{C}[A]^\times$ is a subgroup of $\mathrm{GL}(n,\mathbb{C})$.

2. Prove that $\mathbb{C}[A]^\times$ is connected.
   Hint: for $B, C \in \mathbb{C}[A]^\times$, consider the polynomial $Q(X) = \det(B + X(C-B)) \in \mathbb{C}[X]$, and use the fact that $\mathbb{C} \setminus F$ is connected whenever $F \subset \mathbb{C}$ is finite.

   Solution: Let $B, C \in \mathbb{C}[A]^\times$, and consider the polynomial $Q(X) = \det(B + X(C-B)) \in \mathbb{C}[X]$. Since $Q(0) = \det B \neq 0$, we know that $Q \neq 0$ so $F = \{z \in \mathbb{C} \,|\, Q(z) = 0\}$ is finite. Since $0 \in F$ and $1 \in F$ (because $Q(1) = \det C \neq 0$), there is a continuous path $c : [0,1] \to \mathbb{C} \setminus F$ such that $c(0) = 0$ and $c(1) = 1$. Now consider $\gamma(t) = B + c(t)(C-B)$. This is a continuous path $\gamma : [0,1] \to \mathbb{C}[A]^\times$ such that $\gamma(0) = B$ and $\gamma(0) = C$, so $\mathbb{C}[A]^\times$ is path-connected.

3. Let G be a connected abelian Lie group. Prove that $\exp_G$ is surjective.

   Solution: Since G is abelian, the exponential map $\exp_G : \mathfrak{g} \to G$ is a Lie group morphism from $(\mathfrak{g}, +)$ to G. Its image $\exp_G(\mathfrak{g})$ is an open subgroup of G (because $d_0 \exp_G = \mathrm{Id}_{\mathfrak{g}}$ is invertible), and any open subgroup is closed. It follows that $\exp_G(\mathfrak{g})$ is open and closed in G. Since G is connected, we find that $\exp_G$ is onto.

4. Conclude.

   Solution: Note that $\mathbb{C}[A]^\times$ is an open subset of the vector subspace $\mathbb{C}[A]$ of $\mathcal{M}_n(\mathbb{C})$, so it is a submanifold of $\mathrm{GL}(n,\mathbb{C})$. It follows from question 1. that $\mathbb{C}[A]^\times$ is an embedded Lie subgroup of $\mathrm{GL}(n,\mathbb{C})$

(and its Lie algebra is $\mathbb{C}[A]$). It is also abelian (because $P(A)Q(A) = PQ(A) = QP(A) = Q(A)P(A)$), and connected because of question 3.). According to question 3., there is $B \in \mathbb{C}[A] \subset \mathcal{M}_n(\mathbb{C})$ such that $A = \exp_{\mathbb{C}[A]^\times}(B) = \exp_{\mathrm{GL}(n,\mathbb{C})}(B) = \exp(B)$.

Exercise 2

Let $\xi = (E,p,M)$ be a vector bundle of rank r, and let ∇ be a connection on ξ. We consider the induced connection ∇^{End} on the endomorphism bundle $\mathrm{End}(\xi) = \xi^* \otimes \xi$. Recall that for $\varphi \in \Gamma(\mathrm{End}(\xi))$, $\sigma \in \Gamma(\xi)$ and $X \in \mathcal{X}(M) = \Gamma(TM)$, we get:

$$\left(\nabla_X^{\mathrm{End}}\varphi\right)(\sigma) = \nabla_X\left(\varphi(\sigma)\right) - \varphi\left(\nabla_X \sigma\right)$$

1. For $\varphi \in \Gamma(\mathrm{End}(\xi))$, $\sigma \in \Gamma(\xi)$ and $X, Y \in \mathcal{X}(M) = \Gamma(TM)$, give an expression of $\left(\nabla_X^{\mathrm{End}}\nabla_Y^{\mathrm{End}}\varphi\right)(\sigma)$ that does not involve the connection ∇^{End}.

   Solution:

$$\begin{aligned}\left(\nabla_X^{\mathrm{End}}\nabla_Y^{\mathrm{End}}\varphi\right)(\sigma) &= \nabla_X\left[\left(\nabla_Y^{\mathrm{End}}\varphi\right)(\sigma)\right] - \left[\nabla_Y^{\mathrm{End}}\varphi\right](\nabla_X\sigma)\\ &= \nabla_X\left[\nabla_Y\left(\varphi(\sigma)\right) - \varphi\left(\nabla_Y\sigma\right)\right] - \left[\nabla_Y\left(\varphi(\nabla_X\sigma)\right) - \varphi(\nabla_Y\nabla_X\sigma)\right]\\ &= \nabla_X\nabla_Y\left(\varphi(\sigma)\right) - \nabla_X\left(\varphi(\nabla_Y\sigma)\right) - \nabla_Y\left(\varphi(\nabla_X\sigma)\right) + \varphi(\nabla_Y\nabla_X\sigma)\end{aligned}$$

2. Let $F \in \Omega^2(\mathrm{End}(\xi))$ be the curvature of ∇, and $F^{\mathrm{End}} \in \Omega^2(\mathrm{End}(\mathrm{End}(\xi))$ the curvature of ∇^{End}. Prove that $F^{\mathrm{End}} = \mathrm{ad}(F)$, i.e.

$$\forall x \in M\ \forall u,v \in T_xM\ \forall \varphi \in \mathrm{End}(\xi_x) \quad F_x^{\mathrm{End}}(u,v)\varphi = [F_x(u,v),\varphi]$$

   Solution: The previous computation yields

$$\left(\nabla_X^{\mathrm{End}}\nabla_Y^{\mathrm{End}}\varphi - \nabla_Y^{\mathrm{End}}\nabla_X^{\mathrm{End}}\varphi\right)(\sigma) = \nabla_X\nabla_Y\left(\varphi(\sigma)\right) - \nabla_Y\nabla_X\left(\varphi(\sigma)\right) - \varphi\left(\nabla_X\nabla_Y\sigma - \nabla_Y\nabla_X\sigma\right)$$

   Since $\left(\nabla_{[X,Y]}^{\mathrm{End}}\varphi\right)(\sigma) = \nabla_{[X,Y]}\left(\varphi(\sigma)\right) - \varphi\left(\nabla_{[X,Y]}\sigma\right)$, we find

$$\begin{aligned}\left(F^{\mathrm{End}}(X,Y)\varphi\right)(\sigma) &= F(X,Y)\left(\varphi(\sigma)\right) - \varphi\left(F(X,Y)\sigma\right)\\ &= [F(X,Y),\varphi](\sigma)\end{aligned}$$

Exercise 3

Let G be a Lie group, $\mathfrak{g}$ its Lie algebra, and $H \subset G$ a closed Lie subgroup whose Lie algebra is denoted by $\mathfrak{h}$. Consider $\pi_H : G \to G/H$ and $\pi_{\mathfrak{h}} : \mathfrak{g} \to \mathfrak{g}/\mathfrak{h}$ the canonical projections.

1. Let V be a finite dimensional real vector space, and $\rho : H \to \mathrm{GL}(V)$ a Lie group morphism. Consider the right action $G \times V \curvearrowleft H$ defined by $(g,v).h = (gh, \rho(g^{-1}).v)$. Prove that the quotient $G \times V/H$ is the total space of a vector bundle ξ_ρ over G/H.

Solution: Let $\pi_1 : G \times V \to G$ be the projection on the first factor.
The action of H on $G \times V$ is smooth and free (if $(g,v).h = (g,v)$, then $gh = g$ so $h = e$). It is also proper: if $K \subset G \times V$ is compact, then so is $\pi_1(K)$, and

$$\{h \in H \,|\, Kh \cap K \neq \emptyset\} \subset \{h \in H \,|\, \pi_1(K)h \cap \pi_1(K) \neq \emptyset\}$$

is compact. Let E_ρ be the quotient manifold, and $\pi_\rho : G \times V \to E_\rho$ the projection. The smooth map $\pi_H \circ \pi_1 : G \times V \to G/H$ is H-invariant, so it descends to a smooth map $p : E_\rho \to G/H$.
Since π_1 and π_H are both surjective submersions, the equality $p \circ \pi_\rho = \pi_H \circ \pi_1$ shows that p is also a surjective submersion.

Consider a local section $\sigma : U \to G$ of π_H, where $U \subset G/H$ is open and $\pi_H(e) \in U$. For $x \in U$, consider the map $\varphi_x : V \to p^{-1}(\{x\})$ defined by $\varphi_x(v) = \pi_\rho(\sigma(x), v)$. It is injective: if $\varphi_x(v) = \varphi_x(w)$ for $v, w \in V$, then by definition of π_ρ there is $h \in H$ such that $(\sigma(x), w) = (\sigma(x), v).h$. The first factor yields $h = e$, so the second yields $v = w$. It is also an immersion: if $d_v\varphi_x(w) = 0$ for $v, w \in V$, then $d_{(\sigma(x),v)}\pi_\rho(0, w) = 0$, so the vector $(0, w) \in T_{\sigma(x)}G \times V$ is tangent to the fibre of π_ρ, which is the H-orbit of $(\sigma(x), v)$. Since $T_{(\sigma(x),v)}(\sigma(x), v).H = \{(d_eL_{\sigma}(x)(X), -d_e\rho(X)v) \,\big|\, X \in \mathfrak{h}\}$, we find that $w = 0$.
The map φ_x is surjective: if $z \in p^{-1}(\{x\})$, consider $(g, v) \in G \times V$ such that $z = \pi_\rho(g, v)$. Since $\pi_H(g) = p(z) = x$, there is $h \in H$ such that $g = \sigma(x)h$, and $\varphi_x(\rho(h)v) = z$.
We have shown that φ_x is an immersion and a bijection, so it is a diffeomorphism. So $(\varphi_x)_{x \in U}$ is a trivialisation of $p|_{p^{-1}(U)}$ with respect to V.

Given $g \in G$, we can consider the open set $g.U \subset G/H$ and define $\varphi_x^g(v) = \pi_\rho(g\sigma(g^{-1}.x), v)$ for $x \in g.U$ and $v \in V$. Applying the previous arguments to the local section $x \mapsto g\sigma(g^{-1}.x)$ of π_H, we find that φ_x^g if a diffeomorphism from V to $p^{-1}(\{x\})$ for all $x \in^g .U$.
If $x \in g.U \cap h.U$, there is $k \in H$ such that $g\sigma(g^{-1}x) = h\sigma(h^{-1}x)k$ and the transition map $\left(\varphi_x^h\right)^{-1} \circ \varphi_x^g : V \to V$ is equal to $\rho(k)$, so it is linear, and this shows that $\xi_\rho = (E_\rho, p, G/H)$ is a vector bundle of rank $\dim V$.

2. Let $\rho : H \to GL(\mathfrak{g}/\mathfrak{h})$ be defined by $\rho(h).\pi_{\mathfrak{h}}(X) = \pi_{\mathfrak{h}}(\mathrm{Ad}(h)X)$ for all $h \in H$ and $X \in \mathfrak{g}$. Prove that this defines a Lie group morphism, and that the vector bundle ξ_ρ constructed in the previous question with $V = \mathfrak{g}/\mathfrak{h}$ is isomorphic to the tangent bundle $T(G/H)$.

   Solution: The fact that $\rho(h)$ is well defined comes from $\mathrm{Ad}(h)\mathfrak{h} \subset \mathfrak{h}$. Consider a vector basis $\mathcal{B} = (X_1, \ldots, X_d)$ of $\mathfrak{g}$ such that $(X_1, \ldots, X_k)$ is a basis of $\mathfrak{h}$. Then for all $h \in H$, the matrix of $Ad(h)$ in the basis $\mathcal{B}$ writes as $\begin{pmatrix} A(h) & B(h) \\ 0 & C(h) \end{pmatrix}$. The matrix of $\rho(h)$ in the basis $(\pi_{\mathfrak{h}}(X_{k+1}), \ldots, \pi_{\mathfrak{h}}(X_d))$ of $\mathfrak{g}/\mathfrak{h}$ is $C(h)$ and depends smoothly on h because $Ad(h)$ does. This shows that ρ is smooth, so it is a Lie group morphism.

   Consider the map $\Psi : G \times \mathfrak{g} \to T(G/H)$ defined by $\Psi(g, X) = (\pi_H(g), d_e(\pi_H \circ L_g)(X))$. If $X \in \mathfrak{h}$, then $d_e(\pi_H \circ L_g)(X) = d_e(\pi_H \circ L_g)\left(\frac{d}{dt}\big|_{t=0} \exp_H(tX)\right) = \frac{d}{dt}\big|_{t=0} \pi_H(g \exp_H(tX)) = \frac{d}{dt}\big|_{t=0} \pi_H(g) = 0$. It follows that Ψ descends to a smooth map $\psi : G \times \mathfrak{g}/\mathfrak{h} \to T(G/H)$ which is linear in the second variable.
   Let $g \in G$, $X \in \mathfrak{g}$ and $h \in H$.

$$
\begin{aligned}
\psi(gh, \rho(h^{-1}).\pi_{\mathfrak{h}}(X)) &= \Psi(gh, \mathrm{Ad}(h^{-1})X) \\
&= (\pi_H(gh), d_e(\pi_H \circ L_{gh})(\underbrace{\mathrm{Ad}(h^{-1})}_{=d_e\iota_{h^{-1}}} X)) \\
&= (\pi_H(g), d_e\underbrace{(\pi_H \circ L_{gh} \circ \iota_{h^{-1}})}_{=\pi_H \circ L_g}(X)) \\
&= \psi(gh, \pi_{\mathfrak{h}}(X))
\end{aligned}
$$

It follows that ψ descends to a smooth function $\Phi : E_\rho \to T(G/H)$. In the trivialisation $\varphi_x^g : \mathfrak{g}/\mathfrak{h} \to p^{-1}(\{x\})$ defined in the previous question, we find $\Phi \circ \varphi_x^g(\pi_{\mathfrak{h}}(X)) = (x, d_e(\pi_H \circ L_{g\sigma(g^{-1}.x)})(X))$. Since $\ker d_e(\pi_H \circ L_{g\sigma(g^{-1}.x)}) = \mathfrak{h}$, we find that Φ induces a linear isomorphism from the fibre $p^{-1}(\{x\})$ to $T_x G/H$, so it is a vector bundle isomorphism.

3. Use a homogeneous space to identify $T_V \mathcal{G}_k(\mathbb{R}^d)$ and $\mathrm{Hom}(V, \mathbb{R}^d/V)$ for $V \in \mathcal{G}_k(\mathbb{R}^d)$.

Solution: The action of $G = GL(\mathbb{R}^d)$ on $\mathcal{G}_k(\mathbb{R}^d)$ is transitive, so the differential of the orbit map identifies the tangent space of the homogeneous space $\mathcal{G}_k(\mathbb{R}^d)$ at V with $\mathfrak{g}/\mathfrak{h}$ where $\mathfrak{h}$ is the Lie algebra of the stabiliser $H \subset G$ of V. Since $\mathfrak{h} = \left\{ f \in \mathrm{End}(\mathbb{R}^d) \,\middle|\, f(V) \subset V \right\}$, the map $f \mapsto \pi_V \circ f|_V$ (where $\pi_V = \mathbb{R}^d \to \mathbb{R}^d/V$ is the projection) induces an isomorphism from $\mathfrak{g}/\mathfrak{h}$ to $\mathrm{Hom}(V, \mathbb{R}^d/V)$.