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Prescribing curvature on open surfaces

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0 Introduction

0.1. In this article, we study the problem (sometimes called the Berger-Nirenberg problem) of prescribing the curvature on a Riemann surface (that is on an oriented surface equipped with a conformal class of Riemannian metrics). In the compact case, the theory is well understood, and can be found in [KW1].

Here, we are interested in non compact surfaces. As a byproduct of the results of this paper, we shall prove that, on any connected non compact Riemann surface of finite type, different from $\mathbf{C}$ or $\mathbf{C}^*$, there is no obstruction to construct a conformal metric with prescribed curvature (see A.1). However, the metric given by this result is usually not complete.

Even if we restrict ourselves to complete metrics, the problem of prescribing the curvature may have a continuum of solutions, with variable asymptotic geometries (see the example in [HT]).

Thus we are lead to study *complete* metrics, keeping control on their *asymptotic geometries*. We propose two precise formulations for the problem of prescribing the curvature on open Riemann surfaces (see 0.3). The first will be the natural one for surfaces with finite total curvature, while the second will deal with surfaces having hyperbolic ends.

An announcement of this work has appeared in [HT].

0.2. In order to formulate our first problem, we need to discuss the structure of complete surfaces with finite total curvature. Thanks to the work of Alfred Huber, these surfaces are known to have a natural compactification.

Theorem (Huber). *A complete Riemannian surface (S', g') with finite total curvature is isometric to the regular part of a compact Riemannian surface (S, g) with finitely many singular points $p_i \in S\, (i = 1, \dots, n)$, each admitting a neighbourhood U_i isometric to the disk $D = \{z \in \mathbf{C}: |z| < 1\}$ equipped with the metric*

$$ds^2 = e^{2u_i}|z|^{2\beta_i}|dz|^2 ,$$

where, for $i = 1, \dots, n$, $\beta_i \leqq -1$, *and the functions* u_i *are "sufficiently regular" (see* 2.1*).*

Furthermore, for such surfaces a Gauss-Bonnet formula holds:

$$\frac{1}{2\pi}\int_{S'} K' dA' = \frac{1}{2\pi}\int_{S} K dA = \chi(S) + \sum_{i=1}^{n} \beta_i .$$

This theorem allows us to shift our attention from non compact complete surfaces (with finite total curvature) to compact Riemannian surfaces with isolated singularities.

We shall say that, on a surface S, a metric g having at n points p_i $(i = 1, \dots, n)$ a singularity of the type $e^{2u_i}|z|^{2\beta_i}|dz|^2$ (with u_i "sufficiently regular") *represents* the *divisor* $\boldsymbol{\beta} := \sum_{i=1}^{n} \beta_i p_i$.

0.3. Huber's theorem suggests to formulate the problem of prescribing the curvature on non compact Riemann surfaces with finite total curvature as follows.

Problem 1. Let S be a surface of finite type with a conformal structure, equipped with a divisor $\boldsymbol{\beta} := \sum_{i=1}^{n} \beta_i p_i$. For a given function $K : S \to \mathbf{R}$, find a conformal metric g on S representing $\boldsymbol{\beta}$ and having curvature K.

Our answer to this problem (see 6.1, 7.1, 7.2, and 8.1) can be expressed in a form very similar to the classical results available for compact surfaces. For that purpose, it will be convenient to define the Euler characteristic of a surface with divisor $(S, \boldsymbol{\beta})$ as $\chi(S, \boldsymbol{\beta}) = \chi(S) + \sum_{i=1}^{n} \beta_i$. We then have:

Theorem A. *Let* $(S, \boldsymbol{\beta})$ *be a compact connected Riemann surface with divisor* $\boldsymbol{\beta} = \sum_{i=1}^{n} \beta_i p_i$, *and let* $K : S \to \mathbf{R}$ *be a smooth function. Assume that there exists a number* $p > 1$ *such that, at every* p_i, $(|z - p_i|^{2\beta_i} K(z)) \in L^p$. *Furthermore:*

(a) *if* $\chi(S, \boldsymbol{\beta}) > 0$, *assume* $q\chi(S, \boldsymbol{\beta}) < 2$ *and* $\sup K > 0$ $\left(\textit{where } \frac{1}{p} + \frac{1}{q} = 1\right)$;

(b) *if* $\chi(S, \boldsymbol{\beta}) = 0$, *assume* $K \equiv 0$ *or* $\sup K > 0$ *and* $\int_S K dA_1 < 0$ *(where* dA_1 *is the area element of a flat conformal metric on* S *representing* $\boldsymbol{\beta}$*);*

(c) *if* $\chi(S, \boldsymbol{\beta}) < 0$, *assume* $K \leqq 0$ *and* $K \not\equiv 0$.

Then, there exists a conformal metric g *on* S *with curvature* K, *representing the divisor* $\boldsymbol{\beta}$.

The above is well known for smooth compact Riemannian surfaces (cf. [KW1]). It is also known in the case of the sphere with one point singularity (cf. [Ni1, McO1, McO2]) and for surfaces with conical singularities (cf. [T2]).

However, we might also be interested in surfaces with infinite total curvature, for which there is no analog of Huber's theorem. A convenient way to formulate the problem is then the following.

Problem 2. Let (S, g_1) be a Riemannian surface of finite type. For a given function $K : S \to \mathbf{R}$, find a metric g on S with curvature K, which is conformal and conformally quasi-isometric to g_1.

We say by definition that two conformal metrics g_1 and $g = e^{2u} g_1$ are *conformally quasi-isometric* if the function u is bounded (in the sequel, we will shortly say *quasi-isometric*).

We will see that, for surfaces with finite total curvature, Problem 1 is – in some sense – contained in Problem 2 (2.10).

For negatively curved surfaces, we are – under suitable assumptions – able to solve Problem 2. We can for instance prescribe *cusps* on the surface.

Theorem B (8.1, 8.4, 2.3). *Let (S, g_1) be a compact surface with a (singular) metric g_1 representing some divisor $\boldsymbol{\beta} = \sum_{j=1}^{n} \beta_j q_j$ such that $\chi(S, \boldsymbol{\beta}) < 0$. Set $S' := S \backslash \{q_1, \ldots, q_n\}$ and let $K : S' \to \mathbf{R}$ be a smooth non positive function. Assume that K is negative somewhere, and that there exist two positive constants $a, b > 0$ and a compact set $N \subset S'$ such that*

$$bK(x) \leqq K_1(x) \leqq aK(x) \leqq 0 \quad \textit{for all } x \textit{ in } S' \backslash N,$$

where K_1 is the curvature of g_1.

Then, there exists a unique metric g on S with curvature K which is conformal and (conformally) quasi-isometric to g_1 (in particular, g is complete whenever g_1 is). Moreover g also represents $\boldsymbol{\beta}$.

The theorem holds true even when (S, g_1) is non compact but has finitely many complete hyperbolic ends. For instance, when the prescribed curvature function is negatively pinched at infinity, we have the following:

Theorem C (8.2). *Let S be a connected open Riemann surface of finite topological type, different from $\mathbf{C}$ and $\mathbf{C}^*$. Let $K : S \to \mathbf{R}$ be a smooth non positive function such that*

$$-b \leqq K \leqq -a < 0$$

outside a compact set $N \subset S$.

Then, there exists a unique complete conformal metric g on S with curvature K. Moreover, each end of (S, g) admits a neighbourhood conformally quasi-isometric to the end of the Beltrami pseudo-sphere when parabolic, and to the end of the Poincaré disk when hyperbolic.

Furthermore, if S has no hyperbolic ends, then a generalized Gauss-Bonnet formula holds:

$$\frac{1}{2\pi} \int_S K dA = \chi(S).$$

This result was already known when S is the disk (see [AMcO, BK]).

0.4. Let us shortly describe the methods we use in this article. The technique used to investigate Problems 1 and 2 is by now classical. On our surface S we choose a "base" metric g_1 in the desired conformal class. We also demand that g_1 has a given conformal quasi-isometry type, or that g_1 represents a specified divisor on a compactification $\hat{S}$ of S. We then look for a metric with curvature K in the form $g = e^{2u} g_1$. Then, u has to be a solution of the equation

$$\Delta_1 u = K e^{2u} - K_1,$$

where Δ_1 and K_1 are respectively the Laplace-Beltrami operator and the curvature of g_1.

The surprise is that all the hard work (namely solving the above equation) can be done on the compact surface $\hat{S}$ (rather than on the non compact S) and using only classical tools (namely, the variational method and the method of upper and lower solutions). One has just to be careful, and allow for L^p coefficients in the equation.

0.5. Contents

In the first two sections, we discuss Huber's results on complete surfaces with finite total curvature, and introduce the notion of *simple* singularities. An important example is discussed in (2.2). The result that will serve as foundation for the rest of the paper is stated in (2.9).

An analytic formulation of Problems 1 and 2 and some obstructions are given in Sects. 3 and 4.

The equation to be solved is rapidly studied by means of a variational method in Sect. 5 and the results of this section are used in Sects. 6 and 7 to solve Problem 1 for positive and null surfaces (i.e. when $\chi(S, \boldsymbol{\beta}) \geqq 0$).

In Sect. 8, we discuss the negative case. This discussion rests on the method of upper and lower solutions, and allows us to solve Problem 2 for negatively curved surfaces of finite type (even with hyperbolic ends).

As explained in Appendix A, there exists a description of the class of functions on a non compact surface of finite type which can be prescribed as curvatures if we drop the conformality *or* the completeness requirements (with the exceptions of the once or twice punctured sphere).

In Appendix B, we discuss some relations with uniformization theory. In particular, we show how the uniformization theorems for compact simply connected Riemann surfaces and for Riemann orbifolds follow from our results.

For the convenience of the reader, we finally give in Appendix C a self contained proof of the "method of upper and lower solutions" for the equation $\Delta u = f(x, u)$ on a non compact manifold. Although Ni and Noussair give proofs, their expositions are not self contained and require further reading.

0.6. In this paper, we use standard notations for function spaces L^p, $W^{k,p}\dots$ as in [GT]. Our convention for the sign of the Laplace-Beltrami operator is that $\Delta_g = (-4/\varrho)\partial^2/\partial z \partial \bar{z}$ when $g = \varrho|dz|^2$ (for instance a smooth function ϕ is subharmonic if $\Delta\phi \leqq 0$).

Let us stress that we generally speak about Riemann surfaces though nothing would change for non-orientable surfaces with a given conformal structure. We could even allow for (piecewise-)geodesic boundaries, as in [T2].

1 Complete surfaces with finite total curvature

In the whole section, S' will denote a non compact smooth surface, endowed with a *complete* metric g of class C^2. Under the assumption of finite total curvature, we obtain precise information concerning the topology of S', the conformal type of its ends, and the asymptotic behaviour of the metric at each end. In distinct terms, the following holds:

1.1. Theorem (Huber). *Let (S', g') be a non compact, complete Riemannian surface of class C^2 with finite total curvature $\left(\text{that is } \int_{S'} |K| dA < \infty\right)$. Then:*

(i) *S' is of finite topological type;*
(ii) *each end of (S', g') is parabolic;*
(iii) *in a neighbourhood of each end, the metric can be written as*

$$g = e^{2u}|z|^{2\beta}|dz|^2 \qquad (0 < |z| \leqq 1)$$

where $\beta \leqq -1$, and

$$u \in L^1\,, \quad \text{and} \quad \Delta u \in L^1 \quad \text{in a weak sense.} \tag{$*$}$$

1.2. Remarks. (i) We will show in (1.7) that the function u in (iii) satisfies the nice property that e^u and e^{-u} belong to every $L^q(1 \leqq q < \infty)$.

(ii) For a definition of parabolic ends, see Appendix B (B.2).

(iii) There is a generalization of Theorem 1.1 in higher dimensional Kähler geometry (see [SY, Mok]).

The above result is actually spread in the literature into three theorems, all of them due to Huber.

1.3. Proposition. *Let (S', g') be a complete Riemannian surface such that $\int_{S'} |K^-| dA < \infty$, where K^- is the negative part of the curvature, and dA is the area element of (S', g'). Then:*

(a) *$\int_{S'} |K| dA < \infty$, and S' is homeomorphic to $S \backslash \{p_1, \dots, p_n\}$, where S is a compact surface and p_i $(1 = i, \dots, n)$ are points in S;*

(b) *furthermore, the Cohn-Vossen inequality holds:*

$$\frac{1}{2\pi} \int_{S'} K dA \leqq \chi(S')\,.$$

Part (a) is due to Huber (cf. [H2, Theorem 10, Theorem 13]), and part (b) to Cohn-Vossen. We shall see in (2.9) that the Cohn-Vossen inequality admits a sharp form.

For a nice proof of this proposition, see [W1, W2].

The second step is concerned with the conformal structure of the ends of (S', g').

1.4. Proposition [H2, Theorem 15]. *Let (Ω, g) be a complete Riemannian surface of class C^2, diffeomorphic to $]0, 1] \times \mathbf{S}^1$, and such that $\int_\Omega |K^-| dA < \infty$. Then (Ω, g) is parabolic.*

Karp gave a generalization of this result (see Theorem 3.5 and corollaries in [K]).

We give a proof of this proposition. This will give the flavour of typical arguments used throughout the paper.

Proof. We may assume (cf. B.2) that

$$\Omega = \{z \in \mathbf{C} : r < |z| \leqq 1\} \quad \text{and} \quad g = e^{2u}|dz|^2 ,$$

where $0 \leqq r < 1$, $u \in C^2(\Omega)$, and $\int_\Omega |(\Delta u)^-| dxdy < \infty$. Assuming moreover that r is strictly positive, we will show that (Ω, g) is conformally equivalent to a complete euclidean half-cylinder, and this will lead to a contradiction.

Let $d\mu_-$ be the measure defined on $\mathbf{C}$ by $d\mu_- = (\Delta u)^- \mathbf{1}_\Omega \, dx\, dy$, and u_- be its logarithmic potential defined by

$$u_-(z) = \frac{1}{2\pi} \int_{\mathbf{C}} \log |z - \zeta| d\mu_-(\zeta) .$$

Local elliptic regularity implies that $u_- \in C^{1,\delta}(\Omega)$ for $0<\delta<1$, since $\Delta u_- = (\Delta u)^-$ is locally bounded on Ω. Note also that u_- is subharmonic on $\mathbf{C}$, hence bounded above on Ω.

Let $v = u - u_- \in C^1(\Omega)$, and $h \in C^2(D)$ be the solution of the Neumann problem on $D = \{z \in \mathbf{C} : |z| \leqq 1\}$ defined by:

$$\begin{cases} \Delta h = 0 & \text{on} \quad D , \\ \dfrac{\partial h}{\partial n} = \gamma - \dfrac{\partial v}{\partial n} & \text{on} \quad \partial D , \end{cases}$$

where $\gamma = \dfrac{1}{2\pi} \int_{\partial D} \dfrac{\partial v}{\partial n} |dz|$.

We now introduce the auxiliary metric

$$\tilde{g} = e^{2w}|dz|^2$$

on Ω, where $w = v + h - (1 + \gamma) \log |z|$, and observe the following properties of $\tilde{g}$:

(a) $(\Omega, \tilde{g})$ is *complete*: for $w - u = h - (1 + \gamma) \log |z| - u_-$ is bounded below on $\bar{\Omega} = \{r \leqq |z| \leqq 1\}$ (recall $r > 0$), and $g = e^{2u}|dz|^2$ is complete;

(b) $\partial\Omega$ is *geodesic* in $(\Omega, \tilde{g})$: indeed, the geodesic curvature of $\partial\Omega$ for the metric $\tilde{g}$ is

$$k_{\tilde{g}} = \left(1 + \frac{\partial w}{\partial n}\right) e^{-w} = (1 + \gamma) \left(1 - \frac{\partial}{\partial n} \log |z|\right) e^{-w} = 0$$

on $\partial\Omega = \{z \in \mathbf{C} : |z| = 1\}$;

(c) the curvature of $(\Omega, \tilde{g})$ is *non negative*: since $\Delta w = \Delta v = (\Delta u)^+ \geqq 0$.

Now apply the Cohn-Vossen inequality (cf. 1.3.b) to $(\Omega, \tilde{g})$, (this is valid since w, although not C^2, may be written as the difference of two subharmonic functions (see [H2, p. 15])). No boundary term comes in, for $\partial\Omega$ is geodesic, and we get

$$0 = \chi(\{z \in \mathbf{C} : r < |z| < 1\}) \geqq \int_\Omega K_{\tilde{g}} dA_{\tilde{g}} ,$$

thus (c) implies that

(d) $(\Omega, \tilde{g})$ is *flat*.

From (a), (b), (d), we infer that $(\Omega, \tilde{g})$ is a complete euclidean half-cylinder with geodesic boundary, thus isometric to $\{z \in \mathbf{C} : 0 < |z| \leqq 1\}$ equipped with the metric $|dz/z|^2$. And this contradicts our assumption on r. $\square$

The final step is concerned with the metric properties of the ends of (S', g'). The next proposition describes, for a Riemannian surface admitting a complete parabolic end of finite total curvature, the asymptotic behaviour of the metric at this end. We work on the punctured disk $\Omega = \{z \in \mathbf{C} : 0 < |z| \leqq 1\}$, equipped with a metric $g = e^{2v}|dz|^2$, where $v \in C^2(\Omega)$ and $\Delta v \in L^1$. The function v is defined, up to an harmonic function, by its logarithmic potential; assuming moreover that the metric g is complete at the origin, Huber shows that the singularity of the harmonic term is logarithmic. Thus, the following can be thought of as a "Removability of singularities" result.

1.5. Proposition [H4, Satz 1]. *Let v be a function of class C^2 on the punctured disk $\Omega = \{z \in \mathbf{C} : 0 < |z| \leqq 1\}$, and assume that:*

(i) $\int_\Omega |\Delta v| dxdy < \infty$;

(ii) *for any path γ diverging to the origin,* $\int_\gamma e^{v(z)}|dz| = \infty$.

Then, there exists $\beta \in \mathbf{R}$ and a function h which is harmonic in the whole disk $\Delta = \{z \in \mathbf{C} : |z| \leqq 1\}$, such that

$$v(z) = \beta \log|z| + h(z) - \frac{1}{2\pi} \int_\Omega \log|z - \zeta| \Delta v(\zeta) d\xi d\eta .$$

1.6. Remarks. (a) The hypothesis (ii) may be weakened to:

(ii′) *there exists $\alpha \in \mathbf{R}$ such that, for any path γ diverging to the origin,* $\int_\gamma e^{v(z)}|z|^\alpha |dz| = \infty$.

(b) This result is stated in the original paper for $\Omega' = \{z \in \mathbf{C} : |z| \geqq 1\}$ and with a singularity at infinity. It reads:

$$v(z) = H(z) + c \log|z| - \frac{1}{2\pi} \int_{\Omega'} \log\left|1 - \frac{z}{\zeta}\right| \Delta v(\zeta) d\xi d\eta ,$$

with H harmonic up to infinity. Perform the inversion $z \mapsto 1/z$ to pass from one formulation to the other.

(c) The "regular part" of v, that is the function u defined by

$$u(z) = h(z) - \frac{1}{2\pi} \int_\Omega \log|z - \zeta| \Delta v(\zeta) d\xi d\eta$$

satisfies:

$$u \in L^1, \quad \text{and} \quad \Delta u \in L^1 \quad \text{in a weak sense.} \tag{$*$}$$

For a proof of this proposition, we refer to [H4]. We now state two properties of the functions of the type described in (1.5).

1.7. Proposition. *Let $v \in C^2(\Omega)$ with $\Delta v \in L^1$, and assume that v can be written as*

$$v(z) = \beta \log|z| + h(z) - \frac{1}{2\pi} \int_\Omega \log|z - \zeta| \Delta v(\zeta) d\xi d\eta = \beta \log|z| + u(z) ,$$

where h is harmonic on the whole disk. Then:

(i) *in case $\beta < -1$, the metric $g = e^{2v}|dz|^2$ on Ω is complete at the origin; in case $\beta > -1$, it is not;*

(ii) *e^u and $^{-u}$ belong to every $L^q(\Omega)$ $(1 \leqq q < \infty)$.*

The proof follows from an idea of Rechetjnack (cf. [R2, Theorem 3.1]).

Proof. (i) We first assume that $\beta > -1$; since h is bounded, it suffices to show that the metric $e^{2w}|z|^{2\beta}|dz|^2$, where $w = u - h$ is the Riesz potential of Δv, is not complete at the origin.

Let $\Delta v(\zeta)d\zeta = d\mu^+ + d\mu^-$ be the decomposition of the measure into its positive and negative parts, and set

$$w_\pm(z) = -\frac{1}{2\pi}\int_\Omega \log|z-\zeta|d\mu^\pm(\zeta).$$

The function w_- is subharmonic on $\mathbf{C}$, hence bounded above on Ω. Now let γ be e.g. the segment $[0,1]$. The Hölder inequality yields with $1/p + 1/q = 1$:

$$l(\gamma) \leqq \text{const}\int_\gamma e^{w_+(z)}|z|^\beta|dz| \leqq \text{const}\left(\int_\gamma e^{qw_+(z)}|dz|\right)^{1/q}\cdot\left(\int_\gamma |z|^{p\beta}|dz|\right)^{1/p}.$$

The second integral on the right-hand side is finite for $p\beta > -1$ when q is large enough. The first integral is obviously finite if $\Delta w^+ \equiv 0$. Otherwise set $M = \int_\Omega d\mu^+$, and use Jensen's inequality as follows:

$$\begin{aligned}\int_\gamma e^{qw_+(z)}|dz| &\leqq \int_\gamma \exp\left[\frac{-qM}{2\pi}\int_\Omega \log|z-\zeta|\,\frac{d\mu^+(\zeta)}{M}\right]|dz| \\ &\leqq \int_\gamma\left[\int_\Omega |z-\zeta|^{-qM/2\pi}\,\frac{d\mu^+(\zeta)}{M}\right]|dz| \\ &\leqq \int_\Omega\left[\int_\gamma |z-\zeta|^{-qM/2\pi}|dz|\right]\frac{d\mu^+(\zeta)}{M}.\end{aligned}$$

Working in a smaller disk if necessary, we may assume $-qM/2\pi > -1$, and the integral on the right is finite. Hence, the length of γ is finite, and g is not complete. Assuming now $\beta < -1$, we can choose $1 < p < \infty$ such that $\beta/p < -1$, and write for any segment γ diverging to the origin:

$$\begin{aligned}\infty = \int_\gamma |z|^{\beta/p}|dz| &= \int_\gamma |z|^{\beta/p}e^{w/p}e^{-w/p}|dz| \\ &\leqq \left(\int_\gamma |z|^\beta e^{w(z)}|dz|\right)^{1/p}\left(\int_\gamma e^{-qw(z)/p}|dz|\right)^{1/q}.\end{aligned}$$

The conclusion follows by using the same trick as above.

(ii) Since $u = h + w_- + w_+$, with h and w_- bounded above, it suffices to show that e^{w_+} belongs to every $L^q (1 \leqq q < \infty)$. Once again, write:

$$e^{qw_+(z)} = \exp\left[\int_\Omega -\frac{qM}{2\pi}\log(|z-\zeta|)\,\frac{d\mu^+(\zeta)}{M}\right],$$

and use Jensen's inequality, working in a smaller disk if necessary to make $-qM/2\pi > -2$: thus e^u belongs to L^q. The same argument shows also that e^{-u} belongs to every $L^q (1 \leqq q < \infty)$. □

In view of these results, we are now interested in compact manifolds, equipped with metrics which have finitely many "mild" singularities (that is of the type described in (1.1.iii)). These will be the object of the next section.

2 Compact Riemann surfaces with divisor A Gauss-Bonnet formula

In what follows, S will denote a compact (Riemann) surface. The metrics we will consider on S are "smooth", that is C^2, outside a finite set.

2.1. Definition. A (conformal) metric g on S admits a *simple singularity of order* β at $p \in S$ if it can be locally written as

$$g = e^{2u}|z|^{2\beta}|dz|^2$$

with

$$u \in L^1, \quad \text{and} \quad \Delta u \in L^1 \quad \text{(in a weak sense)}, \tag{$*$}$$

where z is a local (conformal) coordinate such that $z(p) = 0$. Note that this definition does not depend on the particular choice of z, and recall that e^u and e^{-u} belong then to every $L^q(1 \leqq q < \infty)$ (see 1.7).

A simple singularity of order β is said to be:

(i) a *conical singularity* (with total angle $\theta = 2\pi(\beta + 1)$) when $\beta > -1$;

(ii) a *parabolic end* of order β, when $\beta < -1$.

Recall that in the second case, the metric is complete at p, but not in the first one (cf. 1.7).

The order β of the singularity at p describes "first order" properties of the metric (that is provides a best conical approximation of S at p). When β is fixed, the behaviour of u yields "second order" properties and leads to various types of geometries. In particular, since e^u may be unbounded, all these metrics need not be conformally quasi-isometric (see 0.3).

Let us just describe an example of a family of such metrics, in the most striking – though atypical – case (that is $\beta = -1$), for which volume and diameter can be finite or not.

2.2. Example and definition. Let us define on the disk $D = \{z \in \mathbf{C}: 0 \leqq |z| \leqq \frac{1}{2}\}$ the family of metrics

$$g_{(a)} = \frac{|dz|^2}{|z|^2|\log|z||^{2a}} = e^{2u_a}|z|^{-2}|dz|^2 .$$

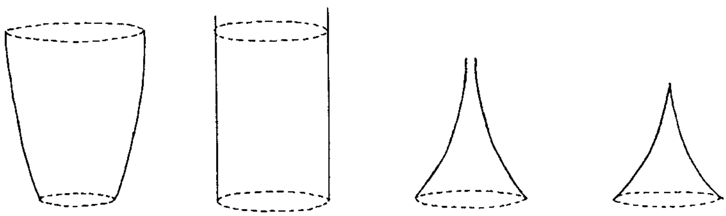
For all values of a, the metric $g_{(a)}$ has got a simple singularity of order -1 at the origin. A computation gives us the curvature K_a of $g_{(a)}$, namely,

$$K_a = -a|\log|z||^{2(1-a)} .$$

In particular, $(D, g_{(0)})$ is an euclidean half-cylinder, and $(D, g_{(1)})$ is isometric to a Beltrami pseudo-sphere. The metric $g_{(1)}$ will be called the *Beltrami metric*.

We sum up some geometric features of the metrics $g_{(a)}$ as follows:

$$\text{diameter: finite iff } a > 1;$$
$$\text{volume: finite iff } a > 1/2;$$
$$\text{sign of curvature: } \begin{cases} a < 0: K > 0; \\ a = 0: K = 0; \\ a > 0: K < 0; \end{cases}$$
$$\text{limit of curvature: } \begin{cases} a < 1: K \to 0; \\ a = 1: K \equiv -1; \\ a > 1: K \to -\infty\,. \end{cases}$$



We will say that a singular point q of a metric g is a *cusp* if g is complete at this point, and has finite volume in a neighborhood of q. Note that if moreover the curvature of g is bounded, then g has at q a simple singularity of order -1 (1.5, 1.7).

Observe that, if $g = e^{2u}g_1$ is conformally quasi-isometric to a metric g_1 having at some point p a simple singularity, then p need not be a simple singularity for g. However, it will be the case whenever g has finite total curvature.

2.3. Lemma. *Let g_1 and $g = e^{2u}g_1$ be two (conformally) quasi-isometric conformal metrics of class C^2 on the punctured disk $\Omega = \{z \in \mathbf{C}: 0 < |z| \leqq 1\}$.*

Assume that g_1 admits a simple singularity of order β at the origin, and that g has finite total curvature.

Then, g also has a simple singularity of order β at the origin.

Proof. By assumption, $g_1 = e^{2\phi_1}|dz|^2$ with $\phi_1 = \beta \log|z| + v_1$, and

$$v_1 \in L^1\,, \qquad \Delta v_1 \in L^1 \quad \text{(in a weak sense)}. \tag{$*$}$$

Write on the other hand $g = e^{2u}g_1 = e^{2\phi}|dz|^2$. Since g_1 has a simple singularity at the origin, and u is bounded, we know by (1.7) that there exists $\alpha \in \mathbf{R}$ such that the metric $|z|^{2\alpha}e^{2\phi}|dz|^2$ is complete at the origin, so that we can apply (1.5) (see 1.6.a), and get

$$\phi = \beta' \log|z| + v\,,$$

where v satisfies ($*$), and $\beta' \in \mathbf{R}$.

Now $u + \phi_1 = \phi$, so that $u + (v_1 - v) = (\beta' - \beta)\log|z|$. Since u is bounded, and $e^{\pm(v_1 - v)}$ belongs to every $L^q (1 \leqq q < \infty)$ by (1.7), we infer that $\beta' = \beta$, so that $u = v - v_1$ satisfies ($*$). $\square$

It will be convenient to control the quasi-isometry type of a simple singularity. For that purpose, we introduce the following.

2.4. Definition. A metric g on S with a simple singularity of order β at p is said to have a *normal* singularity at this point if it can be locally written as

$$g = e^{2u}|z|^{2\beta}|dz|^2$$

with

$$u \in W^{2,p} \quad \text{for some } p > 1\,, \tag{$**$}$$

where, again, $z(p) = 0$. Since $W^{2,p} \subset C^0$, such a metric is conformally quasi-isometric to $|z|^{2\beta}|dz|^2$ in a neighbourhood of p.

In order to deal with several simple singularities, we propose the following:

2.5. Definition. A *divisor* $\boldsymbol{\beta}$ on S is a finite formal sum

$$\boldsymbol{\beta} = \sum_{i=1}^{n} \beta_i p_i\,,$$

where the p_i's are points in S and the β_i's are real numbers. The *support* of the divisor $\boldsymbol{\beta}$ is the finite set $\operatorname{supp}\boldsymbol{\beta} = \{p_1, \dots, p_n\}$. We also define the *Euler characteristic* of $(S, \boldsymbol{\beta})$ as

$$\chi(S, \boldsymbol{\beta}) = \chi(S) + \sum_{i=1}^{n} \beta_i\,.$$

2.6. Definition. A (conformal) metric g on S is said to *represent* the divisor $\boldsymbol{\beta} = \sum_{i=1}^{n} \beta_i p_i$ when:

(i) g is C^2 on $S \backslash \operatorname{supp}\boldsymbol{\beta}$;

(ii) g admits a simple singularity of order β_i at p_i $(1 \leqq i \leqq n)$.

If moreover g admits at each p_i a normal singularity of order β_i, we will say that g represents $\boldsymbol{\beta}$ *normally*.

Examples. (i) For $\alpha \in \mathbf{R}$, $g_\alpha = |z|^{2\alpha}|dz|^2$ is a flat metric on $\mathbf{C} \cup \{\infty\}$, representing normally the divisor $\boldsymbol{\beta} = \alpha \cdot 0 + (-2-\alpha) \cdot \infty$.

(ii) If $\omega = \phi(z)dz$ is a meromorphic differential form on a Riemann surface, then $g = |\omega|^2$ is a flat metric representing normally the divisor $\operatorname{div}(\omega)$.

Other examples are given in [T2, Sect. 1.2].

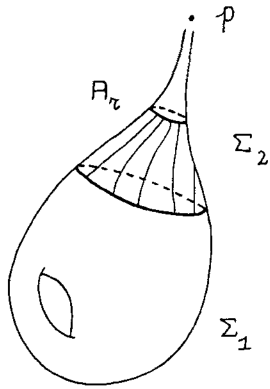
2.7. *A Gauss-Bonnet formula.* As in the smooth case, the total curvature of a compact Riemannian surface with simple singularities does not depend on the metric. It does, however, depend on the divisor.

2.8. Theorem [F]. *Let (S, g) be a compact surface endowed with a metric g representing a divisor $\boldsymbol{\beta}$. Then, the total curvature of g is well defined and satisfies*

$$\frac{1}{2\pi} \int_S K dA = \chi(S, \boldsymbol{\beta})\,.$$

Proof. We indicate the proof in the case $\boldsymbol{\beta} = \beta p$, that is when the metric has a unique singularity. Cut S into two pieces Σ_1 and Σ_2, where $(\Sigma_2 \backslash \{p\}, g)$ is isometric to

$$(\Omega, e^{2u}|z|^{2\beta}|dz|^2) = (\Omega, e^{2v}|dz|^2)\,,$$



where Ω is the punctured disk, and let $A_r = \{r < |z| < 1\} \subset \Sigma_2$ (with $r > 0$).

Since $Ke^{2v} = \Delta v \in L^1$, the total curvature of g is finite and

$$\int_S KdA = \int_{\Sigma_1} KdA + \lim_{r\to 0} \int_{A_r} KdA\,.$$

The Gauss-Bonnet formula for the compact surfaces with boundaries Σ_1 and A_r gives

$$\chi(S) - 1 = \chi(\Sigma_1) = \frac{1}{2\pi}\int_{\Sigma_1} KdA + \frac{1}{2\pi}\int_{\partial\Sigma_1} k_g ds\,,$$

$$0 = \chi(A_r) = \frac{1}{2\pi}\int_{A_r} KdA - \frac{1}{2\pi}\int_{\partial\Sigma_1} k_g ds - \frac{1}{2\pi}\int_{|z|=r} k_g ds\,,$$

where k_g denotes the geodesic curvature. But on $|z| = r$, we have

$$\begin{aligned} k_g ds &= \frac{1}{r}\,|dz| + \frac{\partial v}{\partial n}\,|dz| \\ &= \frac{1}{r}\,|dz| + \beta\,\frac{\partial \log|z|}{\partial n}\,|dz| + \frac{\partial u}{\partial n}\,|dz|\,, \end{aligned}$$

hence the result, since we assumed Δu (in weak sense) to be a L^1 function, which implies:

$$\lim_{r\to 0}\int_{|z|=r} \frac{\partial u}{\partial n}\,ds = 0\,. \quad \square$$

We may now restate the main results of Sect. 1, completed with the Gauss-Bonnet formula, in the language we just introduced.

2.9. Theorem (Huber). *Let (S', g') be a complete Riemannian surface of class C^2 with finite total curvature.*

Then, there exist a compact Riemann surface S, a divisor $\boldsymbol{\beta} = \sum_{i=1}^{n} \beta_i p_i$ on S (with $\beta_i \leqq -1$), and a conformal metric g on S representing $\boldsymbol{\beta}$ such that (S', g') is isometric to $(S\backslash \operatorname{supp}\boldsymbol{\beta}, g)$.

Moreover, the following generalized Gauss-Bonnet formula holds:

$$\frac{1}{2\pi}\int_S K dA = \chi(S,\boldsymbol{\beta}).$$

This Gauss-Bonnet formula is the sharp version of the Cohn-Vossen inequality we mentioned in (1.3).

2.10. Remark. Thanks to Lemma 2.3, we see that if $(S,\boldsymbol{\beta})$ is a compact Riemann surface with a divisor, K a function on S and g_1 a conformal metric representing $\boldsymbol{\beta}$ on S, then any metric g on S solving Problem 2 of the introduction (i.e. such that g is conformal, conformally quasi-isometric to g_1 and has curvature K), and with *finite total curvature* also solves Problem 1 (i.e. g represents $\boldsymbol{\beta}$).

3 The Schwarz lemma and some consequences

The Schwarz lemma is an estimation on a metric derived from an estimation of its curvature.

3.1. Theorem (The Schwarz lemma). *Let g and h be conformal metrics representing a divisor $\boldsymbol{\beta}$ on a Riemann surface S. Assume*

(i) *g is complete at infinity and K_g is bounded below;*
(ii) *we have $K_h \leqq -a < 0$ on the complement of some compact set in S;*
(iii) *$K_h \leqq \min\{0, K_g\}$, and $K_h \not\equiv 0$;*
(iv) *$h = e^{2u}g$ for some continuous function $u : S \to \mathbf{R}$.*

Then $h \leqq g$.

Observe that there is no assumption of completeness on h and no hypothesis on the sign of K_g; the hypothesis (iv) is satisfied e.g. when g and h represent $\boldsymbol{\beta}$ normally (see 2.4).

The proof is obtained by applying the generalized maximum principle of Yau to the equation (4.2) relating K_g and K_h, see [T3, Y, Ah1 and Ah2]. (Our assumptions imply that u is continuous and C^2 outside a compact set, this is good enough to carry on the proof in [T3].)

3.2. Corollary (Liouville's theorem, compare [Os]). *There is no conformal metric on $\mathbf{C}$ or $\mathbf{C}^*$ with curvature K satisfying*

$$K \leqq 0 \quad \textit{and} \quad K \leqq -a < 0 \quad \textit{outside a compact subset.}$$

Proof. Suppose that such a metric h exists. Denote by g the canonical (flat and complete) metric on $\mathbf{C}$ (or $\mathbf{C}^*$). The Schwarz lemma tells us that

$$h \leqq c \cdot g$$

for all $c > 0$, which is absurd. □

This result has been generalized by Sattinger [Sa] and Oleinik [Ol]. They prove:

3.3. Theorem. *There is no conformal metric on $\mathbf{C}$ with curvature K satisfying $K(z) \leqq 0$ on $\mathbf{C}$ and $K(z) \leqq -c|z|^{-2}$ on $|z| \geqq R$ (c and R being any positive constants).*

Theorem 3.3 has an obvious extension on $\mathbf{C}^*$ (namely there is no conformal metric on $\mathbf{C}^*$ with curvature K satisfying $K(z) \leqq -c(\log|z|)^{-2}$ for all $z \in \mathbf{C}^*$).

It is still an open problem to describe all negative functions K on $\mathbf{C}$ that are curvature of some conformal metric. Ni proves in [Ni1] that any function K such that

$-c^2|z|^{-2-\varepsilon} \leqq K(z) \leqq 0$ is the curvature of (many) conformal (complete) metrics on $\mathbf{C}$ (a result recovered by our Theorem 8.2).

On the other hand, Ni also gives some obstructions that are more refined than (3.2) (see (3.6) and (3.11) in [Ni1]).

The reader should be aware that these results are special to $\mathbf{C}$ and $\mathbf{C}^*$, and are false for other open Riemann surfaces (see Theorem A.1). Also, there is no Liouville theorem on $\mathbf{C}$ if K is allowed to be positive even in a small region (think of a spherical cap glued to a Beltrami pseudo-sphere).

We now draw some consequences of Liouville's theorem that we shall need later.

3.4. Proposition. *Let g be a conformal metric on a Riemann surface Ω homeomorphic to the half closed annulus $]0,1] \times \mathbf{S}^1$. Assume that the curvature K of g satisfies*

$$\sup K < 0 \quad \textit{and} \quad \int_\Omega K dA = -\infty .$$

Then Ω is hyperbolic.

Proof. Assume *ad absurdum* that (Ω, g) is parabolic. Then, by (B.2), we may suppose that $\Omega = \bar{D}\backslash\{0\} = \{z \in \mathbf{C} : 0 < |z| \leqq 1\}$ and $g = e^{2v}|dz|^2$. Let us set

$$\gamma := -\int_{|z|=1} \left(1 + \frac{\partial v}{\partial n}\right) |dz| = -\int_{|z|=1} k_g ds .$$

Case 1. $\gamma \geqq 0$. Choose any function $f \in C_0^\infty(\Omega)$ such that $f \leqq 0$ and

$$\int_D f dx\, dy = -\gamma .$$

Then, there exists a function $u \in C^2(\bar{D})$ which solves the Neumann problem:

$$\begin{cases} \Delta u = f & \text{on} \quad D; \\ \dfrac{\partial u}{\partial n} = -\left(1 + \dfrac{\partial v}{\partial n}\right) = -k_g e^v & \text{on} \quad \partial D . \end{cases} \tag{3.5}$$

And the metric $g_0 = e^{2u} g$ on Ω satisfies:

(i) $\partial\Omega = \{z : |z| = 1\}$ is geodesic for g_0;

(ii) $\sup K_0 \leqq \sup(K e^{2u}) < 0$.

Now, g_0 induces a C^2 conformal metric $\tilde{g}_0$ on $\mathbf{C}^*$ with $\sup \tilde{K}_0 < 0$ by reflection ($\tilde{g}_0 = g_0$ on Ω and $\tilde{g}_0 = \varphi^*(g_0)$ on $\{z : |z| \geqq 1\}$ with $\varphi(z) = 1/\bar{z}$). The surface $(\mathbf{C}^*, \tilde{g}_0)$ is obtained by gluing two copies of (Ω, g_0) along their geodesic boundaries.

We have thus constructed a conformal metric $\tilde{g}_0$ on $\mathbf{C}^*$ with $\sup \tilde{K}_0 < 0$, in contradiction to Theorem 3.2.

Case 2. $\gamma < 0$. From the hypothesis, we see that we can easily construct a function $f \in C_0^\infty(\Omega)$ such that

$$f \leqq \frac{1}{2}|K|e^{2v} \quad \text{and} \quad \int_D f dx dy = -\gamma .$$

We may again find a function u on $\bar{D}$ satisfying (3.5) and conclude as in the previous case. $\square$

3.5. Corollary. *Let g be a complete conformal metric on the annulus $\Omega = \{z \in \mathbf{C}: 0 < |z| \leqq 1\}$, and K be its curvature. Assume that* $\sup K < 0$.

Then, g is the restriction to Ω of a metric on the disk $\bar{D} = \Omega \cup \{0\}$ having a simple singularity of order -1 at the origine. Also, $\int_\Omega |K| dA < \infty$.

Proof. By the previous proposition, we necessarily have $\int_\Omega |K| dA < \infty$ and, since g is complete, (1.5) implies that g is the restriction to Ω of a metric on the disk D having a simple singularity of order β at the origin. By (1.7.i), the order β of this singularity does not exceed -1 (since g is complete). On the other hand, $\beta \geqq -1$, for otherwise $\int_\Omega |K| dA \geqq -(\sup K) \cdot \int_\Omega dA = \infty$. □

4 Discussion of our assumptions on K

Given a function K on a Riemann surface with divisor $(S, \boldsymbol{\beta})$, our task is to construct a conformal metric g with curvature K representing $\boldsymbol{\beta}$. If S has got hyperbolic ends, we will also require that g be complete at these ends.

4.1. To attack this problem, we first choose a *smooth* conformal metric g_0 on S and then a singular metric $g_1 = \varrho g_0$ representing $\boldsymbol{\beta}$. We will usually require from g_1 to have special features (e.g., that g_1 represents normally $\boldsymbol{\beta}$ and/or that g_1 is complete at each hyperbolic end of the surface).

Writing the desired metric as $g = e^{2u} g_1$, we have to solve the non linear PDE:

(4.2) $$\Delta_1 u = K e^{2u} - K_1 ,$$

which is equivalent to the more convenient equation

(4.3) $$\Delta_0 u = K \varrho e^{2u} - K_1 \varrho ,$$

where Δ_i is the Laplacian and K_i the curvature of the metric g_i. The latter is an elliptic equation on the *smooth* compact Riemannian manifold (S, g_0).

We will always work in a fixed conformal quasi-isometry class. Therefore, we are interested in bounded solutions u of (4.3). If the point $p \in \operatorname{supp} \boldsymbol{\beta}$ is to be a normal singularity of g, then we have to require $u \in W^{2,p}$ in some neighbourhood of p.

4.4. In order to solve Eq (4.3), we will have to assume that K behaves reasonably near $\operatorname{supp} \boldsymbol{\beta}$. Since the curvature of the desired metric g should be regular enough to define a measure, a natural hypothesis on K should be $K\varrho \in L^1_{\mathrm{loc}}(S, g_0)$. However, this would be too weak for our methods to apply, and we will usually assume that in some neighbourhood U of a point $p_i \in \operatorname{supp} \boldsymbol{\beta}$ which is a normal singularity of g_1,

(H) $$K\varrho \in L^p(U, g_0) \quad \text{for some } p > 1 .$$

Recall that if z is a coordinate in U, then $\varrho = O(|z|^{2\beta})$. Hence, if $\beta_i \leqq -1$, then (H) implies that $\liminf_{z \to p_i} |K| = 0$, whereas if $\beta_i > -1$ then any bounded function satisfies (H).

4.5. Working under hypothesis (H) is convenient, but the metrics constructed under this assumption will always represent $\boldsymbol{\beta}$ *normally*. This is sometimes too restrictive since, for instance, we might want to construct metrics having cusps (which are not normal singularities).

When dealing with negative curvature, this difficulty may be overcome by assuming the following alternative "pinching" hypothesis for the functions K and K_1 in some neighbourhood U of a point $q_i \in \operatorname{supp}\boldsymbol{\beta}$:

(P) $\qquad bK_1 \leqq K \leqq aK_1 \leqq 0$ for some positive constants $a, b > 0$.

4.6. In the case where S is a non compact surface, we will always assume (P) to hold in the complement $S \backslash N$ of some compact set $N \subset S$.

Let us finally indicate that we will use hypothesis (H) to study Problem 1 on a compact $(S,\boldsymbol{\beta})$ with $\chi(S,\boldsymbol{\beta}) \geqq 0$ (in Sects. 6 and 7), and a combination of (H) and (P) to study Problem 2 on a compact $(S,\boldsymbol{\beta})$ with $\chi(S,\boldsymbol{\beta}) < 0$ or on a non compact $(S',\boldsymbol{\beta})$ with finitely many hyperbolic ends (in Sect. 8).

5 Variational theory of the equation $\Delta u = he^{2u} - h_1$

The discussion of the preceeding section leads us to study the quasilinear elliptic equation:

(5.1) $$\Delta_0 u = he^{2u} - h_1\,,$$

on a compact smooth Riemannian surface (S, g_0) with h, $h_1 \in L^p(S, g_0)$ for some $p > 1$.

In this section, we give a short description of the variational method for solving (5.1). Further details can be found in [KW1] and [T2].

Before constructing a variational scheme, we recall a basic "non-linear" property of the Sobolev space $H = W^{1,2}(S, g_0)$:

5.2. Trudinger's nequality. *Let* $p, q > 1$ *with* $\frac{1}{p} + \frac{1}{q} = 1$, $0 < b < 4\pi$ *and* $\psi \in L^r(S, g_0)$ *with* $\int_S \psi dA_0 = 1$ *for* $r > 1$.

Let I be the functional defined on H by

$$I(u) = \frac{q}{p} \int_S \|\nabla_0 u\|^2 dA_0 + 2 \int_S \psi u \, dA_0\,.$$

Then, there exists a constant $c > 0$ such that for all $\varphi \in L^p(S, g_0)$ the following inequality holds

$$\int_S e^{2u} \varphi \, dA_0 \leqq c \|\varphi\|_{L^p} \exp I(u)\,.$$

A reference is [T2, Sect. 4.8, Theorem 8]. Trudinger proved this inequality for small values of $b > 0$. The sharp form we give here is due to Moser and Cherrier.

5.3. Corollary. *The embedding*

$$\begin{aligned} H &\to L^q(S, g_0) \\ u &\to e^{2u} \end{aligned}$$

is compact for all $1 \leqq q < \infty$.

A proof can be found e.g. in [T2, Sect. 4.10, Proposition 11].

For $u \in H$, we now define

$$\mathscr{F}(u) := \int_S \|\nabla_0 u\|^2 \, dA_0 + 2 \int_S h_1 u \, dA_0 \quad \text{and} \quad \mathscr{G}(u) := \int_S e^{2u} h \, dA_0 .$$

Observe that $\mathscr{F}$ and $\mathscr{G}$ define functionals of class C^1 on H whenever h and h_1 belong to $L^p(S, g_0)$ for some $p > 1$. To solve (5.1), we try and minimize the functional $\mathscr{F}$ on the hypersurface of H defined by the "constraint"

$$\mathscr{G}(u) = \gamma := \int_S h_1 \, dA_0 .$$

When $\gamma = 0$ set $\tilde{H} = \left\{ u \in H : \int_S u \, dA_0 = 0 \right\}$, and $\tilde{H} = H$ otherwise.

5.4. Theorem [T2, Sect. 5.3, Theorem 1]. *Let (S, g_0) be a smooth compact Riemannian surface. Let h, $h_1 \in L^p(S, g_0)$ (for some $p > 1$) and assume there exists a number $m \in \mathbf{R}$ such that*

(i) *the set $B := \{u \in \tilde{H} : \mathscr{G}(u) = \gamma$ and $\mathscr{F}(u) \leqq m\}$ is not empty;*

(ii) *$\mathscr{F}$ is bounded below on B;*

(iii) *B is a bounded subset of $\tilde{H}$.*

Then, there exists $\lambda \in \mathbf{R}$ and $u_* \in \tilde{H}$ such that $\mathscr{G}(u_*) = \gamma$, and u_* satisfies

$$\Delta u_* = \lambda h e^{2u_*} - h_0 . \tag{5.5$_\lambda$}$$

Moreover, u_* belongs to $W^{2,p}(S, g_0)$ and is of class $C^{2,\delta}$ on any domain on which h and h_1 are C^δ $(0 < \delta < 1)$.

Remark . When $\gamma \neq 0$, integrating (5.5$_\lambda$) shows that $\lambda = 1$.

Proof (sketch, see [T2] *for details).* Corollary 5.3 implies that $\mathscr{G}$ is continuous for the weak topology on H. On the other hand, $\mathscr{F}$ is lower semi-continuous (in the weak topology). This implies that $\mathscr{F}$ achieves its minimum μ on B. Thus, there exists $u_* \in B$ such that $\mathscr{F}(u_*) = \mu$ and u_* satisfies the Euler-Lagrange equation of our variational problem which is but (5.5$_\lambda$). Standard elliptic regularity theory applies and proves the last assertions. $\square$

Applying the above variational theory, we obtain the following "three-headed" existence theorem for (5.1). We first consider the case $\gamma > 0$:

5.6$^+$. Theorem. *Assume h, $h_1 \in L^p(S, g_0)$ for some $p > 1$ and $\gamma = \int_S h_1 \, dA_0 > 0$. If $0 < q\gamma < 4\pi$ (where $1/p + 1/q = 1$) and h is positive on some domain of S, then* (5.1) *has a solution $u \in W^{2,p}(S, g_0)$. Moreover, u is $C^{2,\delta}$ wherever h and h_1 are C^δ.*

Proof. We can clearly find $v \in C^1(S)$ with $\mathscr{G}(v) = \gamma$. Set $\tilde{H} = H$ and $m = \mathscr{F}(v)$, then

(i) the set $B := \{u \in \tilde{H} : \mathscr{G}(u) = \gamma$ and $\mathscr{F}(u) \leqq m\}$ is not empty. Choose now $b \in \mathbf{R}$ such that $q\gamma < b < 4\pi$ and define the functional I on $\tilde{H}$ by

$$I(u) = \frac{q}{b} \int_S \|\nabla_0 u\|^2 \, dA_0 + \frac{2}{\gamma} \int_S h_1 u \, dA_0 .$$

By (5.2), we have for all $u \in B$

$$\gamma = \int_S e^{2u} h \, dA_0 \leqq c \|h\|_{L^p} \exp I(u) ,$$

and I is thus bounded below on B. The main fact is that I is dominated by $\mathscr{F}$:

$$\frac{1}{\gamma} \mathscr{F}(u) = I(u) + \left(\frac{1}{\gamma} - \frac{q}{b}\right) \int_S \|\nabla_0 u\|^2 dA_0 \geqq I(u)$$

(since $1/\gamma > q/b$). Hence

(ii) $\mathscr{F}$ is bounded below on B.

Now, using Poincaré inequality as in [T2, Sect. 5.7], we can show that

(iii) B is a bounded subset of H,

and apply (5.5) to conclude. □

We now turn to the case $\gamma = 0$. If $h \equiv 0$, the equation reduces to $\Delta_0 u + h_1 = 0$ and has a solution as soon as $\int h_1 \, dA_0 = 0$. If $h \not\equiv 0$, we will only need to solve (5.1) with $h_1 \equiv 0$ (see [T2, Sect. 5.5] for the general case). In this situation, necessary and sufficient conditions were given by Kazdan and Warner (see [KW1]).

5.6⁰. Theorem. *Assume $h \in L^p(S, g_0)$ for some $p > 1$ and $h \not\equiv 0$. Then the equation $\Delta_0 u = he^{2u}$ has a solution if and only if:*

(i) *h is positive on some domain;*

(ii) *$\int_S h dA_0 < 0$.*

Such a solution belongs to $W^{2,p}(S, g_0)$ and is $C^{2,\delta}$ wherever $h \in C^\delta$.

Proof. The necessity of the first condition is obvious. As for the second, we have for any solution:

$$\int_S h \, dA_0 = \int_S e^{-2u} he^{2u} \, dA_0 = \int_S e^{-2u} \Delta_0 u \, dA_0 = -2 \int e^{-2u} \|\nabla u\|^2 < 0 .$$

Set $\tilde{H} = \left\{u \in H : \int_S u \, dA_0 = 0\right\}$; since h changes sign, we may find $v \in C^1$ with $\mathscr{G}(v) = 0$, hence $B := \{u \in \tilde{H} : \mathscr{G}(u) = 0 \text{ and } \mathscr{F}(u) \leqq m\}$ is not empty for $m = \mathscr{F}(v)$. We may show, using Poincaré inequality, that B is a bounded subset of H on which $\mathscr{F}$ is bounded.

By (5.1), there exists a function u_* such that $\Delta_0 u_* = \lambda h e^{2u_*}$. An integration by parts shows that condition (ii) above implies $\lambda > 0$. We check that $u := u_* + \frac{1}{2} \log \lambda$ solves our equation. □

When $\gamma < 0$, the variational approach is fruitful only if h is everywhere negative. (Otherwise, one has to use the method of upper and lower solutions (see [KW1, Sect. 10])).

5.6⁻. Theorem. *Assume h, $h_1 \in L^p(S, g_0)$ for some $p > 1$ and $\gamma = \int_S h_1 \, dA_0 < 0$. If $h \leqq -\varepsilon < 0$ on S, then* (5.1) *has a unique solution $u \in W^{2,p}(S, g_0)$. This solution is $C^{2,\delta}$ wherever h and $h_1 \in C^\delta$.*

The proof is easy, see [Be] or [T2, Sect. 5.6].

6 The case $\chi(S,\boldsymbol{\beta}) > 0$

We have the following existence result.

6.1. Theorem. *Let $(S,\boldsymbol{\beta})$ be a compact Riemann surface equipped with a divisor $\boldsymbol{\beta} = \sum_{i=1}^{n} \beta_i p_i$, such that $\chi(S,\boldsymbol{\beta}) > 0$.*

Let $K : S \to \mathbf{R}$ be locally Hölder continuous on $S\setminus\operatorname{supp}\boldsymbol{\beta}$, and assume the following:

(i) *K is positive somewhere;*
(ii) (H) *$K\varrho \in L^p(S, g_0)$ for some $p > 1$;*
(iii) $0 < q\,\chi(S,\boldsymbol{\beta}) < 2$, $\left(\textit{where } \frac{1}{p} + \frac{1}{q} = 1\right)$.

Then, there exists a conformal metric g on S, representing $\boldsymbol{\beta}$ normally, and with curvature K.

Proof. Let g_0 be a smooth conformal metric on S, and $g_1 = \varrho g_0$ be a conformal metric representing normally $\boldsymbol{\beta}$.

From (5.6$^+$), the equation $\Delta_0 u = K\varrho\, e^{2u} - K_1\varrho$ admits a solution $u \in W^{2,p}$ since (see 2.8):

$$\frac{4\pi}{q} > \gamma := \int_S K_1\varrho\, dA_0 = \int_S K_1 dA_1 = 2\pi\chi(S,\boldsymbol{\beta}) > 0\,.$$

The metric $g = e^{2u} g_1$ is the desired one. Since K is locally Hölder continuous, g is $C^{2,\delta}_{\mathrm{loc}}$ on $S\setminus\operatorname{supp}\boldsymbol{\beta}$. □

When looking for complete metrics, this forces $\beta_i \leqq -1$, for $\chi(S,\boldsymbol{\beta}) > 0$. Hence, $(S,\boldsymbol{\beta})$ is a sphere with a unique singularity. In this case we have the following.

6.2. Corollary. *Let $\boldsymbol{\beta} = \beta p$ be a divisor on $\mathbf{S}^2$ with $-2 < \beta < 0$. Let $K : S\setminus\{p\} \to \mathbf{R}$ be locally Hölder continuous, positive somewhere, and such that in the neighbourhood of p*

$$K(z) = O(|z-p|^{\ell}) \quad \textit{with} \quad \ell > -\beta\,.$$

Then, K is the curvature of a conformal metric on $\mathbf{S}^2$ representing $\boldsymbol{\beta}$ normally.

This result has been previously obtained by MacOwen [McO2, Theorem 1]. Compare to [Av].

Proof. We have $K\varrho = O(|z-p|^{2\beta+\ell})$ in a neighbourhood of p. Now, since $\ell > -\beta$, we can find $p > 1$ such that

$$\chi(\mathbf{S}^2,\boldsymbol{\beta}) = 2+\beta < 2/q = 2(1-1/p) < 2+2\beta+\ell\,.$$

Thus $K\varrho \in L^p$ (for $(2\beta+\ell) > -2/p$) and $0 < q\chi(\mathbf{S}^2,\boldsymbol{\beta}) < 2$. □

When dealing with conical singularities, Theorem 6.1 recovers a previous result of Troyanov [T2, Sect. 5.6, Theorem 5].

6.3. Corollary. *Let $(S,\boldsymbol{\beta})$ be a compact Riemann surface equipped with a divisor $\boldsymbol{\beta} = \sum_{i=1}^{n} \beta_i p_i$, with $\beta_i > -1$. Assume $0 < \chi(S,\boldsymbol{\beta}) < \tau(S,\boldsymbol{\beta}) := \inf\{2+2\beta_i, 2\}$.*

Let $K : S \to \mathbf{R}$ be bounded and locally Hölder continuous outside $\operatorname{supp}\boldsymbol{\beta}$ and positive on a domain. Then, there exists a conformal metric g on S representing $\boldsymbol{\beta}$ normally, and with curvature K.

Proof. Indeed, if all the β_i's are nonnegative, $K\varrho$ belongs to L^∞ (with $\varrho = O(|z-p|^{2\beta})$ again), and our condition reads

$$\chi(S,\boldsymbol{\beta}) < 2 = \tau(S,\boldsymbol{\beta}).$$

If not, $K\varrho$ belongs to $L^p(p>1)$ if and only if

$$-1/p < \inf \beta_j =: \underline{\beta},$$

and our condition reads again:

$$\chi(S,\boldsymbol{\beta}) < 2(1+\underline{\beta}) = \tau(S,\boldsymbol{\beta}). \quad \square$$

6.4. Remark. When $\tau(S,\boldsymbol{\beta}) < 2$, Theorem 6.1 shows that our problem has a solution as long as $0 < \chi(S,\boldsymbol{\beta}) < 2$, provided p is large enough. Compare to [M].

7 The case $\chi(S,\boldsymbol{\beta}) = 0$

We first study the case where the prescribed curvature function is identically zero (see also [T1]). It is easy to see that any *flat* metric representing a divisor actually represents it *normally.*

7.1. Theorem. *Let $(S,\boldsymbol{\beta})$ be a compact Riemann surface equipped with a divisor $\boldsymbol{\beta} = \sum_{i=1}^{n} \beta_i p_i$, such that $\chi(S,\boldsymbol{\beta}) = 0$.*

Then, there exists a flat conformal metric g on S, representing $\boldsymbol{\beta}$ normally. Such a metric is unique up to homothety.

Proof. Let g_0 be a smooth conformal metric on S, and $g_1 = \varrho g_0$ be a conformal metric representing $\boldsymbol{\beta}$ normally.

We are looking for a solution to the equation $\Delta_0 u = -K_1\varrho$. We know that $K_1\varrho \in L^p(S,g_0)$ for some $p>1$, and by (2.8) that

$$\int_S K_1\varrho\, dA_0 = \int_S K_1\, dA_1 = 2\pi\chi(S,\boldsymbol{\beta}) = 0\,.$$

Hence, our equation admits a solution $u \in W^{2,p}(S,g_0)$. The metric $g = e^{2u}g_1$ has got locally Hölder continuous curvature, hence is C^2 on $S\backslash \operatorname{supp}\boldsymbol{\beta}$.

Now, the difference of two solutions is a bounded harmonic function on $S\backslash \operatorname{supp}\boldsymbol{\beta}$, hence a constant. Uniqueness follows. $\square$

We now turn to the case of variable curvature. Necessary and sufficient conditions for existence are available.

7.2. Theorem. *Let $(S,\boldsymbol{\beta})$ be a compact Riemann surface equipped with a divisor $\boldsymbol{\beta} = \sum_{i=1}^{n} \beta_i p_i$, such that $\chi(S,\boldsymbol{\beta}) = 0$.*

Let $K : S : \to \mathbf{R}$ be locally Hölder continuous on $S\backslash \operatorname{supp}\boldsymbol{\beta}$.

We assume that K is not identically zero, and that

(H) $$K\varrho \in L^p(S,g_0) \quad \textit{for some } p>1.$$

Then, there exists a conformal metric g, representing normally $\boldsymbol{\beta}$ and of curvature K if and only if:

(i) *K is positive somewhere;*

(ii) $\int_S K\,dA_1 = \int_S K\varrho\,dA_0 < 0$, *where* dA_1 *is the area element of a flat conformal metric* g_1 *representing* $\boldsymbol{\beta}$.

Proof. Let g_0 be a smooth conformal metric on S, and $g_1 = \varrho g_0$ be a flat conformal metric representing $\boldsymbol{\beta}$ normally.

We must solve the equation $\Delta u = K\varrho\, e^{2u}$. The result is then a straightforward consequence of (5.6^0), since (2.8) yields

$$\int_S K_1\varrho\,dA_0 = \int_S K_1\,dA_1 = \chi(S,\boldsymbol{\beta}) = 0\,. \quad \Box$$

7.3. Remarks. (i) This result has been previously obtained by Kazdan-Warner in the smooth compact case (cf. [KW1, Theorem 6.1]), by Troyanov for compact surfaces with conical singularities (that is with $\beta_i > -1$), and by MacOwen in the case of the plane – that is on $\mathbf{S}^2$ with a unique singularity of order $\beta = -2$ (see [T2, Sect. 3,Theorem 3] and [McO2, Theorem 2]).

(ii) When looking for complete metrics, this forces $\beta_i \leqq -1$, hence the non compact surface $S' = S\backslash\operatorname{supp}\boldsymbol{\beta}$ we are dealing with in this case are homeomorphic either to the plane or to the cylinder.

8 Negative surfaces, and surfaces with hyperbolic ends

For non positive curvature, we will construct our metrics using the method of upper and lower solutions, as in [KW1].

Geometrically speaking, it relies on the following fact: let g_1 and g_2 be two conformal metrics on a (maybe non compact) Riemann surface S, and such that:

$$0 < \varepsilon \leqq \frac{g_1}{g_2} \leqq 1\,, \quad \text{and} \quad K_1 \leqq K_2$$

(where K_i is the curvature of g_i). Then, for any function K on S which satisfies $K_1 \leqq K \leqq K_2$, there exists a conformal metric g, conformally quasi-isometric to g_1 and g_2, and with curvature K.

For example, any function K on the unit disk satisfying $-b \leqq K \leqq -a < 0$ is the curvature of a conformal metric g which is conformally quasi-isometric to the Poincaré metric (in particular, g is complete).

The statement of our first theorem is quite technical, the reader should compare it with Theorems B and C in the introduction.

8.1. Theorem. *Let* $(S,\boldsymbol{\beta})$ *be a Riemann surface of finite type equipped with a divisor*

$$\boldsymbol{\beta} = \sum_{i=1}^{n} \beta_i p_i + \sum_{j=1}^{m} \beta'_j q_j\,.$$

Assume either that S is compact and $\chi(S,\boldsymbol{\beta}) < 0$, *or that S is non compact and with all its ends hyperbolic.*

Let g_0 *be a smooth conformal metric on* S, $g_1 = \varrho g_0$ *be a conformal metric representing* $\boldsymbol{\beta}$, *having normal singularities at each of the* p_i'*s, and assume that* g_1 *is complete at each (hyperbolic) end of* S, *and is* $C^{2,\alpha}_{\mathrm{loc}}$ *on* $S\backslash\operatorname{supp}\boldsymbol{\beta}$. *We denote by* K_1 *the curvature of* g_1.

Let $K : S \to \mathbf{R}$ *be locally Hölder continuous on* $S\backslash\operatorname{supp}\boldsymbol{\beta}$ *and assume the following:*

(i) *K is non positive on S, and strictly negative on a domain of S;*
(ii) *in a neighbourhood of each p_i, we suppose that*

$$\text{(H)} \qquad K\varrho \in L^p(S, g_0) \quad \textit{for some } p > 1;$$

(iii) *in a neighbourhood of each q_j and of the hyperbolic ends, we suppose that:*

$$\text{(P)} \qquad bK \leqq K_1 \leqq aK \leqq 0 \quad \textit{holds, for some positive constants } a \textit{ and } b.$$

Then, there exists a conformal metric g on S, conformally quasi-isometric to g_1, and of curvature K. Moreover, g represents $\boldsymbol{\beta}$.

Note that, at a point where assumption (P) is used, we can construct metrics having cusps.

Remarks. Partial results have been previously obtained by several people. In the smooth compact case, this theorem is due to Berger (when $K < 0$) and to Kazdan and Warner (when $K \leqq 0$). Later on, Ni and MacOwen, have studied the case of the plane (that is $\mathbf{S}^2$ with one parabolic end). Aviles and MacOwen, and Bland and Kalka treated the case of the disk. Finally, Troyanov worked out the case of a compact surface with conical singularities and negative Euler characteristic. See [AMcO, Be, BK, KW1, McO1, McO3, Ni1, and T2].

Proof of Theorem 8.1. Case 1. S is compact. To prove existence, we solve the equation

$$\text{(E)} \qquad \Delta_0 u = K\varrho\, e^{2u} - K_1\varrho\,,$$

by showing it admits upper and lower solutions. The first step is to consider the modified equation

$$\text{(E}'\text{)} \qquad \Delta_0 u = f e^{2u} - f_1\,,$$

obtained from (E) by cutting $K\varrho$ and $K_1\varrho$ as follows.

By assumption, there exists a compact set $N \subset S\backslash\{q_1, \dots, q_m\}$ such that:
(i) $\int_N K_1\varrho\, dA_0 < 0$ (Gauss-Bonnet formula 2.8);
(ii) K is negative on a domain of N;
(iii) $bK \leqq K_1 \leqq aK \leqq 0$ on $S\backslash N$.

Define then f and f_1 by

$$\begin{aligned} &\text{on} \quad N: f = K\varrho\,, \qquad f_1 = K_1\varrho \\ &\text{on} \quad S\backslash N: f = f_1 = 0 \end{aligned}$$

where we gain f_1, $f \in L^p(S, g_0)$ for some $p > 1$. However, since the supremum of f may be zero, (5.6^-) does not apply directly to (E$'$). We proceed as follows.

Lower solutions: Since $f - 1 \leqq -1 < 0$, Theorem 5.6^- applies now to produce a solution $\phi \in W^{2,p}(S)$ for the equation

$$\Delta_0\phi = (f - 1)e^{2\phi} - f_1\,.$$

We claim that, for $\alpha > 0$ large enough, $u_- = \phi - \alpha$ is a lower solution for (E). Indeed we have on N

$$\Delta_0 u_- = \Delta_0\phi = (f - 1)e^{2\phi} - f_1 \leqq f e^{2\alpha} e^{2u_-} - f_1\,,$$

hence, since $f \leqq 0$ and $\alpha < 0$,

$$\Delta_0 u_- \leqq f e^{2u_-} - f_1 \leqq K\varrho\, e^{2u_-} - K_1\varrho\,.$$

While on $S\backslash N$

$$\Delta_0 u_- = 0 \leqq K\varrho e^{2\phi}\left(e^{-2\alpha} - \frac{K_1}{K}e^{-2\phi}\right)$$

holds as soon as $\alpha \geqq \|\phi\|_\infty - \frac{1}{2}\log a$, that is

$$\Delta_0 u_- \leqq K\varrho e^{2u_-} - K_1\varrho .$$

Upper solutions: Fix $\mu > 0$ large enough such that

$$\lambda := \frac{\int_S (\mu f - f_1) dA_0}{\int_S dA_0} < 0 .$$

Then, there exists a solution $\psi \in W^{2,p}(S)$ of the equation

$$\Delta_0 \psi = \mu f - f_1 - \lambda$$

(for $\mu f - f_1 - \lambda \in L^p(S, g_0)$ for some $p > 1$).

Now we claim that for $\gamma > 0$ large enough, $u_+ = \psi + \gamma$ provides an upper solution for (E).

Namely on N

$$\begin{aligned} \Delta_0 u_+ &= \Delta_0 \psi = \mu f - f_1 - \lambda \geqq \mu f - f_1 \\ &\geqq \mu e^{-2u_+}(K\varrho e^{2u_+}) - K_1\varrho \\ &\geqq K\varrho e^{2u_+} - K_1\varrho \end{aligned}$$

when $\gamma \geqq \|\psi\|_\infty + \frac{1}{2}\log\mu$.

And on $S\backslash N$

$$\Delta_0 u_+ = -\lambda \geqq 0 \geqq K\varrho e^{2u_+} - K_1\varrho$$

as soon as $\gamma \geqq \|\psi\|_\infty + \frac{1}{2}\log b$.

Finally, since ϕ and ψ are bounded, we conclude that for α and γ large enough:

(i) $u_- = \phi - \alpha$ is a lower solution for (E);
(ii) $u_+ = \psi + \gamma$ is an upper solution for (E);
(iii) $u_\pm \in W^{1,2} \cap C^0(S)$;
(iv) $1 + u_- \leqq u_+$.

By (iv), there exists a smooth function $w \in C^\infty(S)$ such that $u_- \leqq w \leqq u_+$. Theorem C.4 produces a bounded classical solution $u \in C^2(S\backslash \operatorname{supp}\boldsymbol{\beta})$ for our equation. The metric $g = e^{2u}g_1$ is conformal, conformally quasi-isometric to g_1 and of curvature K.

We now turn to the *divisor*.

Since g is conformally quasi-isometric to g_1, our assumption on K ensures that g has finite total curvature in a neighbourhood of $\operatorname{supp}\boldsymbol{\beta}$. Now, (2.3) implies that g represents $\boldsymbol{\beta}$. This achieves the proof in the compact case.

Case 2. S is non compact. By (B.3), there exists a compact Riemann surface $\hat{S}$ such that $S \subset \hat{S}$, and $\hat{S}\backslash S$ is a finite union of disks.

We may assume that g_0 extends to a smooth metric on $\hat{S}$ (still denoted by g_0). To prove existence, we must find a bounded solution u for the equation

$$\Delta_0 u = K\varrho e^{2u} - K_1\varrho \quad \text{on} \quad S$$

$$\Delta_0 u = 0 \quad \text{on} \quad \hat{S}\backslash S .$$

Since K_1 is non positive in a neighbourhood of the hyperbolic ends, we infer from (1.4) that

$$\int_S K_1 \, dA_1 = -\infty .$$

Hence, there exists a compact subset N of $\hat{S}$, with $N \subset S \backslash \{q_1, \ldots , q_m\}$, such that:

(i) $\int_N K_1 \varrho \, dA_0 < 0$;

(ii) K is negative on a domain of N;

(iii) $bK \leqq K_1 \leqq aK \leqq 0$ on $S \backslash N$.

Now, we define f and f_1 by

$$\begin{aligned} &\text{on} \quad N : f = K\varrho , \qquad f_1 = K_1 \varrho \\ &\text{on} \quad \hat{S} \backslash N : f = f_1 = 0 \end{aligned}$$

where f_1, $f \in L^p(\hat{S}, g_0)$ for some $p > 1$, then proceed as in Case 1, and exhibit a lower solution u_- and an upper solution u_+ for (E), with $1 + u_- \leqq u_+$, and $u_\pm$ bounded (the same construction holds). Applying (C.4) again yields a bounded solution u for our equation. The desired metric is given on S by $g = e^{2u} g_1$.

That g represents $\boldsymbol{\beta}$ can be proved as in Case 1. □

When we assume the function K to be pinched between two negative constants at infinity, we have uniqueness of the metric with curvature K:

8.2. Theorem. *Let S be a connected Riemann surface of finite topological type equipped with a divisor $\boldsymbol{\beta} = \sum_{i=1}^{n} \beta_i p_i$. Assume that either $\chi(S, \boldsymbol{\beta}) < 0$ or that S has a hyperbolic end.*

Let $K : S \to \mathbf{R}$ be locally Hölder continuous on $S \backslash \operatorname{supp} \boldsymbol{\beta}$ and assume the following:

(i) K is non positive on S, and strictly negative on a domain of S;

(ii) in a neighbourhood of each p_i, we suppose that

(H) $$K|z|^{2\beta_i} \in L^p \quad \text{for some } p > 1;$$

(iii) in the complement of a compact subset $N \subset S$, we suppose that:

(P) $$-b \leqq K \leqq -a < 0 \quad \text{holds, for some positive constants } a \text{ and } b.$$

Then, there exists a unique conformal metric g on S, complete at infinity, representing $\boldsymbol{\beta}$ normally and having curvature K.

Moreover, each end of (S, g) admits a neighbourhood conformally quasi-isometric to the end of the Beltrami pseudo-sphere when parabolic, and to the end of the Poincaré disk when hyperbolic.

Finally, if S has no hyperbolic ends, then we have:

$$\frac{1}{2\pi} \int_S K \, dA = \chi(S, \boldsymbol{\beta}) .$$

Proof. The uniqueness follows from the Schwarz lemma (3.1).

To prove the existence, add an ideal point at each parabolic end of S to obtain a Riemann surface $\hat{S}$. Let $q_1, q_2, \ldots , q_n$ be the ideal points of $\hat{S}$ and set $\hat{\boldsymbol{\beta}} := \boldsymbol{\beta} + (-1)q_1 + (-1)q_2 + \cdots + (-1)q_n$.

Now choose a conformal metric g_1 on $\hat{S}$ such that

(i) g_1 has a normal singularity of order β_i at p_i;

(ii) g_1 coincides with the Beltrami metric (see 2.2) in a neighbourhood of each q_j;

(iii) g_1 coincides with the standard Poincaré metric of the disk (at infinity) in a neighbourhood of each hyperbolic end of $\hat{S}$.

In particular, g_1 is complete and represents the divisor $\hat{\boldsymbol{\beta}}$.

By Theorem 8.1, there exists a conformal metric $\hat{g}$ on $\hat{S}$, conformally quasi-isometric to g_1, representing $\hat{\boldsymbol{\beta}}$ and of curvature K. The restriction g of $\hat{g}$ to S is the desired metric.

To prove the last statement, observe that $\chi(S, \boldsymbol{\beta}) = \chi(\hat{S}, \hat{\boldsymbol{\beta}})$ and apply the Gauss-Bonnet formula (2.8). □

8.3. Other results on uniqueness. In the general case, (i.e. when some singularities are not normal), the Schwarz lemma is not available and we can only prove uniqueness in a given conformal quasi-isometry class. We still have to assume $-b \leqq K \leqq -a$ at infinity.

8.4. Proposition. *Let $(S, \boldsymbol{\beta})$ be a Riemann surface of finite type with divisor. Let g and $g' = e^{2v}g$ be two conformally quasi-isometric conformal metrics, complete at infinity, on S. Suppose that g and g' represent $\boldsymbol{\beta}$ and have the same curvature $K \leqq 0$, $K \not\equiv 0$. Assume moreover that*

$$-b \leqq K \leqq -a < 0 \quad \textit{outside a compact set } N \subset S .$$

Then $g = g'$.

For the proof, it will be convenient to introduce a smooth conformal metric g_0 on S (say whose curvature $K_0 \equiv -1$ outside N).

We will need the following result:

8.5. Lemma. *We have $v \in W^{1,2}_{\text{loc}}(S, g_0)$.*

Proof. We may work in a neighbourhood U of a singular point, identified w.l.o.g. with $D = \{z \in \mathbf{C} : |z| \leqq 1\}$. Since $\Delta v \in L^1$ in a weak sense, elliptic regularity shows that the first weak derivatives of v on the disk D are functions; it remains to prove that these functions are square integrable.

Let $\Omega = \{z \in \mathbf{C} : 0 < |z| \leqq 1\}$. We know that $v \in C^2(\Omega)$, and that v is uniformly bounded, and $\Delta v \in L^1$ in a weak sense. The result will easily follow from Green's theorem applied to a sequence of smooth approximations of v. □

We now prove Proposition 8.4:

Let us choose a smooth conformal metric g_0 on S such that g_0 is isometric to the Poincaré metric in a neighbourhood of the (hyperbolic) ends of S. A localization of the Schwarz lemma shows that $g = e^{2u}g_0$ and $g' = e^{2u'}g_0$ are conformally quasi-isometric to g_0 at infinity (i.e. u and u' are bounded on $S \backslash N$).

Moreover, we know from (8.5) that $v = u' - u \in W^{1,2}_{\text{loc}}(S, g_0)$. Since v is smooth off a finite set, v is bounded and $\nabla v \in L^2$, we infer that $v^2 \in W^{1,2}_{\text{loc}}(S, g_0)$. Now,

$$\begin{aligned} \frac{1}{2}\, \Delta_0(v^2) &= v\Delta_0 v - |\nabla^0 v|^2 \\ &= K(u' - u)(e^{2u'} - e^{2u}) - |\nabla^0 v|^2 \\ &\leqq 0 \end{aligned}$$

so that v^2 is subharmonic.

In particular, v^2 is upper semicontinuous and satisfies the maximum principle.

Case 1. The maximum of v^2 is achieved at a point $x \in S$. By the maximum principle, v^2 is a constant. Since g and $g' = e^{2v}$ have the same curvature $K \not\equiv 0$, we see that in fact $v \equiv 0$.

Case 2. The maximum of v^2 is not achieved. Then there exists a divergent sequence of points $x_n \in S$ with

$$\lim_{n\to\infty} v^2(x_n) = \sup(v^2).$$

We want to show that $\sup(v^2) = 0$ (hence $v \equiv 0$). Since v^2 is uniformely bounded and of class C^2 outside a compact set, and since (S, g_0) has bounded curvature, we can use the generalized maximum principle (see e.g. [Au, 8.4]). It asserts the existence of a divergent sequence of points $y_n \in S$ such that

$$\lim_{n\to\infty} v^2(y_n) = \sup(v^2), \quad \text{and} \quad \overline{\lim_{n\to\infty}}\, (\Delta_0(v^2))(y_n) \geqq 0.$$

Assuming $\sup(v^2) > 0$ yields

$$\frac{1}{2}(\Delta_0(v^2))(y_n) \leqq Kv(e^{2u'} - e^{2u}) \leqq \text{const} < 0$$

for n large enough since $K \leqq -a < 0$ and u, u' are bounded, a contradiction. $\square$

Appendix A: Further results on open surfaces

So far, we have been looking for conformal metrics on a Riemann surface having a specified geometry (e.g., being complete, and representing some divisor or lying in some conformal quasi-isometry class), and having a prescribed curvature.

If we drop the control on the geometry, the problem of prescribing the curvature has a straightforward answer as we now show.

A.1. Theorem. *Let S' be a non compact Riemann surface of finite type. Assume S' is neither conformally equivalent to $\mathbf{C}$ nor to $\mathbf{C}^*$.*

Let $K: S' \to \mathbf{R}$ be any bounded locally Hölder-continuous function. Then, there exists a conformal metric g on S' with curvature K.

Remarks. (i) As can be seen from the proof, this metric is far from being unique.

(ii) It follows from Theorem 3.2 that the above result is clearly false for $\mathbf{C}$ or $\mathbf{C}^*$.

(iii) When S' is of genus less than two and has a hyperbolic end, the result is due to Kazdan and Warner [KW2, Theorem 3.4].

Proof. By (B.3), there exists a compact Riemann surface S such that $S' \subset S$ and $S \backslash S'$ is a disjoint union of (finitely many) points and disks.

The strategy of the proof will be best illustrated in case S' has a hyperbolic end.

Case 1. $S \backslash S'$ contains an open set U. Choose points $p_1, p_2, \dots, p_n \in U$ and numbers $\beta_1, \beta_2, \dots, \beta_n > -1$ such that $\chi(S) + \sum_{i=1}^{n} \beta_i = 0$.

Let now g_1 be a flat metric on S representing the divisor $\boldsymbol{\beta} = \sum_{i=1}^{n} \beta_i p_i$ (known to exist by (7.1)) and choose a function $\tilde{K}: S \to \mathbf{R}$ such that

(i) $\tilde{K}$ is bounded on S and $\tilde{K}|_{S'} = K$;
(ii) $\tilde{K}$ is positive on some domain;
(iii) $\int_S \tilde{K}\, dA_1 < 0$.

Then, Theorem 7.2 provides us with a conformal metric $\tilde{g}$ on S having curvature $\tilde{K}$ (and representing $\boldsymbol{\beta}$). The desired metric on S' is the restriction $g := \tilde{g}|_{S'}$.

In the next five cases, S' is assumed to have only parabolic ends, that is there exists a compact Riemann surface S such that $S' \subset S$ and $S \backslash S' = \{p_1, p_2, \dots, p_n\}$ is a finite set. Choosing a suitable divisor $\boldsymbol{\beta}$ with support in this set we may achieve a compatibility between the signs of K and of $\chi(S, \boldsymbol{\beta})$. We will work only with conical singularities ($\beta_i > -1$) unless $K \equiv 0$, thus condition (H) will automatically hold.

Case 2. S is a sphere and $\sup K > 0$. In case S is a sphere, our assumption is that $S \backslash S' = \{p_1, p_2, \dots, p_n\}$ contains at least three points. Let us choose a real number β such that $-1 < -2/n < \beta < 0$ and set $\boldsymbol{\beta} = \sum_{i=1}^{n} \beta p_i$. Then

$$0 < \chi(S, \boldsymbol{\beta}) = 2 + n\beta < \tau(S, \boldsymbol{\beta}) = 2 + 2\beta,$$

and (6.3) gives us the desired metric.

Case 3. S is not a sphere and $\sup K > 0$. Now $\chi(S) \leqq 0$, thus we may choose $\beta \in \mathbf{R}$ such that $0 \leqq -\chi(S) < \beta < 2 - \chi(S)$. We observe that, if $\boldsymbol{\beta} = \beta p_1$, then $0 < \chi(S, \boldsymbol{\beta}) = \chi(S) + \beta < 2$ and conclude as above.

Case 4. $K \equiv 0$. Let $\beta := -\chi(S)$ and $\boldsymbol{\beta} := \beta p_1$. Proposition 7.1 applies.

Case 5. S is a sphere, $K \not\equiv 0$ and $K \leqq 0$. Again $S \backslash S' = \{p_1, \dots, p_n\}$ with $n \geqq 3$. Choose β such that $-1 < \beta < -2/n$ and set $\boldsymbol{\beta} = \sum_{i=1}^{n} \beta p_i$. Then, $\chi(S, \boldsymbol{\beta}) = 2 + n\beta < 0$ and Theorem 8.2 gives us the desired metric.

Case 6. S is not a sphere, $K \not\equiv 0$ and $K \leqq 0$. Pick some number $\beta \in]-1, 0[$ and set $\boldsymbol{\beta} = \beta p_1$. Observe that $\chi(S, \boldsymbol{\beta}) < 0$ and apply Theorem 8.2. □

Alternatively, we may drop the condition that g be in some specified conformal class and hope to control the geometry of g. The following theorem gives a characterization of curvature functions on *complete* open surfaces of finite type. This result has been proved by Kazdan and Warner for surfaces homeomorphic to the plane and then extended to the general case by Burago (see [KW2, Theorem 4.1] and [Bu]).

A.2. Theorem (Burago-Kazdan-Warner). *Let K be a smooth function on an open surface S' of finite type. Then, the following conditions are necessary and sufficient for K to be the curvature of some complete metric on S':*
(i) $\liminf_{x \to \infty} K(x) \leqq 0$ *at each end of S';*
(ii) $\begin{cases} \textit{when } \chi(S') < 0, & \textit{assume also } \inf K < 0; \\ \textit{when } \chi(S') = 0, & \textit{assume also } \inf K < 0 \textit{ or } K \equiv 0. \end{cases}$

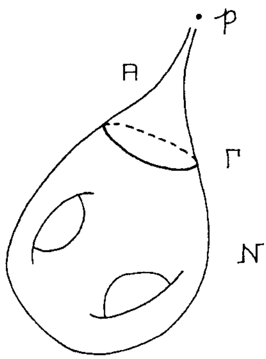
Since the proof (for $\chi(S') \leqq 0$) is only available in Russian, we give it below.

Proof. The necessity of condition (i) follows from the Bonnet-Myers theorem (cf. e.g. [GHL, 3.85]) on the existence of conjugate points on geodesics in positively curved Riemannian manifolds, and that of condition (ii) follows from the Cohn-Vossen inequality (1.3.b). Indeed, if a complete surface has $K \geqq 0$, then $0 = \int |K^-| < \infty$ and Proposition 1.3 implies $0 \leqq \int K \leqq 2\pi\chi(S')$. This is possible only if $\chi(S') > 0$ or $\chi(S') = 0$ and $K \equiv 0$.

The ingredient used to prove the sufficiency of conditions (i) and (ii) is a combination of the methods used by Kazdan and Warner to treat the compact case and the case of the plane, with a clever lemma due to Burago.

Case 1. $\chi(S') > 0$. Then, S' is diffeomorphic to the plane and we refer to [KW2, Theorem 4.1].

Case 2. $\chi(S') < 0$. Recall that S' is diffeomorphic to a compact surface S with a finite numbers of points removed: $S' = S\backslash\{p_1, p_2, \dots, p_n\}$. We may decompose S' into a compact domain with smooth boundary N and a finite number of (disjoint) annuli $A_1, A_2, \dots, A_n$ (such that the closure of A_i in S is a neighbourhood of p_i). The function K satisfies condition (ii) of the theorem, thus we may choose our decomposition in such a way that N and each A_i meet the set $\{x \in S' : K(x) < -\mu\}$ for some $\mu > 0$. We will also assume that N meets A_i on a circle $\Gamma_i := N \cap A_i = \partial N \cap \partial A_i$.



The desired metric will be constructed separately on N and the A_i's in such a way that it can be glued to give a smooth complete metric on S'.

Since $\chi(N) = \chi(S') < 0$ and $\inf_N(K) < 0$, there exists a metric g on N which has curvature K and for which ∂N is geodesic. This follows from [KW1, Theorem 11.6] applied to the double $M = N \cup \check{N}$ of N (observing that all constructions in the proof may be performed in a σ-invariant way, where σ is the natural involution on M exchanging N and $\check{N}$, giving thus a symmetric metric for which ∂N is obviously geodesic).

Let $(t, \theta)\,(-\varepsilon < t \leqq 0; \theta \in \mathbf{R}/(\ell_i \mathbf{Z}))$ be radial exponential coordinates (for the metric g) in a neighbourhood of Γ_i in N. Thus, ℓ_i is the length of Γ_i, θ is the length parameter along Γ_i, and we have in this neighbourhood

$$g = dt^2 + a_i^2(t, \theta)\, d\theta^2 \,,$$

where $a_i(t, \theta)$ is a function such that $a_i(0, \theta) = 1$ and $(\partial a_i / \partial t)(0, \theta) = 0$, since Γ_i is geodesic.

Let us now parametrize A_i by $(t, \theta) \in [0, \infty[\times \mathbf{R}/(\ell_i \mathbf{Z})$ in such a way that this parametrization is a smooth continuation of the one defined above. We shall need the following.

A.3. Fact. *There exists a diffeomorphism ψ_i of A_i which is the identity in a neighbourhood of $\Gamma_i = \partial A_i$ in A_i, and such that the initial value problem*

$$\text{(A.4)} \qquad \frac{\partial^2 b}{\partial t^2}(t, \theta) + (K \circ \psi_i(t, \theta))\, b(t, \theta) = 0 \,, \qquad b(0, \theta) = 1 \,, \qquad \frac{\partial b}{\partial t}(0, \theta) = 0 \,,$$

admits a positive solution b_i.

Assuming this fact, we continue the proof of Theorem A.2. Define a metric $\bar{g}$ on S' by setting

$$\bar{g} = \begin{cases} dt^2 + b_i^2(t,\theta) & \text{on} \quad A_i; \\ g & \text{on} \quad N. \end{cases}$$

Let us denote by ψ the diffeomorphism of S' which is the identiy on N and satisfies $\psi = \psi_i$ on A_i.

A simple check shows that $\psi_* \bar{g}$ is a complete smooth metric with curvature K on S'. This finishes the proof when $\chi(S') < 0$.

Case 3. S' is a cylinder. If $K \equiv 0$, then take the usual euclidean metric on $S' \cong \mathbf{R}^2/\mathbf{Z}$. If $K \not\equiv 0$, then decompose S' as $S' = A_1 \cup A_2$ and use Fact A.3 to construct the desired metric separatedly on A_1 and A_2.

Case 4. S' is a Möbius band. Then work equivariantly on its orientation cover $\tilde{S} \cong \mathbf{R}^2/\mathbf{Z}$.

The theorem will thus be proved for all surfaces once Fact A.3 is established. To this aim, we will need two technical lemmas. We have to understand which conditions on a function $f \in C^\infty([0,\infty[)$ insure that the solution u of

$$u''(t) + f(t)u(t) = 0 \quad \text{with} \quad u(0) = 1\,, \qquad u'(0) = 1$$

never vanishes. The first lemma deals with "short time" behaviour of the solution, and will allow us to start with "better" initial data (namely $u' > 0$) in the above differential equation.

A.5. Lemma. *Let $f : [0,T] \to \mathbf{R}$ be a smooth function. Then, for all $B > 0$, there exists an $\varepsilon > 0$ such that if*

$$\|f\|_{L^\infty} \leqq B^2 \quad \textit{and} \quad \operatorname{mes}\{t \in [0,T] : f(t) > -\mu\} \leqq \varepsilon\,,$$

where $\mu := 4T^{-2}$, then the solution u of the initial value problem

$$\text{(A.6)} \qquad u'' + fu = 0\,, \qquad u(0) = 1\,; \qquad u'(0) = 0$$

is positive for all $t \in [0,T]$ and satisfies $u'(T) \geqq 1/(2T)$.

Proof. We will prove the lemma with $\varepsilon = \frac{1}{4}\min\{1; T; B^{-2}e^{-BT}; T^{-1}B^{-2}e^{-BT}\}$. We first show that $u \geqq \frac{1}{2}$ on $[0,\varepsilon]$. By the Sturm comparison theorem, we have $u \leqq e^{BT}$, hence integrating (A.6) once gives us

$$\text{(A.7)} \qquad u'(t) \geqq -tB^2e^{BT}\,.$$

Integrating this inequality yields for $t \leqq \varepsilon$

$$u(t) \geqq 1 - \frac{t^2}{2}\,B^2e^{BT} \geqq \frac{1}{2}$$

since $\varepsilon^2 B^2 e^{BT} \leqq \frac{1}{16} < 1$.

Then, we prove that $u > \frac{1}{4}$ on $[\varepsilon, T]$. Indeed, assume that there exists $t \in [0,T]$ such that $u(t) \leqq \frac{1}{4}$ and set $t_0 = \inf\{t \in [0,T] : u(t) \leqq \frac{1}{4}\}$. In particular $\varepsilon < t_0 \leqq T$. Now, on the interval $[0,t_0]$, we have

$$u'' = -fu \geqq \begin{cases} -B^2e^{BT} & \text{everywhere;} \\ \dfrac{\mu}{4} & \text{everywhere in } [0,t_0] \text{ minus a set of measure } \leqq \varepsilon\,. \end{cases}$$

We also have $u'(0) = 0$, hence, for $s \in [\varepsilon, t_0]$

$$u'(s) \geqq \frac{\mu}{4}(s-\varepsilon) - \varepsilon B^2 e^{BT}. \tag{A.8}$$

Integrating (A.8) from ε to t_0 yields

$$u(t_0) - u(\varepsilon) \geqq \frac{\mu}{8}(t_0-\varepsilon)^2 - (t_0-\varepsilon)\varepsilon B^2 e^{BT},$$

hence, since $u(\varepsilon) \geqq \frac{1}{2}$ and $u(t_0) = \frac{1}{4}$,

$$0 < \frac{\mu}{8}(t_0-\varepsilon)^2 \leqq u(t_0) - u(\varepsilon) + (t_0-\varepsilon)\varepsilon B^2 e^{BT} \leqq \frac{1}{4} - \frac{1}{2} + \varepsilon T B^2 e^{BT}.$$

This is impossible since $\varepsilon T B^2 e^{BT} \leqq \frac{1}{4}$.

We have thus established that $u > \frac{1}{4}$ on $[0, T]$. Now, from (A.8) we have

$$u'(T) \geqq \frac{\mu}{4}(T-\varepsilon) - \varepsilon B^2 e^{BT},$$

since $\frac{1}{4}\mu(T-\varepsilon) \geqq 3/(4T)$ and $\varepsilon B^2 e^{BT} \leqq 1/(4T)$, we conclude

$$u'(T) \geqq \frac{1}{2T}. \quad \square$$

The second lemma deals with "long time" behaviour of u. It is similar to Lemma 4.3 in [KW2], and is again based on a comparison theorem for differential equations.

A.9. Lemma. *Let $f : [T, \infty[\to \mathbf{R}$ be a smooth function ($T > 0$). Let u be a solution of the equation $u'' + fu = 0$ such that*

$$u'(T) \geqq \gamma u(T) > 0, \quad \textit{for some } \gamma > 0.$$

Assume that there exists $\alpha \geqq \max\{4, 2/(\gamma T)\}$ such that for any $t \geqq T$

$$\int_t^\infty f^+(s)ds \leqq \frac{1}{\alpha t},$$

where $f^+(s) = \max\{f(s), 0\}$.

Then, u does not vanish on $[T, \infty[$.

Proof. By the comparison theorem for differential equations (see [D, XIV 7.4]), it suffices to exhibit a function F on $[T, \infty[$ which satisfies $f \leqq F$ and such that there exists a non vanishing function v on $[T, \infty[$ with

$$v'' + Fv = 0, \qquad v(T) > 0, \qquad v'(T) > 0, \qquad \frac{v'(T)}{v(T)} \leqq \frac{u'(T)}{u(T)}.$$

Now, let $z(t) = \int_t^\infty f^+(s)ds + \frac{1}{\alpha t} \leqq 2/(\alpha t)$, and set $F(t) := -z'(t) - z^2(t)$. A

$F(t) \leqq$

$$-f^+(t) + \frac{1}{\alpha^2 t^2}(4-\alpha) \leqq -f(t)$$

since $\alpha > 4$.

Let us set $v(t) = \exp \int_T^t z(s)ds$. Then, v satisfies $v'' + Fv = 0$, $v(T) = 1$, $v'(T) = z(T) \leqq 2/(\alpha T)$, and v does not vanish on $[T, \infty[$.

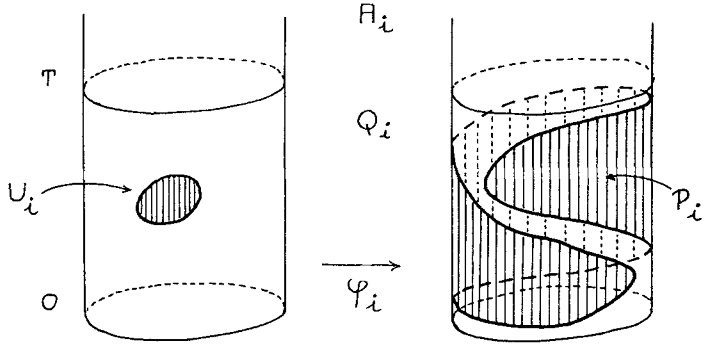
Hence, since $\gamma \geqq 2/(\alpha T)$, the comparison theorem applies, and shows that u is everywhere positive on $[T, \infty[$. □

We can now complete the proof of Theorem A.2. To prove Fact A.3, we have to construct a diffeomorphism ψ_i of A_i which is the identity in a neighbourhood of $\Gamma_i = \partial A_i$ and such that the initial value problem

$$\text{(A.4)} \qquad \frac{\partial^2 b}{\partial t^2}(t, \theta) + (K \circ \psi_i(t, \theta))\, b(t, \theta) = 0\,, \qquad b(0, \theta) = 1\,, \qquad \frac{\partial b}{\partial t}(0, \theta) = 0\,,$$

admits a *positive* solution b_i.

Recall that $K \leqq -\mu$ in a neighbourhood of some point of A_i. Thus, for every $T > 0$, there exists a non empty region $U_i \Subset]0, T[\times \mathbf{R}/\ell_i\mathbf{Z} =: Q_i \subset A_i$ on which $K \leqq -\mu$. We shall choose $T = \sqrt{4/\mu}$. Now take a simply connected domain $P_i \Subset Q_i$ with smooth boundary and such that each "generatrix" $\{\theta = \text{const}\}$ intersects P_i on a set of t-measure $\geqq T - \varepsilon$, where ε is small enough for Lemma A.5 to apply (with μ, T defined above and $B^2 = \sup |K|$).



Now, choose also a diffeomorphism φ_i of Q_i mapping U_i onto P_i and such that φ_i is the identity on a neighbourhood of ∂Q_i.

Then, by Lemma A.5, the solution b_i of

$$\frac{\partial^2 b}{\partial t^2}(t, \theta) + (K \circ \varphi_i(t, \theta))\, b(t, \theta) = 0\,, \qquad b(0, \theta) = 1\,, \qquad \frac{\partial b}{\partial t}(0, \theta) = 0$$

is positive on $[0, T]$ and satisfies $(\partial b_i/\partial t)(T, \theta) > 0$.

Let us now set $\gamma := \inf\{b_i'(\theta, T)/b_i(\theta, T) : \theta \in \mathbf{R}/\ell_i\mathbf{Z}\}$, recall that $\liminf K < 0$ and construct as in [KW2, Lemma 4.7] a diffeomorphism φ_i' of $A_i \backslash Q_i = [T, \infty[\times \mathbf{R}/\ell_i\mathbf{Z}$ which is the identity near the boundary $\{t = T\}$ and such that for any θ and any $t \geqq T$

$$\int_t^\infty K^+ \circ \varphi_i'(s, \theta)\, ds \leqq \frac{1}{\alpha t}\,,$$

for $\alpha = \max\{4, 2/(\gamma T)\}$.

The desired diffeomorphism ψ_i of A_i is given by $\psi_i = \varphi_i$ on Q_i and $\psi_i = \varphi_i'$ on $A_i \backslash Q_i$. Indeed, Lemma A.9 applies and shows that the solution b_i of (A.3) is always positive. This completes the proof of Theorem A.2. □

Appendix B: Relations with uniformization theory

First, we recall the Poincaré-Koebe uniformization theorem for open simply connected Riemann surfaces.

B.1. Theorem. *Let S be a non compact simply connected Riemann surface. Then, S is either conformally equivalent to the plane $\mathbf{C}$, or to the unit disk $\{z \in \mathbf{C} : |z| < 1\}$.*

See [Ah2] or [Fis, Sect. 2.2] for a proof.

The surface S is called *parabolic* in the first case, and *hyperbolic* in the second one. Uniformization theory may be defined as the study of the consequences of this theorem.

B.2. Corollary. *Let Ω be a Riemann surface homeomorphic to the annulus $]0,1[\times S^1$. Then, either Ω is conformally equivalent to $\mathbf{C}^*$ or there exists a well defined number $r \in [0,1[$ such that Ω is conformally equivalent to $\{z \in \mathbf{C} : r < |z| < 1\}$.*

The surface Ω is called *parabolic* if it is conformally equivalent to $\mathbf{C}^*$ or $D^* = \{z \in \mathbf{C} : 0 < |z| < 1\}$, and *hyperbolic* otherwise.

Proof (sketch). It is convenient to replace the unit disk by the (conformally equivalent) upper-half plane $\mathscr{H} := \{z \in \mathbf{C} : \mathrm{Im}(z) > 0\}$.

By (B.1), the simply connected covering of Ω is $\tilde{\Omega} \cong \mathscr{H}$ or $\tilde{\Omega} \cong \mathbf{C}$.

If $\tilde{\Omega} \cong \mathscr{H}$, then $\Omega = \mathscr{H}/\Gamma$ where Γ is a discrete infinite cyclic subgroup of $\mathrm{PSL}_2(\mathbf{R})$ (the group of conformal transformations of $\mathscr{H}$).

It is easy to prove that such a group is – up to conjugation – generated by $z \mapsto z+1$ or $z \mapsto \lambda z \ (\lambda > 1)$.

In the first case, $z \mapsto \exp(2i\pi z)$ is an isomorphism between $\mathscr{H}/\Gamma$ and $\{z \in \mathbf{C} : 0 < |z| < 1\}$. In the second case, $z \mapsto z^{2i\pi\mu} = \exp(2i\pi \log(z)/\log(\lambda))$ (principal determination) is an isomorphism between $\mathscr{H}/\Gamma$ and $\{z \in \mathbf{C} : r < |z| \leqq 1\}$ (where $\mu = 1/\log\lambda$ and $r = e^{-2\pi^2\mu}$).

Now if $\tilde{\Omega} \cong \mathbf{C}$, then $\Omega = \mathbf{C}/\Gamma$, where Γ is generated by $z \mapsto z+1$, and $z \mapsto \exp(2i\pi z)$ is an isomorphism between $\mathbf{C}/\Gamma$ and D^*.

Since an end of a surface of finite topological type can be represented by a closed subset diffeomorphic to the annulus $]0,1] \times S^1$, the above corollary has the following important consequence.

B.3. Corollary. *Let S be a Riemann surface of finite type. Then, there exists a compact Riemann surface $\hat{S}$ such that S is isomorphic to $\hat{S}$ minus a disjoint union of (finitely many) points and disks.*

An end of S is said to be *parabolic* if it corresponds to a point in $\hat{S}$ and *hyperbolic* if it corresponds to a disk in $\hat{S}$.

Uniformization theory is thus relevant to geometry through the distinction between parabolic and hyperbolic ends. But, on the other hand, the geometry pays its tribute to the uniformization theory in offering alternative proofs of some important theorems.

B.4. Theorem. *Let S be a compact Riemann surface;*

if $\chi(S) > 0$, then S is isomorphic to $\mathbf{C} \cup \{\infty\}$;
if $\chi(S) = 0$, then S is isomorphic to $\mathbf{C}/\Lambda$ (where Λ is a lattice in $\mathbf{C}$);
if $\chi(S) < 0$, then S is isomorphic to $\mathscr{H}/\Gamma$ (where Γ is a Fuchsian group).
Furthermore, Γ and Λ are unique up to conjugation.

Recall that a lattice in $\mathbf{C}$ is a subgroup generated by two complex numbers whose ratio is not real and that a Fuchsian group is a discrete subgroup of $\mathrm{PSL}_2\mathbf{R}$.

Proof. If $\chi(S) > 0$, then S is homeomorphic to a sphere and $\chi(S) = 2$. Choose a point p in S and consider the divisor $\boldsymbol{\beta} := -(2)p$. By (7.1), there exists a flat metric g on S representing $\boldsymbol{\beta}$. It is easy to see that (S, g) is isometric (hence conformally equivalent) to $\mathbf{C} \cup \{\infty\}$ endowed with its usual flat metric $|dz|^2$.

If $\chi(S) = 0$, then S admits (7.1) a smooth conformal flat metric g, thus $(\tilde{S}, g)$ is isometric to $(\mathbf{C}, |dz|^2)$, and the conclusion follows from the classification of discrete groups of isometries of the euclidean plane.

If $\chi(S) < 0$, then S admits by (8.2) a smooth conformal metric g of constant curvature -1, thus $(\tilde{S}, g)$ is isometric to $\mathscr{H}$ equipped with its Poincaré metric $|dz/\operatorname{Im}(z)|^2$.

Uniqueness for Γ (or Λ) is standard. □

For open surfaces, we also have by (8.2):

B.5. Theorem. *Let S be an open Riemann surface of finite type different from $\mathbf{C}$ and $\mathbf{C}^*$. Then, S is isomorphic to $\mathscr{H}/\Gamma$ for some Fuchsian group Γ.*

The above theorems state in particular that to each Riemann surface S with either $\chi(S) < 0$, or with an hyperbolic end, corresponds a Fuchsian group well defined up to conjugation. On the other hand, given a Fuchsian group of finite type Γ, we can consider the quotient $Q := \mathscr{H}/\Gamma$. This quotient Q is generally not a Riemann surface, but it is an *oriented 2-dimensional orbifold endowed with a conformal structure* (see [Th, Sc] for these notions). To such an orbifold, we can associate a Riemann surface (of finite type) with a divisor $(S, \boldsymbol{\beta})$ in a canonical way: let S be a Riemann surface such that the regular part Q_0 of Q is isomorphic to $S' = S \backslash \{p_1, \ldots, p_n\}$, and $\boldsymbol{\beta} := \sum_{i=1}^{n} \beta_i p_i$ where $\beta_i = -1$ if p_i corresponds to a parabolic end of Q and $\beta_i = (1/\nu_i) - 1$ if p_i is an orbifold singularity of local group $\mathbb{Z}/\nu_i\mathbb{Z}$.

It is clear that the Poincaré metric on $Q := \mathscr{H}/\Gamma$ gives rise to a metric g of curvature -1 representing $\boldsymbol{\beta}$ on S (such that each hyperbolic end is complete). It is also clear that $\chi(Q)$ (in orbifold sense) is equal to $\chi(S, \boldsymbol{\beta})$.

With these preparations, we can state the following.

B.6. Theorem. *There are canonical one-to-one correspondances between the following three classes:*

$\mathscr{F}$: the class of Fuchsian groups of finite type up to conjugation;

$\mathscr{O}$: the class of oriented 2-dimensional orbifolds of finite type with a conformal structure and such that $\chi(Q) < 0$ or with an hyberbolic end, up to isomorphism;

$\mathscr{S}$: the class of finite type Riemann surfaces with divisors $(S, \boldsymbol{\beta})$ such that $\beta_i = (1/\nu_i) - 1$ for $\nu_i \in \mathbb{N} \cup \{\infty\}$, and with $\chi(S, \boldsymbol{\beta}) < 0$ or with an hyperbolic end, up to isomorphism.

Proof. The bijection $\mathscr{O} \leftrightarrow \mathscr{S}$ has just been described. The map $\mathscr{F} \to \mathscr{O}$ is given by $\Gamma \mapsto Q = \mathscr{H}/\Gamma$ and is clearly injective.

It only remains to prove that the compound map $\mathscr{F} \to \mathscr{O} \to \mathscr{S}$ is onto, a fact which follows from our Theorem 8.2. □

Appendix C: The method of upper and lower solutions on a non compact manifold

This appendix is devoted to the so-called "method of upper and lower solutions" used in Sect. 8 for solving on a non compact Riemannian manifold (M, g) the non linear equation

$$\Delta u = he^{2u} - h_1 = f(x, u),$$

where h and h_1 are assumed to be locally Hölder continuous.

In the compact case, a proof has been given by Kazdan and Warner [KW1, Sect. 9], and in the non compact case by Ni and Noussair. We basically follow their exposition [Ni2, Theorem 2.10; Nou].

Let us first describe the method. Assuming we are already provided with upper and lower solutions u_+ and u_-, i.e., that

$$\Delta u_- \leqq f(x, u_-), \qquad \Delta u_+ \geqq f(x, u_+), \quad \text{with} \quad u_- \leqq u_+,$$

we follow an iterative scheme based on a compact exhaustion of M, and construct a solution u of the equation

$$\Delta u = f(x, u)$$

which satisfies $u_- \leqq u \leqq u_+$.

The main result is stated in (C.4). We will first set the method of upper and lower solutions for a non linear Dirichlet problem on a compact Riemannian manifold with boundary (C.2). For this purpose, we need the following existence result for a *linear* Dirichlet problem.

C.1. Lemma. *Let $(N, \partial N)$ be a compact smooth riemannian manifold with boundary, $f \in L^\infty(\bar{N})$ and $w \in C^{2,\delta}(\bar{N})$ for some $0 < \delta < 1$, and $c > 0$ be a constant. Then, the Dirichlet problem*

$$\Delta u = -cu + f(x), \qquad u|_{\partial N} = w$$

has a unique solution $u \in W^{2,2}(\bar{N})$.

If furthermore $f \in C^\delta(\bar{N})$, then $u \in C^{2,\delta}(\bar{N})$.

Proof. We better set $v = u - w$ and solve the Dirichlet problem:

$$\Delta v = -cv + f_1, \qquad v|_{\partial N} = 0,$$

with $f_1 = f - cw - \Delta w \in L^\infty(\bar{N})$.

Fix a real number b with $2cb \geqq (1 + \|f_1\|_\infty^2)$ and, for $v \in W_0^{1,2}(\bar{N})$, define

$$J(v) = \int_N \left(\frac{1}{2}\,|\nabla v|^2 + \frac{1}{2}\,cv^2 - f_1 v + b\right) dv_g\,.$$

From the choice of b, we have $|v| \leqq \frac{1}{2}\,cv^2 - f_1 v + b$, hence

$$\|\nabla v\|_2 + \|v\|_1 \leqq 2J(v),$$

and from the Poincaré inequality we get

$$\|v\|_{W^{1,2}} \leqq CJ(v),$$

where C is a constant.

Let $\alpha := \inf\{J(v) : v \in W_0^{1,2}(\bar{N})\}$, and let $(v_j) \in W_0^{1,2}(\bar{N})$ be a minimizing sequence for J. Since $\|v_j\|_{W^{1,2}}$ is bounded, there exists a subsequence which converges weakly to v_* in $W_0^{1,2}$, hence strongly to v_* in L^2. Now, the Dirichlet integral $v \mapsto \int_N |\nabla v|^2$ being lower semi-continuous with respect to the weak topology on $W_0^{1,2}$, we have $J(v_*) = \alpha$.

Thus, for any $\phi \in C_0^\infty(\bar{N})$, we have

$$0 = \frac{d}{dt} J(v_* + t\phi)|_{t=0} = \int_N \langle \nabla v_*, \nabla \phi \rangle + cv_*\phi - f_1\phi\,,$$

that is v_* is a weak solution of $\Delta v_* = -cv_* + f_1$.

Thus $v_* \in W^{2,2} \cap W_0^{1,2}$, hence by the embedding theorems of Sobolev, $v_* \in C^\delta(\bar{N})$. In case $f \in C^\delta(\bar{N})$, we also have $\Delta v_* \in C^\delta(\bar{N})$, so that the Schauder regularity yields $v_* \in C^{2,\delta}(\bar{N})$. Uniqueness follows from the Hopf maximum principle (cf. [GT, 8.1]). □

We can now state and prove the result for a compact manifold with boundary.

C.2. Proposition. *Let $(N, \partial N)$ be a smooth compact Riemannian manifold with boundary, $w \in C^{2,\delta}(\bar{N})$ for some $\delta \in]0,1[$, and $f : \bar{N} \times \mathbf{R} \to \mathbf{R}$ be such that:*

(i) *f is locally Hölder continuous on $\bar{N} \times \mathbf{R}$;*

(ii) *for any $-\infty < a < b < \infty$, there exists a positive constant $c = c(a,b)$ such that, for $a \leqq t_1 \leqq t_0 \leqq b$, we have $f(x,t_0) - f(x,t_1) \geqq -c(t_0 - t_1)$.*

Assume there exist upper and lower solutions $u_\pm \in W^{1,2}(N) \cap C^0(\bar{N})$:

$$\Delta u_- \leqq f(x,u_-)\,, \qquad \Delta u_+ \geqq f(x,u_+)\,,$$
$$u_{-|\partial N} \leqq w \qquad\qquad u_{+|\partial N} \geqq w$$

with $u_- \leqq u_+$. Then, there exists a solution $u \in C^2(\bar{N})$ of the Dirichlet problem

$$\text{(D)} \qquad \Delta u = f(x,u)\,, \qquad u|_{\partial N} = w$$

which satisfies $u_- \leqq u \leqq u_+$.

Proof. Let $u_-(\bar{N})$, $u_+(\bar{N}) \subset [a,b]$, $c = c(a,b)$ and assume w.l.o.g. f is C^δ on $\bar{N} \times [a,b]$. Let $u_1 \in W^{2,2}(\bar{N})$ be the (unique) solution of

$$\Delta u_1 = -cu_1 + (f(x,u_+) + cu_+)\,, \qquad u_{1|\partial N} = w$$

(see C.1). We claim the following:

Assertion. u_1 satisfies $u_- \leqq u_1 \leqq u_+$ and is an upper solution for (D).

For $\Delta(u_1 - u_+) \leqq \Delta u_1 - f(x,u_+) \leqq -c(u_1 - u_+)$, hence $u_1 \leqq u_+$ (if not, the maximum of $(u_1 - u_+)$ would be positive and achieved at a point where $\Delta(u_1 - u_+)$ is negative, a contradiction to the maximum principle, see [GT, 8.1]).

Argue similarly with

$$\begin{aligned}\Delta(u_- - u_1) &\leqq f(x,u_-) - \Delta u_1\\ &\leqq (f(x,u_-) - f(x,u_+)) + c(u_1 - u_+)\\ &\leqq c(u_+ - u_-) + c(u_1 - u_+) \quad \text{by hypothesis (ii)}\\ &\leqq -c(u_- - u_1)\,,\end{aligned}$$

hence $u_- \leqq u_1$.

Finally, $\Delta u_1 - f(x, u_1) = f(x, u_+) - f(x, u_1) + c(u_+ - u_1) \geqq 0$, (by hypothesis (ii) and the choice of c), so that u_1 is a supersolution.

Now Lemma C.1 provides us recursively with a decreasing sequence $(u_j)_{j \geqq 2}$ of supersolutions $u_j \in W^{2,2}(\bar{N})$ for (D) by setting

$$\begin{aligned} \Delta u_j &= -cu_j + f(x, u_{j-1}) + cu_{j-1} \\ u_{j|\partial N} &= w \qquad j \geqq 2 . \end{aligned}$$

Since $W^{2,2} \subset C^\delta$, $f(x, u_{j-1}) \in C^\delta$ hence $u_j \in C^{2,\delta}(\bar{N})$ $\quad (j > 1)$. By the above arguments, $u_- \leqq \cdots \leqq u_j \leqq u_{j-1} \leqq u_+$ and $\Delta u_j \geqq f(x, u_j)$ on N.

Since $\|u_j\|_\infty$ is uniformly bounded, so is $\|\Delta u_j\|_\infty$. Moreover, the u_j's have fixed boundary values, hence elliptic regularity [GT, 9.17] yields that

$$\forall p > 1 , \qquad \|u_j\|_{W^{2,p}(N)} \quad \text{is bounded.}$$

By the embedding theorems of Sobolev [GT, 7.26], we know that (u_j) is bounded in some $C^{1,\alpha}(\bar{N})\,(0 < \alpha < 1)$, hence $\|\Delta u_j\|_{C^\delta(\bar{N})}$ is again bounded.

Now, the Schauder estimates [GT, 6.6] prove that (u_j) is bounded in $C^{2,\delta}(\bar{N})$, hence contains a subsequence which converges in $C^2(\bar{N})$. Let u be its limit. Then

$$u_- \leqq u \leqq u_+ , \qquad \Delta u = f(x, u) , \qquad u|_{\partial N} = w ,$$

thus u is the desired solution. □

We finally set an a priori estimate which will be useful in the iteration procedure of Theorem C.4.

C.3. Lemma. *We work under the same assumptions as in* (C.2). *Let $K \Subset N$ be a compact domain with smooth boundary contained in the interior of N. Then, there exists a constant $C = C(K)$ such that if $u \in C^2(\bar{N})$ satisfies:*

(i) $u_- \leqq u \leqq u_+$,

(ii) $\Delta u = f(x, u)$,

then $\|u\|_{C^{2,\delta}(K)} \leqq C$.

Proof. Let $u_\pm(\bar{N}) \subset [a, b]$ and $f(\bar{N} \times [a, b]) \subset [-B, B]$. Since $u_- \leqq u \leqq u_+$, we have $\|f(x, u)\|_\infty \leqq B$, hence $\|\Delta u\|_\infty \leqq B$.

Now, let K_1 be a compact domain with $K \Subset K_1 \Subset N$. The interior L^p estimates [GT, 9.11] yield for any $p > 1$

$$\|u\|_{W^{2,p}(K_1)} \leqq C(p, B) .$$

And we derive from the Sobolev embedding theorems (cf. [GT, 7.26])

$$\|u\|_{C^{1,\alpha}(K_1)} \leqq C'(p, B) , \quad \text{for some } \alpha > 0 ,$$

hence

$$\|\Delta u\|_{C^\delta(K_1)} \leqq C''(p, B)$$

(since u satisfies $\Delta u = f(x, u)$).

Now, the Schauder interior estimates (cf. [GT, 6.2]) show that

$$\|u\|_{C^{2,\delta}(K)} \leqq C(K) . \quad \square$$

The proof of the main result of this section is now straightforward.

C.4. Theorem. *Let (M, g) be a non compact Riemannian manifold, exhausted by a sequence $(N_i, \partial N_i)$ of compact smooth manifolds with boundaries, that is with*

$$N_i \Subset N_{i+1}\,, \qquad \bigcup_{i\in N} N_i = M\,.$$

Let $f: M \times \mathbf{R} \to \mathbf{R}$ be such that:

(i) *f is locally Hölder continuous on $M \times \mathbf{R}$;*

(ii) *for any $-\infty < a < b < \infty$, and any compact subset $N \subset M$, there exists a positive constant $c = c(a, b, N)$ such that, for $a \leqq t_1 \leqq t_0 \leqq b$*

$$f(x, t_0) - f(x, t_1) \geqq -c(t_0 - t_1)$$

holds. Assume there exist upper and lower solutions $u_\pm \in W^{1,2}_{\mathrm{loc}} \cap C^0(M)$, and $w \in C^{2,\delta}_{\mathrm{loc}}(M)$, with

$$\Delta u_- \leqq f(x, u_-)\,, \qquad \Delta u_+ \geqq f(x, u_+)\,, \qquad u_- \leqq w \leqq u_+\,.$$

Then, there exists a solution $u \in C^2(M)$ of the equation $\Delta u = f(x, u)$, which satisfies $u_- \leqq u \leqq u_+$.

Proof. Let $v_j \in C^2(\overline{N_j})\,(j \in \mathbf{N})$ be a solution of the following Dirichlet problem on N_j:

$$\Delta v_j = f(x, v_j)\,, \qquad v_{j|\partial N_j} = w\,, \quad \text{with} \quad u_- \leqq v_j \leqq u_+\,,$$

known to exist by (C.2). These functions satisfy the a priori estimates (C.3):

$$\text{for} \quad 0 \leqq k \leqq j-1 : \|v_j\|_{C^{2,\alpha(k)}(\bar N_k)} \leqq C(k)\,,$$

where $\alpha(k) \in]0, 1[$ and $C(k)$ depend only on k.

Since the embedding $C^{2,\alpha}(\overline{N_1}) \subset C^2(\overline{N_1})$ is compact, there exists a subsequence (v_{j_1}) which converges in $C^2(\overline{N_1})$ to some function $u_1 \in C^2(\overline{N_1})$.

By induction, we produce subsequences $(v_{j_{k+1}}) \subset (v_{j_k})\,(k \geqq 1)$, such that $(v_{j_{k+1}})$ converges in $C^2(\overline{N_{k+1}})$ to $u_{k+1} \in C^2(\overline{N_{k+1}})$, and with $u_- \leqq u_{k+1} \leqq u_+$, $u_{k+1|N_k} = u_k$.

Thus, we can define a function $u \in C^2(M)$ as follows:

$$\text{for} \quad k \geqq 1 : u|_{N_k} = u_k\,,$$

so that the diagonal sequence (v_{j_j}) converges to u in the compact-C^2 topology.

Hence u is a (classical) solution of the equation $\Delta u = f(x, u)$, and satisfies moreover $u_- \leqq u \leqq u_+$. □

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