Harmonic quasi-isometries of pinched Hadamard surfaces are injective

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Abstract

We prove that a harmonic quasi-isometric map between pinched Hadamard surfaces is a quasi-conformal diffeomorphism.

1 Introduction

1.1 Main result

The main result of this paper is the following.

Theorem 1.1 *Let $h : S_1 \to S_2$ be a harmonic quasi-isometric map between pinched Hadamard surfaces. Then, h is a quasi-conformal diffeomorphism.*

A *pinched Hadamard manifold* is a complete simply-connected Riemannian manifold whose curvature satisfies $-b^2 \leq K \leq -a^2$ for some positive constants $0 < a \leq b$. For instance, the hyperbolic disk $\mathbb{D}$ is a pinched Hadamard surface with constant curvature -1.

A map $f : M_1 \to M_2$ between two metric spaces is *quasi-isometric* if there exists a constant $c \geq 1$ such that, for every $x, x' \in M_1$,

$$c^{-1}\, d(x,x') - c \leq d(f(x), f(x')) \leq c\, d(x,x') + c. \tag{1.1}$$

A smooth map $h : M_1 \to M_2$ between Riemannian manifolds is *harmonic* if it is a critical point for the Dirichlet energy integral $E(h) = \int |Dh|^2 dv_{M_1}$ with respect to variations with compact support.

A diffeomorphism $h : M_1 \to M_2$ between n-dimensional Riemannian manifolds is *quasi-conformal*, if there exists a constant $C > 0$ such that $\|Dh\|^n \leq C\,|\mathrm{Jac}(h)|$ where $\mathrm{Jac}(h) := \det(Dh)$ is the Jacobian of h.

2020 Math. subject class. 53C43 ; Secondary 30C62, 58E20

Key words Harmonic map, Quasi-isometric map, Boundary map, Negative curvature, Quasi-conformal diffeomorphism

1.2 A few comments

The special case of Theorem 1.1 where both S_1 and S_2 are the hyperbolic disk $\mathbb{D}$, is due to Li-Tam [19] and Markovic [20].

The main issue in Theorem 1.1 is the injectivity of h. The quasi-conformality of h is but our way to prove injectivity.

In Theorem 1.1 we only deal with surfaces. Indeed the analog in higher dimension is not true. A counterexample due to Farrell, Ontaneda and Raghunathan is given in [9].

Given two pinched Hadamard surfaces S_1 and S_2, there exist many harmonic quasi-isometric maps from S_1 to S_2 (see [4] or Theorem 2.2 below). Theorem 1.1 asserts that all these maps are injective.

Theorem 1.1 extends the Schoen-Yau injectivity theorem in [22] which says that a harmonic map between two compact Riemannian surfaces with negative curvature, when homotopic to a diffeomorphism, is also a diffeomorphism. This injectivity theorem is used in the parametrization due to J. Sampson and M. Wolf of the Teichmuller space by the Hopf quadratic differentials, see [24] and [15].

From a historical point of view, the first injectivity theorem for harmonic maps is due to Rado-Kneser-Choquet, almost 100 years ago. It states that, in the Euclidean plane, the harmonic extension of an homeomorphism of the unit circle is a diffeomorphism of the unit disk, see [14, Lemma 5.1.10]. The analog statement in dimension $d \geq 3$ is not true. A counterexample is given by R. Laugesen in [17]. Later on, injective harmonic maps between surfaces were studied by H. Lewy in [18] who proved that their Jacobian does not vanish, by R. Heinz in [12] and by J. Jost and H. Karcher in [16, Chapter 7] who found a lower bound for their Jacobian. There is also an extension of the Schoen-Yau injectivity theorem by J. Jost and R. Schoen that allows some positive curvature in [16, Chapter 11].

1.3 Structure of the paper

In Chapter 2, we recall classical facts concerning Hadamard surfaces, quasi-isometric maps and harmonic maps between surfaces. We will see that we can assume that the source S_1 is the hyperbolic disk $\mathbb{D}$. Recall that the special case of Theorem 1.1 where the target S_2 is the hyperbolic disk $\mathbb{D}$ is due to Li–Tam and Markovic.

In Chapter 3 we give an overview of the proof of Theorem 1.1. This proof uses a deformation (g_t) of the metric on S_2, starting with the hyperbolic metric, and a deformation (h_t) of the harmonic map h. The key point will be to obtain a uniform upper bound for the norm of the differential of h_t and a uniform lower bound for the Jacobian of h_t.

In Chapter 4, we gather compactness results for Hadamard surfaces and harmonic maps.

In Chapter 5, we obtain a uniform lower bound for the Jacobian of harmonic quasi-conformal diffeomorphisms.

In Chapter 6, we prove that the family (h_t) varies continuously with t and we complete the proof of Theorem 1.1.

In Chapter 7, we include a short new proof of the special case of Theorem 1.1 where $S_1 = S_2 = \mathbb{D}$.

This paper is as self-contained as possible, the main tools being the Bland-Kalka uniformization theorem in [5], the Bochner equations for harmonic maps between surfaces in [15], the existence and uniqueness of quasi-isometric harmonic maps in [4], and the PDE elliptic regularity in [11].

2 Background

We recall well-known properties of pinched Hadamard surfaces, quasi-isometric maps and harmonic maps between surfaces.

2.1 Pinched Hadamard surfaces

The first example of a pinched Hadamard surface is the hyperbolic disk $\mathbb{D} = (D, \mathrm{g_{hyp}})$, where $D = \{|z| < 1\} \subset \mathbb{C}$ is the unit disk equipped with the hyperbolic metric $\mathrm{g_{hyp}} = \rho^2(z)|dz|^2$ with conformal factor $\rho^2 = 4(1-|z|^2)^{-2}$. It is a Hadamard manifold with constant curvature -1.

Any pinched Hadamard surface is conformal to the disk, namely reads as $(D, \sigma^2(z)|dz|^2)$. Moreover the conformal factors ρ^2 and σ^2 are in a bounded ratio : if the curvature K of this surface satisfies $-b^2 \le K \le -a^2 < 0$, then $a^2\sigma^2 \le \rho^2 \le b^2\sigma^2$. See Proposition 3.1.

Also observe that, for maps defined on a Riemannian surface S_1, the Dirichlet energy functional is invariant under a conformal change of metric on S_1. Hence, the harmonicity of such a map depends only on the conformal class of the source surface.

We infer from this discussion that, to prove Theorem 1.1, we can assume that S_1 is the hyperbolic disk $\mathbb{D}$.

2.2 Quasi-isometric maps

Let $S = (D, \sigma^2(z)|dz|^2)$ be a pinched Hadamard surface. It is a proper Gromov hyperbolic space (a general reference for Gromov hyperbolic spaces is [10]). The boundary at infinity $\partial_\infty S$ of S is defined as the set of equivalence classes of geodesic rays, where two geodesic rays are identified whenever they remain within bounded distance from each other. The union $\overline{S} = S \cup \partial_\infty S$ provides a compactification of S (see [1]).

The boundary at infinity $\partial_\infty \mathbb{D}$ naturally identifies with the boundary $\mathbb{S}^1 = \{z \in \mathbb{C}\,,\ |z| = 1\}$ of D. Since the identity map $\mathrm{Id} : D \to D$ is a quasi-isometry between the hyperbolic disk $\mathbb{D} = (D, \rho^2(z)|dz|^2)$ and the surface $S = (D, \sigma^2(z)|dz|^2)$, the boundary at infinity $\partial_\infty S$ also identifies canonically with $\partial_\infty \mathbb{D} = \mathbb{S}^1$.

A quasi-isometric map $f : \mathbb{D} \to S$ admits a boundary value at infinity $\partial_\infty f : \partial_\infty \mathbb{D} \to \partial_\infty S$, that we read as $\partial_\infty f : \mathbb{S}^1 \to \mathbb{S}^1$ through the above identifications. Two quasi-isometric maps share the same boundary value at infinity if and only if they remain within bounded distance from each other. The maps $\varphi : \mathbb{S}^1 \to \mathbb{S}^1$ that appear as boundary values at infinity of quasi-isometric maps $f : \mathbb{D} \to S$ are exactly the quasi-symmetric homeomorphisms. For convenience, we identify $\mathbb{S}^1$ with $\mathbb{R}/2\pi\mathbb{Z}$.

Definition 2.1 *Let $k \geq 1$. An homeomorphism $\varphi : \mathbb{S}^1 \to \mathbb{S}^1$ is a k-quasi-symmetric map if*

$$\frac{1}{k} \leq \frac{\varphi(\theta + \alpha) - \varphi(\theta)}{\varphi(\theta) - \varphi(\theta - \alpha)} \leq k \tag{2.1}$$

holds for every θ, α with $0 < \alpha \leq \pi$.

Note that any quasi-isometric map $f : \mathbb{D} \to S$ is actually a quasi-isometry. Namely, there exists $C > 0$ such that $d(y, f(\mathbb{D})) \leq C$ holds for all y in S. Indeed, the inverse φ^{-1} of its boundary map is also a quasi-symmetric homeomorphism, hence φ^{-1} is the boundary map of a quasi-isometric map $f' : S \to \mathbb{D}$, and the map $f \circ f' : S \to S$ is within bounded distance from the identity map.

In a previous paper, we studied harmonic quasi-isometric maps between pinched Hadamard manifolds. Our result, when specialized to surfaces, asserts that any quasi-isometric map $f : \mathbb{D} \to S$ has the same boundary value at infinity as a unique harmonic quasi-isometric map. In other words, the following holds.

Theorem 2.2 [4] *Let $S = (D, \sigma^2(z)|dz|^2)$ be a pinched Hadamard surface and $\varphi : \mathbb{S}^1 \to \mathbb{S}^1$ be a quasi-symmetric map. Then, there exists a unique harmonic quasi-isometric map $h : \mathbb{D} \to S$ such that $\partial_\infty h = \varphi$.*

2.3 Harmonic maps between surfaces

We introduce some notation that will be used throughout the paper, and recall some classical results concerning harmonic maps between surfaces. A general reference for this section is Jost [15].

Let $h : \mathbb{D} \to S$ be a smooth map from the hyperbolic disk $\mathbb{D} = (D, \rho^2(z)|dz|^2)$ to a pinched Hadamard surface $S = (D, \sigma^2(z)|dz|^2)$ with pinching condition $-b^2 \leq K \leq -a^2 < 0$. Recall that the curvature K of S is given by

$$K = -\sigma^{-2}\,\Delta_e \log \sigma$$

where $\Delta_e = 4\partial_z\partial_{\bar z}$ is the Euclidean Laplacian. For such a map h, we introduce as usual the functions $h_z, h_{\bar z} : \mathbb{D} \to \mathbb{C}$ defined by

$$h_z = \frac{1}{2}(h_x - ih_y), \quad h_{\bar z} = \frac{1}{2}(h_x + ih_y)$$

where the conformal parameter reads as $z = x + iy$, and the subscript x or y indicates a directional derivative. The map h is holomorphic (or anti-holomorphic) if $h_{\bar z} = 0$ (or $h_z = 0$). It is worth noting that $\overline{h_{\bar z}} = \bar h_z$.

Proposition 2.3 [15, Section 3.6] *The map $h : \mathbb{D} \to S$ is harmonic if and only if it satisfies*

$$h_{z\bar z} + 2\,(\frac{\sigma_z}{\sigma} \circ h)\, h_z h_{\bar z} = 0\,.$$

If the map h is either holomorphic, or anti-holomorphic, then it is harmonic. Introduce the square norms of the complex derivatives of h :

$$H = \|\partial h\|^2 := \frac{\sigma^2 \circ h}{\rho^2}\,|h_z|^2 \quad \text{and} \quad L = \|\bar\partial h\|^2 := \frac{\sigma^2 \circ h}{\rho^2}\,|h_{\bar z}|^2\,,$$

so that one has $\|Dh\|^2 = H + L$. Observe that h is a local diffeomorphism if the Jacobian $J = H - L$ does not vanish, and is moreover orientation preserving if $J > 0$.

Lemma 2.4 [15, Section 3.10] *Let $h : \mathbb{D} \to S$ be a harmonic map. On the open subsets where they are non zero, the functions H and L satisfy the Bochner equations*

$$(1/2)\,\Delta \log H \;=\; (-K \circ h)\, J - 1\,, \tag{2.2}$$
$$(1/2)\,\Delta \log L \;=\; (\,K \circ h)\, J - 1\,. \tag{2.3}$$

Here $\Delta = 4\,\rho^{-2}\partial_z\partial_{\bar z}$ is the Laplace operator relative to the hyperbolic metric.

On the open set $\Omega := \{h_z \neq 0\}$, we introduce the conformal distortion $\mu : \Omega \to \mathbb{C}$ by letting $h_{\bar z} = \mu h_z$, so that one has the useful equalities

$$|\mu|^2 = L/H\ ,\ \ 1 - |\mu|^2 = J/H \ \text{ and }\ \frac{1-|\mu|^2}{1+|\mu|^2} = \frac{J}{\|Dh\|^2}\,. \tag{2.4}$$

3 A family of metrics and harmonic maps

In this section we explain the continuity method that will be used to prove Theorem 1.1.

Let $S = (D, \sigma^2(z)|dz|^2)$ be a pinched Hadamard surface, with curvature bounds $-b^2 \leq K \leq -a^2 < 0$. Choose an increasing quasi-symmetric homeomorphism $\varphi : \mathbb{S}^1 \to \mathbb{S}^1$, and let $h : \mathbb{D} \to S$ be the unique harmonic quasi-isometric map with boundary value at infinity $\partial_\infty h = \varphi$. We want to prove that h is a quasi-conformal diffeomorphism.

In case the surface S is the hyperbolic disk, that is for a harmonic quasi-isometric map $h : \mathbb{D} \to \mathbb{D}$, the result is due to Li-Tam and Markovic (see Chapter 7 for a proof). To prove it for a harmonic map $h : \mathbb{D} \to S$ with values in a general pinched Hadamard surface S, we use the method of continuity, involving a family of pinched Hadamard surfaces $S_t = (D, e^{2u_t}\mathrm{g}_{\mathrm{hyp}})$, for $0 \le t \le 1$, starting with $S_0 = \mathbb{D}$ and such that $S_1 = S$.

3.1 Construction of the metrics g_t

We construct the metric g_t by prescribing its curvature.

More specifically, we introduce for $0 \le t \le 1$ the unique complete conformal metric $g_t = e^{2u_t}\mathrm{g}_{\mathrm{hyp}}$ on the unit disk D with curvature $K_t := -(1-t)+tK$. Each function K_t being pinched between two negative constants, the existence and uniqueness of such a metric is granted by the following.

Proposition 3.1 [5] *Let k be a smooth function on the unit disk D such that $-\beta^2 \le k \le -\alpha^2$ for some constants $0 < \alpha \le \beta$. Then, there exists a unique complete conformal metric $g = e^{2u}\mathrm{g}_{\mathrm{hyp}}$ on D with curvature k. Moreover, the conformal factor e^{2u} is controlled, with $\beta^{-2} \le e^{2u} \le \alpha^{-2}$.*

We do not reproduce here the proof that is given in [5] and that relies on the sub-supersolution method for the curvature equation

$$\Delta u = (-k)\, e^{2u} - 1\,, \tag{3.1}$$

where, as above, Δ is the Laplace operator for the hyperbolic metric $\mathrm{g}_{\mathrm{hyp}}$.

The proof also uses the generalized maximum principle of Yau in [25]. We will need later a light form of this principle that reads as follows.

Lemma 3.2 *Let $v : S \to \mathbb{R}$ be a smooth function defined on a pinched Hadamard surface S. Assume that v is bounded above.*

Then, there exists a sequence (x_n) in S such that

$$v(x_n) \to \sup_S v, \quad |\nabla v|(x_n) \to 0 \ \textit{and} \ \ \limsup \Delta v(x_n) \le 0. \tag{3.2}$$

Proof We can assume that $\sup_S v = 1$. We fix a point $x_0 \in S$ where this supremum is not achieved and we introduce the function v_n on S given by $v_n(x) = v(x)\, e^{-d(x,x_0)/n}$. This function is smooth, except maybe at x_0, and it achieves its supremum at a point $x_n \neq x_0$ for n large. This sequence (x_n) satisfies (3.2) since $v_n(x_n) \to 1$, $\nabla v_n(x_n) = 0$ and $\Delta v_n(x_n) \le 0$. □

3.2 Construction of the harmonic maps h_t

We construct the harmonic map h_t by prescribing its boundary map.

By construction, one has $\mathbb{D} = (D, g_0)$ and $S = (D, g_1)$. For $0 \le t \le 1$, we let $h_t : \mathbb{D} \to S_t$ be the unique harmonic quasi-isometric map whose boundary value at infinity is $\varphi : \mathbb{S}^1 \to \mathbb{S}^1$. Recall that the existence and uniqueness of those h_t are granted by Theorem 2.2.

Here are some basic information concerning these harmonic maps h_t. For $0 \le s, t \le 1$, let $d(h_s, h_t) := \sup_{z \in D} d(h_s(z), h_t(z))$ denote the uniform distance between these two maps, where the distance is taken with respect to the hyperbolic metric g_{hyp} on the target.

Lemma 3.3 *There exists $c_* > 0$ such that, for all $t \in [0,1]$, the map h_t is c_*-quasi-isometric, one has $d(h_t, h_0) \le c_*$, and the map h_t is c_*-Lipschitz.*

Remark that, since the functions u_t are uniformly bounded (Proposition 3.1), it was not really necessary to specify with respect to which one of the metrics g_t the above distances were being estimated.

Proof As explained in Section 2.2, there exists a c-quasi-isometric map $f : \mathbb{D} \to \mathbb{D}$ whose boundary value at infinity is our quasi-symmetric map $\partial_\infty f = \varphi$. By taking a larger constant c, we may assume that each map $f : \mathbb{D} \to S_t$ (that is, the same map f now seen with values in one of the Riemannian surfaces S_t, with $t \in [0,1]$) is c-quasi-isometric.

Thus the main result of [4] asserts that there exists a constant $C > 0$ such that $d(f, h_t) \le C$. This constant C depends only on c and on the pinching constants a and b, hence it does not depend on $t \in [0,1]$. Thus the first two claims hold if $c_* \ge 2c + 2C$.

The map f being c-quasi-isometric, each harmonic map $h_t : \mathbb{D} \to S_t$ sends any ball $B(z,1) \subset \mathbb{D}$ with radius 1 inside the ball $B(h_t(z), R) \subset S_t$ with radius $R = 2c + 2C$. Now the uniform Lipschitz continuity of the maps h_t follows from the Cheng lemma, that we recall below. $\square$

Lemma 3.4 [8] *Let S be a Hadamard surface with $-b^2 \le K \le 0$. There exists a constant κ, that depends only on b, such that if a harmonic map $h : \mathbb{D} \to S$ satisfies $h(B(z,1)) \subset B(h(z), R)$ for some radius R, then*

$$\|Dh(z)\| \le \kappa\, R\,.$$

3.3 An injectivity criterion

The following lemma tells us that a uniform lower bound for the Jacobian $J_t = \text{Jac}(h_t)$ is enough to ensure that h_t is a quasi-conformal diffeomorphism.

Lemma 3.5 *If $\inf_{z \in \mathbb{D}} J_t(z) > 0$ then h_t is a quasi-conformal diffeomorphism.*

Proof By assumption the Jacobian J_t does not vanish, hence the map $h_t : \mathbb{D} \to S_t$ is a local diffeomorphism. By construction, the map h_t is quasi-isometric, hence it is a proper map. It thus follows that h_t is a covering map. Hence, since S is simply connected, the map h_t is a diffeomorphism. Since, by Lemma 3.4, h_t is Lipschitz, the lower bound for its Jacobian J_t ensures that h_t is quasi-conformal. □

3.4 Strategy of proof of Theorem 1.1

We will need the following two propositions.

Proposition 3.6 *There exists $j_* > 0$ such that, for all $t \in [0,1]$ for which h_t is a quasi-conformal diffeomorphism, one has $J_t \geq j_*$.*

Proposition 3.6 is a straightforward consequence of Proposition 5.2 that will be proven in Chapter 5. Indeed Lemma 3.3 ensures that the maps h_t are c_*-Lipschitz.

Let $C_b(\mathbb{D}, \mathbb{R})$ be the space of bounded continuous functions ψ endowed with the sup norm : $\|\psi\|_\infty = \sup_{z \in \mathbb{D}} |\psi(z)|$.

Proposition 3.7 *The map $t \in [0,1] \to J_t \in C_b(\mathbb{D}, \mathbb{R})$ is continuous.*

Proposition 3.7 will be proven in Chapter 6 as part of Proposition 6.2 .

Proof of Theorem 1.1 using Propositions 3.6 and 3.7 Let A be the set of parameters $t \in [0,1]$ such that the harmonic map $h_t : \mathbb{D} \to S_t$ is a quasi-conformal diffeomorphism. We want to prove that $1 \in A$. We already know that $0 \in A$ (this is Theorem 7.1 due to Li-Tam and Markovic). It is enough to check that A is open and closed. Let j be the function on $[0,1]$ given by

$$j(t) := \inf_{z \in \mathbb{D}} J_t(z) \in \mathbb{R}.$$

By Proposition 3.7 the function j is continuous. By Lemma 3.5 and Proposition 3.6, one has both $A = j^{-1}(]0, \infty[)$ and $A = j^{-1}([j_*, \infty[)$. Hence A is both open and closed. □

4 Sequences of metrics and harmonic maps

In order to obtain the uniform lower bounds in Chapter 5, or the continuity properties in Chapter 6, we will have to consider sequences of conformal metrics on the unit disk D, and sequences of harmonic maps. In this chapter, we state compactness results for such sequences.

These compactness results also hold in higher dimension (see [21], or [4]). Since we will only deal here with conformal metrics on the disk D, the complex parameter $z \in D$ naturally provides a global harmonic chart for these metrics so that the statements and the proofs are more elementary.

4.1 Sequence of Hadamard surfaces

Let us begin with sequences of conformal Riemannian structures on the unit disk D.

Convergence in the following lemma is a special case of the Gromov-Hausdorff convergence for isometry classes of pointed proper metric spaces using the base point $0 \in D$. See [4, §5.3] or [7] for a short introduction to this notion.

Lemma 4.1 *Let $g_n = e^{2u_n} \mathrm{g}_{\mathrm{hyp}}$ be a sequence of complete conformal metrics on the unit disk D with curvature $-b^2 \leq K_n \leq -a^2 < 0$. Then there is a subsequence of (u_n) that converges to a C^1 function u_∞ in the C^1_{loc} topology.*

The limit metric $g_\infty = e^{2u_\infty} \mathrm{g}_{\mathrm{hyp}}$ is a C^1 complete conformal metric on D, and $S_\infty := (D, g_\infty)$ is a CAT-space with curvature between $-b^2$ and $-a^2$.

Proof Proposition 3.1 ensures that the logarithms $u_n : D \to \mathbb{R}$ of the conformal factors are uniformly bounded. The curvature equation

$$\Delta u_n = (-K_n)\, e^{2u_n} - 1 \tag{3.1$_n$}$$

for g_n ensures that the Laplacians Δu_n are also uniformly bounded.

Pick $0 \leq \alpha < 1$. We may now apply to the sequence (u_n) the following first Schauder estimates (see [11, Theorem 3.9] or [21, Theorem 70]). These estimates state that there exists a constant c such that, for any smooth function $v : \mathbb{D} \to \mathbb{R}$ on the hyperbolic disk, the inequality

$$\|v\|_{C^{1,\alpha}(B_1)} \leq c_\alpha (\|\Delta v\|_{C^0(B_2)} + \|v\|_{C^0(B_2)}) \tag{4.1}$$

holds for any pair of concentric hyperbolic balls $B_1 \subset B_2 \subset \mathbb{D}$ with respective radii 1 and 2. This provides a uniform local bound for the norms $\|u_n\|_{C^{1,\alpha}}$. Going if necessary to a subsequence, we may thus assume that the sequence (u_n) converges in the C^1_{loc} topology. Let $u_\infty = \lim u_n$ and $g_\infty = e^{2u_\infty} \mathrm{g}_{\mathrm{hyp}}$ and introduce $S_\infty = (D, g_\infty)$. As a limit of such, the length space S_∞ is a CAT-space with curvature between $-b^2$ and $-a^2$ (see [6, Corollary II.3.10] and [7, Theorem 10.7.1]). □

Remark Under the hypothesis of Lemma 4.1, after extraction, the sequence of bounded functions $K_n : D \to \mathbb{R}$ converges weakly to a bounded measurable function $K_\infty : D \to \mathbb{R}$ with $-b^2 \leq K_\infty \leq -a^2$, and the C^1 function u_∞ is a weak solution of

$$\Delta u_\infty = (-K_\infty)\, e^{2u_\infty} - 1\,. \tag{3.1$_\infty$}$$

4.2 Sequence of harmonic maps

Now turn to sequences of maps between such Riemannian surfaces.

Lemma 4.2 *Let $S_n = (D, g_n)$ be a sequence converging to $S_\infty = (D, g_\infty)$ as in Lemma 4.1. Let $c > 0$, and let $h_n : \mathbb{D} \to S_n$ be c-Lipschitz maps satisfying $d_n(h_n(0), 0) \leq c$. Then there is a subsequence of (h_n) that converges locally uniformly to a c-Lipschitz map $h_\infty : \mathbb{D} \to S_\infty$.*
a) If all the maps h_n are C-quasi-isometric, then h_∞ is C-quasi-isometric.
b) If all the maps h_n are harmonic, then h_∞ is C^2 and is harmonic too.

Proof Observe that, on any fixed compact set, the maps $h_n : \mathbb{D} \to S_\infty$ are c_n-Lipschitz for some constants c_n converging to c. Indeed these are the initial maps h_n, albeit with the limit metric on the target. Since we assumed that $d_n(h_n(0), 0) \leq c$, these maps h_n are locally uniformly bounded (this means locally in z and uniformly in n). It thus follows from the Ascoli lemma that we may assume the sequence (h_n) to converge uniformly on compact sets to a c-Lipschitz map $h_\infty : \mathbb{D} \to S_\infty$.

a) If all $h_n : \mathbb{D} \to S_n$ are C-quasi-isometric, then, on any fixed compact set, the maps $h_n : \mathbb{D} \to S_\infty$ are C_n-quasi-isometric for some constant C_n converging to C, and so h_∞ is C-quasi-isometric.

b) Now assume that each map $h_n : \mathbb{D} \to S_n$ is harmonic, namely that each function $h_n : \mathbb{D} \to D \subset \mathbb{C}$ satisfies the equation

$$(h_n)_{z\bar{z}} + 2\left((u_n)_z \circ h_n\right)(h_n)_z (h_n)_{\bar{z}} = 0 \,. \tag{4.2}$$

We want to prove that h_∞ is harmonic, namely that it is C^2 and satisfies

$$(h_\infty)_{z\bar{z}} + 2\left((u_\infty)_z \circ h_\infty\right)(h_\infty)_z (h_\infty)_{\bar{z}} = 0 \,. \tag{4.3}$$

The maps $h_n : \mathbb{D} \to S_n$ are c-Lipschitz, so that all the derivatives $(h_n)_z$ and $(h_n)_{\bar{z}}$ are locally uniformly bounded. We have seen in the proof of Lemma 4.1 that the gradients ∇u_n are locally uniformly bounded, hence $(u_n)_z \circ h_n$ are locally uniformly bounded. Then (4.2) ensures that the functions Δh_n are also locally uniformly bounded. We apply the first Schauder estimates (4.1) to the functions $v = h_n$. This implies that, for $0 < \alpha < 1$, the functions h_n are uniformly bounded in the $C^{1,\alpha}_{\mathrm{loc}}$ topology.

Plugging this information in (4.2), and remembering from the proof of Lemma 4.1 that the gradients ∇u_n are also uniformly bounded in the C^α_{loc} topology, we see that the functions Δh_n are uniformly bounded in the C^α_{loc} topology. We will now apply the second Schauder estimates to the functions $v = h_n$ (see [21, Theorem 70]). With the same notation as (4.1), these estimates state

$$\|v\|_{C^{2,\alpha}(B_1)} \leq c_\alpha\big(\|\Delta v\|_{C^\alpha(B_2)} + \|v\|_{C^0(B_2)}\big). \tag{4.4}$$

Hence the functions h_n are uniformly bounded in the $C^{2,\alpha}_{\mathrm{loc}}$ topology.

Therefore (h_n) admits a subsequence which converges in the $C^2_{\rm loc}$ topology. This proves that h_∞ is C^2 and going to the limit in (4.2) ensures that the limit map h_∞ is harmonic, as claimed. □

5 A lower bound for the Jacobian

In this section we provide a lower bound for the Jacobian J_t of h_t when h_t is a quasi-conformal diffeomorphism (Proposition 3.6).

The notation are those of Section 2.3 : S is a pinched Hadamard surface and $h : \mathbb{D} \to S$ is an harmonic map. We assume moreover that h is an orientation preserving diffeomorphism. The Jacobian of h, which is $J = H - L$ with $H := \|\partial h\|^2$ and $L := \|\overline{\partial} h\|^2$, is positive. The function $w := \frac{1}{2}\log H$ satisfies Equation (2.2), that we may also write as

$$\Delta w = (-K \circ h)\,(1 - |\mu|^2)\,e^{2w} - 1\,, \tag{5.1}$$

where $\mu := h_{\bar z}/h_z$ is the conformal distortion. By (2.4) the diffeomorphism h is quasi-conformal if and only if there exists a $\delta < 1$ such that $|\mu| \leq \delta$.

5.1 Controlling the norm of the differential

The next lemma tells us that the norm of the differential $\|Dh\|$ of a harmonic quasi-conformal diffeomorphism is uniformly bounded below (see also [23]).

Lemma 5.1 *Let $h : \mathbb{D} \to S$ be a quasi-conformal harmonic diffeomorphism, where S is a pinched Hadamard surface with curvature $-b^2 \leq K \leq -a^2 < 0$. Then one has $e^{2w} \geq b^{-2}$.*

Proof Introduce the conformal metric $\tilde g = e^{2w} {\rm g}_{\rm hyp}$ on D. We first prove that $\tilde g$ is complete with pinched negative curvature. Proposition 3.1 will then provide the lower bound on w.

Let $S = (D, \sigma^2(z)|dz|^2)$. The map $h : \mathbb{D} \to S$ being a diffeomorphism and S being complete, the pull back metric $G = h^*(\sigma^2(z)|dz|^2)$ is complete. This pull-back metric reads as $G = (\sigma^2 \circ h)|h_z|^2|dz + \mu d\bar z|^2$. Since one has $\tilde g = (\sigma^2 \circ h)|h_z|^2|dz|^2$ and $|\mu| \leq 1$, one easily checks that $G \leq 4\tilde g$. This ensures that the metric $\tilde g$ is complete.

Comparison of Equation (5.1) satisfied by w and the curvature equation (3.1) yields that the metric $\tilde g$ has curvature $\tilde K = (K \circ h)(1 - |\mu|^2)$. It follows that $-b^2 \leq \tilde K \leq -a^2(1 - \delta^2) < 0$, where $\delta := \|\mu\|_\infty < 1$. Proposition 3.1 thus ensures that w satisfies $b^{-2} \leq e^{2w} \leq a^{-2}(1 - \delta^2)^{-1}$. □

5.2 Controlling the Jacobian

The following proposition tells us that the Jacobian of a harmonic quasi-conformal diffeomorphism is controlled by its Lipschitz constant.

Proposition 5.2 *Let $0 < a \leq b$. Then, for every $c > 0$, there exists $j_* = j_*(a, b, c) > 0$ such that, if S is a pinched Hadamard surface with curvature $-b^2 \leq K \leq -a^2$, the Jacobian J of any c-Lipschitz quasi-conformal harmonic diffeomorphism $h : \mathbb{D} \to S$ satisfies $J \geq j_*$.*

Proof Assume by contradiction that there exist a sequence of pinched Hadamard surfaces $S_n = (D, e^{2u_n} \mathrm{g}_{\mathrm{hyp}})$ with curvatures $-b^2 \leq K_n \leq -a^2$, a sequence $h_n : \mathbb{D} \to S_n$ of c-Lipschitz harmonic quasi-conformal diffeomorphisms and a sequence (x_n) of points of D such that the Jacobian J_n of h_n satisfy $J_n(x_n) \to 0$.

Choosing sequences (γ_n) and (γ'_n) of isometries of the hyperbolic disk such that $\gamma_n(x_n) = 0$ and $\gamma'_n(h_n(x_n)) = 0$, and replacing u_n by $u_n \circ {\gamma'_n}^{-1}$ and h_n by $\gamma'_n h_n \gamma_n^{-1}$, we can assume that $x_n = 0$ and $h_n(x_n) = 0$.

By Lemmas 4.1 and 4.2, going to a subsequence, one may assume that :
– the sequence (u_n) converges to a C^1 function u_∞ in the C^1_{loc} topology.
– the sequence (h_n) converges to a C^2 map h_∞ in the C^2_{loc} topology.

Recall from (2.4) that $J_n = (1 - |\mu_n|^2) e^{2w_n}$ where $\mu_n = (h_n)_{\bar{z}}/(h_n)_z$ is the conformal distorsion and where $e^{2w_n} = \|\partial h_n\|^2$. Lemma 5.1 ensures that

$$e^{2w_\infty} = \lim_{n\to\infty} e^{2w_n} \geq b^{-2} . \tag{5.2}$$

Thus $(h_\infty)_z$ does not vanish. Hence the functions μ_n also converge to a C^1 functions μ_∞ in the C^1_{loc} topology, and one has $\|\mu_\infty\|_\infty = 1$ and $|\mu_\infty(0)| = 1$.

First step We claim that $|\mu_\infty| \equiv 1$.
Indeed, we introduce the non negative C^1 functions $\ell_n := -\log |\mu_n|^2$ defined on $\Omega_n := \{\mu_n \neq 0\}$ and their limit $\ell_\infty := -\log |\mu_\infty|^2$, which is defined on $\Omega_\infty := \{\mu_\infty \neq 0\}$. By assumption, the function ℓ_∞ is a non-negative function that achieves its minimum $\ell_\infty(0) = 0$ at the origin. We will prove that the set $\{\ell_\infty = 0\}$ is open in Ω_∞, so that $\ell_\infty \equiv 0$ as claimed.

The function ℓ_n satisfies the equation on Ω_n, difference of (2.2) and (2.3):

$$\Delta \ell_n = 4 \, (-K_n \circ h_n) \, (1 - e^{-\ell_n}) \, e^{2w_n} . \tag{5.3}$$

Since $|K_n| \leq b^2$, $1 - e^{-\ell_n} \leq \ell_n$ and $e^{2w_n} \leq c^2$, we infer that

$$\Delta \ell_n \leq 4b^2 c^2 \, \ell_n .$$

Hence ℓ_∞ is a C^1 function on Ω_∞ that satisfies in the weak sense

$$\Delta \ell_\infty \leq 4b^2 c^2 \, \ell_\infty .$$

In particular, one has bounds $\Delta_e \ell_\infty \leq C_K \ell_\infty$ on compact sets K of Ω_∞ and, by Lemma 5.3 below, the set $\{\ell_\infty = 0\}$ is open. This proves $|\mu_\infty| \equiv 1$.

Second step We reach a contradiction.
We recall that the functions w_n satisfy (5.1), namely

$$\Delta w_n = (-K_n \circ h_n)\,(1 - |\mu_n|^2)\, e^{2w_n} - 1\,.$$

Since the functions $(-K_n \circ h_n)$ and e^{2w_n} are uniformly bounded and since $\lim_{n\to\infty} |\mu_n| = 1$, the limit function $w_\infty = \lim w_n$ satisfies $\Delta w_\infty = -1$ in the weak sense. In particular w_∞ is smooth. Note also that (5.2) yields the lower bound $w_\infty \geq \log b^{-2}$.

In conclusion, w_∞ is a smooth function on $\mathbb{D}$ which is bounded below and satisfies $\Delta w_\infty = -1$. By the generalized maximum principle of Lemma 3.2, such a function w_∞ does not exist. Contradiction. $\square$

In the previous proof we have used the following lemma as in [12].

Lemma 5.3 *Let $C > 0$ and ℓ be a non-negative continuous function on an open set $U \subset \mathbb{R}^2$ such that $\Delta_e \ell \leq C\ell$ weakly. Then the set $\{\ell = 0\}$ is open.*

Proof We can assume that $\ell(0) = 0$. By a standard convolution argument, in a small ball $B(0,R) \subset \Omega$, we can write ℓ as a uniform limit of non-negative C^2-functions ℓ_n that also satisfy

$$\Delta_e \ell_n \leq C\ell_n\,. \tag{5.4}$$

We introduce the mean values of ℓ_n and ℓ on circles of radius $r \leq R$,

$$M_n(r) := \tfrac{1}{2\pi}\int_0^{2\pi} \ell_n(r\,e^{i\theta})\,d\theta \quad \text{and} \quad M(r) := \tfrac{1}{2\pi}\int_0^{2\pi} \ell(r\,e^{i\theta})\,d\theta.$$

The Green representation formula (see Hörmander [13, p.119]) gives

$$\ell_n(0) = M_n(r) - \frac{1}{2\pi}\int_{B(0,r)} \Delta_e \ell_n(y)\,\log\frac{r}{|y|}\,dy\,.$$

Since ℓ_n converges uniformly to ℓ and $\ell(0) = 0$ we infer, using (5.4), that

$$M(r) \leq \frac{C}{2\pi}\int_{B(0,r)} \ell(y)\,\log\frac{r}{|y|}\,dy\,,$$

so that, for every $r \leq R$,

$$M(r) \leq \frac{C\,R^2}{4}\,\sup_{[0,R]} M(t)\,.$$

Choosing $R^2 < 4/C$, we obtain that $\ell \equiv 0$ on the ball $B(0,R)$. $\square$

6 Continuity of the Jacobian

In this section we prove that the metrics g_t, the harmonic maps h_t and their Jacobians J_t depend continuously on t, thus proving Proposition 3.7.

6.1 A continuous family of metric

In Chapter 3, we introduced pinched Hadamard surfaces $S_t = (D, e^{2u_t}\mathrm{g}_{\mathrm{hyp}})$ with curvature $K_t = (t-1) + tK$, where $-b^2 \le K \le -a^2 < 0$ ($t \in [0,1]$). In particular, $S_0 = \mathbb{D}$. We have seen that all the metrics g_t are uniformly bi-Lipschitz to each other. This means that the functions $u_t : D \to \mathbb{R}$ are uniformly bounded.

Lemma 6.1 tells us that they are uniformly bounded in norm C^1 and that the map $t \in [0,1] \to u_t \in C^1$ is continuous. Here the gradients ∇, as well as their norms, are taken with respect to the hyperbolic metric $\mathrm{g}_{\mathrm{hyp}}$.

Lemma 6.1 *There exists a constant c such that, for every* $0 \le t \le 1$

$$\|u_t\|_\infty + \|\nabla u_t\|_\infty \le c \tag{6.1}$$

$$\|u_t - u_s\|_\infty + \|\nabla(u_t - u_s)\|_\infty \le c\,|t-s|\,. \tag{6.2}$$

Proof We argue as in the proof of Lemma 4.1. Let us first prove (6.1). Each conformal factor e^{2u_t} is solution of the curvature equation (3.1), here

$$\Delta u_t = (-K_t)\, e^{2u_t} - 1\,. \tag{6.3}$$

Since the metrics g_t are complete, and the K_t satisfy a uniform pinching condition $-B^2 \le K_t \le -A^2 < 0$ for all $0 \le t \le 1$, Proposition 3.1 ensures that the functions u_t are uniformly bounded. Plugging into (6.3), we infer that the Laplacians Δu_t are also uniformly bounded. Hence the Schauder estimates (4.1) with $\alpha = 0$ and $v = u_t$ yield the uniform bound (6.1).

We now prove (6.2). Using the curvature equations (6.3) satisfied by u_s and u_t ($0 \le s < t \le 1$), we obtain

$$\Delta(u_t - u_s) = (K_s - K_t)e^{2u_t} + K_s(e^{2u_s} - e^{2u_t})$$

that we rewrite as :

$$\Delta(u_t - u_s) = (s-t)(1+K)e^{2u_t} + (-K_s)\,(e^{2u_t} - e^{2u_s})\,. \tag{6.4}$$

Since the functions u_t are uniformly bounded, there exists a constant $m_0 > 0$, such that one has $|u_t - u_s| \le m_0\,|e^{2u_t} - e^{2u_s}|$ for all s, t in $[0,1]$.

The generalized maximum principle applied to $u_t - u_s$ combined with (6.4) ensures the existence of a constant c such that $\|u_t - u_s\|_\infty \le c\,|t-s|$ for every s, t in $[0,1]$.

Plugging this information into (6.4) yields a similar bound for $\Delta(u_t - u_s)$, and (6.2) follows from the Schauder estimates (4.1) with $v = u_t - u_s$. $\square$

Remark Since the curvature function K is smooth, one could improve Lemma 6.1 and prove that all u_t are smooth and that, for all $p \geq 2$ the maps $t \in [0,1] \to u_t \in C^p_{\mathrm{loc}}$ is continuous. But the p^{th} derivatives of u_t might not be bounded.

6.2 A continuous family of harmonic maps

Recall that we have natural identifications $\partial_\infty S_t \simeq \mathbb{S}^1$. We fix an increasing quasi-symmetric homeomorphism $\varphi : \mathbb{S}^1 \to \mathbb{S}^1$. In Chapter 3, we introduced the unique harmonic quasi-isometric map $h_t : \mathbb{D} \to S_t$ with boundary value at infinity $\partial_\infty h_t = \varphi$.

Here are the continuity properties of this family of maps h_t that we used in the proof of Theorem 1.1.

Proposition 6.2 *(a) The map $t \in [0,1] \to h_t \in C(\mathbb{D}, \mathbb{D})$ is continuous.*
(b) The map $t \in [0,1] \to J_t \in C_b(\mathbb{D}, \mathbb{R})$ is continuous.

This means that $\lim\limits_{s \to t} d(h_s, h_t) = 0$ and $\lim\limits_{s \to t} \|J_s - J_t\|_\infty = 0$, for all $t \in [0,1]$.

Proof Assume this is not the case. Then there exist a sequence (t_n) in $[0,1]$ and a sequence (x_n) of points in $\mathbb{D}$ such that

$$\lim_{n\to\infty} d(h_{t_n}(x_n), h_t(x_n)) > 0 \quad \text{or} \quad \lim_{n\to\infty} |J_{t_n}(x_n) - J_t(x_n)| > 0\,. \tag{6.5}$$

We want to get a contradiction by applying Lemmas 4.1 and 4.2 to recentered surfaces and recentered harmonic maps. We thus choose sequences (γ_n) and (γ'_n) of isometries of the hyperbolic disk $\mathbb{D}$ such that $\gamma_n(x_n) = 0$ and $\gamma'_n(h_t(x_n)) = 0$. Let $S_n = (D, g_n)$ and $S'_n = (D, g'_n)$ be the conformal surfaces where $g_n = e^{2u_n} g_{\mathrm{hyp}}$ and $g'_n = e^{2u'_n} g_{\mathrm{hyp}}$ with

$$u_n := u_t \circ {\gamma'_n}^{-1} \quad \text{and} \quad u'_n := u_{t_n} \circ {\gamma'_n}^{-1}\,.$$

By Lemma 4.1 we may assume, after extraction, that the sequence (u_n) converges to a C^1 function u_∞ in the C^1_{loc} topology, and that the limit C^1 metric space $S_\infty := (D, e^{2u_\infty})$ is a CAT space with pinched curvature $-b^2 \leq K_\infty \leq -a^2 < 0$.

By Lemma 6.1, one has

$$\lim_{n\to\infty} \|u'_n - u_n\|_\infty + \|\nabla u'_n - \nabla u_n\|_\infty = 0\,.$$

Hence the sequence (u'_n) also converges in the C^1_{loc} topology to the function u_∞. We now introduce the sequence of maps

$$h_n := \gamma'_n \circ h_t \circ \gamma_n^{-1} \quad : \quad \mathbb{D} \to S_n\,, \tag{6.6}$$

$$h'_n := \gamma'_n \circ h_{t_n} \circ \gamma_n^{-1} \quad : \quad \mathbb{D} \to S'_n\,. \tag{6.7}$$

These maps h_n and h'_n are harmonic and (6.5) can be rewritten as

$$\lim_{n\to\infty} d(h_n(0), h'_n(0)) > 0 \quad \text{or} \quad \lim_{n\to\infty} |J_n(0) - J'_n(0)| > 0\,, \qquad (6.8)$$

where J_n is the Jacobian of h_n and J'_n the Jacobian of h'_n. By Lemma 3.3, all these maps h_n and h'_n are uniformly Lipschitz and uniformly quasi-isometric. Hence Lemma 4.2 ensures that, after extraction, the sequences (h_n) and (h'_n) converge respectively, in the $C^2_{\rm loc}$ topology, to harmonic quasi-isometric maps $h_\infty, h'_\infty : \mathbb{D} \to S_\infty$.

Since Lemma 3.3 also asserts that $d(h_n, h'_n) \leq 2\,c_*$ for all n, the limit harmonic quasi-isometric maps $h_\infty, h'_\infty : \mathbb{D} \to S_\infty$ are within bounded distance from each other. Then the uniqueness theorem for quasi-isometric harmonic maps in [4, §5] ensures that $h_\infty = h'_\infty$. This contradicts (6.8). □

This also ends the proof of both Proposition 3.7 and Theorem 1.1.

7 The injectivity theorem in constant curvature

This chapter is an appendix in which we prove the injectivity theorem 7.1 that we used as a starting point in the proof of our main theorem 1.1.

7.1 The Li-Tam-Markovic injectivity theorem

Theorem 7.1 *Let $\mathbb{D}$ be the hyperbolic disk. Any harmonic quasi-isometric map $h : \mathbb{D} \to \mathbb{D}$ is a quasi-conformal harmonic diffeomorphism.*

This theorem is an output of Markovic solution of the Schoen conjecture in [20]. It relies on a previous injectivity result of Li-Tam in [19] when the boundary map of h is smooth, which is Proposition 7.4 below. The proof of Li-Tam itself relies on the Schoen-Yau injectivity theorem in [22].

We would like to give in this appendix a short new proof of Theorem 7.1 that does not rely on this Schoen-Yau theorem and that uses instead a continuity method combined with a simple topological fact (Lemma 7.8).

Proof The proof will last till the end of this appendix. We know (see Section 2.2) that the boundary value $\varphi = \partial_\infty h : \mathbb{S}^1 \to \mathbb{S}^1$ is a k-quasi-symmetric homeomorphism of $\mathbb{S}^1 = \partial_\infty \mathbb{D}$, where k depends only on the constant c of quasi-isometry of h. For $k \geq 1$, we introduce the set

$$\mathcal{M}_k \quad = \quad \{\ k\text{-quasi-symmetric homeomorphism } \varphi : \mathbb{S}^1 \to \mathbb{S}^1\}$$

equipped with the uniform distance $d(\varphi_1, \varphi_2) = \sup_{\xi\in\mathbb{S}^1} |\varphi_1(\xi) - \varphi_2(\xi)|$.

We also know that, for all φ in $\mathcal{M}_k$, there exists a unique harmonic quasi-isometric map $h_\varphi : \mathbb{D} \to \mathbb{D}$ whose boundary map is φ. We want to prove that all these maps h_φ are quasiconformal diffeomorphisms. This will follow from the next Lemma 7.2, Proposition 7.3 and Proposition 7.4. □

Lemma 7.2 *The k-quasi-symmetric C^1 diffeomorphisms are dense in $\mathcal{M}_k$.*

Proof Choose a smooth approximation of unity (α_n) on $\mathbb{S}^1$. For φ in $\mathcal{M}_k$, each function $\alpha_n * \varphi$ is a k-quasi-symmetric C^1 diffeomorphism while the sequence $(\alpha_n * \varphi)$ converges uniformly to φ. □

Proposition 7.3 *Let $\mathcal{F}_k$ be the set of those $\varphi \in \mathcal{M}_k$ such that h_φ is a quasi-conformal diffeomorphism. Then $\mathcal{F}_k$ is a closed subset of $\mathcal{M}_k$.*

The proof of Proposition 7.3 will be given in Section 7.3. It relies on continuity properties of the boundary map $h \mapsto \partial_\infty h$ proven in Section 7.2.

Proposition 7.4 *When φ is a C^1 diffeomorphism of $\mathbb{S}^1$, its quasi-isometric harmonic extension $h_\varphi : \mathbb{D} \to \mathbb{D}$ is a quasi-conformal diffeomorphism.*

The proof of Proposition 7.4 will be given in Section 7.5. It uses a deformation φ_t of φ starting with the identity. Let G be the group of isometries of $\mathbb{D}$ acting on $\mathbb{S}^1$. The proof relies on the fact that the only homeomorphisms which are limits of elements of $G\varphi_t G$ belong to G. This is Lemma 7.8 which will be proven in Section 7.4.

7.2 Continuity of the boundary map

Let $c > 1$. Endow the space $\mathcal{Q}_c$ of c-quasi-isometric maps $f : \mathbb{D} \to \mathbb{D}$ with the topology of uniform convergence on compact sets, and the space $\mathcal{C}$ of continuous maps $\varphi : \mathbb{S}^1 \to \mathbb{S}^1$ with the topology of uniform convergence.

Lemma 7.5 *The map $f \in \mathcal{Q}_c \to \partial_\infty f \in \mathcal{C}$ is continuous.*

Proof We use the quasi-invariance of the Gromov product under quasi-isometric maps. We fix a point 0 in $\mathbb{D}$. For $n \in \mathbb{N} \cup \{\infty\}$, let $f_n \in \mathcal{Q}_c$ be c-quasi-isometric maps, with boundary values at infinity φ_n . Assume that the sequence (f_n) converges uniformly to f_∞ on compact sets. In particular, the quantity $R := \sup_n d(f_n(0), 0)$ is finite. We want to prove that the sequence (φ_n) converges uniformly to the boundary map φ_∞ of f_∞.

For $\xi \in \mathbb{S}^1$, denote by $t \in [0, \infty[\to x_\xi^t \in \mathbb{D}$ the geodesic ray with origin 0 and endpoint ξ. By [10, Proposition 5.15], there exists a constant $\lambda > 1$ such that the following lower bound for the Gromov product seen from 0

$$(f_n(x_\xi^t), f_n(x_\xi^s))_0 \geq (x_\xi^t, x_\xi^s)_0/\lambda - \lambda = t/\lambda - \lambda$$

holds when $s \geq t > 0$ and $n \in \mathbb{N} \cup \{\infty\}$. Letting $s \to \infty$, we obtain

$$(f_n(x_\xi^t), \varphi_n(\xi))_0 \geq t/\lambda - \lambda$$

for $n \in \mathbb{N} \cup \{\infty\}$. Since $\mathbb{D}$ is δ-hyperbolic for a constant $\delta > 0$, each Gromov product $(\varphi_n(\xi), \varphi_\infty(\xi))_0$ is bounded below by

$$\min[\,(\varphi_n(\xi), f_n(x_\xi^t))_0\,,\ (f_n(x_\xi^t), f_\infty(x_\xi^t))_0\,,\ (f_\infty(x_\xi^t), \varphi_\infty(\xi))_0\,] - 2\delta$$

for every $\xi \in \mathbb{S}^1$ and $n \in \mathbb{N}$ (see [10, Chap. 2]). The sequence (f_n) converging uniformly to f_∞ on compact sets there exists, for all $t > 0$, an integer $n_t \geq 1$ such that one has, for $n \geq n_t$ and $\xi \in \mathbb{S}^1$,

$$d(f_n(x_\xi^t), f_\infty(x_\xi^t)) \leq 1\ , \quad \text{and hence}$$

$$(\varphi_n(\xi), \varphi_\infty(\xi))_0 \ \geq\ \min[t/\lambda - \lambda\,;\, t/c - c - R - 1/2] - 2\delta\,.$$

This proves that the sequence (φ_n) converges uniformly to φ_∞. □

7.3 A continuous inverse to the boundary map

The following lemma is a variation of Lemma 3.3. Fix $k \geq 1$.

Lemma 7.6 *There exist a compact subset $L_k \subset \mathbb{D}$ and a constants c_k such that the harmonic quasi-isometric extension h_φ of any $\varphi \in \mathcal{M}_k$ is c_k-quasi-isometric, the point $h_\varphi(0)$ is in L_k, and the map h_φ is c_k-Lipschitz.*

Proof We introduce the Douady-Earle extension $f_\varphi : \mathbb{D} \to \mathbb{D}$ of φ and we recall some of their properties that can be found in J. Hubbard's book [14, §5.1]. By definition, the image $f_\varphi(z)$ of $z \in \mathbb{D}$ is the barycenter of the measure $\varphi_*(m_z)$ where m_z is the visual measure on $\mathbb{S}^1$ seen from z. This map f_φ is smooth, and is C_k-quasi-isometric for some constant that depends only on k (it is even δ_k-quasi-conformal or some constant that depends only on k). The map $\varphi \to f_\varphi$ is continuous hence, since $\mathcal{M}_k$ is compact, the points $f_\varphi(0)$ belong to a fixed compact set of $\mathbb{D}$.

By the main result of [20] or [3], the distance $d(h_\varphi, f_\varphi)$ is bounded by a constant M_k that depends only on C_k. The first two claims follow. The Lipschitz continuity of h_φ then follows from the Cheng lemma 3.4. □

Corollary 7.7 *The map $\varphi \in \mathcal{M}_k \to h_\varphi \in C^2(\mathbb{D}, \mathbb{D})$ is continuous in the C^2_{loc} topology.*

Proof Let (φ_n) be a sequence in $\mathcal{M}_k$ converging to φ. By Lemma 7.6, the harmonic maps $h_n := h_{\varphi_n}$ are uniformly locally bounded and uniformly Lipschitz. By Lemma 4.2, after extraction, the sequence (h_n) converges in the C^2_{loc} topology to a harmonic quasi-isometric map $h_\infty : \mathbb{D} \to \mathbb{D}$. To reach the conclusion, we need to prove that such a limit h_∞ is always equal to h_φ. Since the maps h_n are uniformly quasi-isometric, the continuity lemma 7.5 yields that the limit φ of the boundary maps φ_n of h_n must be the boundary map of h_∞. This proves that $h_\infty = h_\varphi$. □

Proof of Proposition 7.3 Let (φ_n) be a sequence in $\mathcal{M}_k$ converging to φ such that all the harmonic quasi-isometric extensions h_{φ_n} are quasiconformal diffeomorphisms. We want to prove that the harmonic map h_φ is also a quasiconformal diffeomorphism.

Corollary 7.7 ensures that the sequence (h_{φ_n}) converges to h_φ in the $C^2_{\rm loc}$ topology. Lemma 7.6 ensures that these maps h_{φ_n} are uniformly Lipschitz. Hence, by Proposition 5.2, there exists a uniform lower bound $j_* > 0$ for the Jacobians of all these harmonic quasi-isometric diffeomorphisms h_{φ_n}. Therefore h_φ is also a Lipschitz harmonic map whose Jacobian is bounded below by j_*. Hence, by the injectivity criterion in Lemma 3.5, the harmonic map h_φ is also a quasiconformal diffeomorphism. □

7.4 Orbit closure in the group of homeomorphisms of $\mathbb{S}^1$

Recall that $\mathbb{D}$ is the hyperbolic disk and $\mathbb{S}^1$ is its boundary at infinity. Let G be the group of isometries of $\mathbb{D}$ acting on $\mathbb{S}^1$. It is isomorphic to $\mathrm{PGL}(2,\mathbb{R})$.

In order to prove Proposition 7.4 in the next section we will need the following lemma.

Lemma 7.8 *Let φ_n be a sequence of C^1 diffeomorphisms of $\mathbb{S}^1$ converging in the C^1 topology to a C^1 diffeomorphism φ_∞ of $\mathbb{S}^1$. Let γ_n and γ'_n be two unbounded sequences in G such that the sequence $\psi_n := \gamma'_n \circ \varphi_n \circ \gamma_n^{-1}$ converges to an homeomorphism ψ_∞ of $\mathbb{S}^1$. Then this limit ψ_∞ belongs to G.*

Proof We recall the Cartan decomposition $G = KA^+K$ of G where K is the group $PO(2,\mathbb{R})$ and $A^+ = \{\mathrm{diag}(s, s^{-1}) \text{ with } s \geq 1\}$. Since K is compact, we can assume that both γ_n and γ'_n are in A^+. We write

$$\gamma_n = \mathrm{diag}(s_n^{1/2}, s_n^{-1/2}) \text{ and } \gamma'_n = \mathrm{diag}({s'_n}^{1/2}, {s'_n}^{-1/2})$$

with both s_n and s'_n converging to ∞. Here it will be convenient to use the identification $\mathbb{S}^1 \simeq \mathbb{R} \cup \{\infty\}$ given by the upper half-plane model of $\mathbb{D}$, so that, for x in $\mathbb{R}$, one has $\gamma_n(x) = s_n x$ and $\gamma'_n(x) = s'_n x$.

We notice that $\varphi_\infty(0) = 0$. Indeed if this were not the case, we would have $\psi_\infty(x) = \infty$ for all $x \in \mathbb{R}$, contradicting the injectivity of ψ_∞.

Similarly we have $\psi_\infty(\infty) = \infty$. Indeed if this were not the case, we would have $\varphi_\infty(x) = 0$ for all $x \in \mathbb{R}$, contradicting the injectivity of φ_∞.

Since the sequence φ_n converges in the C^1 topology to φ_∞, we can write for all $n \geq 1$ and all $x \in \mathbb{R}$ with $|x| \leq 1$

$$\varphi_n(x) = \alpha_n + (\beta_n + r_n(x))x \quad \text{with} \quad \lim_{x\to 0} \sup_{n\in\mathbb{N}} |r_n(x)| = 0\,. \tag{7.1}$$

Since $\varphi_\infty(0) = 0$ and $\beta_\infty := \varphi'_\infty(0)$ is non zero, one has

$$\lim_{n\to\infty} \alpha_n = 0 \text{ and } \lim_{n\to\infty} \beta_n = \beta_\infty > 0\,. \tag{7.2}$$

Therefore we can write for all $n \geq 1$ and all $x \in \mathbb{R}$ with $|x| \leq s_n$

$$\psi_n(x) = s'_n \alpha_n + (\beta_n + r_n(\tfrac{x}{s_n}))\tfrac{s'_n}{s_n} x \quad \text{with} \quad \lim_{n\to\infty} |r_n(\tfrac{x}{s_n})| = 0\,. \tag{7.3}$$

Since the sequences $\psi_n(0)$ and $\psi_n(1)$ converge, the following limits exist

$$\alpha'_\infty := \lim_{n\to\infty} s'_n \alpha_n \in \mathbb{R} \;\text{ and }\; \beta'_\infty := \lim_{n\to\infty} \beta_n \tfrac{s'_n}{s_n} > 0\,, \tag{7.4}$$

Hence one has $\psi_\infty(x) = \alpha'_\infty + \beta'_\infty x$ for all $x \in \mathbb{R}$, and ψ_∞ belongs to G. □

Remark - As can be seen in the proof, the assumption on ψ_n can be weakened: it is sufficient to assume that there are three points ξ_0, ξ_1, ξ_∞ in $\mathbb{S}^1$ whose images $\psi_n(\xi_0)$, $\psi_n(\xi_1)$, $\psi_n(\xi_\infty)$ converge to three distinct points. This ensures that the sequence ψ_n converges uniformly to an element ψ_∞ of G.
- However, it is important to assume that the limit φ_∞ is of class C^1 and that the convergence $\varphi_n \to \varphi_\infty$ is in the C^1 topology.

Here is a direct corollary of Lemma 7.8 in the spirit of [2].

Corollary 7.9 *For all C^1 diffeomorphism φ of $\mathbb{S}^1$, one has the equality*
$$\overline{G\varphi G} \cap \mathcal{H}omeo(\mathbb{S}^1) \;=\; G\varphi G \cup G.$$

7.5 When the boundary map is a C^1 diffeomorphism

We now conclude the proof of Theorem 7.1 by giving the last argument:

Proof of Proposition 7.4 Let φ be a C^1 diffeomorphism of $\mathbb{S}^1$. We want to prove that the harmonic quasi-isometric extension h_φ of φ is a quasi-conformal diffeomorphism. For convenience we identify here $\mathbb{S}^1$ with $\mathbb{R}/2\pi\mathbb{Z}$. For $t \in [0,1]$, we introduce the C^1 diffeomorphism φ_t given by

$$\varphi_t(\xi) = \xi + (\varphi(\xi) - \xi)\, t \quad \text{for all } \xi \text{ in } \mathbb{S}^1.$$

This is well defined since the map $\xi \to \varphi(\xi) - \xi$ lifts as a map from $\mathbb{S}^1$ to $\mathbb{R}$.

We argue as in Section 3.4. For $t \in [0,1]$ we introduce the harmonic quasi-isometric extension $h_t = h_{\varphi_t} : \mathbb{D} \to \mathbb{D}$ of φ_t. Let A be the set of parameters $t \in [0,1]$ for which h_t is a quasi-conformal diffeomorphism. By the injectivity criterion of Lemma 3.5, one has

$$A = \{t \in [0,1] \mid \inf_{z\in\mathbb{D}} J_t(z) > 0\}$$

where J_t is the Jacobian of h_t. We want to prove that $1 \in A$. We already know that $0 \in A$ because h_0 is the identity. Since the maps φ_t are uniformly quasi-symmetric, Proposition 7.3 tells us that A is closed. Therefore it is enough to check that A is open.

Assume by contradiction that there exists a sequence $t_n \notin A$ converging to $t_\infty \in A$. By assumption there exists a sequence (z_n) in $\mathbb{D}$ such that $\liminf_{n\to\infty} J_{t_n}(z_n) \leq 0$. After extraction we are in one of the two cases:

First case The sequence (z_n) converges to a point $z_\infty \in \mathbb{D}$.
Since the maps φ_t are uniformly quasi-symmetric, Corollary 7.7 ensures that the map $t \in [0,1] \to h_t \in C^2(\mathbb{D},\mathbb{D})$ is continuous in the $C^2_{\rm loc}$ topology. Therefore, one has $J_{t_\infty}(z_\infty) = \lim\limits_{n\to\infty} J_{t_n}(z_n) \leq 0$, and t_∞ is not in A. Contradiction.

Second case The sequence (z_n) goes to infinity.
To simplify we set $\varphi_n = \varphi_{t_n}$ and $h_n = h_{t_n}$ for all $n \in \mathbb{N} \cup \{\infty\}$. By Lemma 7.6, the sequence $h_n(z_n)$ goes to infinity. We choose sequences (γ_n) and (γ'_n) in G with $\gamma_n(z_n) = 0$ and $\gamma'_n(h_n(z_n)) = 0$. We introduce the harmonic maps

$$h'_n := \gamma'_n \circ h_n \circ \gamma_n^{-1} \quad : \quad \mathbb{D} \to \mathbb{D}$$

and their boundary values $\psi_n := \gamma'_n \circ \varphi_n \circ \gamma_n^{-1}$. By construction, one has

$$h'_n(0) = 0 \;\text{ and }\; \liminf_{n\to\infty} J'_n(0) \leq 0\,, \tag{7.5}$$

where J'_n is the Jacobian of h'_n. Moreover by Lemma 7.6, these maps h'_n are uniformly Lipschitz. Therefore, after extraction, they converge in the $C^2_{\rm loc}$ topology to a harmonic quasi-isometric map h'_∞. By the continuity lemma 7.5, the sequence of boundary maps ψ_n converge to the boundary map ψ_∞ of h'_∞. Now, by Lemma 7.8, this limit ψ_∞ belongs to G. Therefore the harmonic map h'_∞ is an isometry and its Jacobian is $J'_\infty \equiv 1$. This contradicts (7.5). □

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