Quasicircles and the conformal group

Yves Benoist & Dominique Hulin

Abstract

We prove that a Jordan curve in the 2-sphere is a quasicircle if and only if the closure of its orbit under the action of the conformal group contains only points and Jordan curves.

1 Introduction

The 2-sphere $\mathbb{S}^2$, oriented and equipped with its standard conformal structure, is isomorphic to the complex projective line $\mathrm{P}^1\mathbb{C} \simeq \mathbb{C} \cup \{\infty\}$.

Let $K \geq 1$. By definition (see [16] or paragraph 2.1), a K-quasicircle $c \subset \mathbb{S}^2$ is the image $c = f(c_0)$ of a circle $c_0 \subset \mathbb{S}^2$ under a K-quasiconformal homeomorphism $f : \mathbb{S}^2 \to \mathbb{S}^2$.

Our aim in this paper is to characterize those Jordan curves in $\mathbb{S}^2$ that are quasicircles in terms of their orbit under the action of the conformal group $G := \mathrm{Conf}^+(\mathbb{S}^2) \simeq \mathrm{PSl}_2\mathbb{C}$.

Let $\mathcal{K}$ denote the set of nonempty compact subsets of $\mathbb{S}^2$ equipped with the Hausdorff distance. This space $\mathcal{K}$ is a compact metric space. Observe that, when $C \subsetneq \mathbb{S}^2$ is any proper compact subset of $\mathbb{S}^2$, the closure $\overline{G\,C} \subset \mathcal{K}$ of its orbit in $\mathcal{K}$ contains all singletons in $\mathbb{S}^2$. We thus also introduce $\mathcal{K}_0 \subset \mathcal{K}$, the set of compact subsets of $\mathbb{S}^2$ distinct from a singleton.

Theorem 1.1 *Let $K \geq 1$. The set of all K-quasicircles of $\mathbb{S}^2$ is a closed G-invariant subset of $\mathcal{K}_0$.*

Conversely, any closed G-invariant subset of $\mathcal{K}_0$ which consists only of Jordan curves is included in the set of K-quasicircles for some $K \geq 1$.[1]

A straightforward consequence is the following topological characterization of quasicircles.

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[1]After publication of this preprint, we learned that its main result was already included in the nice paper by V. Aseev and D. Kuzin "Continua of bounded turning: chain conditions and infinitesimal connectedness", Siberian Math. Journal, **41** 801-810 (2000).

Corollary 1.2 *A Jordan curve $c \subset \mathbb{S}^2$ is a quasicircle if and only if its orbit closure $\overline{G\,c}$ in $\mathcal{K}$ consists only of points and Jordan curves.*

In other words, quasicircles are characterized among Jordan curves by the fact that "when zooming in, one sees nothing but Jordan curves".

Since they were first introduced by Pfluger and Tienari in the early 60's, a number of various characterizations of quasicircles has been progressively discovered. This history leading to an impressive list of equivalent definitions is the subject of the nice and recent book "The ubiquitous quasidisk" by Gehring and Hag [9]. Our new characterization has yet a different flavour.

The paper is organized as follows. In Section 2, we briefly recall the definition of quasiconformal maps and quasicircles, and prove that the limit in $\mathcal{K}$ of a convergent sequence of K-quasicircles is either a point or a K-quasicircle. This result, which is the first part of Theorem 1.1, follows readily from a standard compactness result for K-quasiconformal homeomorphisms of $\mathbb{S}^2$. We also recall the so-called Ahlfors' arc condition, which is a criterion for a Jordan curve in $\mathbb{S}^2$ to be a quasicircle. In section 3 we outline the proof of the second part of Theorem 1.1. It involves three intermediate results : Propositions 3.1, 3.3 and 3.4. We fill in on the details of these three propositions in Section 4 where we address topology of the plane, in Section 5 where we consider maximal disks in Jordan domains of $\mathbb{S}^2$, and in Section 6 where we examine finite sequences of real numbers. We wrap up the proof of Theorem 1.1 in Section 7.

In Section 8, we will explain an analog of Theorem 1.1 where Jordan curves are replaced by Cantor sets (Theorem 8.1). In Section 9, we give an elementary proof of a technical result (Proposition 5.6) which is needed in our proof.

2 Limits of K-quasicircles

In this section, we prove the first part of Theorem 1.1. It relies on the classical compactness property of K-quasiconformal maps (Theorem 2.1). We also recall Ahlfors' characterization of quasicircles (Theorem 2.4).

2.1 Quasiconformal maps and quasicircles

Let us first recall the definition of quasiconformal maps (see [2] or [13]).

A quadrilateral Q is a Jordan domain in $\mathbb{S}^2$, together with a cyclically ordered quadruple of boundary points. We say that two quadrilaterals Q and Q' are conformally equivalent when there exists an homeomorphism

$\varphi : \overline{Q} \to \overline{Q'}$ between their closures that sends the vertices of Q to the vertices of Q', and whose restriction $\varphi : Q \to Q'$ is a conformal map.

Any quadrilateral Q is conformally equivalent to a rectangle R with vertices $(0, x, x+iy, iy)$ where x and y are positive. The conformal modulus of the quadrilateral Q is then defined as $m(Q) = m(R) := x/y$.

A homeomorphism $f : \mathbb{S}^2 \to \mathbb{S}^2$ is said to be K-quasiconformal ($K \geq 1$) if the inequalities

$$K^{-1} m(Q) \leq m(f(Q)) \leq K m(Q)$$

hold for any quadrilateral $Q \subset \mathbb{S}^2$.

A conformal homeomorphism $f : \mathbb{S}^2 \to \mathbb{S}^2$ is 1-quasiconformal. It can be proved that the converse is true, that is, a 1-quasiconformal homeomorphism is actually conformal ([13], Theorem I.5.1). It follows immediately from the definition that, when $f_i : \mathbb{S}^2 \to \mathbb{S}^2$ are K_i-quasiconformal homeomorphisms ($i = 1, 2$), the composed map $f_1 \circ f_2 : \mathbb{S}^2 \to \mathbb{S}^2$ is $K_1 K_2$-quasiconformal.

As already mentionned, a K-quasicircle $c \subset \mathbb{S}^2$ is the image $c = f(c_0)$ of a circle $c_0 \subset \mathbb{S}^2$ under a K-quasiconformal homeomorphism $f : \mathbb{S}^2 \to \mathbb{S}^2$.

2.2 Compactness for quasiconformal maps

Equip the 2-sphere $\mathbb{S}^2$ with its canonical Riemannian metric d. The main property of quasiconformal maps that will be used in this paper is the following fundamental compactness theorem.

Theorem 2.1 [13, Theorems II.5.1 and II.5.3] *Let $K \geq 1$, z_1, z_2, z_3 be three distinct points in $\mathbb{S}^2$, and $f_n : \mathbb{S}^2 \to \mathbb{S}^2$ be a sequence of K-quasiconformal homeomorphisms such that the three sequences $(f_n(z_i))_{n\geq 1}$ converge to three distinct points. Then, there exists a subsequence (f_{n_k}) and a K-quasiconformal homeomorphism $f : \mathbb{S}^2 \to \mathbb{S}^2$ such that $f_{n_k} \to f$ uniformly on $\mathbb{S}^2$.*

We will infer that the limit of a sequence of K-quasicircles that converges to a compact set which is not a point is also a K-quasicircle. Moreover, we will prove that one can choose parameterizations for this sequence of quasicircles that converge to a parameterization of the limit. This means that a convergent sequence of K-quasicircles cannot fold several times over its limit, that is, the configuration in Figure 1 is forbidden.

Let $\mathbb{S}^1 := \mathbb{R} \cup \{\infty\}$ denote the standard circle in $\mathbb{S}^2 \simeq \mathbb{C} \cup \{\infty\}$.

Proposition 2.2 *Let $c_n \subset \mathbb{S}^2$ be a sequence of K-quasicircles that converges in $\mathcal{K}$ to a compact set $c_\infty \subset \mathbb{S}^2$ which is not a point. After going to a subsequence if necessary, there exist K-quasiconformal homeomorphisms $f_n : \mathbb{S}^2 \to \mathbb{S}^2$ and $f_\infty : \mathbb{S}^2 \to \mathbb{S}^2$ with $c_n = f_n(\mathbb{S}^1)$, $c_\infty = f_\infty(\mathbb{S}^1)$, and such that $f_n \to f_\infty$ uniformly on $\mathbb{S}^2$.*

Proof Let h_n be a K-quasiconformal homeomorphism of $\mathbb{S}^2$ such that $c_n = h_n(\mathbb{S}^1)$. The limit $c_\infty = \lim c_n$, as a limit of compact connected sets, is also compact and connected. Since c_∞ is not a singleton, it contains at least two, hence three distinct points x_∞, y_∞ and z_∞.

For each $n \in \mathbb{N}$, one can pick three distinct points x_n, y_n, z_n in $\mathbb{S}^1$ such that $h_n(x_n) \to x_\infty$, $h_n(y_n) \to y_\infty$ and $h_n(z_n) \to z_\infty$. Let $\gamma_n \in \mathrm{PSl}_2\mathbb{R} \subset \mathrm{PSl}_2\mathbb{C}$ be the conformal transformation of $\mathbb{S}^2$ that preserves $\mathbb{S}^1$ and that sends 0 to x_n, 1 to y_n and ∞ to z_n. Each map $f_n := h_n \circ \gamma_n : \mathbb{S}^2 \to \mathbb{S}^2$ is still a K-quasiconformal homeomorphism of $\mathbb{S}^2$ such that $c_n = f_n(\mathbb{S}^1)$. We now have $f_n(0) \to x_\infty$, $f_n(1) \to y_\infty$ and $f_n(\infty) \to z_\infty$. Thus Theorem 2.1 applies to the sequence (f_n), and yields the result. □

This proves the first part of Theorem 1.1. To prove the second part of Theorem 1.1, we will use the following characterization of quasicircles due to Ahlfors.

2.3 Ahlfors' arc condition

We recall that a Jordan arc $a \subset \mathbb{S}^2$ is a subset of $\mathbb{S}^2$ which is homeomorphic to the closed interval $[0, 1]$.

Definition 2.3 *A Jordan curve $c \subset \mathbb{S}^2$ satisfies the arc condition with constant $A \geq 1$ if, for any pair of points $x, z \in c$ delimiting two Jordan arcs a_+, a_- on c, their diameters satisfy*

$$\min(\mathrm{diam}(a_+), \mathrm{diam}(a_-)) \leq A\, d(x, z)\,.$$

Theorem 2.4 (Ahlfors, see [1] and [9, Theorem 2.2.5]) *A Jordan curve $c \subset \mathbb{S}^2$ is a quasicircle if and only if it satisfies the arc condition.*

The implied constants depend only on each other.

3 An overview of the proof of Theorem 1.1

We sketch the proof of the second part of Theorem 1.1. It will consist of four propositions that will be proved in the following sections.

We assume that

$\mathcal{F}$ is a closed G-invariant subset of $\mathcal{K}_0$, which consists only of Jordan curves and such that, for all $A \geq 1$, there exists a Jordan curves in $\mathcal{F}$ that does not satisfy Ahlfors' arc condition with constant A. (3.1)

Under this assumption, we want to find a contradiction. The first step is the following and will be completed in Section 4.

Proposition 3.1 Two threads with the same limit. *Assume* (3.1)*. Then, there exist a sequence* (c_n) *in* $\mathcal{F}$ *that converges to a Jordan curve* $c_\infty \in \mathcal{F}$*, a Jordan arc* $a_\infty \subset c_\infty$ *and, for each* $n \in \mathbb{N}$*, two disjoint Jordan arcs* a_n *and* a'_n *in* c_n *such that both sequences* (a_n) *and* (a'_n) *converge to* a_∞ *when* $n \to \infty$*.*

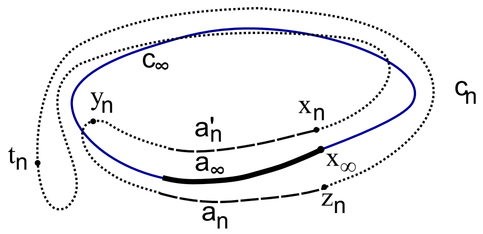

Figure 1: Two threads with the same limit. Note that this cannot happen in a convergent sequence of K-quasicircles.

We will assume from now on, without loss of generality, that the Jordan arc a_∞ lies in $\mathbb{C} \subset \mathbb{C} \cup \{\infty\} \sim \mathbb{S}^2$. What we have in mind to obtain our contradiction is now to zoom in, that is to replace each c_n by another Jordan curve $\gamma_n c_n \in \mathcal{F}$ where $\gamma_n \in \mathrm{PSl}_2\mathbb{C}$ fixes the point ∞, and examin the behaviour of the arcs $\gamma_n a_n$ and $\gamma_n a'_n$.

To achieve this goal, we will first associate to each one of the Jordan curves c_n a "pearl necklace", that is a sequence of disks roughly joining the endpoints of a_∞ and channelled by the arcs a_n and a'_n (see Figure 2). This necklace will grow thinner as $n \to \infty$. The precise statement, which constitutes our second step and will be proved in Section 5, is as follows.

Definition 3.2 *Let* $U \subsetneq \mathbb{C}$ *be an open subset. A necklace* $N = (D(i) | i \in I)$ *in* U *is a sequence of open disks* $D(i) \subset U$ *of the complex plane, indexed by a finite interval* $I \subset \mathbb{Z}$*, and that satisfy the following conditions :*

1. *two consecutive disks* $D(i)$ *and* $D(i+1)$ *intersect orthogonally*
2. *when* $|i - j| \geq 2$*, the disks* $D(i)$ *and* $D(j)$ *do not intersect*
3. *for each three consecutive disks, the set* $\partial D(i) \setminus (D(i-1) \cup D(i+1))$ *is a disjoint union of two arcs that both meet the boundary* ∂U*.*

The thickness of the necklace N *is the ratio*

$$\max_{i \in I} \mathrm{diam} D(i) / \mathrm{diam}(\cup_{i \in I} D(i))$$

where diam *denotes the diameter with respect to the Euclidean distance on* $\mathbb{C}$*. The necklace is said to be* ε*-thin if its thickness is bounded by* ε*.*

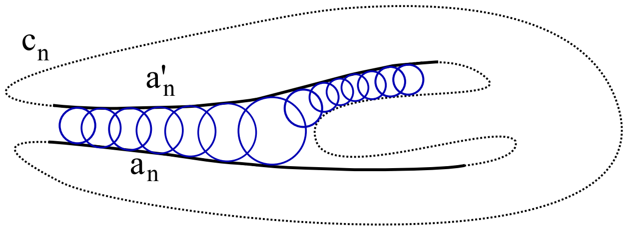

Figure 2: A pearl necklace

Proposition 3.3 Thin pearl necklaces. *Assume* (3.1)*. Then, for each $\varepsilon > 0$, there exist a Jordan curve $c_\varepsilon \in \mathcal{F}$ and an ε-thin necklace $N_\varepsilon = (D_\varepsilon(i)|i \in I_\varepsilon)$ in the domain $U_\varepsilon = \mathbb{C} \setminus c_\varepsilon$.*

As mentioned above, the last step of the proof will consist in zooming in on a well chosen pearl of each of these necklaces to obtain a contradiction. To this effect, we associate to each necklace N_ε provided by Proposition 3.3 the sequence

$$x_\varepsilon = (x_\varepsilon(i)|i \in I_\varepsilon)\,, \text{ where } x_\varepsilon(i) = \log \mathrm{diam} D_\varepsilon(i)\,,$$

of the logarithms of the diameters of the $D_\varepsilon(i)$'s.

Our third proposition is a very general statement on families of sequences of real numbers and will be proved in Section 6.

Proposition 3.4 Finite sequences of real numbers. *Let $\mathcal{S}$ be a family of sequences $x = (x(i)\,|\,i \in I_x)$ of real numbers, indexed by finite intervals $I_x \subset \mathbb{Z}$. Then at least one of the following three possibilities occurs.*

1. There are pipes in $\mathcal{S}$ *: there exists a thickness $h_1 > 0$ such that, for any length ℓ, there exists a sequence $x \in \mathcal{S}$ and a subinterval $J \subset I_x$ with length $|J| = \ell$ such that*

$$|x(i) - x(j)| \leq h_1 \quad \textit{for any } i, j \in J.$$

2. There are wells in $\mathcal{S}$ *: for any depth $h_2 > 0$, there exist a sequence $x \in \mathcal{S}$ and $i < j < k$ in I_x such that*

$$x(j) < x(i) - h_2 \ \text{ and } \ x(j) < x(k) - h_2.$$

3. All sequences in $\mathcal{S}$ are slides *: there exist a height $h_3 \in \mathbb{R}$ and a slope $\sigma > 0$ such that for any sequence $x \in \mathcal{S}$ and any index $i_0 \in I_x$ such that $x(i_0) = \max\{x(i),\ i \in I_x\}$, one has*

$$x(i) \leq x(i_0) + h_3 - \sigma|i - i_0| \quad \textit{for any } i \in I_x.$$

By applying Proposition 3.4 to the family $\mathcal{S}_{\mathcal{F}}$ of finite sequences associated to the necklaces in Proposition 3.3 we will obtain Proposition 3.5 which provides us with the desired contradiction.

Proposition 3.5 Excluding the three cases. *Assume* (3.1)*. Then, there are no pipes in* $\mathcal{S}_{\mathcal{F}}$*. There are no wells in* $\mathcal{S}_{\mathcal{F}}$*. Not all the sequences in* $\mathcal{S}_{\mathcal{F}}$ *are slides.*

Proposition 3.5 will be proved in Section 7.

4 Constructing the two threads

Proof of Proposition 3.1 Recall that $\mathcal{F} \subset \mathcal{K}_0$ is a closed G-invariant subset of $\mathcal{K}_0$ which consists only of Jordan curves, and that the Jordan curves in $\mathcal{F}$ do not satisfy a uniform Ahlfors' arc condition. This means that we can find a sequence $c_n \in \mathcal{F}$ and, for each $n \in \mathbb{N}$, a cyclically ordered quadruple (x_n, y_n, z_n, t_n) of points on c_n such that

$$\min(d(x_n, y_n), d(x_n, t_n)) \geq n\, d(x_n, z_n)\,.$$

Note that this forces $d(x_n, z_n) \to 0$. Replacing if necessary each Jordan curve c_n by $\gamma_n(c_n) \in \mathcal{F}$, where $\gamma_n \in \mathrm{PSl}_2\mathbb{C}$ is a suitable zoom in on the point x_n, we may assume moreover that

$$r := \inf_{n\in\mathbb{N}} \min(d(x_n, y_n), d(x_n, t_n)) \;\text{ is positive}\,.$$

We want to prove that there exist, for each $n \in \mathbb{N}$, two disjoint Jordan arcs a_n and a'_n in c_n that converge to the same Jordan arc a_∞ when $n \to \infty$.

The sphere $\mathbb{S}^2$ and the space $\mathcal{K}$ are compact metric spaces. Going to a subsequence, we may thus also assume that :

– there exist three points x_∞, y_∞ and t_∞ in $\mathbb{S}^2$ with $x_\infty \neq y_\infty$ and $x_\infty \neq t_\infty$, and such that

$$x_n \to x_\infty,\; z_n \to x_\infty,\; y_n \to y_\infty,\; t_n \to t_\infty \quad \text{when } n \to \infty$$

– the sequence of Jordan curves (c_n) converges in $\mathcal{K}$. Its limit c_∞ is not a singleton since it contains the points $x_\infty \neq y_\infty$. Since $\mathcal{F}$ is a closed subset of $\mathcal{K}_0$, the limit c_∞ belongs to $\mathcal{F}$: it is a Jordan curve.

Consider the four Jordan arcs

$$[x_n, y_n],\; [x_n, t_n],\; [z_n, y_n] \text{ and } [z_n, t_n].$$

We shorten each of them, keeping the first endpoint x_n or z_n, in order to get a Jordan arc whose diameter is exactly $r/2$. Going again to a subsequence, we may assume that each one of these four sequences of shortened arcs converges in $\mathcal{K}$. Their respective limits α_1, α_2, α_3 and α_4 are compact connected subsets of c_∞ that contain x_∞ and have diameter $r/2$ so that they all are Jordan arcs in c_∞. Thus there exists a Jordan arc $a_\infty \subset c_\infty$ that contains x_∞ as an endpoint and that is a subarc of at least two of the limit sets α_i (see Figure 1). Proposition 3.1 now follows from Lemma 4.1. □

Lemma 4.1 *Let $\alpha_n \subset \mathbb{S}^2$ be a sequence of Jordan arcs converging to a Jordan arc α_∞. Let a_∞ be a Jordan subarc of α_∞. Then, there exists a sequence of Jordan subarcs $a_n \subset \alpha_n$ such that a_n converges to a_∞.*

Proof Using Jordan theorem, one may assume that $\alpha_\infty \subset \mathbb{S}^1$ and $a_\infty = [-1, 1]$. The proof in this case is left to the reader. □

5 Pearl necklaces

5.1 Normal disks and Thurston's stratification

Recall that $\mathbb{S}^2$ is equipped with its canonical Möbius structure. Let $U \subset \mathbb{S}^2$ be a connected domain that avoids at least two points. W. Thurston introduced a stratification of the domain U associated to the family of maximal disks $D \subset U$ sitting in U. We briefly recall the construction of this stratification, and the facts we will be using in this paper.

Any disk $D \subset \mathbb{S}^2$ carries a conformal hyperbolic metric, whose geodesics are arcs of circles that cut the boundary ∂D of D orthogonally. The convex hull, for this metric, of a subset $A \subset \partial D$ will be denoted by $\mathrm{conv}_D(A) \subset D$.

Definition 5.1 *An open disk $D \subset U$ is normal when its boundary ∂D meets ∂U in at least two points. When $D \subset U$ is a normal disk, define*

$$C(D) = \mathrm{conv}_D(\partial D \cap \partial U)\,.$$

Note that a normal disk $D \subset U$ is maximal among the disks sitting in U.

Proposition 5.2 (W. Thurston) *For any point $p \in U$, there exists a unique normal disk $D_p \subset U$ such that $p \in C(D_p)$. This disk D_p depends continuously on the point $p \in U$.*

This means that the convex hulls $C(D)$ of all the normal disks provide a stratification of U.

A proof of this proposition is given in [11, Theorem 1.2.7], [12], [7] or [3, Chapter 4]. See also [15] and [8] for other applications of this construction. For the convenience of the reader we include a short and elementary proof.

Proof of Proposition 5.2 We may assume $U \subset \mathbb{C}$.

Uniqueness Just notice that, for any two open disks D_1 and D_2 in $\mathbb{C}$, the convex hulls $\mathrm{Conv}_{D_1}(\partial D_1 \setminus D_2)$ and $\mathrm{Conv}_{D_2}(\partial D_2 \setminus D_1)$ do not meet.

Existence Let $p \in U$. We will use the inversion $j_p : z \mapsto (z-p)^{-1}$ of the sphere. We introduce the compact subset $K_p := j_p(\mathbb{S}^2 \setminus U)$ of $\mathbb{C}$. Recall that there exists a unique closed disk Δ_p of $\mathbb{C}$ with minimal radius that contains K_p. Moreover the intersection $\partial\Delta_p \cap \partial K_p$ is not included in an open arc of $\partial\Delta_p$ whose endpoints are diametrically opposed. Thus the open disk $D_p := j_p^{-1}(\mathbb{S}^2 \smallsetminus \Delta_p)$ is a normal disk of U and the point p belongs to the convex hull $C(D_p)$.

Continuity Since the compact set K_p depends continuously on the point p, the disks Δ_p and D_p also depend continuously on p. $\square$

Notation 5.3 *When $p \in U$, we will denote by $C_p := C(D_p) \subset U$ the stratum that contains the point p.*

When $D \subset U$ is a normal disk, let $\lambda(D) \subset U$ denote the boundary in the disk D of the convex hull $C(D)$. This boundary $\lambda(D)$ has a finite or countable number of connected components, which are arcs of circles.

Assume from now on the domain U to be simply connected. Then, when $D \subset U$ is a normal disk, $U \setminus C(D)$ is not connected. More precisely, we have the following lemma whose proof is left to the reader.

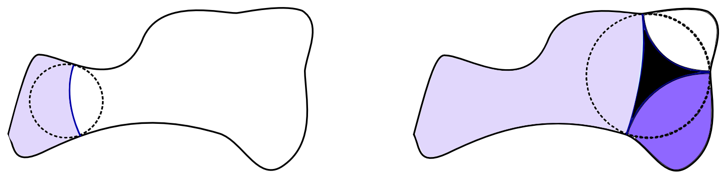

Figure 3: A normal disk D and the connected components of $U \setminus C(D)$

Lemma 5.4 *Let $U \subset \mathbb{S}^2$ be a simply connected domain that avoids at least two points, and $D \subset U$ be a normal disk. If $\partial D \cap \partial U$ contains only two points, then $U \setminus C(D)$ has two connected components. If $\partial D \cap \partial U$ contains at least three points then, there is a natural bijection between the set of connected components Ω of $U \setminus C(D)$ and the set of connected components of $\lambda(D)$. It is given by $\Omega \longrightarrow \partial\Omega \cap U$.*

5.2 Monotonous paths

We now introduce monotonous paths in the simply connected domain U.

Definition 5.5 *Let x, y, z be three points of U. We say that y lies between x and z if one cannot find a connected component of $U \setminus C_y$ that contains both x and z. A path $\gamma : [0,1] \to U$ is monotonous if, for any parameters $0 \le r \le s \le t \le 1$, the point $\gamma(s)$ lies between $\gamma(r)$ and $\gamma(t)$.*

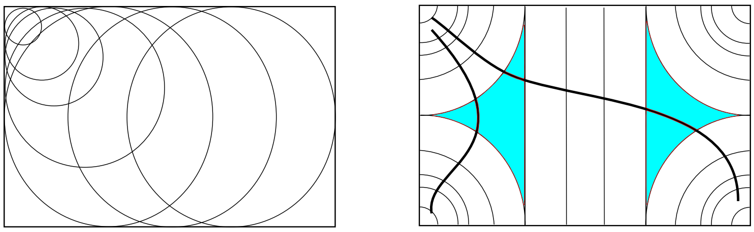

Figure 4: Maximal disks, the stratification and monotonous paths in a rectangle

Proposition 5.6 Monotonous paths. *Let $U \subset \mathbb{S}^2$ be a simply connected domain that avoids at least two points. Let $p, q \in U$. Then there exists a monotonous path $\mu : [0,1] \to U$ with $\mu(0) = p$ and $\mu(1) = q$.*

A proof of this proposition is given in [12, Section 11.1], where μ is obtained as a geodesic for the Thurston metric. Recall that the Thurston metric is a complete $C^{1,1}$ metric on U with non positive curvature, for which the strata C_p $(p \in U)$ are convex (see [12, Section 5] or [5]; see also [10, Chapter 5] for more general constructions of monotonous paths).

We will give an elementary proof of Proposition 5.6 in Section 9.

5.3 Pearl necklaces

To a monotonous path between $p, q \in U$, we will associate a pearl necklace.

Proposition 5.7 Pearl necklaces. *Let $U \subset \mathbb{S}^2$ be a simply connected domain that avoids at least two points. Let $p, q \in U$. Then there exists a pearl necklace $(D(i))_{1 \leq i \leq n}$ in U that joins the points p and q, that is, such that $D(1) = D_p$ and $D(n) \cap D_q \neq \emptyset$.*

The proof will follow from a series of lemmas that describe the behaviour of the normal disks along a monotonous path.

Notation 5.8 *Let $\mu : [0,1] \to U$ be a monotonous path and $s \in [0,1]$, We let Ω_s^- (resp. Ω_s^+) be the connected component of $U \setminus C_{\mu(s)}$ that meets $\mu([0,s])$ (resp. $\mu([s,1])$) if such a connected component does exist. Otherwise we let Ω_s^- (resp. Ω_s^+) be the empty set.*

Roughly, starting at time s, the past of μ lies in Ω_s^- and its future in Ω_s^+.

Lemma 5.9 *When $0 \leq r \leq s \leq t \leq 1$, we have the inclusions*

$$D_{\mu(t)} \setminus D_{\mu(s)} \subset \Omega_s^+ \quad \textit{and} \quad D_{\mu(r)} \setminus D_{\mu(s)} \subset \Omega_s^- .$$

Proof It suffices to prove the first assertion. Assume that $D_{\mu(s)} \neq D_{\mu(t)}$. Then, $C_{\mu(t)}$ is connected and disjoint from $C_{\mu(s)}$. Since $\mu(t) \in C_{\mu(t)}$, it follows that $C_{\mu(t)} \subset \Omega_s^+$. The result follows since $D_{\mu(t)} \setminus D_{\mu(s)}$ is connected and $C_{\mu(t)}$ does not entirely lie in $D_{\mu(s)}$. □

Lemma 5.10 *Let* $0 \leq r \leq s \leq t \leq 1$*.*
a) We have the inclusion $D_{\mu(r)} \cap D_{\mu(t)} \subset D_{\mu(s)}$ *.*
b) If $D_{\mu(s)}$ *meets both* $D_{\mu(r)}$ *and* $D_{\mu(t)}$ *and is equal to none of them, the set* $\partial D_{\mu(s)} \setminus (D_{\mu(r)} \cup D_{\mu(t)})$ *is a disjoint union of two arcs, each of them meeting the boundary* ∂U*.*

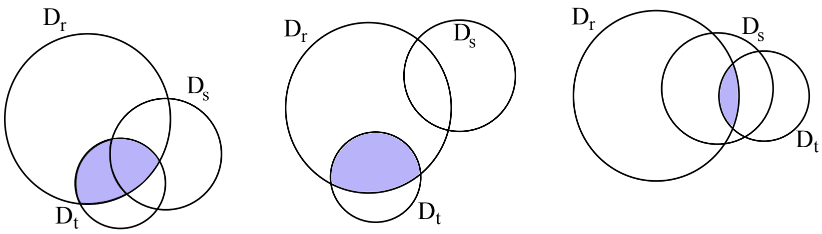

Figure 5: Two forbidden configurations, and a legit one ($r < s < t$)

Proof a) Lemma 5.9 ensures that $(D_{\mu(r)} \cap D_{\mu(t)}) \setminus D_{\mu(s)}$ lies in both Ω_s^- and Ω_s^+, hence is empty.

b) If $\partial D_{\mu(s)} \cap \partial U$ were included in only one arc of $\partial D_{\mu(s)} \setminus (D_{\mu(r)} \cup D_{\mu(t)})$, then the points $\mu(r)$ and $\mu(t)$ would belong to the same connected component of $U \setminus C_{\mu(s)}$. A contradiction to the monotonicity of μ. □

Lemma 5.11 *Let* $r \in [0, 1]$*. Then, there exists at most one disk* $D_{\mu(s)}$*, with* $s \geq r$*, that is orthogonal to the disk* $D_{\mu(r)}$*.*

Proof Assume that $r < s < t$ and that $D_{\mu(r)}$ is orthogonal to both $D_{\mu(s)}$ and $D_{\mu(t)}$. Lemma 5.10 forces the inclusion $D_{\mu(t)} \subset D_{\mu(s)}$ of these maximal disks, hence the equality $D_{\mu(t)} = D_{\mu(s)}$. □

Proof of Proposition 5.7 Let $p, q \in U$. According to Proposition 5.6, there exists a monotonous path $\mu : [0, 1] \to U$ from p to q. Let $t_1 = 0$ and define recursively, when it is possible, $t_{i+1} \in [t_i, 1]$ as the only parameter in the future of t_i for which the disks $D(i) := D_{\mu(t_i)}$ and $D(i + 1) := D_{\mu(t_{i+1})}$ are orthogonal (Lemma 5.11).

We end up with a chain of normal disks $(D(i))$, where $i \geq 1$. Since the distance of the image $\mu([0, 1]) \subset \mathbb{S}^2$ to ${}^cU$ is non zero, the diameters of these disks are uniformly bounded below, hence the orthogonality of consecutive disks ensures that this chain has finite cardinality n.

By construction, there is no parameter $t \in [t_n, 1]$ with $D_{\mu(t)}$ orthogonal to $D(n)$. Hence, since the normal disk D_x depends continuously on the point $x \in U$, it follows that the disk $D(n)$ intersects $D_{\mu(t)}$ for every $t \in [t_n, 1]$. In particular, $D(n)$ meets D_q.

We must now prove that this chain is a necklace in U. Condition 1 and 3 in Definition 3.2 follow from the construction and from Lemma 5.10. We check now Condition 2. Let $1 \leq i < j \leq n$, with $j - i \geq 2$, we want to prove that $D(i)$ and $D(j)$ are disjoint. According to Lemma 5.10, the intersection $D(i) \cap D(j)$ is included in $D(i) \cap D(i+2)$. This is empty since both $D(i)$ and $D(i+2)$ intersect $D(i+1)$ orthogonally and since, by the same Lemma 5.10, the set $\partial D(i+1) \setminus (D(i) \cup D(i+2))$ is a union of two disjoint arcs. □

5.4 Thin necklaces

We prove Proposition 3.3. Recall that we assume that the arc a_∞ provided by Proposition 3.1 lies in $\mathbb{C}$.

As mentionned in the previous paragraph, there always exist pearl necklaces in any domain $U \subsetneq \mathbb{C}$. However the thickness of these necklaces (see Definition 3.2) may well be uniformly bounded below. It is the case for example when U is a triangle. On the contrary, one can find arbitrarily thin necklaces in a domain whose boundary is a piecewise C^1 curve that admits a cusp.

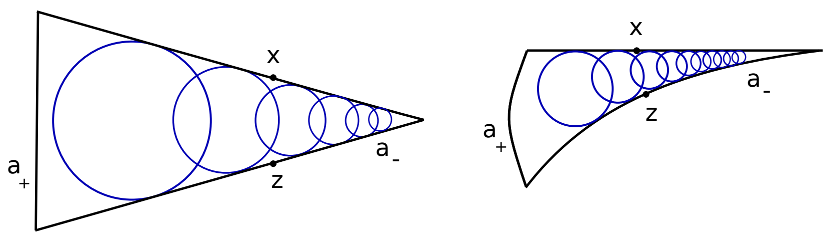

Figure 6: A triangle satisfies the Ahlfors' arc condition, and it does not contain arbitrarily thin necklaces. A Jordan domain with a cusp does not satisfy the Ahlfors' arc condition, and it contains arbitrarily thin necklaces.

Let (c_n) be the sequence of Jordan curves in $\mathcal{F}$ provided by Proposition 3.1. We will use Proposition 5.7, to construct a necklace of $\mathbb{S}^2 \setminus c_n$, that is drawn within a small neighbourhood of the arc a_∞ and that roughly joins its endpoints. This necklace will grow thinner when $n \to \infty$. We begin with a general lemma.

Lemma 5.12 *Let $a \subset \mathbb{C}$ be a Jordan arc. Let $\varepsilon > 0$.*

1. *There exists $\eta > 0$ such that the diameter of any disk lying in the η-neighbourhood of a is at most ε.*

2. *There exists $r > 0$ such that, for any Jordan curve Γ sitting in the r-neighbourhood of a, the bounded connected component of $\mathbb{C} \setminus \Gamma$ lies in the η-neighbourhood of the arc a.*

Proof 1. If $V_\eta(a)$ denotes the η-neighbourhood of a, one has $a = \cap_{\eta>0} V_\eta(a)$. Proceed by contradiction and assume that there exists $r_0 > 0$ and a sequence of disks $D(x_n, r_0)$ of center $x_n \in \mathbb{C}$ and radius r_0 that are included in $V_{1/n}a$. The sequence $(x_n)_{n\geq 1}$ being bounded, we may assume that it converges to x_∞. We would then have $D(x_\infty, r_0) \subset a$, a contradiction.

2. The statement is obvious when $a = [0,1]$ is a segment. The Jordan's theorem yields an homeomorphism $\varphi : \mathbb{C} \to \mathbb{C}$ with $\varphi(a) = [0,1]$. The result follows, since both φ and φ^{-1} are locally uniformly continuous. □

We may now proceed with our construction.

Proof of Proposition 3.3. Let (c_n) be the sequence of Jordan curves in $\mathcal{F}$ and a_∞ the arc of the limit curve c_∞ provided by Proposition 3.1. Let p, q denote the endpoints of a_∞. Let $\varepsilon > 0$ be very small with respect to $d(p,q)$, and r, η as in Lemma 5.12. Note that $r \leq \eta \leq \varepsilon$. By construction, for n large enough, there exist disjoint arcs a_n and a'_n of c_n that lie in the r-neighborhood of a_∞ and meet both spheres $S(p,r)$ and $S(q,r)$.

Cutting out both ends of a_n and a'_n if necessary, we may moreover assume that a_n (resp. a'_n) has an endpoint x_n (resp. x'_n) on the sphere $S(p,r)$, an endpoint y_n (resp. y'_n) on the sphere $S(q,r)$, and is otherwise drawn in $\mathbb{S}^2 \setminus \overline{D(p,r)} \cup \overline{D(q,r)}$. Choose an arc $\alpha_{n,p}$ on $S(p,r)$ joining x_n and x'_n, and an arc $\alpha_{n,q}$ on $S(q,r)$ joining y_n and y'_n. Then the union $\Gamma_n := a_n \cup a'_n \cup \alpha_{n,p} \cup \alpha_{n,q}$ is a Jordan curve which lies in the r-neighbourhood of a_∞. It thus follows from Lemma 5.12 that the bounded component B_n of $\mathbb{C} \setminus \Gamma_n$ lies in the η-neighbourhood of a_∞.

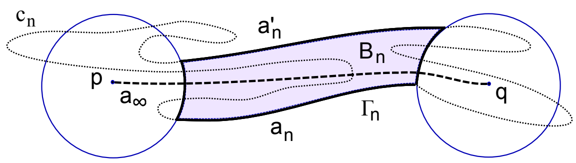

Figure 7: The arcs a_n and a'_n on the Jordan curves c_n and Γ_n, and the box B_n

Observe that some arcs of c_n may enter the box B_n. We thus introduce the connected component U_n of $B_n \setminus c_n$ whose closure $\overline{U_n}$ contains the arc a_n. We claim that such a connected component U_n does exist. This fact is easy when the quadrilateral B_n is a rectangle, and the general case follows since the Jordan's theorem provides us with an homeomorphism between $\overline{B_n}$ and a rectangle. Note that, since $\mathbb{S}^2 \setminus U_n$ is connected, the domain U_n is simply connected. A glance at Figure 7 may be useful, but keep in mind that the closure $\overline{U_n}$ does not always contain the arc a'_n.

Choose two points p_n and q_n in U_n such that $d(p, p_n) = r + \varepsilon$ and $d(q, q_n) = r + \varepsilon$. Then, Proposition 5.7 provides us with a pearl necklace $\mathcal{N}_n = (D_n(i) \mid i \in I_n)$ in the simply connected domain U_n, joining p_n to q_n. Since $U_n \subset B_n$, Lemma 5.12 ensures that the diameters of all the disks that constitute this necklace are at most ε.

We are not done yet, since $\mathcal{N}_n$ is only a necklace in the domain U_n. Indeed the contact points of the disks $D_n(i)$ with the boundary ∂U_n, which occur in the condition 3 of Definition 3.2, may either lie in c_n (which is good), or on one of the arcs $\alpha_{n,p}$ or $\alpha_{n,q}$. Hence we choose a subnecklace

$N_n \subset \mathcal{N}_n$, whose first disk meets the sphere $S(p, r+2\varepsilon)$ and whose last disk meets the sphere $S(q, r+2\varepsilon)$, and that is minimal with respect to these properties. This necklace N_n is also a necklace in $\mathbb{S}^2 \setminus c_n$. This necklace is $\varepsilon/(d(p,q) - 4\,\varepsilon)$-thin as required. □

6 Finite sequences of real numbers

We prove here an elementary fact on finite sequences of real numbers.

Proof of Proposition 3.4 Assume that there are no wells. This means that there exists $h_2 > 0$ such that, for any sequence $x = (x(i))_{i \in I_x} \in \mathcal{S}$ and $i < j < k$, one has either $x(i) \le x(j) + h_2$ or $x(k) \le x(j) + h_2$.

Assume that there are no pipes either. Choose $h_1 = 2h_2$. Then, there exists a length ℓ such that any subsequence $(x(i))_{i \in J} \subset (x(i))_{i \in I_x}$ – where $x \in \mathcal{S}$ and $J \subset I_x$ is a subinterval of length at least ℓ – has an oscillation greater than $2h_2$: there exist $i, j \in J$ with $|x(i) - x(j)| \ge 2h_2$.

We will prove that each sequence in $\mathcal{S}$ is a slide, with height $h_3 := h_2$ and slope $\sigma := \frac{h_2}{\ell}$. Let thus $x = (x(i))_{i \in I} \in \mathcal{S}$ and choose i_0 such that $x(i_0) = \max_{i \in I_x} x(i)$. Let us work for example in the future of i_0.

Assume that $[i_0, i_0 + \ell] \subset I_x$. Since there are no pipes and all $x(i)$ are bounded by $x(i_0)$, there exists $i_1 \in [i_0, i_0 + \ell]$ such that $x(i_1) \le x(i_0) - 2h_2$. Since there are no wells, this implies that $x(i) \le x(i_0) - h_2$ for any $i \in I_x$ with $i \ge i_1$.

Assume that $[i_1, i_1 + \ell] \subset I_x$. Denying again the existence of pipes and wells yields an $i_2 \in [i_1, i_1 + \ell]$ with $x(i_2) \le x(i_0) - 3h_2$, and ensures that $x(i) \le x(i_0) - 2h_2$ for $i \ge i_2$. We go on and, as long as $[i_{k-1}, i_{k-1} + \ell] \subset I_x$, we get an integer i_k in $[i_{k-1}, i_{k-1} + \ell]$ such that,

$$x(i) \le x(i_0) - k\,h_2 \quad \text{for any } i \in I_x \text{ with } i \ge i_k.$$

For all $i \ge i_0$ in I_x, one can choose k such that $i \in [i_k, i_k + \ell]$. Note that, by construction, $i_k \le i_0 + k\ell$. Hence this k is larger than $\frac{i - i_0 - \ell}{\ell}$, and one has as required

$$x(i) \le x(i_0) - \tfrac{i - i_0 - \ell}{\ell}\, h_2 \le x(i_0) + h_2 - \tfrac{h_2}{\ell}\, |i - i_0| \,. \qquad \square$$

7 Pipes, wells and slides

We put together the results of the previous sections 5.4 and 6 to finally prove Proposition 3.5, and hence Theorem 1.1.

Proof of Proposition 3.5 Proposition 3.3 provides us, for all $\varepsilon > 0$, with a Jordan curve $c_\varepsilon \in \mathcal{F}$ and an ε-thin necklace $N_\varepsilon = ((D_\varepsilon(i)|\, i \in I_\varepsilon)$ in $\mathbb{S}^2 \setminus c_\varepsilon$. We let $\mathcal{S}_\mathcal{F}$ denote the family of sequences $x_\varepsilon = (x_\varepsilon(i)|\, i \in I_\varepsilon)$ associated to the necklaces N_ε, with $x_\varepsilon(i) = \log \operatorname{diam} D_\varepsilon(i)$ and apply Proposition 3.4 to $\mathcal{S}_{\mathcal{F}}$.

$\triangleright$ Suppose that there exist pipes in $\mathcal{S}_{\mathcal{F}}$. Shortening and shifting the intervals I_ε, we may extract from the family $\{c_\varepsilon\,, \varepsilon > 0\}$ a sequence of Jordan curves $(c_n)_{n\in\mathbb{N}}$ and corresponding necklaces $N_n = (D_n(i) \mid |i| \leq n)$ such that the ratios

$$\operatorname{diam} D_n(i)/\operatorname{diam} D_n(j) \quad (\text{for } n \in \mathbb{N} \text{ and } |i|, |j| \leq n)$$

of the diameters of these disks are uniformly bounded between $1/\delta$ and δ for some $\delta > 1$.

Applying a suitable conformal transformation of $\mathbb{S}^2$ that fixes ∞, we furthermore assume that the middle disk $D_n(0)$ of each necklace is always the unit disk $D(0,1) \subset \mathbb{C}$. Together with condition 1, this implies that, for a fixed $i \in \mathbb{Z}$, all the disks $D_n(i)$ (where $n \geq |i|$) live in a compact set of disks of the complex plane. Using a diagonal argument we may thus assume that, for each $i \in \mathbb{Z}$, the sequence $(D_n(i))_{n\geq|i|}$ converges to a disk $D_\infty(i)$ with center ω_i and diameter between $1/\delta$ and δ. Going again to a subsequence, we may also assume that the sequence $(c_n)_{n\in\mathbb{N}}$ converges to $c_\infty \in \mathcal{K}$ when $n \to \infty$. The bounds on the diameters of the $D_n(i)$'s and condition 2 force $|\omega_i| \to \infty$ when $|i| \to \infty$, so that the broken line $L = \cup_{i\in\mathbb{Z}}[\omega_i, \omega_{i+1}] \subset \mathbb{C}$ yields a proper embedding of $\mathbb{R}$ into $\mathbb{C}$. By Jordan's theorem, $\mathbb{C} \setminus L$ has two connected components. As a consequence of condition 3, the limit curve c_∞ visits both connected components of $\mathbb{C} \setminus L$. This ensures that the limit c_∞ is not a singleton, hence is a Jordan curve. However since, for all n, c_n avoids the open set $\cup_i D_n(i)$, the limit curve c_∞ does not meet L and hence can not be a Jordan curve (see the first drawing in Figure 8) : a contradiction.

$\triangleright$ Assume now that there are wells in $\mathcal{S}_{\mathcal{F}}$. We proceed as in the previous case. This time, we shift the intervals so that the bottom of each well occurs for the index 0, and apply a conformal normalisation so that the corresponding disk $D_n(0)$ is always the unit disk $D(0,1)$.

We obtain this time a sequence of Jordan curves $c_n \in \mathcal{F}$, a non-decreasing sequence of finite intervals I_n that contains 0, and a sequence of necklaces $N_n = (D_n(i) \mid i \in I_n)$ in $\mathbb{S}^2 \setminus c_n$ such that $D_n(0) = D(0,1)$, and such that the logarithms of the diameters of these disks, $x_n(i) := \log \operatorname{diam} D_n(i)$, satisfy :

$$x_n(0) = 0,\ x_n(i) \geq 0 \text{ for } i \text{ in } I_n, \text{ and } x_n(i) \geq n \text{ for both endpoints } i \in I_n.$$

Let $I \subset \cup_n I_n$ be the maximal subinterval containing 0 such that the sequence $n \mapsto x_n(i)$ is bounded when i is an interior point of I. Note that this interval may be finite or infinite. By construction for any endpoint i of I the sequence $n \mapsto x_n(i)$ is unbounded. After going through a diagonal process, the sequence of disks $D_n(i)$ converges, when $n \to \infty$, to a disk $D_\infty(i)$ when i is an interior point of I and to a half-plane $D_\infty(i)$ when i is an endpoint of I.

The conclusion follows as in the previous case : the limit $c_\infty = \lim_{n\to\infty} c_n$ avoids the shaded area and visits both components of its complementary set and hence can not be a Jordan curve (see the second and third drawings in Figure 8) : a contradiction.

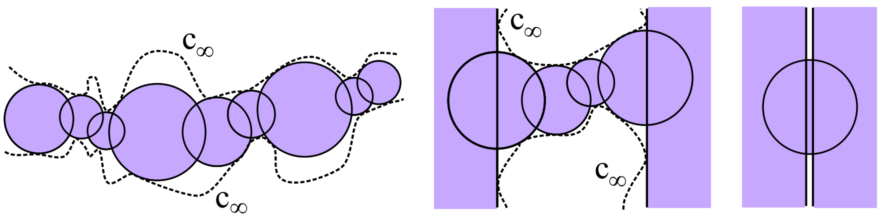

Figure 8: The limit curve and the limit of necklaces when respectively $I = \mathbb{Z}$, I is finite, and $I = \{-1, 0, 1\}$ in which case $\partial D_\infty(-1) = \partial D_\infty(1)$

▷ Finally, if all sequences in $\mathcal{S}_\mathcal{F}$ were slides with height $h_3 \geq 0$ and slope $\sigma > 0$, the ratio inverse of the thickness would be uniformly bounded,

$$\operatorname{diam}(\cup_{i\in I_\varepsilon} D_\varepsilon(i)) / \max_{i\in I_\varepsilon} \operatorname{diam}(D_\varepsilon(i)) \leq 2\sum_{i=0}^{\infty} e^{h_3} e^{-\sigma i} \leq 2e^{h_3}/(1-e^{-\sigma}) \ :$$

a contradiction. □

This also ends the proof of Theorem 1.1.

8 Cantor sets and the conformal group

In this section we prove an analog of Theorem 1.1 where Jordan curves are replaced by Cantor sets.

8.1 Quasi-middle-third Cantor sets

Recall that a non-empty compact set C is called a Cantor set if it is perfect and totally disconnected. The main example is the middle-third Cantor set

$$C_0 := \{\textstyle\sum_{n\geq 1} a_n 3^{-n} \mid a_n = 0 \text{ or } 2\} \subset [0,1].$$

Let $K \geq 1$. We will say that a Cantor set $C \subset \mathbb{S}^2$ is a K-quasi-middle-third Cantor set if C is the image $C = f(C_0)$ of the middle-third Cantor set $C_0 \subset \mathbb{S}^2$ under a K-quasiconformal homeomorphism $f : \mathbb{S}^2 \to \mathbb{S}^2$.

The following theorem is an analog of Theorem 1.1.

Theorem 8.1 *Let $K \geq 1$. The set of all K-quasi-middle-third Cantor sets of $\mathbb{S}^2$ is a closed G-invariant subset of $\mathcal{K}_0$.*

Conversely, any closed G-invariant subset of $\mathcal{K}_0$ which consists only of Cantor sets is included in the set of K-quasi-middle-third Cantor sets for some $K \geq 1$.

Corollary 8.2 *A Cantor set $C \subset \mathbb{S}^2$ is a quasi-middle-third Cantor set if and only if its orbit closure $\overline{GC}$ in $\mathcal{K}$ consists only of points and Cantor sets.*

We will just sketch the proof of Theorem 8.1 which is much shorter than the proof of Theorem 1.1. It follows from the three Lemmas 8.5, 8.6 and 8.7, combined with the MacManus' condition described below.

8.2 MacManus' condition

Here is an analog of Ahlfors's arc condition (Theorem 2.4) for Cantor sets. We recall that d denotes the canonical Riemannian metric on $\mathbb{S}^2$ and we denote by $B(x,r)$ the balls and by diam_d the diameter with respect to this metric.

Definition 8.3 *Let $A > 1$. A compact set $C \subset \mathbb{S}^2$ is A-uniformly perfect if, for all x in C and all $0 < r < \mathrm{diam}_d(C)$, one has*

$$B(x,r) \cap C \not\subset B(x, r/A).$$

A compact set $C \subset \mathbb{S}^2$ is A-uniformly disconnected if, for all x in C and all $r > 0$, the connected component of x in the r/A-neighborhood of C is included in $B(x,r)$.

Theorem 8.4 (MacManus, see [14, Theorem 3]) *A compact subset $C \subset \mathbb{S}^2$ is a quasi-middle-third Cantor set if and only if it is uniformly perfect and uniformly disconnected. The implied constants depend only on each other.*

Let us also mention two related results : another characterization of quasi-middle-third Cantor sets (see [4, Corollary 2.1]), and a similar characterization of all compact metric spaces that are quasisymmetric to the middle-third Cantor set (see [6, Chapter 15]).

8.3 Limits of Cantor sets

The direct implication in Theorem 8.1 is a special case of the following analog of Proposition 2.2.

Lemma 8.5 Limits of quasi-middle-third Cantor sets. *Let $(C_n)_{n\geq 1}$ be a sequence of K-middle-third Cantor sets in $\mathbb{S}^2$ that converges in $\mathcal{K}$ to a compact set $C_\infty \subset \mathbb{S}^2$ which is not a point. After going to a subsequence if necessary, there exist K-quasiconformal homeomorphisms $f_n : \mathbb{S}^2 \to \mathbb{S}^2$ and $f_\infty : \mathbb{S}^2 \to \mathbb{S}^2$ with $C_n = f_n(C_0)$, $C_\infty = f_\infty(C_0)$, and such that $f_n \to f_\infty$ uniformly on $\mathbb{S}^2$.*

Proof Same as Proposition 2.2 using the "uniformly perfect" part of MacManus condition to know that the limit C_∞ contains at least 3 points. □

The converse implication in Theorem 8.1 follows from the MacManus' condition and the following two lemmas.

Lemma 8.6 Non uniformly perfect sequences of compact sets.
Let $(C_n)_{n\geq 1}$ be a sequence of compact subsets of $\mathbb{S}^2$ such that C_n is not n-uniformly perfect. After going to a subsequence if necessary, there exist elements $\gamma_n \in G$ such that $(\gamma_n C_n)$ converges in $\mathcal{K}$ to a non-perfect set C_∞ containing at least two points.

Proof By assumption, there exist x_n in C_n and $0 < r_n < \mathrm{diam}(C_n)$ such that $B(x_n, r_n) \cap C \subset B(x_n, r_n/n)$. Fix x_0 in $\mathbb{S}^2$ and choose $\gamma_n \in G$ such that $\gamma_n x_n = x_0$ and $\gamma_n(B(x_n, r_n)) = B(x_0, 1)$. Going to a subsequence, $(\gamma_n C_n)$ converges to a compact set C_∞ of $\mathbb{S}^2$ which contains x_0 as the only point in $B(x_0, 1)$, and is not a singleton. Hence the set C_∞ is not perfect. □

The second lemma is very similar to the first one.

Lemma 8.7 Non uniformly disconnected sequences of compact sets. *Let $(C_n)_{n\geq 1}$ be a sequence of compact subsets of $\mathbb{S}^2$ such that C_n is not n-uniformly disconnected. After going to a subsequence if necessary, there exist elements $\gamma_n \in G$ such that $(\gamma_n C_n)$ converges in $\mathcal{K}$ to a non-totally disconnected set C_∞.*

Proof The argument is also very similar. By assumption, there exist x_n in C_n, $0 < r_n < \pi$ and a finite subset $F_n \subset C_n$ containing x_n such that the r_n/n-neighborhood of F_n is connected and meets the sphere $S(x_n, r_n)$. Fix x_0 in $\mathbb{S}^2$ and choose $\gamma_n \in G$ such that $\gamma_n x_n = x_0$ and $\gamma_n(B(x_n, r_n)) = B(x_0, 1)$. Going to a subsequence, $(\gamma_n C_n)$ converges to a compact set C_∞ of $\mathbb{S}^2$ and $(\gamma_n F_n)$ converges to a compact set $F_\infty \subset C_\infty$ which contains x_0, which is connected and which meets the sphere $S(x_0, 1)$. Hence the set C_∞ is not totally disconnected. □

This ends the proof of Theorem 8.1.

9 Thurston's lamination and monotonous paths

We give in this last section an elementary and self-contained proof of the existence of monotonous paths (Proposition 5.6).

In all this section, $U \subset \mathbb{S}^2$ will denote a simply connected domain that avoids at least two points. We keep the notation of Sections 5.1 and 5.2.

9.1 Thurston's lamination

Thurston's lamination is the lamination Λ of U by the arc of circles equal to the connected component of $\lambda(D_p)$ for some $p \in U$ (see Lemma 5.4).

Definition 9.1 *Let A be an arc of the lamination Λ. Let $a \in A$ be a point on this arc. A transverse τ to (A, a) is a non trivial segment on a geodesic of D_p that meets A orthogonally at the point a and that admits this point a as an endpoint.*

The following proposition will be useful for the construction of monotonous paths.

Proposition 9.2 *Let $A \in \Lambda$ be an arc of the lamination, Ω be one of the connected components of $U \setminus A$ and $a_1, a_2 \in A$ be two points on this arc. Then, there exist*
- *transverses τ_i to (A, a_i) with second endpoints x_i in Ω (for $i = 1, 2$) with $C_{x_1} = C_{x_2}$*
- *and a path $\gamma \subset C_{x_1}$ from x_1 to x_2*

satisfying the following property. Let $\mathcal{B} \subset \Omega$ be the closed region bounded by the segments $[a_1, a_2] \subset A$, the transverses τ_1 and τ_2 and the path γ. Then each stratum C_m that intersects $\mathcal{B}$ also intersects both τ_1 and τ_2.

The closed region $\mathcal{B}$ is called a well-combed box for $([a_1, a_2], \Omega)$.

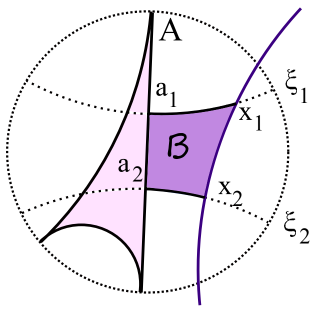

Figure 9: A box

We begin with a lemma.

Lemma 9.3 *Let A be an arc of the lamination Λ, $a \in A$ and τ be a transverse to (A, a). Assume that $C_a \cap \tau = \{a\}$.*

Let p, q denote the endpoints of A. Let $\varepsilon > 0$. Then, when $x \in \tau$ is close to a, the set of contact points $\partial D_x \cap \partial U$ meets both balls $B(p, \varepsilon)$ and $B(q, \varepsilon)$, and lies in their union.

In particular, if (a_n) is a sequence of points on the transverse τ that converges to a, then, the sequence of convex hulls (C_{a_n}) converges to A.

Note that when $C_a \cap \Omega \neq \emptyset$ and the transverse τ is short enough, then τ lies in C_a. In this case, $C_x = C_a$ for any $x \in \tau$.

Proof Let $p, q \in \partial D_a \cap \partial U$ denote the endpoints of the arc A. Let (a_n) be a sequence of points of τ that converges to a.

To begin with, we assume that each point a_n belongs to an arc A_n of the lamination Λ. Both endpoints of A_n lie in ∂U, thus the arc A_n intersects the boundary ∂D_a in two points p_n and q_n. Going to a subsequence, we may assume that (p_n) and (q_n) respectively converge to $\tilde{p} \in \partial D_a$ and $\tilde{q} \in \partial D_a$. We claim that $\{p, q\} = \{\tilde{p}, \tilde{q}\}$.

Indeed, the sequence of disks (D_{a_n}) converges to D_a (see Proposition 5.2), thus

$$\mathrm{conv}_{D_{a_n}}(p_n, q_n) \longrightarrow \mathrm{conv}_{D_a}(\tilde{p}, \tilde{q})\,.$$

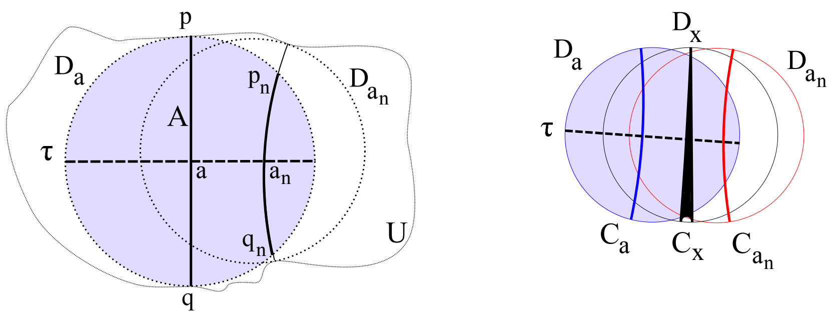

Figure 10: Transverses

Since the points a_n belong to $\mathrm{conv}_{D_{a_n}}(p_n, q_n)$ and since the point a belongs to $\mathrm{conv}_{D_a}(p, q)$ while, by construction, the points p_n and q_n belong to the same connected component of $\partial D_a \setminus \{p, q\}$, we infer that $\{p, q\} = \{\tilde{p}, \tilde{q}\}$.

We now turn to the general case. Note that we can find a sequence $\alpha_n \in \tau$ such that $\alpha_n \neq a$, (α_n) converges to a and α_n belongs to an arc A_{α_n} of the lamination Λ. For x sitting in the segment $]a, \alpha_n[\subset \tau$, the portion $C_x \cap D_a$ is included in the region delimited in D_a by the arcs A and A_{α_n}. When n is large, we just proved that A and A_{α_n} are close. The result follows as above, arguing that the disk D_x depends continuously on x. □

Proof of Proposition 9.2 Note that we have $D_{a_1} = D_{a_2}$ and $C_{a_1} = C_{a_2}$.

The result is obvious when $A \neq C_{a_1}$ and Ω is the connected component of $U \setminus A$ that intersects C_{a_1} since, in this case, we may find a box $\mathcal{B}$ that lies in C_{a_1}.

Assume now that Ω does not intersect C_{a_1}, and let $[a_1, \xi_1[$ and $[a_2, \xi_2[$ respectively denote the geodesic rays of D_{a_1} that are orthogonal to A at the points a_1 (resp. a_2), and point towards Ω (see Figure 9).

Lemma 9.3 provides a point $x_1 \in]a_1, \xi_1[$ such that all the C_x's, where $x \in [a_1, x_1] \subset \tau_1$, are close enough to A so that they intersect $]a_2, \xi_2[$. We may even assume that C_{x_1} is an arc and let $x_2 := C_{x_1} \cap T_2$. If x_1 is close to a_1, then x_2 is close to a_2 and all the C_y's – where $y \in [a_2, x_2] \subset \tau_2$ – do intersect τ_1. We may then choose $\tau_1 = [a_1, x_1]$ and $\tau_2 = [a_2, x_2]$. □

9.2 Piecewise monotonous paths

Recall that a path $\gamma : [0,1] \to U$ is monotonous if, for any parameters $0 \le r \le s \le t \le 1$, the point $\gamma(s)$ lies between $\gamma(r)$ and $\gamma(t)$ (see Definition 5.5). To strengthen our intuition, we first observe the following.

Lemma 9.4 *A monotonous path $\gamma : [0,1] \to U$ with endpoints x and z crosses a convex hull C_y if and only if y lies between x and z.*

Proof By definition of monotony, we know that $\gamma(t)$ lies between x and z for any $0 \le t \le 1$. Conversely, let y be a point of U that lies between x and z and such that $C_x \ne C_y$ and $C_y \ne C_z$. This means that x and y belong to two different connected components of $U \setminus C_y$, hence the path γ, which goes from x to z, must cross C_y. □

Our aim in this paragraph is to prove that U is monotonous-path-connected (this is Proposition 5.6). To prove this result, we will use the following alternative definition of monotony.

Proposition 9.5 *A path $\gamma : [0,1] \to U$ is monotonous if and only if it cuts each C_p only once namely if, for any point $p \in U$, the set $\gamma^{-1}(C_p) \subset [0,1]$ is connected.*

Proof Let γ be monotonous. We proceed by contradiction, and assume that there exist $r < s < t$ in $[0,1]$ such that both $x = \gamma(r)$ and $z = \gamma(t)$ lie in C_x, while $y = \gamma(s) \notin C_x$. Since γ is monotonous, we know that x and z lie in two distinct connected components of $U \setminus C_y$. This is a contradiction since $C_x \subset U \setminus C_y$ is connected.

Assume that $\gamma : [0,1] \to U$ is not monotonous ; we may assume with no loss of generality that both points $x = \gamma(0)$ and $z = \gamma(1)$ lie in the same connected component Ω of $U \setminus C_y$, where $y := \gamma(1/2)$. Define

$$t_1 = \inf\{t > 0 \mid \gamma(t) \in C_y\}, \quad b_1 = \gamma(t_1)$$
$$t_2 = \sup\{t < 1 \mid \gamma(t) \in C_y\}, \quad b_2 = \gamma(t_2).$$

It follows from the assumption and Lemma 5.4 that the points b_1 and b_2 belong to the same arc $A \subset \lambda(D_y)$ of the lamination Λ. Pick two points a_1 and a_2 on A so that the segment $[b_1, b_2] \subset A$ lies in the open segment $]a_1, a_2[\subset A$. Proposition 9.2 provides a well-combed box $\mathcal{B}$ for $([a_1, a_2], \Omega)$.

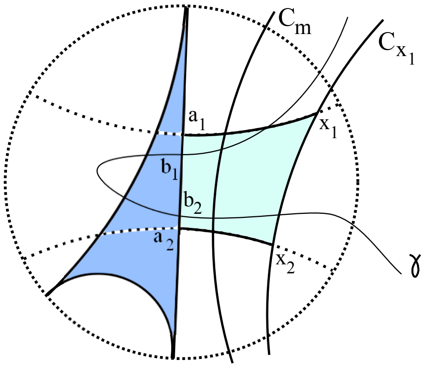

Figure 11: Equivalent definition of monotony

Observe that any C_m that meets the transverse τ_1 at an interior point disconnects the box $\mathcal{B}$. Choosing $C_m \subset \Omega$ close enough to A, this ensures that there exist $s_1 \in]0, t_1[$ and $s_2 \in]t_2, 1[$ such that $\gamma(s_1)$ and $\gamma(s_2)$ both lie in C_m, hence that $\gamma^{-1}(C_m)$ is not connected. $\square$

Corollary 9.6 *Let $A \in \Lambda$ be an arc of the lamination and $a \in A$. Let τ and τ' be transverses to (A, a) corresponding to each connected component of $U \setminus A$. If these transverses are short enough, their union $\tau \cup \tau'$ is the image of a monotonous path.*

Proof Proposition 9.2 implies that both τ and τ' are images of monotonous paths. The equivalent definition of monotony provided by Proposition 9.5 implies that their union $\tau \cup \tau'$ is also the image of a monotonous path. $\square$

Lemma 9.7 *The domain U is locally monotonous-path-connected : each point $x \in U$ admits arbitrarily small neighbourhoods V such that any pair of points $y, z \in V$ may be joined by a monotonous path.*

Proof Assume first that x belongs to the interior of C_x. Then, we may take for V any neighbourhood of x included in C_x.

Suppose now that x belongs to an arc A of the lamination. Pick two points a_1 and a_2 on A such that x lies in the open segment $]a_1, a_2[\subset A$. For the segment $[a_1, a_2]$, and each one of the connected components Ω of $U \setminus A$, Proposition 9.2 provides us with a well-combed box. The union of these two boxes is a neighbourhood V of x. It can be made arbitrarily small by choosing the points a_i close to x, and the transverses short. We claim that V is monotonous-path-connected.

Let indeed $y, z \in V$. Choose paths $\gamma_y \subset C_y \cap V$ and $\gamma_z \subset C_z \cap V$ that respectively join the points y and z to points y_1 and z_1 that lie on one of the transverses to (A, a_1) that bound the domain V. It follows from Corollary 9.6 that the concatenated path $\mu = \gamma_z^{-1} * [y_1, z_1] * \gamma_y$ is monotonous. $\square$

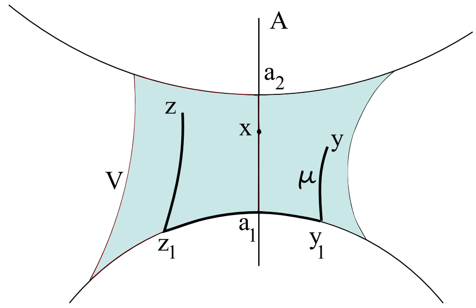

Figure 12: A neighbourhood of x that is monotonous-path-connected

Definition 9.8 *A path $\gamma : [0,1] \to U$ is piecewise-monotonous if there exists an interval subdivision $0 = t_0 < t_1 \cdots < t_n = 1$ such that each restriction $\gamma_{|[t_i,t_{i+1}]}$ is a monotonous path for $i = 0, \cdots, n-1$.*

Corollary 9.9 *The domain U is piecewise-monotonous-path connected.*

Proof Immediate consequence of Lemma 9.7, since U is connected. □

9.3 Monotonous paths

Proof of Proposition 5.6 We just proved that U is piecewise-monotonous-path connected. The fact that U is monotonous-path-connected will thus be an immediate consequence of the following Proposition. □

Proposition 9.10 *Let x, y and z be three points in U. Assume that there exists a monotonous path γ_1 from x to y, and a monotonous path γ_2 from y to z. Then, there exists a monotonous path from x to z.*

Proof Let both paths γ_i ($i = 1, 2$) be parameterized by $[0, 1]$.

1. Assume first that y lies in between x and z. We claim that, in this case, the concatenated path $\gamma := \gamma_2 * \gamma_1$ is monotonous. Were it not the case, Proposition 9.5 would provide a point $p \in U$ (with $p \notin C_y$) such that $\gamma^{-1}(C_p)$ is not connected. Since both paths γ_1 and γ_2 are monotonous, this forces both $\gamma_1^{-1}(C_p)$ and $\gamma_2^{-1}(C_p)$ to be non-empty. Hence the stratum C_p, which is connected, would intersect two distinct connected components of $U \setminus C_y$. This is a contradiction.

2. We now assume that x lies in between y and z. Since the path γ_2 is monotonous, Lemma 9.4 ensures that there exists $t_0 \in [0, 1]$ with $\gamma_2(t_0) \in C_x$. Choose a path $\gamma_3 : [0, 1] \to C_x$ from x to $\gamma_2(t_0)$. Then, the concatenated path $(\gamma_2)_{|[t_0,1]} * \gamma_3$ joins x to z and is monotonous.
In case z lies in between x and y, the proof is similar.

3. We now proceed with the last configuration, where none of the points x, y, z lies in between the other two. We want to produce a monotonous path from x to z.

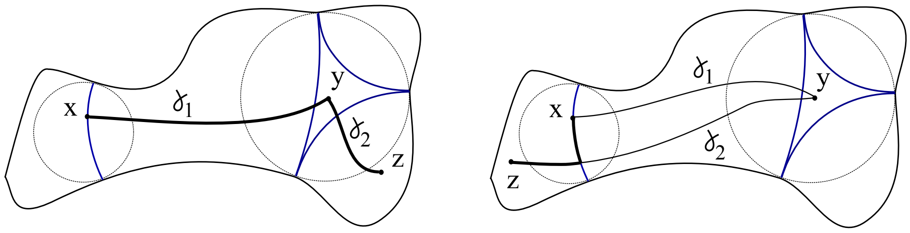

Figure 13: A piecewise-monotonous path yields a monotonous one (1 and 2)

As above, we will construct our monotonous path from x to z by cutting out a subpath of $\gamma_2 * \gamma_1$, and replacing it by a shortcut that lies in a stratum. Lemma 9.4 ensures that

$$J_1 := \{s \in [0,1] \mid \exists t \in [0,1] \text{ with } C_{\gamma_1(s)} = C_{\gamma_2(t)}\}$$

is a sub-interval of $[0,1]$ containing 1 and that

$$J_2 := \{t \in [0,1] \mid \exists s \in [0,1] \text{ with } C_{\gamma_1(s)} = C_{\gamma_2(t)}\}$$

is a sub-interval containing 0. Define $T_1 = \inf J_1$ and $T_2 = \sup J_2$.

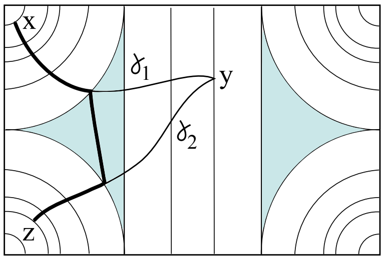

Figure 14: A piecewise-monotonous path yields a monotonous one (3)

Since the normal disk D_x depends continuously on $x \in U$ (see Proposition 5.2), it follows that $D_{\gamma_1(T_1)} = D_{\gamma_2(T_2)}$, so that $C_{\gamma_1(T_1)} = C_{\gamma_2(T_2)}$. Let now γ_3 be a path drawn in the stratum $C_{\gamma_1(T_1)}$, and that goes from $\gamma_1(T_1)$ to $\gamma_2(T_2)$. The choice of T_1 and T_2 and Proposition 9.5 ensure that the path $\mu := \gamma_{2|[T_2,1]} * \gamma_3 * \gamma_{1|[0,T_1]}$, that goes from x to z, is monotonous. $\square$

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Y. Benoist and D. Hulin
Université Paris-Sud, Orsay 91405, France
yves.benoist@math.u-psud.fr & dominique.hulin@math.u-psud.fr