The arborification-coarborification transform:
analytic, combinatorial, and algebraic aspects.

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Abstract: This expository paper is devoted to the so-called *arborification-coarborification transform* which, by automatically carrying out suitable regroupings, often manages to restore convergence in multiple expansions that, in raw form, would seem hopelessly divergent. We first unravel the underlying combinatorics. Then we review 14 applications to complex analysis and holomorphic dynamics. Lastly, we present some new algebraic material: a bevy of some twenty richly structured "\(\sigma\)-functions", which are defined simultaneously on all symmetric groups \(S_r\). Since all these objects originate in arborification, their ‘distinctiveness’ rubs off on that particular transform, reinforcing its privileged status among all possible alternatives.

Résumé: Nous tâchons de faire le point sur l’arborification-coarborification. Il s’agit là d’une transformation générale qui effectue, au sein de séries multiples divergentes, des regroupements judicieux susceptibles d’instituer la
convergence. Nous examinons la méthode tour à tour sous trois angles : combinatoire, analyse, algèbre. La partie algébrique présente une multitude de “σ-fonctions” (i.e. de fonctions définies simultanément sur tous les groupes de permutations) apparemment nouvelles et aux propriétés très riches. Tous ces objets, liés qu’ils sont à l’arborification, confirment indirectement le statut privilégié de cette dernière parmi toutes les transformations concurrentes.

1 Arborification-coarborification as a special case of fusion-fission.

1.1 Introduction. Why arborify?

Analysis often presents us with so-called mould-comould expansions, i.e. infinite series of the form:

\[ SS := \sum \sum A^\bullet B^\bullet = \sum \omega \sum A^{\omega_1,\ldots,\omega_r} B_{\omega_1,\ldots,\omega_r} \]

which, despite being divergent, somehow ought to converge, or at least to be re-arrangeable into convergent shape. But let us be a bit more specific. These expansions SS typically involve three ingredients:

- a highly multiple indexation, with “•” running through an infinite set of sequences \( r = r(\bullet) \) of arbitrary lengths.
- a mould part \( A^\bullet \), usually consisting of scalars, or scalar functions of some variables \( x_i \) or parameters \( t_i \).
- a comould part \( B^\bullet \), usually consisting of operators, which most of the time are ordinary differential operators in the variables \( x_i \), but of high degree \( d \).

Unfortunately, as pointed out, these mould-comould expansions SS tend to be normally divergent even when there are strong reasons to suspect that the corresponding power series \( S_i := SS.x_i \) do, in fact, have positive convergence radii. No contradiction here: since a great many terms \( A^\bullet B^\bullet \) in \( SS \) contribute to any given Taylor coefficient of \( S_i \), there is ample scope for mutual cancellations or compensations within each Taylor coefficient. The challenge, therefore, is to regroup – preferably, in a conceptually appealing and universally valid manner – the terms in \( SS \) so as to make the suspected cancellations manifest. Clearly, these regroupings should be carried out adroitly,

\(^1\)usually, “•” runs through a monoid freely generated by a countable index reservoir \( \Omega \) such as \( \mathbb{N} \) or \( \mathbb{Z} \) or \( \mathbb{N}^\nu \) or \( \mathbb{Z}^\nu \).

\(^2\)quite often, the \( B_{\omega_1,\ldots,\omega_r} \) are simple products \( B_{\omega_r}\ldots B_{\omega_1} \) of first-order differential operators, in which case length and degree coincide: \( r = d \).

\(^3\)i.e. \( \sum |A^\bullet| \cdot \|B^\bullet\| = +\infty \) for any reasonable norm or semi-norm \( \|\cdot\| \).
and be exactly the right size: *neither too vast*, for then we would get unmanageably large expressions and the mechanisms responsible for compensation would remain as opaque as they are “inside” the Taylor coefficients of the \( S_i \), *nor too constricted*, for in that case there would be no opportunity for compensation to take place.

One extremely general way of re-ordering our expansions \( SS \) to achieve promising re-groupings is to move from the “\( \bullet \)”-indexation by *totally ordered sequences* to some “\( \# \)”-indexation by *partially ordered sequences*, for some specified type of partial order.

The idea translates into the general *fusion-fission transform*:

\[
SS = \sum_{\bullet} A^\bullet B_\bullet \longrightarrow SS = \sum_{\#} A^\# B_\# \tag{2}
\]

with dual rules for the mould and comould parts:

\[
\text{Fusion rule :} \quad A^\# := \sum_{\bullet} F^\#_\bullet A^\bullet := \sum_{\#} A^\bullet \tag{3}
\]

\[
\text{Fission constraint :} \quad B_\bullet := \sum_{\#} F^\#_\# B_\# := \sum_{\bullet \geq \#} B_\# \tag{4}
\]

which automatically ensure that \( SS \) remains globally unchanged. Here, the coefficients \( F^\#_\bullet \) are either 1 or 0 and the notation \( \bullet \geq \# \) says that, while both sequences \( \bullet \) and \( \# \) consist of exactly the same elements \( \omega_i \) with exactly the same multiplicities, the second sequence has on it a partial\(^4\) order *weaker than, but compatible with* the total order of the first.

As a special case, we have the *arborification-coarborification transform*:

\[
SS = \sum_{\bullet} A^\bullet B_\bullet \longrightarrow SS = \sum_{\prec} A^{\prec} B_{\prec} \tag{5}
\]

with the dual rules:

\[
\text{Arborification rule :} \quad A^{\prec} := \sum_{\bullet} F^{\prec}_\bullet A^\bullet := \sum_{\# \geq \prec} A^\bullet \tag{6}
\]

\[
\text{Coarborification constraint :} \quad B_\bullet := \sum_{\prec} F^{\prec}_{\#} B_{\prec} := \sum_{\bullet \geq \prec} B_{\prec} \tag{7}
\]

which correspond to the choice of *arborescent orders*. In other words, we work here with partially ordered sequences \( \prec \), each element \( \omega_i \) of which possesses

\(^4\)non-strictly, of course: that partial order may on occasion be total!\n
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at most one antecedent, which we denote $\omega_i$. Minimal elements, or roots, are not assumed to be unique.\textsuperscript{5}

There are three distinct angles - analytic, combinatorial, algebraic - for approaching our “regrouping” transforms, and all three point to the same conclusion: among all fusion-fission transforms, arborification-coarborification, for innumerable reasons, towers as the most important and the most useful. The present paper is devoted to showing why this is so, by successively adopting the three viewpoints:

- **Analysis**, of course, remains the main raison d’être for these regrouping techniques. In §4, we shall review no less than fourteen genuinely distinct situations, ranging from holomorphic dynamics to KAM theory to resurgence calculus, where arborification can be harnessed to great effect – and often must.

- **Combinatorics**, on the other hand, lays bare the mechanisms at work, and explains why the technique succeeds. Here, the mould-comould duality is very helpful in sorting out the difficulties. As we shall see in §3, it is the comould part that leads us, rather naturally, to single out the coarborification constraints (7) among all fusion constraints (4). But it is in the mould part that the really subtle phenomena, those that hold the key to compensation, do occur, as will be shown in §2 on some rich mould material

- **Algebra** here is something of a side-show, but a fascinating one. As we shall see, to each fusion-fission transform one may attach a string of algebraic objects, mainly arithmetical moulds and $\sigma$-functions (i.e functions that are defined, simultaneously and uniformly, on all permutation groups $S_r$) which encapsulate all that is most distinctive about each given transform. Now, the first surprise is that the particular moulds and $\sigma$-functions attached to arborification-coarborification (they constitute what we call the haukian family) are replete with structure, symmetries, and all manner of highly improbable properties, which are listed in §5 and illustrated in the tables of §7. And the second surprise is that all this structure comes crashing down as soon as we move on to the moulds or $\sigma$-functions associated with the other transforms: unlike the haukian prototypes, they seem to be utterly unremarkable.

\textsuperscript{5}so that, technically, our arborescent sequences $\prec$ must be viewed as “weighted forests” rather than “weighted trees”.

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The arborification-coarborification technique has been around for quite some time; so here we merely present a unified treatment, catalogue some typical applications, and refer to the literature for details. The algebraic part, on the other hand, is quite new, or appears to be\(^6\), but here the material is so abundant that the exposition had to be both sketchy (with only the barest hints at proofs) and lacunary (with many developments left out). Thus, damaging as the admission may sound, the present paper is partly a review, and partly a formulary. But this is all that the limited format allowed. And there will be, circumstances permitting, a sequel.

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### 1.2 Straight / contracting arborification-coarborification.

A brief reminder about mould calculus has been prefaced to the next section (§2). Here we simply recall how moulds and comoulds from the basic symmetry types interact in (5) and what sort of objects they produce:

1: \((A^\bullet, B^\bullet) = (\text{symmetrel}, \text{cosymmetrel}) \Rightarrow SS = \text{formal diffeomorphism}\)
2: \((A^\bullet, B^\bullet) = (\text{alternel}, \text{cosymmetrel}) \Rightarrow SS = \text{formal derivation}\)
3: \((A^\bullet, B^\bullet) = (\text{symmetrel}, \text{cosymmetrel}) \Rightarrow SS = \text{formal diffeomorphism}\)
4: \((A^\bullet, B^\bullet) = (\text{alternel}, \text{cosymmetrel}) \Rightarrow SS = \text{formal derivation}\)
5: \((A^\prec, B^\prec) = (\text{separative}, \text{coseparative}) \Rightarrow SS = \text{formal diffeomorphism}\)
6: \((A^\prec, B^\prec) = (\text{atomic}, \text{coseparative}) \Rightarrow SS = \text{formal derivation}\)

As it happens, depending on the symmetry types involved (whether they are of the straight sort, with the vowel \(a\), or of the contracting sort, with the vowel \(e\)) one should resort to one or the other of two slightly different variants of arborification-coarborification:

**Straight arborification-coarborification: for case 1 or 2**

\[
\text{Arborification rule : } A^\prec := \sum_{\prec} F^\prec A^\bullet \\
\text{Coarborification constraint : } B^\bullet := \sum_{\prec} F^\prec B^\prec \\
\text{Standard coarborification rule : } B^\prec := \sum_{\prec} \text{Stan}^\prec B^\bullet
\]  

\(^6\)we cannot vouch for its newness, because the literature on groups and group functions is bottomless. But so far all our checks and inquiries have drawn a blank. Yet if some reader knows of previous connections, we would appreciate hearing from him.
Here, the arborification tensor $F_{\omega}^{\omega} \prec \omega$ is equal to 1 if there exists a bijection of $\omega \prec \omega$ into $\omega$ which:

(i) respects\textsuperscript{7} the order on $\omega \prec \omega$ and $\omega$

(ii) leaves the indices $\omega_i$ unchanged

and $F_{\omega}^{\omega} \prec \omega := 0$ in all other cases. Thus (8) translates into such relations as:

\[
\begin{align*}
A(\omega_1 \omega_2 \rightarrow \omega_3) \prec \omega &:= A_{\omega_1, \omega_2, \omega_3} + A_{\omega_2, \omega_1, \omega_3} + A_{\omega_2, \omega_3, \omega_1} \\
A(\omega_1 \omega_2 \omega_3 \rightarrow \omega_4) \prec \omega &:= A_{\omega_1, \omega_2, \omega_3, \omega_4} + A_{\omega_1, \omega_2, \omega_4, \omega_3}
\end{align*}
\]

Whereas the arborification rule (8) completely defines $A \prec$, the dual relation (9) merely constrains $B \prec$. However, in the important case when the comoulds are differential operators, there is a natural\textsuperscript{8} way to define $B \prec$ which not only agrees with the constraints (9), but also meets the conditions $C_3, C_4$ below, which ensure the transparent (term by term) conservation of the nature (i.e. being a derivation or an automorphism) of the expansion $SS$\textsuperscript{9}. When the comoulds belong to free associative algebras, there exists no such compelling choice, but several competing possibilities (see §1.5-9).

Let us sum up the pattern for case 1 and 2:

\begin{itemize}
  \item \textbf{C$_1$}: \textit{Straight arborification:} \hspace{1em} $A^* = \text{symmetral}$ \hspace{1em} $\Rightarrow \hspace{1em} A \prec = \text{separative}$
  \item \textbf{C$_2$}: \textit{Straight arborification:} \hspace{1em} $A^* = \text{alternal}$ \hspace{1em} $\Rightarrow \hspace{1em} A \prec = \text{atomic}$
  \item \textbf{C$_3$}: \textit{Standard coarborification:} \hspace{1em} $B^* = \text{cosymmetral}$ \hspace{1em} $\Rightarrow \hspace{1em} B \prec = \text{coseparative}$
  \item \textbf{C$_4$}: \textit{Standard coarborification:} \hspace{1em} $B^* = \text{coalternal}$ \hspace{1em} $\Rightarrow \hspace{1em} B \prec = \text{coatomic}$
\end{itemize}

\textbf{Contracting arborification-coarborification: for case 3 or 4.}

\begin{align*}
\text{Contracting arborification rule} : \hspace{2em} A \prec &:= \sum \cdot CF^\omega \prec \cdot A^* \tag{11} \\
\text{Contracting coarborification constraint} : \hspace{2em} B_\prec &:= \sum \cdot CF^\omega \prec \cdot B_\prec \tag{12} \\
\text{Standard coarborification rule} : \hspace{2em} B_\prec &:= \sum \cdot Stan^\omega_\prec \cdot B_\prec \tag{13}
\end{align*}

Here, the arborification tensor $CF_{\omega}^{\omega} \prec \omega$ is equal to 1 if there exists a \textit{surjection} of $\omega \prec \omega$ onto $\omega$ which:

\textsuperscript{7}non-comparable elements in $\omega \prec \omega$ may become comparable in $\omega$, but comparable elements have to remain so.

\textsuperscript{8}even canonical, up to the choice of variables $x_i$.

\textsuperscript{9}its \textit{global} conservation is not an issue: it automatically follows from the dualness of the rules (8) and (9).
(i) respects the order on $\omega^\prec$ and $\omega$

(ii) contracts the indices, in the sense that each $\omega_i$ in $\omega$ has to be the sum of all its pre-images $\omega_j$ in $\omega^\prec$

and $CF_{\omega^\prec} := 0$ in all other cases. Thus (11) translates into such relations as:

$$A^{(\omega_1 - \omega_3)} := A^{\omega_1, \omega_2, \omega_3} + A^{\omega_2, \omega_1, \omega_3} + A^{\omega_3, \omega_1, \omega_2} + A^{\omega_1, \omega_2, \omega_3 + \omega_2, \omega_1 + \omega_3}$$

$$A^{(\omega_1 - \omega_2 > \omega_3)} := A^{\omega_1, \omega_2, \omega_3, \omega_4} + A^{\omega_1, \omega_2, \omega_4, \omega_3} + A^{\omega_1, \omega_2, \omega_3, \omega_4}$$

Here again, we have to supply some coarborification rule compatible with the constraints (12) and, if possible, with conditions $C'_3, C'_4$. Remarkably, it turns out that, in the case of differential operators at any rate, one and the same standard arborification rule (cf §1.4 and §3) applies equally in both contexts: straight or contracting.

Summing up, here is the general pattern for case 3 and 4:

- $C'_1$: Contracting arborification: $A^* = \text{symmetrel} \Rightarrow A^\prec = \text{separative}$
- $C'_2$: Contracting arborification: $A^* = \text{alternel} \Rightarrow A^\prec = \text{atomic}$
- $C'_3$: Standard coarborification: $B_* = \text{cosymmetrel} \Rightarrow B_\prec = \text{coseparative}$
- $C'_4$: Standard coarborification: $B_* = \text{coalternel} \Rightarrow B_\prec = \text{coatomic}$

1.3 The reason why arborification-coarborification works.

As far as analytic applications are concerned, the whole point of arborification-coarborification is to re-arrange expansions of the form $\sum A^* B_*$, which are usually hopelessly divergent, because they typically admit no better bounds than:

$$|A^*| \leq a_1 a_2^r \quad ; \quad \|B_*\|_D \leq r! a_3 a_4^r \quad (\text{with} \quad r := r(\bullet)) \quad (14)$$

into formally identical expansions $\sum A^\prec B_\prec$, which are often convergent, because they usually admit bounds of the form:

$$|A^\prec| \leq c_1 c_2^r \quad ; \quad \|B_\prec\|_D \leq c_3 c_4^r \quad (\text{with} \quad r := r(\prec)) \quad (15)$$

with fixed constants $c_1, c_2$, but with adjustable constants $c_3, c_4$ that depend on a neighbourhood $D$ of the origin, and go to 0 as this neighbourhood shrinks.

The key here is not so much the disappearance of the factorial in the comould estimates as its non-appearance in the mould estimates. The disappearance is not really surprising, because the coarborification constraints
enable us to spread the ‘load’ of any given $B_\bullet$ more or less evenly among a
great many $B_\prec$. What calls for an explanation is the non-appearance of $r!$
in $A^\prec$, since under the arborification rule (8) or (11), and for very weakly
ordered arborescent sequences, $A^\prec$ is equal to a sum of almost $r!$ distinct $A^\bullet$,
which have no a priori reason of cancelling or compensating each other, and
in fact don’t cancel nor compensate for moulds $A^\bullet$ picked “at random”. But
for moulds of “natural origin”, i.e. for the ones that spontaneously occur
in the expansions $\sum A^\bullet B_\bullet$ that originate, not in our whims, but in analysis,
such cancellations, on the contrary, tend to take place with providential reg-
ularity. Why so? Because of case-specific identities, which ensure that the
norms of natural moulds don’t increase significantly under arborification. A
more precise mechanism, which accounts for this small miracle, is the fre-
quent phenomenon of form preservation: after arborification, many moulds
retain their outward analytical expression, except that in this expression all
sums, differences, etc, of indices $\omega_i$ have to be re-interpreted in terms of the
new arborescent order. But the ultimate reason lies is the fact that “useful”
or “natural” moulds almost invariably conform to some “template” (usually,
one or several relations involving some of the many operations that are de-
defined on moulds) and that arborification ordinarily preserves the “template”
in question, for the simple reason that nearly all mould operations “arborify”,
i.e. extend painlessly to arborescent moulds.

Summing up, we may say that the arborification technique works so well
because arborification usually respects “norm”, “form”, and “template”, with
usually almost meaning whenever needed.

The section §2 infra enumerates a long list of natural moulds, which shall
all be required for the applications to analysis of section §4, and which, bar-
ing two (explainable) exceptions, all possess the above properties. But take
any of these moulds, and tinker ever so slightly with its definition, and ev-
erything immediately unravels: arborification no longer preserves norm, nor
form, nor template. To grasp this stark dichotomy between the behaviour
of natural-useful and artificial-random moulds, we may reach for an anal-
ogy: whereas a random Taylor series with convergence radius one will, with
probability one, possess a natural boundary on the unit circle, most series
encountered in real life tend, on the contrary, to possess only isolated singular-
ities and endless continuability.
1.4 Standard coarborification.

Pending the precise description of coarborification in §3 (with the exact bounds), let us give a rough description with the heuristics behind it. Consider what is perhaps the most frequent situation. Take some comould $B_\bullet$ consisting of differential operators, with the following factorisation property:

$$B_\omega = B_{\omega_1, \omega_2, \ldots, \omega_r} \cdot B_{\omega_r} \cdots B_{\omega_2} B_{\omega_1} = x^{n_r} B_{\omega_r}^* \cdots x^{n_2} B_{\omega_2}^* x^{n_1} B_{\omega_1}^*$$  \hspace{1cm} (16)

with each factor $B_{\omega_i}$ separating into a homogeneous monomial $x^{n_i}$ and a differential operator $B_{\omega_i}^*$ of homogeneity 0:

$$B_{\omega_i} = x^{n_i} B_{\omega_i}^*, \quad \text{with} \quad B_{\omega_i} : x^m \mathbb{C} \to x^{m+n_i} \mathbb{C}; \quad B_{\omega_i}^* : x^m \mathbb{C} \to x^m \mathbb{C}$$  \hspace{1cm} (17)

In view of the Leibniz rules, a natural way to coarborify our comould is to define the action of the sought-after operator $B_\omega \prec$ on any test function $\varphi(x)$ as follows. We write $B_\omega \prec \varphi(x) = (x^{n_r} B_{\omega_r}^* \cdots x^{n_2} B_{\omega_2}^* x^{n_1} B_{\omega_1}^*) \prec \varphi(x)$ and decree that:

(i) if $\omega_i$ is a root of $\omega \prec$, then $B_{\omega_i}^*$ should act on $\varphi(x)$ alone

(ii) if $\omega_i$ has an immediate antecedent $\omega_i^-$ in $\omega \prec$, then $B_{\omega_i}^*$ should act on the homogeneous monomial $x^{n_i^-}$ that accompanies the corresponding $B_{\omega_i^-}$.

If we start from a cosymmetral comould $B_\bullet$ with factor operators that are first-order derivations, then the Leibniz rules clearly ensure the desired decomposition $B_\omega = \sum_{\omega < \omega} B_\omega \prec \omega$. But that decomposition also holds, less obviously so, when we start from a cosymmetrel comould.

1.5 Quadratic coarborification.

It applies above all to the case of comoulds with values in free associative algebras. Its true significance lies in the fact that it clears the way for the algebraic developments of section §5. But it also has analytic implications, namely for the notion of free-analyticity in §6.2.

Its quickest definition is by means of the tensor contractions\textsuperscript{10}:

$$B_{\prec} := B_\bullet K_{\bullet}^* F_{\prec}^* \quad \text{with} \quad K_{\bullet}^* := (H_{\bullet}^*)^{-1}, \quad H_{\bullet}^* := F_{\bullet}^* F_{\prec}^*$$  \hspace{1cm} (18)

where $F_{\bullet}^* F_{\prec}^*$, short for $\sum_{\prec} F_{\bullet}^* F_{\prec}^*$, denotes the symmetric tensor obtained by contracting both $\prec$ and leaving the two $\bullet$ alone. Viewed as a square matrix, the tensor $H_{\bullet}^*$ so produced is invertible, with real-positive spectrum, and admits an inverse $K_{\bullet}^*$.

\textsuperscript{10}covariant indices contract with contravariant ones in proximate positions.
There is a more conceptual characterisation of quadratic coarborification: it is the one that minimises the quadratic ‘coarborification norm’

\[ \|B\|_{\text{coarb}}^2 := \sum_{\prec \leq \cdot} < B_{\prec}, B_{\prec} > \]  

for the natural scalar product on the free algebra generated by the \( B_{\omega} \).

1.6 Instances of over- and undershooting.

Overshooting: We may take all possible orders. But the regroupings are then too large to be helpful or to illuminate the compensation mechanisms.

Undershooting: We may take all laminations, i.e. all partial orders that allow to each element at most one successor and at most one predecessor. A lamination clearly splits a set into subsets (“branches”) which (i) are mutually non-comparable (ii) carry each a total order. Here, the regroupings are too small to permit compensation to come into its own, at least if we insist that to each \( d \)-branched lamination there should correspond an operator of differential order \( d \). But despite its uselessness as far as restoring convergence is concerned, lamination has interesting combinatorial-algebraic aspects. We shall briefly review two instances in §1.8 and §1.9. For now, let us note in passing that laminations lead to a decomposition of the space \( \mathbb{B}_r \) spanned by all \( r! \) products of \( r \) distinct, non-commuting operators \( B_i \):

(i) first into subspaces \( ^d \mathbb{B}_r \), consisting of derivations of order \( d \)

(i) then into subspaces \( \mathbb{B}_{r_1, r_2, \ldots, r_k} \) spanned by associative products of \( d_1 \) Lie elements of homogeneity \( r_1 \), \( d_2 \) Lie elements of homogeneity \( r_2 \), etc...

\[ \mathbb{B}_r = \bigoplus_{1 \leq d \leq r} ^d \mathbb{B}_r = \bigoplus_{d_1 + \ldots + d_k = r} \mathbb{B}_{r_1, r_2, \ldots, r_k} \]  

Since these cells correspond one-to-one to the sets of all order-respecting laminations \( r^\# \) of \( r := (1, \ldots, r) \) which have \( d_1 \) branches of length \( r_1 \), \( d_2 \) branches of length \( r_2 \), etc, the corresponding dimensions clearly are:

\[ \dim\left( \frac{d_1}{r_1}, \frac{d_2}{r_2}, \ldots, \frac{d_k}{r_k} \right) := \dim\left( \mathbb{B}_{r_1, r_2, \ldots, r_k} \right) = \frac{r!}{\prod d_i! \prod r_i^{d_i}} \]  

\[ \dim_{r,d} := \dim\left( \frac{d}{\mathbb{B}_r} \right) = \sum_{k \geq 1} \sum_{d_1 + \ldots + d_k = d} \sum_{r_1 + \ldots + r_k = r} \dim\left( \frac{d_1}{r_1}, \frac{d_2}{r_2}, \ldots, \frac{d_k}{r_k} \right) \]

\[ = \# \{ r^\prec : r^\prec \leq r, r^\prec \text{ with } d \text{ roots} \} \]  

\[ ^{11} \text{with } < B_{\omega}, B_{\omega'} > := \delta_{\omega, \omega'}. \]
The reason for the last identity is that \( d^d \mathbb{B}_r \) also possesses a basis whose elements correspond one-to-one to the various order-respecting arborescent orders \( r^\prec \) on the sequence \( r := (1, \ldots, r) \).

### 1.7 Lamination-colamination on a free algebra.

We consider the associative algebra \( \mathbb{B} \) freely generated by the symbols \( B_1, B_2, \ldots \) viewed as formal, order-one derivations, and we use the customary notations: 
\[
B_n = B_{n_1, \ldots, n_r} := B_{n_r} \cdots B_{n_1}.
\]

Whereas \( \mathbb{B} \) admits a unique filtration
\[
\text{associative algebra } \mathbb{B} = \mathbb{B} = \mathbb{B} = \mathbb{B} = \mathbb{B} = \mathbb{B} = \mathbb{B} = \text{Lie algebra}
\]
into the subspaces \( d^d \mathbb{B} \) consisting of formal derivations of order at most \( d \), there exist several more or less natural ways of converting this into a gradation
\[
\oplus \ d^d \mathbb{B}_n \text{ with privileged projections } \mathbb{B} \to d^d \mathbb{B}:
\]

\[
\begin{align*}
\mathbb{B}_* &= 1^d \mathbb{B}_* \oplus 2^d \mathbb{B}_* \oplus 3^d \mathbb{B}_* \cdots \quad \text{with } 1^d \mathbb{B}_* = 1^d \mathbb{B}, \quad d+1^d \mathbb{B}_* \sim d+1^d \mathbb{B} / d^d \mathbb{B} \quad (25) \\
B_n &= 1 B_n + 2 B_n + 3 B_n \cdots \quad \forall B_n = B_{n_1, \ldots, n_r} := B_{n_r} \cdots B_{n_1} \quad (26) \\
B_n &\mapsto d^d B_n = \sum_{n'} d^d H_{n'}^n B_{n'} \in d^d \mathbb{B}_n \quad \text{with } d^d H_{n'}^n \in \mathbb{Q} \quad (27)
\end{align*}
\]

depending on which set of conditions \( C_i \) we impose:

**C\(_1\): economy:**
The projection tensors \( d^d H_{n'}^n \) should vanish unless the sequences \( n \) and \( n' \) have same length \( r \), same elements \( n_i \) and \( n'_i \) (with the same multiplicities in case of repetitions), and differ only as to the order of these elements.

**C\(_2\): isotropy (or universality):**
The projection tensors should depend only on the permutation \( \sigma \) that turns the ordered sequence \( n \) into \( n' \), ie:
\[
d^d H_{n_1, \ldots, n_r}' \equiv d^d h(\sigma) \quad \text{with } n_i' \equiv n_{\sigma(i)} \quad \forall i \quad (28)
\]

**C\(_3\): symmetry:**
The projection tensors should be symmetric: \( d^d H_n' \equiv d^d H_n \).
In combination with condition \( C_2 \), this translates into: \( d^d h(\sigma) \equiv d^d h(\sigma^{-1}) \).

**C\(_4\): orthogonality:**
The gradation subspaces \( d^d \mathbb{B}_n \) should be pairwise orthogonal, relative to the
natural scalar product:

\[ < B_n, B_{n'} > := 1 \quad (\text{resp} \ 0) \quad \text{if} \quad n = n' \quad (\text{resp} \ n \neq n') \quad (29) \]

\( C_5: \) order-compatibility:
The first projection tensor \( \overset{1}{H}^n_{n'} \) should depend only on the number of compatibilities/ incompatibilities in the orders of \( n \) and \( n' \). More concretely, and assuming condition \( C_2 \), this means that \( \overset{1}{h}(\sigma) \) should depend only on the numbers \( p \) and \( q \) of + and − signs in the sequence \( \sigma(i + 1) - \sigma(i) \).

\( C_6: \) lamination-compatibility:
The higher projection tensors \( \overset{d}{H}^n_{n'} \) should be simply deducible from the first one. Ideally, we should have:

\[ \overset{d}{B}_n = \frac{1}{d!} \sum \overset{1}{B}_{n_1} \overset{1}{B}_{n_2} \ldots \overset{1}{B}_{n_d} \quad (30) \]

leading to a natural co-lamination \( \overset{d}{B}_n \equiv \sum_{n^\# \leq n} B_{n^\#} \)

\( C_1, C_2 \) are minimum demands in this free algebra context but, as it turns out, there is some incompatibility between the further conditions.

1.8 Uniform lamination-colamination.

Imposing \( C_1, C_2 \) and \( C_5 \) (order compatibility) totally fixes the first projection tensor. If the sequences \( n, n' \) are repetition-free, we get:

\[ \overset{1}{H}^n_{n'} = \overset{1}{h}(\sigma) = (-1)^q \frac{p! q!}{(p + q + 1)!} = (-1)^q \frac{p! q!}{r!} \quad (31) \]

If \( n, n' \) involve repetitions, with multiplicities \( k_1, k_2, \ldots \), we must consider all \( k_1! k_2! \ldots \) sequences \( n, n' \) that coincide with \( n, n' \), except that identical terms are now regarded as distinct, in all possible ways, and then set \( \overset{1}{H}^n_{n'} := \sum \overset{1}{H}^{n^\#}_{n^\#} \) with \( H^{n^\#}_{n^\#} \) calculated according to the rule (31). Then condition \( C_6 \) is automatically fulfilled, in its strong form (30), leading to a natural colamination. But we have neither \( C_3 \) (symmetry) nor \( C_4 \) (orthogonality).

1.9 Quadratic lamination-colamination.

If we now add \( C_3 \) (orthogonality) to \( C_1, C_2 \), all projection tensors \( \overset{d}{H}^n_{n'} \) are fixed at once. Although they lack simple, closed expressions, the associated \( \sigma \)-function \( \overset{d}{h}(\sigma) \), especially the first one \( (d = 1) \) possess remarkable properties
(see §5.18 and §7.9). Condition $C_3$ (symmetry) is then automatically fulfilled (the implication is non-trivial), as well as a weaker form of $C_6$: the right-hand side of (30) may involve partial sequences $n^i$ which are not always order-compatible with $n$.

2 Combinatorial aspects of arborification.

2.1 Basic mould operations.

Moulds are functions of a variable number of variables: they depend on sequences $\omega := (\omega_1, \ldots, \omega_r)$ of arbitrary length $r = r(\omega)$. The sum $\|\omega\|$ of a sequence is simply $\sum_1^r \omega_i$. Sequences are systematically written in boldface, with upper indexation when such is called for, and with the product denoting concatenation: e.g. $\omega = \omega^1.\omega^2$. The elements $\omega_i$ which make up these sequences are written in normal print, with lower indexation. The sequences themselves are affixed to the moulds as upper indices $A^\omega = \{A^\omega\}$, since moulds are meant to be contracted

$$A^\bullet, B^\bullet \mapsto <A^\bullet, B^\bullet> := \sum A^\omega B_\omega$$

with dual objects (often differential operators or elements of an associative algebra), the so-called comoulds $B^\bullet = \{B_\omega\}$, which carry their own indices in lower position. Moulds may be added, multiplied, composed.

Mould addition is what you expect: components of equal length get added. Mould multiplication ($\mu$ or $\times$) is associative, but non-commutative:

$$C^\bullet = A^\bullet \times B^\bullet \iff C^\omega = \sum_{\omega = \omega^1.\omega^2} A^{\omega^1} B^{\omega^2}$$

(32)

(This includes the trivial decompositions $\omega = \omega.\emptyset$ and $\omega = \emptyset.\omega$).

Mould composition ($\circ$) too is associative and non-commutative:

$$C^\bullet = (A^\bullet \circ B^\bullet) \iff C^\omega = \sum_{\omega = \omega^1 \ldots \omega^s} A^{\|\omega^1\| \ldots \|\omega^s\|} B^{\omega^1} \ldots B^{\omega^s}$$

(33)

with a sum extending to all possible decompositions of $\omega$ into $s \leq r(\omega)$ non-empty factor sequences $\omega^i$.

The operations ($+, \times, \circ$) on moulds interact in exactly the same way as their namesakes for power series. Thus $(A^\bullet \times B^\bullet) \circ C^\bullet \equiv (A^\bullet \circ C^\bullet) \times (B^\bullet \circ C^\bullet)$. 

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2.2 Basic mould symmetries.

Nearly all useful moulds fall into a few basic symmetry types. A mould \( A^* \) is said to be symmetral (resp. alternal) iff:

\[
\sum_{\omega \in \text{sha}(\omega^1, \omega^2)} A^\omega = A^{\omega^1} A^{\omega^2} \quad (\text{resp. } 0) \quad \forall \omega^1 \neq \emptyset, \forall \omega^2 \neq \emptyset
\]  

(34)

A mould \( A^* \) is said to be symmetrel (resp. alternel) iff:

\[
\sum_{\omega \in \text{she}(\omega^1, \omega^2)} A^\omega = A^{\omega^1} A^{\omega^2} \quad (\text{resp. } 0) \quad \forall \omega^1 \neq \emptyset, \forall \omega^2 \neq \emptyset
\]  

(35)

Here \( \text{sha}(\omega^1, \omega^2) \) (resp. \( \text{she}(\omega^1, \omega^2) \)) denotes the set of all sequences \( \omega \) obtained from \( \omega^1 \) and \( \omega^2 \) under ordinary (resp. contracting) shuffling. In a contracting shuffle, two adjacent indices \( \omega_i \) and \( \omega_j \) stemming from \( \omega^1 \) and \( \omega^2 \) respectively may coalesce to \( \omega_{ij} := \omega_i + \omega_j \).

Thus, for a sequence \( \omega^1 := (\omega_1) \) of length 1 and a sequence \( \omega^2 := (\omega_2, \omega_3) \) of length 2, the symmetrality (resp alternality) condition reads:

\[
A^{\omega^1, \omega^2, \omega_3} + A^{\omega^2, \omega_1, \omega^3} + A^{\omega^1, \omega_2, \omega_3} \equiv A^{\omega^1} A^{\omega^2, \omega_3} \quad (\text{resp. } \equiv 0)
\]

(36)

and the symmetrelity (resp alternelity) condition reads:

\[
A^{\omega^1, \omega^2, \omega_3} + A^{\omega^2, \omega_1, \omega^3} + A^{\omega^1, \omega_2, \omega_3} + A^{\omega^2, \omega_1, \omega_3} \equiv A^{\omega^1} A^{\omega^2, \omega_3} \quad (\text{resp. } \equiv 0)
\]

(37)

These concepts are particularly relevant for arbomoulds, i.e. moulds \( A^* \) with an arborescent order on their indices, two new symmetries come into play: separativity and atomicity.

**Separativity** means that whenever \( \omega^\prec \) is many-rooted, with one-rooted subsequences \( \omega_1^\prec \), the arbomould factors accordingly:

\[
A^{\omega^\prec} \equiv \prod A^{\omega_1^\prec} \quad \text{if} \quad \omega^\prec = \bigoplus \omega_1^\prec \quad \text{with} \quad \omega_1^\prec \quad \text{one-rooted}
\]

(36)

**Atomicity** means that whenever \( \omega^\prec \) has more than one root, the arbomould vanishes:

\[
A^{\omega^\prec} \equiv 0 \quad \text{if} \quad \omega^\prec \quad \text{is many-rooted}
\]

(37)

**Mould-comould contractions.**

Let \( B_\omega \) be the homogeneous components of some local-analytic, \( \nu \)-dimensional vector field \( X \) (resp of the postcomposition operator \( F \) associated with some local-analytic \( \nu \)-dimensional diffeomorphism \( f \)) and let

\[
B_\omega = B_{\omega_1, \ldots, \omega_r} := B_{\omega_r} \ldots B_{\omega_1} \quad (\text{return})
\]

(38)
The comould $B$ so defined is said to be co-symmetrical (resp co-symmetrical) if its action on a product $\varphi_1 \varphi_2$ obeys the Leibniz rule:

$$B_\omega (\varphi_1 \varphi_2) = \sum (B_{\omega_1} \varphi_1) (B_{\omega_2} \varphi_2)$$

with a sum extending to all pairs $(\omega_1, \omega_2)$ such that $\omega \in sha(\omega_1, \omega_2)$ (resp $\omega \in she(\omega_1, \omega_2)$).

The four main symmetry types admit a simple characterisation in terms of mould-comould contractions:

$$A^* : B^* \mapsto C^* \text{ with } C_{\omega_0} := \sum_{\|\omega\|=\omega_0} A^* B_\omega$$

Indeed:

- $A^* : B^* \mapsto C^*$
- alternating : field $\mapsto$ field
- symmetrical : field $\mapsto$ diffeo
- alternating : diffeo $\mapsto$ field
- symmetrical : diffeo $\mapsto$ diffeo

Most stability properties follow from this interpretation. Thus:

- symmetrical$^* \times$ symmetrical$^* =$ symmetrical$^*$
- symmetrical$^* \times$ symmetrical$^* =$ symmetrical$^*$
- alternating$^* \circ$ alternating$^* =$ alternating$^*$
- symmetrical$^* \circ$ symmetrical$^* =$ symmetrical$^*$

### 2.3 Constant-type moulds.

<table>
<thead>
<tr>
<th>mould</th>
<th>value</th>
<th>symmetry type</th>
<th>associated series</th>
</tr>
</thead>
<tbody>
<tr>
<td>$1^*$</td>
<td>1</td>
<td>symmetrical</td>
<td>1</td>
</tr>
<tr>
<td>$1^*$</td>
<td>1</td>
<td>alternating</td>
<td>$x$</td>
</tr>
<tr>
<td>log$^*$</td>
<td>$(-1)^{r-1} \frac{1}{r}$</td>
<td>alternating</td>
<td>$\log(1 + x)$</td>
</tr>
<tr>
<td>exp$^*$</td>
<td>$\frac{a^r}{r!}$</td>
<td>symmetrical</td>
<td>$e^{ax}$</td>
</tr>
<tr>
<td>tu$^*$</td>
<td>$\frac{\Gamma(r-a)}{\Gamma(-a)} \frac{(-1)^r (r-a)}{r!}$</td>
<td>symmetrical</td>
<td>$(1 + x)^a$</td>
</tr>
</tbody>
</table>

### 2.4 Difference-type flat moulds.

For any $t = (t_1, \ldots, t_r) \in \mathbb{R}^r, t_i \neq t_j$, we set $p := \sum_{t_i < t_{i+1}} 1$, $q := \sum_{t_i > t_{i+1}} 1$ and define the symmetrical mould $sad^*_a$ (special case: $sad^*_a = sad^*_1$) and the
2.5 Difference-type polar moulds.

\[
\begin{align*}
\text{tas}^0_{a,b} & := 1 ; \quad \text{tas}^{t_1}_{a,b} := \frac{a - b}{(a - t_1)(t_1 - b)} \\
\text{tas}^{t_1 \ldots t_r}_{a,b} & := \frac{a - b}{(a - t_1)(t_1 - t_2) \ldots (t_{r-1} - t_r)(t_r - b)} \\
\text{tas}^0_* & := 0 ; \quad \text{tas}^{t_1}_* := \frac{1}{(-t_1)(t_1)} \\
\text{tas}^{t_1 \ldots t_r}_* & := \frac{1}{(-t_1)(t_1 - t_2) \ldots (t_{r-1} - t_r)(t_r)} \\
\text{tas}^0_{**} & := 0 ; \quad \text{tas}^{t_1}_{**} := 1 \\
\text{tas}^{t_1 \ldots t_r}_{**} & := \frac{1}{(t_1 - t_2) \ldots (t_{r-1} - t_r)} \\
\text{tas}^*_{a,b} \times \text{tas}^*_{b,c} & = \text{tas}^*_{a,c} \\
\text{tas}^*_{a,b} \times \text{tas}^*_{b,a} & = 1^*
\end{align*}
\]

2.6 Sum-type flat moulds.

We first settle some notations, then define our moulds:

\[
\begin{align*}
\mathbf{x} & := (x_1, \ldots, x_r) \\
\tilde{x}_i & := x_1 + \cdots + x_i \\
\hat{x}_i & := x_i + \cdots + x_r \\
\|\mathbf{x}\| & := x_1 + \cdots + x_r = \tilde{x}_1 = \hat{x}_r \\
\sigma_+(x) & := 1 \text{ if } x > 0 \quad (\text{resp } := 0 \text{ if } x < 0) \\
\sigma_-(x) & := 1 \text{ if } x < 0 \quad (\text{resp } := 0 \text{ if } x > 0) \\
\delta(x) & := 1 \text{ if } x = 0 \quad (\text{resp } := 0 \text{ if } x \neq 0)
\end{align*}
\]
sofo\(\pm\) := \((-1)^r\) \(\sigma_\pm(\hat{x}_1)\ldots\sigma_\pm(\hat{x}_r)\)
antisofo\(\pm\) := \((-1)^r\) \(\sigma_\pm(\hat{x}_1)\ldots\sigma_\pm(\hat{x}_r)\)
sefo\(\pm\) := \((-1)^{r-1}\sigma_\pm(\hat{x}_1)\ldots\sigma_\pm(\hat{x}_{r-1})\) \(\sigma_\pm(\hat{x}_r)\)
antisefo\(\pm\) := \((-1)^{r-1}\sigma_\pm(\hat{x}_1)\ldots\sigma_\pm(\hat{x}_{r-1})\) \(\sigma_\pm(\hat{x}_r)\)
lefo\(\pm\) := \((-1)^r\) \(\sigma_\pm(\hat{x}_1)\ldots\sigma_\pm(\hat{x}_{r-1})\) \(\delta(\hat{x}_r)\)
antinefo\(\pm\) := \((-1)^r\) \(\delta(\hat{x}_1)\) \(\sigma_\pm(\hat{x}_{r-1})\ldots\sigma_\pm(\hat{x}_r)\)

2.7 Sum-type polar moulds. The “organic” family.

\(\text{sa}_a^\omega := \prod_{i=1}^{r=\hat{x}_i} \omega_i^{\hat{x}_i} \quad \text{musa}_a^\omega := (-1)^r \prod_{i=1}^{i=r} \omega_i^{\hat{x}_i}\)
\(\text{romo}_a^\omega := \prod_{i=1}^{i=r}(a \omega_i^{\hat{x}_i} - 1) \quad \text{antiromo}_a^\omega := \prod_{i=1}^{i=r}(a \omega_i^{\hat{x}_i} - 1)\)
\(\text{remo}_a^\omega := a \omega_i^{\hat{x}_i} \prod_{i=1}^{i=r-1}(a \omega_i^{\hat{x}_i} - 1) \quad \text{antiremo}_a^\omega := a \omega_i^{\hat{x}_i} \prod_{i=1}^{i=r-2}(a \omega_i^{\hat{x}_i} - 1)\)

\[
\text{somo}_{a,b}^* := \text{remo}_a^* \times \text{antiromo}_{1-b}^* \quad (48)
\]
\[
:= \text{romo}_a^* \times \text{antiremo}_{1-b}^* \quad (49)
\]
\[
=: \text{remo}_{a/b}^* \times \text{remo}_b^* \quad (50)
\]
\[
\text{somo}_{c,d}^* := \text{somo}_{a,b}^* \quad (51)
\]
\[
\text{somo}_{a,1}^* := \text{somo}_{a,b}^* \quad (52)
\]

2.8 Main properties.

Symmetry types: \(^{12}\)

All the above moulds fall into one or the other of the main symmetry types.

\textit{Alternate}: \(\text{lad}^*, \text{tas}_a^*, \text{tas}_a^*\)
\textit{Symmetrical}: \(\text{exp}_a^*, \text{sad}^*, \text{sad}_a^*, \text{tas}_{a,b}^*, \text{sa}^*, \text{musa}^*\)
\textit{Alternate}: \(\text{log}^*, \text{lefo}^*, \text{redo}^*, \text{redom}^*\)
\textit{Symmetrical}: \(\text{tu}_a^*, \text{sofo}_{\pm}^*, \text{sfeo}_{\pm}^*, \text{romo}_a^*, \text{remo}_a^*, \text{somo}_{a,b}^*\)

\(^{12}\)Flat moulds should be regarded as distribution-valued: for them the symmetries hold \textit{almost everywhere}, not necessarily \textit{everywhere}.

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All pairs \((\text{mould}^\bullet, \text{antimould}^\bullet)\) have the same symmetry type.

Useful identities and closure properties:

\[
\text{sofo}^\bullet_+ \times \text{sefo}^\bullet_- = 1^\bullet \quad (53)
\]

\[
\text{antisofo}^\bullet_+ \times \text{antisefo}^\bullet_- = 1^\bullet \quad (54)
\]

\[
\text{remo}^\bullet_a \times \text{antiromo}^\bullet_{1-a} = 1^\bullet \quad (55)
\]

\[
\text{romo}^\bullet_a \times \text{antiremo}^\bullet_{1-a} = 1^\bullet \quad (56)
\]

\begin{itemize}
    \item \textit{multiplicative inverse}:
        \[
        \text{somo}^\bullet \begin{bmatrix}
            a & b \\
            c & d 
        \end{bmatrix} \leftrightarrow \text{somo}^\bullet \begin{bmatrix}
            c & a \\
            b & d 
        \end{bmatrix} \quad (a, c \text{ exchanged})
        \]
    \item \textit{composition inverse}:
        \[
        \text{somo}^\bullet \begin{bmatrix}
            a & b \\
            c & d 
        \end{bmatrix} \leftrightarrow \text{somo}^\bullet \begin{bmatrix}
            b & a \\
            d & c 
        \end{bmatrix} \quad (\text{columns exchanged})
        \]
    \item \textit{sequence reversal}:
        \[
        \text{somo}^\bullet \begin{bmatrix}
            a & b \\
            c & d 
        \end{bmatrix} \leftrightarrow \text{somo}^\bullet \begin{bmatrix}
            c & a \\
            b & d 
        \end{bmatrix} \quad (\text{rows exchanged})
        \]
    \item \textit{multiplication}:
        \[
        \text{somo}^\bullet_{a_1,a_2} \times \text{somo}^\bullet_{a_2,a_3} = \text{somo}^\bullet_{a_1,a_3}
        \]
    \item \textit{composition}:
        \[
        \text{somo}^\bullet_{a_1,b_1} \circ \text{somo}^\bullet_{a_2,b_2} = \text{somo}^\bullet_{(a_2-b_2)a_1+b_2,(a_2-b_2)b_1+b_2}
        \]
    \item \textit{multiplication}:
        \[
        \text{somo}^\bullet_{a_1,b_1} \times \text{somo}^\bullet_{a_1,b_1} = \text{somo}^\bullet_{a_1,b_1}
        \]
    \item \textit{composition}:
        \[
        \text{somo}^\bullet_{a_1,b_1} \circ \text{somo}^\bullet_{a_1,b_1} = \text{somo}^\bullet_{a_1,b_1}
        \]
\end{itemize}

2.9 Smooth and form-preserving arborification.

Smooth or size-preserving arborification.

All the above moulds possess the property of smooth arborification (meaning that their arborified variants admit essentially the same type of bounds) the only exception being the moulds \(\log^\bullet\) and \(\text{tu}^\bullet_1\) for \(a \notin \mathbb{Z}\) and in particular for \(a = 1/2\). This is in relation with the fact that the standard alien derivations (which admit \(\log^\bullet\) as their left-lateral mould) and the standard or median convolution average (which admits \(\text{tu}^\bullet_{1/2}\) as its right- and left-lateral mould) are not well-behaved.\(^{13}\)

Of course, for alternal or symmetrical (resp alternel or symmetrel) moulds, one should take the ordinary (resp contracting) form of arborification.

Form-preserving arborification.

All the sum-type moulds listed above, i.e. all those moulds whose definition

\(^{13}\)See §4.10, §4.11 and [E11].
involves forward sums $\hat{x}_i$ or $\hat{\omega}_i$ (resp backward sums $\check{x}_i$ or $\check{\omega}_i$) have the stronger and very useful property of form-preserving arborification. This means that they retain their outward analytical expression, except that the sums $\hat{x}_i$ or $\hat{\omega}_i$ (resp $\check{x}_i$ or $\check{\omega}_i$) are now relative to the arborescent (resp antiarborescent) order. The same holds for the difference-type moulds $\text{tas}^*_{\omega,\omega}$ and $\text{tas}^*_{\omega,\omega}$.

Thus, it is an easy matter to check that for any arborescent sequence $\omega \prec$ (resp antiarborescent sequence $\omega \succ$) we still have:

$$s_a^{\omega} := \prod_{i=1}^{r} \frac{\omega_i}{\check{\omega}_i} \quad \text{musa}^{\omega} := (-1)^r \prod_{i=1}^{r} \frac{\omega_i}{\hat{\omega}_i}$$

exactly as in §2.1.10, except that $\hat{\omega}_i$ (resp $\check{\omega}_i$) now denotes the sum of all indices $\omega_j$ that follow (resp precede) $\omega_i$ inside $\omega \prec$ (resp $\omega \succ$). Of course, as in the case of totally ordered sequences, $\omega_i$ itself should be included in that sum.

### 2.10 Mould mixing and arborification.

For any pair $A^\bullet, B^\bullet$ of moulds carrying real indices $\omega_i$, the mould mixture $C^\bullet := A^\bullet \text{mix} B^\bullet$ is defined by:

$$C^{\omega_1,\ldots,\omega_r} := \sum_{\pi \in S_r} \sum_{1 \leq m \leq r} \text{MIX}^{\omega_1,\ldots,\omega_r}_{\pi,m} \check{B}^{\omega_{\pi(1)},\ldots,\omega_{\pi(m)}} A^{\omega_{\pi(m+1)},\ldots,\omega_{\pi(r)}}$$ (57)

with a sum extending to all permutations $\pi$ of the sequence $(1,\ldots,r)$. This sum involves the mould $A^\bullet$ itself and the conjugate $\check{B}^\bullet$ of the mould $B^\bullet$:

$$\check{B}^{\omega_1,\ldots,\omega_r} := (-1)^r \check{B}^{\omega_r,\ldots,\omega_1}$$ (58)

as well as a ‘disorder coefficient’ which is defined as follows:

$$\text{MIX}^{\omega_1,\ldots,\omega_r}_{\pi,m} := \epsilon_1 \epsilon_2 \ldots \epsilon_r \sigma_{\epsilon_1}(\check{\omega}_1) \sigma_{\epsilon_2}(\check{\omega}_2) \ldots \sigma_{\epsilon_r}(\check{\omega}_r)$$ (59)

and assumes the values $0, \pm 1$. Here, the sign function $\sigma_{\pm}$ and the forward sums $\check{\omega}_1 := \omega_1 + \ldots + \omega_r$ are as in §2.6, and the signs $\epsilon_i$ are given by:

$$\epsilon_1 := + \quad \text{if} \quad m < \pi^{-1}(1)$$
$$:= - \quad \text{if} \quad m \geq \pi^{-1}(1)$$

$$\epsilon_i := + \quad \text{if} \quad \pi^{-1}(i-1) < \pi^{-1}(i) \quad \text{(for } i > 1)$$
$$:= - \quad \text{if} \quad \pi^{-1}(i-1) > \pi^{-1}(i) \quad \text{(for } i > 1)$$

The usefulness of mix derives from the automatic sign separation which it brings about in the index sequences. Indeed, the sum on the right-hand side of (57) involves only terms of the form $A^{\alpha_1,\ldots,\alpha_r}$ and $B^{\beta_1,\ldots,\beta_r}$ such that:

$$\check{\alpha}_i := \alpha_1 + \cdots + \alpha_r \geq 0 \quad ; \quad \check{\beta}_i := \beta_1 + \cdots + \beta_r \leq 0$$ (60)
Mould mixing also respects symmetrality (in particular, self-mixing leaves symmetral moulds unchanged) and commutes with pre-multiplication by a third mould:

\[
\{A^* \text{ and } B^* \text{ symmetral}\} \implies \{A^* \text{ mix } B^* \text{ symmetral}\} \quad (61)
\]

\[
\{A^* \text{ symmetral}\} \implies \{A^* \text{ mix } A^* = A^*\} \quad (62)
\]

\[
(C^* \times A^*) \text{ mix } (C^* \times B^*) = C^* \times (A^* \text{ mix } B^*) \quad (63)
\]

Moreover – and this is essential for the sequel – the mixing operation retains its form under arborification. Indeed, if we construct \(C^* := A^* \text{ mix } B^*\) as in (57), then the standard (non-contracting) arborification \(C^\prec\) is given by a straightforward variant of (57):

\[
C(\omega^1, \ldots, \omega^r) \prec := \sum_{\pi \in S_r} \sum_{1 \leq m \leq r} \text{MIX}^{(\omega^1, \ldots, \omega^r)}_{\pi, m} B^{\omega^1_1(1), \ldots, \omega^m_1(m)} A^{\omega^1(m+1), \ldots, \omega^r_1(r)} \quad (64)
\]

with disorder coefficients \(\text{MIX}^{\omega^1_1, \ldots, \omega^r_1}_{\pi, m}\) still given by (59), except that the forward sums \(\hat{\omega}_i\) are now relative to the arborescent order on \(\omega^\prec\), and with a suitable redefinition of the signs \(\epsilon_i\):

\[
\epsilon_1 := + \quad \text{if} \quad m < \pi^{-1}(1) \quad \text{(for } i \text{ root of } \omega^\prec)\]
\[
\epsilon_1 := - \quad \text{if} \quad m \geq \pi^{-1}(1)\]
\[
\epsilon_i := + \quad \text{if} \quad \pi^{-1}(i_-) < \pi^{-1}(i) \quad \text{(for } i \text{ not a root } \omega^\prec)\]
\[
\epsilon_i := - \quad \text{if} \quad \pi^{-1}(i_-) > \pi^{-1}(i) \quad \text{and } i_- \text{ antecedent of } i\]

2.11 Mould flattening and arborification.

Let us also mention two more mould transforms which turn alternel (resp symmetrel) moulds \(A^*\) into alternel (resp symmetrel) moulds \(B^*\). The first transform is quite elementary and applies to all cases. The second transform is more subtle, but also more relevant to the present investigation. It applies only to moulds \(A^*\) with indices \(n_i\) in \(\mathbb{N}\) and turns them into ‘flat’ or ‘piecewise-constant’ moulds \(B^*\) with indices \(t_i\) in \(\mathbb{R}\). Both transforms respect multiplication in the sense that \(\text{transf}(A^*_1 \times A^*_2) \equiv \text{transf}(A^*_1) \times \text{transf}(A^*_2)\).

Here is how they are defined:

First mould transform:

\[
\text{direct} : \quad A^* \mapsto B^* := A^* \circ \exp^* \quad (65)
\]

\[
\text{inverse} : \quad B^* \mapsto A^* := B^* \circ \log^* \quad (66)
\]

Second mould transform:
\[ A^\bullet \leftrightarrow B^\bullet \quad \text{with} \quad B^{t_1, \ldots, t_r} := SA^{t_1, \ldots, t_r-1, +} \quad \text{and} \]
\[ \epsilon_1 := \text{sign}(t_1 - t_2), \ldots, \epsilon_{r-1} := \text{sign}(t_{r-1} - t_r) \]
\[ SA^+ := -A^1 \]
\[ SA^{+, +} := +A^{1,1} \]
\[ SA^{1, +} := +A^{1,1,1} + A^2 \]
\[ SA^{+, +} := -A^{1,1,1} - A^{1,2} \]
\[ SA^{1, +} := -A^{1,1,1} - A^{2,1} \]
\[ SA^{1, +} := -A^{1,1,1} - A^{1,2} - A^{2,1} - A^3 \]
\[ \text{etc} \ldots \quad \text{Generally:} \]
\[ \text{direct:} \quad SA^{\epsilon_1, \ldots, \epsilon_s} := (-1)^r \sum^s A^{n_1, \ldots, n_s} \]
\[ \text{inverse:} \quad A^{r_1, \ldots, r_s} := (-1)^s \sum^{**} \epsilon_1 \ldots \epsilon_r \ SA^{r_1, \ldots, r_s} \]

In the last but one identity, all sign subsequences \( \epsilon^i \) consist of \((r_i - 1)\) initial - signs and one final + sign \((r_i \text{ may be } =1)\) and \(\sum^*\) extends to all integer sequences \(n^i\) of sum \(r_i\), whereas in the last (reverse) identity the sum \(\sum^{**}\) extends to all \(\epsilon_j \in \{+,-\}\) except when \(j \in \{r_1, r_1 + r_2, \ldots, r_1 + \ldots + r_s\}\), in which case \(\epsilon_j\) has to be +.

3 Combinatorial aspects of coarborification.

3.1 The standard coarborification rule.

Let \( \{B_\omega, \omega \in \Omega\} \) be any system of ordinary differential operators in the variables \(x_1, \ldots, x_\nu\) and define the comould \(B^\bullet\) as usual by setting:

\[ B_{\omega_1, \ldots, \omega_r} := B_{\omega_r} \ldots B_{\omega_1} \quad (67) \]

Then there exists a privileged arborescent comould \(B^\prec\), the so-called standard or homogeneous coarborification of \(B^\bullet\), which is entirely characterised by the following three properties:

\textbf{P1} \(B^\prec\) is coseparative\(^{14}\) i.e.:

\[ B_\omega (\varphi_1 \varphi_2) \equiv \sum_{\omega^1 \oplus \omega^2 = \omega} B_{\omega^1} (\varphi_1) B_{\omega^2} (\varphi_2) \quad (68) \]

\(^{14}\) \(\omega^1 \oplus \omega^2\) denotes the tree obtained by juxtaposition of \(\omega^1\) and \(\omega^2\), with no other order relations than those inherited from the sub-trees \(\omega^i\). The sum (68) extends also to the trivial juxtapositions, with one summand \(\omega^i\) equal to \(\omega\) and the other one empty.
P2 If \( \text{deg}(\omega^\prec) = d \) i.e. if the tree \( \omega^\prec \) has exactly \( d \) roots, then the operator is homogeneous in the \( \partial_i := \partial_{x_i} \) with total degree \( d \).

P3 If \( \omega = \omega_1 \omega^* \) (in other words, if \( \omega \) is of degree one, with a root element \( \omega_1 \) followed by some arborescent sequence \( \omega^\prec \)) the corresponding operator factors as:

\[
B_{\omega^\prec} x_j \equiv B_{\omega^\prec} B_{\omega_1} \log x_j \quad (j = 1, 2, \ldots, \nu)
\] (69)

Moreover, if \( B_\bullet \) is cosymmetrical \(^{15}\) (resp cosymmetrel \(^{16}\)), then \( B_\bullet \) and \( B_\prec \bullet \) are indeed correlated according to \( B_\omega := \sum_{\omega^\prec < \omega} B_{\omega^\prec} \) (resp \( B_\omega := \sum_{\omega^\prec < \omega} B_{\omega^\prec} \)). In other words, whereas symmetrical and symmetrel moulds obey different arborification rules (simple/contracting), the standard co-arborification rules are exactly the same for a cosymmetrical comould and a cosymmetrel one.

Let us check, by induction on the length \( r \) of \( \omega^\prec \), the fact that P1, P2, P3 together do determine \( B_{\omega^\prec} \).

Either \( \text{deg}(\omega^\prec) = 1 \), which means that \( \omega^\prec \) is of the form (70), in which case \( B_{\omega^\prec} \) is as in (71) below:

\[
\omega^\prec = (\omega_1, \omega^*) \quad \text{(70)}
\]

\[
B_{\omega^\prec} = \sum_{1 \leq i \leq \nu} (B_{\omega^\prec} . B_{\omega_1} . \log x_j)(x_j \partial_j) \quad \text{(71)}
\]

Or \( \text{deg}(\omega^\prec) = d \geq 2 \), which means that \( \omega^\prec \) is of the form (72), with \( s \) clusters of \( d_1, \ldots, d_s \) identical, irreducible summands \( \omega^{i_1^\prec}, \ldots, \omega^{i_s^\prec} \), in which case \( B_{\omega^\prec} \) is as below:

\[
\omega^\prec = \omega^{i_1^\prec} \oplus \cdots \oplus \omega^{i_s^\prec} \quad (\omega^{i_1^\prec} \neq \emptyset, \text{deg}(\omega^{i_1^\prec}) = 1)
\]

\[
= (\omega^{i_1^\prec})^{\oplus d_1} \oplus \cdots \oplus (\omega^{i_s^\prec})^{\oplus d_s} \quad (d_1 + \cdots + d_s = d) \quad \text{(72)}
\]

\[
B_{\omega^\prec} = \frac{1}{d_1! \cdots d_s!} \sum_{1 \leq i_1 \leq d_1} \ldots \sum_{1 \leq i_s \leq d_s} (B_{\omega^\prec} . \log x_{j_1}) \ldots (B_{\omega^\prec} . \log x_{j_d}) (x_{j_1} \partial_{j_1}) \ldots (x_{j_d} \partial_{j_d})
\]

3.2 Interpretation for cosymmetrical/el comoulds.

To see how one and the same operation works equally well in the seemingly different contexts of cosymmetricality and cosymmetrelity, the reader may

\(^{15}\) See §3.2.

\(^{16}\) See §3.2.
examine the simplest non-trivial examples of cosymmetral and cosymmetrel comoulds, with one variable \( x \) only and the factorisation property:

\[
B_n^{(a)} : \quad n_1, ..., n_r := B_{n_1}^{(a)} \cdots B_{n_r}^{(a)}
\]

\[
B_n^{(e)} : \quad n_1, ..., n_r := B_{n_1}^{(e)} \cdots B_{n_r}^{(e)}
\]

\[
B_n^{(a)} := b_n x^{n+1} \partial_n \quad B_n^{(e)} : x^m \mathbb{C} \to x^{m+n} \mathbb{C} \quad (\forall m, n)
\]

\[
\sum_{n \geq 0} B_n^{(e)} := \exp \left( \sum_{n \geq 1} b_n x^{n+1} \partial_n \right)
\]

and then use the corresponding cosymmetries\(^\text{17}\):

\[
B_n^{(a)} \xrightarrow{\text{coproduct}} 1 \otimes B_n^{(a)} + B_n^{(a)} \otimes 1 \quad \text{(cosymmetry)}
\]

\[
B_n^{(e)} \xrightarrow{\text{coproduct}} \sum_{n_1 + n_1 = n} B_{n_1}^{(e)} \otimes B_{n_2}^{(e)} \quad \text{(cosymmetry)}
\]

and verify the distinct coarborification constraints (9) and (12).

### 3.3 Standard coarborification and norm reduction.

**Coarborification automatically diminishes comould norms.** This of course is its main property, its main justification, and the reason for its usefulness in analysis. The phenomenon takes place for any reasonable norm on local differential operators, for instance:

\[
\| B \| = \| B \|_{D_1, D_2} := \sup_{\varphi \neq 0} \frac{\| B \varphi \|_{D_1}}{\| \varphi \|_{D_2}} \quad \text{with} \quad 0 \in D_1, D_1 \subset D_2 \subset \mathbb{C}^\nu
\]

with \( D_1, D_2 \) two small open neighbourhoods of 0 and \( \| \varphi \|_{D_1} \) the uniform norm on \( D_1 \). To illustrate norm reduction, i.e. the improvement from (76) to (77):

\[
\| B_{\omega} \| \leq r(\omega^\zeta) \| \omega^\zeta \| \| B_{\omega_1} \| \cdots \| B_{\omega_r} \| \quad (76)
\]

\[
\| B_{\omega} \| \leq C^N(\omega^\zeta) \| B_{\omega_1} \| \cdots \| B_{\omega_r} \| \quad (77)
\]

let us fix a non-resonant spectrum \( \lambda \in \mathbb{C}^\nu \) and consider first-order differential operators of the form:

\[
B_{\omega_i} := x_n \omega_i^* \quad \text{with} \quad \omega_i^* := \sum_{1 \leq j \leq \nu} b_{\omega_i}^j \partial x_j, \quad \omega_i := < n_i, \lambda >, \quad b_{\omega_i}^j \in \mathbb{C}
\]

\(^{17}\)Since \( B_0^{(e)} = 1 \) the sum (74) includes as extreme terms \( B_n^{(e)} \otimes 1 \) and \( 1 \otimes B_n^{(e)} \).
Next, let us carry out homogeneous coarborification for three extreme types of arborescent sequences:

\[ \omega := (\omega_1, \ldots, \omega_r) ; \text{ total order } ; \text{ all } \omega_i \text{ distinct} \]
\[ \omega' := (\omega_1, \ldots, \omega_r) ; \text{ total order } ; \text{ all } \omega_i \text{ distinct} \]
\[ \omega'' := (\omega_1 \oplus \cdots \oplus \omega_r) ; \text{ no order } ; \text{ all } \omega_i \text{ distinct} \]
\[ \omega''' := (\omega_1 \oplus \cdots \oplus \omega_r) ; \text{ no order } ; \text{ all } \omega_i \text{ identical} \]

We find:

\[ B_\omega := B_{\omega_r} \cdots B_{\omega_1} \quad (78) \]
\[ B_{\omega'} := x^{n_r} (B_{\omega_r} x^{n_{r-1}}) (B_{\omega_{r-1}} x^{n_{r-2}}) \cdots (B_{\omega_3} x^{n_2}) (B_{\omega_2} x^{n_1}) B_{\omega_1} \quad (79) \]
\[ B_{\omega''} := x^{n_1+\cdots+n_r} B_{\omega_1}^* \cdots B_{\omega_r}^* \quad (80) \]
\[ B_{\omega'''} := \frac{1}{r!} x^{n_1+\cdots+n_r} B_{\omega_1}^* \cdots B_{\omega_r}^* \quad (81) \]

and in all three cases we observe the disappearance of the factor \( r! \), though for rather distinct reasons:

– in (79) we have a first-order differential operator \( B_{\omega_1}^* \) preceded by innocuous scalar factors \( B_{\omega_i} x^{n_i-1} \)

– in (80) we have a differential operator \( B_{\omega_1}^* \cdots B_{\omega_r}^* \) (all terms commute) of order \( r \) and of factorially large norm, but with a more than factorially small front factor \( x \parallel n \parallel \) since \( x \) is small and \( \| n \| \geq \text{const.} r^{1+\frac{1}{r}} \)

– in (81) we have again a differential operator \( B_{\omega_1}^* \cdots B_{\omega_r}^* \) (all terms are equal) of order \( r \) and of factorially large norm, but with a multiplicity factor \( \frac{1}{r!} \) in front.

4 The arborification-coarborification transform. Fourteen applications to analysis.

4.1 Application 1: Linearisation of vector fields with diophantine spectra.

A local analytic vector field \( X \) with diophantine, non-resonant spectrum \( \lambda := (\lambda_1, \ldots, \lambda_\nu) \):

\[ X := X^{\text{lin}} + \sum B_n \quad \text{with} \]
\[ X^{\text{lin}} := \sum_{1 \leq i \leq \nu} \lambda_i x_i \partial_{x_i} \quad \text{and} \]
\[ B_n := \text{homog. part of deg. } n = (n_1, \ldots, n_\nu) (n_i \geq -1, \text{ at most one neg. } n_i) \]
admits a formal linearisation $\Theta_{\text{ent}}$ which in operatorial form reads:

$$X = \Theta_{\text{ent}} X^{\text{lin}} \Theta_{\text{ent}}^{-1}$$  \hspace{1cm} (83)$$

$$\Theta_{\text{ent}} := \sum Sa^\bullet B^\bullet \equiv \sum Sa^\prec B^\prec$$  \hspace{1cm} (84)$$

$$\Theta_{\text{ent}}^{-1} := \sum \text{inv} Sa^\bullet B^\bullet \equiv \sum \text{inv}Sa^\prec B^\prec$$  \hspace{1cm} (85)$$

$$Sa^\omega := (-1)^r \prod_{1 \leq i \leq r} \frac{1}{\omega_1 + \cdots + \omega_i}$$  \hspace{1cm} (86)$$

$$\text{inv}Sa^\omega := \prod_{1 \leq i \leq r} \frac{1}{\omega_1 + \cdots + \omega_r}$$  \hspace{1cm} (87)$$

Prior to arborification, the normalising series (84),(85) are usually divergent. After arborification, they are always convergent, because both moulds $Sa^\bullet$ and $\text{inv}Sa^\bullet$ suffer no significant norm increase. And the reason why they don’t is that one of them, namely $\text{inv}Sa^\bullet$ actually retains its form, i.e. its outward analytical expression, under arborification. 18

N.B. Here and in the sequel, we take advantage of the non-resonance of the $\lambda_i$’s to substitute an indexation by $\omega_i =< \lambda, n_i > \in \mathbb{C}$ for the original indexation by $n_i \in \mathbb{Z}^r$.

4.2 Application 2: Linearisation of diffeos with diophantine spectra.

A local analytic diffeo $X$ with diophantine and (multiplicatively) non-resonant spectrum $l := (l_1, \ldots, l_\nu)$:

$$F := (1 + \sum B_n).F^{\text{lin}}$$  \hspace{1cm} (88)$$

$$F^{\text{lin}} := \varphi(x_1, \ldots, x_\nu) \mapsto \varphi(l_1 x_1, \ldots, l_\nu x_\nu)$$

$$B_n := \text{homog. part of deg. } n = (n_1, \ldots, n_\nu) \text{ (n_i \geq -1, at most one neg. } n_i)$$

18For details, see [E3],[E9], also [Sie1],[Sie2],[Br] for the historical background.
admits a formal entire linearisation $\Theta_{\text{ent}}$ which in operatorial form reads:

$$ F = \Theta_{\text{ent}} F^{\text{lin}} \Theta_{\text{ent}}^{-1} \quad \text{with} \quad (89) $$

$$ \Theta_{\text{ent}} := \sum S e^\bullet \cdot B. \equiv \sum S e^< \cdot B_< \quad (90) $$

$$ \Theta_{\text{ent}}^{-1} := \sum \text{inv} S e^\bullet \cdot B. \equiv \sum \text{inv} S e^< \cdot B_< \quad (91) $$

$$ S e^\omega := (-1)^r \prod_{1 \leq i \leq r} \frac{e^{-\omega_i}}{1 - e^{-\omega_1,\ldots,\omega_r}} \quad (92) $$

$$ \text{inv} S e^\omega := \prod_{1 \leq i \leq r} \frac{1}{e^{\omega_i+\cdots+\omega_r}-1} \quad (93) $$

As before, and for the same reasons, arborification restores convergence in the normalising series $\Theta_{\text{ent}}^{\pm 1}$. 19

4.3 Application 3: Normalisation of vector fields with resonant spectra.

Here normalisation rather than linearisation is the order of the day, with normalising transformations $\Theta_{\text{res}}$ that are generally divergent but resurgent. To simplify, assume the resonance to be of degree 1 (only one relation between the $\lambda_i$’s), in which case one single ‘normal’ variable $z$ bears the whole burden of divergence and resurgence.

$$ X := X^{\text{nor}} + \sum B_n \quad \text{with} \quad (94) $$

$$ X^{\text{nor}} := \sum_{1 \leq i \leq \nu} \lambda_i x_i \partial_{x_i} + x^m \sum_{1 \leq i \leq \nu} \tau_i x_i \partial_{x_i} \quad \text{with} \quad < m, \tau > = -1 \quad \text{and} $$

$$ B_n := \text{homog. part of deg. } n = (n_1, \ldots, n_\nu) (n_i \geq -1, \text{ at most one neg. } n_i) \quad $$

In operatorial form, the resurgent normalising transformations $\Theta_{\text{res}}^{\pm 1}$ read:

$$ X = \Theta_{\text{res}} X^{\text{nor}} \Theta_{\text{res}}^{-1} \quad \text{with} \quad (95) $$

$$ \Theta_{\text{res}} := \sum \mathcal{V} e(z)^\bullet \cdot B. \equiv \sum \mathcal{V} e(z)^< \cdot B_< \quad (96) $$

$$ \Theta_{\text{res}}^{-1} := \sum \text{inv} \mathcal{V} e(z)^\bullet \cdot B. \equiv \sum \text{inv} \mathcal{V} e(z)^< \cdot B_< \quad (97) $$

with mould elements $\mathcal{V} e(z)^\omega, \mathcal{V} e(z)^\omega$ that are elementary resurgent monomials. The normalising transformations being usually divergent, the only

19For details, see [E3],[E9], also [Sie1],[Sie2],[Br],[Rü] for the background.
The question that arises is of course whether the $\Theta_{res}^{\pm 1}$ are convergent as series of resurgent functions. Sometimes they already are, prior to arborification; sometimes arborification is called for.\textsuperscript{20}

4.4 Application 4: Normalisation of diffeos with resonant spectra.

The picture is much the same as in the previous example.

\[ F := (1 + \sum B_n) \cdot F_{nor} \quad \text{with} \quad F_{nor} := \varphi(x_1, \ldots, x_\nu) \mapsto \varphi(l_1 x_1, \ldots, l_\nu x_\nu) \]

\[ B_n := \text{homog. part of deg. } n = (n_1, \ldots, n_\nu) (n_i \geq -1, \text{ at most one neg. } n_i) \]

with resurgent normalising transformations $\Theta_{res}^{\pm 1}$ of the form:

\[ F = \Theta_{nor} F_{lin} \Theta_{nor}^{-1} \quad \text{with} \quad \Theta_{nor} := \sum \mathcal{W}e(z)^{B_\bullet} \equiv \sum \mathcal{W}e(z)^{<}\ B_\prec \]

\[ \Theta_{ent}^{-1} := \sum \text{inv}\mathcal{W}e(z)^{B_\bullet} \equiv \sum \text{inv}\mathcal{W}e(z)^{<}\ B_\prec \]

and with suitable resurgent monomials $\mathcal{W}e(z)^{\bullet}$ and $\text{inv}\mathcal{W}e(z)^{\bullet}$.\textsuperscript{21}

4.5 Application 5: Ramified linearisation of vector fields with quasi-resonant spectra.

Here, we assume pure quasi-resonance. In other words, we have no (exact) resonance, but a violation of Bryuno’s classical diophantine condition.

\[ X := X_{lin} + \sum B_n \quad \text{with} \quad X_{lin} := \sum_{1 \leq i \leq \nu} \lambda_i x_i \partial_{x_i} \]

\[ B_n := \text{homog. part of deg. } n = (n_1, \ldots, n_\nu) (n_i \geq -1, \text{ at most one neg. } n_i) \]

Quasi-resonance doesn’t prevent formal entire linearisation, but it usually renders $\Theta_{ent}^{\pm 1}$ divergent. To get hold of something convergent, we must harness the phenomenon of compensation and work with ramified transformations $\Theta_{ram}^{\pm 1}$. These are ‘ramified’ in the sense that they involve positive,

\textsuperscript{20}For details, see [E2],[E3],[E5].

\textsuperscript{21}For details, see [E2],[E3],[E5].
irrational powers of at least one, but sometimes two or three variables $x_i$. Moreover, instead of being defined on ordinary (uniform) neighbourhoods of the origin $0 \in \mathbb{C}^\nu$, they are defined in spiral-like, ramified neighbourhoods. The operatorial expansions for $\Theta_{\text{ram}}^{\pm 1}$ are always of the form:

\[
X = \Theta_{\text{ram}} X^{\text{lin}} \Theta_{\text{ram}}^{-1} \quad \text{with} \quad \Theta_{\text{ram}} := \Theta_{\text{ent}} \Theta_{\text{colin}}^{-1} = \sum Sa_{\text{ram}}(z) B_\bullet \equiv \sum Sa_{\text{ram}}^\text{\textless}(z) B_\text{\textless} \quad (104)
\]

\[
\Theta_{\text{ram}}^{-1} := \Theta_{\text{colin}} \Theta_{\text{ent}}^{-1} = \sum \text{inv} Sa_{\text{ram}}(z) B_\bullet \equiv \sum \text{inv} Sa_{\text{ram}}^\text{\textless}(z) B_\text{\textless} \quad (105)
\]

but the analysis very much depends on the ‘badness’ of the quasiresonance.

**Case 1: Real, semi-mixed spectrum:**
This is the case when $\lambda_1 < 0$ but $0 < \lambda_2, \lambda_3, \ldots, \lambda_\nu$.
Then *one* ramification suffices:

\[
z := x_1^{-1/\lambda_1}
\]

\[
Sa_{\text{ram}}(z) := Sa_{\text{co}}(z) := (\text{inv} Sa_{\text{co}} z_{\parallel}) \times \text{inv} Sa_{\text{co}}(z) = \text{inv} Sa_{\text{ram}}^\text{\textless}(z) B_\text{\textless} \equiv \text{inv} Sa_{\text{co}}(z)_1 \text{ mix } \text{inv} Sa_{\text{co}}(z)_2 (110)
\]

with $z_{\parallel}$ used as short-hand for $z_{\parallel} : = \sum \omega_i$. Here the expansions for $\Theta_{\text{ram}}^{\pm 1}$ are already convergent before arborification.

**Case 2: Real, mixed spectrum:**
This is the case when we have at least two negative and two positive $\lambda_i$. Here, *two* ramifications become necessary, attached to two eigenvalues of our own choosing, but of opposite signs, say $\lambda_1 < 0 < \lambda_2$, and we must resort to the sophisticated operation of *mould mixing*, which is described in §2.10. The mould ingredients for $\Theta_{\text{ram}}^{\pm 1}$ now read:

\[
z_1 := x_1^{-1/\lambda_1}, \quad z_2 := x_2^{-1/\lambda_2}
\]

\[
\text{inv} Sa_{\text{ram}}(z) := \text{inv} Sa_{\text{co}}(z_1) \text{ mix } \text{inv} Sa_{\text{co}}(z_2) \equiv \text{inv} Sa_{\text{ram}}^\text{\textless}(z_1) \text{ mix } \text{inv} Sa_{\text{co}}(z_2) (111)
\]

but the novelty is that now $\Theta_{\text{ram}}^{\pm 1}$ requires arborification to become convergent (in a suitable space of ramified functions, of course).

---

\textsuperscript{22} of course, they remain so after arborification: arborification is sometimes *unnecessary*, but never *harmful.*
Case 3: Full-blown quasiresonance with complex spectrum.
The same approach as above applies, but with three ramifications and more intricate forms of mixing. Here again, one cannot avoid arborification.

Link with the so-called ‘compensators’.
The mould ingredients $\text{Sa}_{\text{ram}}^*(x)$ and $\text{invSa}_{\text{ram}}^*(x)$ which enter the construction of $\Theta_{\text{ram}}^\pm$ are actually sums of compensators of the form:

$$z^{\sigma_0, \sigma_1, \ldots, \sigma_r} := \sum_{0 \leq i \leq r} z^{\sigma_i} \prod_{j \neq i} \frac{1}{\sigma_i - \sigma_j} \quad (z \in \mathbb{C}, \sigma_i \in \mathbb{R}^+) \quad (112)$$

which remain bounded even when the $\sigma_i$’s get dangerously close to one another. This simple remark underpins the whole theory of compensation. \(^{23}\)

4.6 Application 6: “Correction” of vector fields with resonant spectra.

This section and the two that follow deal with a remarkable, often misunderstood phenomenon: the non-appearance of supermultiple small denominators\(^{24}\) when resonance interacts with diophantine small denominators. The present section tackles the phenomenon in its purest form and at the simplest level. Take a resonant vector field $X$ with diophantine spectrum. Since resonance generally precludes linearisation (even formal), that leaves two options. In the first one, we add a resonant series to the linear part $X_{\text{lin}}$ to get a normal or prenormal form, leading to an entire, but divergent and resurgent conjugation of $X$ to that normal form, as in § 4.3. In the second option, we subtract a resonant series (the ‘correction’) from the field $X$ to force formal conjugation with $X_{\text{lin}}$. But this time, despite the deceptive symmetry of the two approaches, the formal conjugation turns out to be analytic as well.

$$X \sim X_{\text{lin}} \quad \text{(non-resonant case)} \quad (113)$$

$$X \sim X_{\text{lin}} + X_{\text{pre}} \quad \text{with} \quad [X_{\text{pre}}, X_{\text{lin}}] = 0 \quad \text{(resonant case)} \quad (114)$$

$$X - X_{\text{cor}} \sim X_{\text{lin}} \quad \text{with} \quad [X_{\text{cor}}, X_{\text{lin}}] = 0 \quad \text{(resonant case)} \quad (115)$$

\(^{23}\)For details, see [E8], also [E6].

\(^{24}\)very roughly: small denominators with such abnormally high multiplicities that their presence would automatically thwart convergence.
Translating the second option into mould expansions, we find:

\[ X - X^{\text{cor}} = \Theta^{\text{cor}} X \Theta^{-1}\text{cor} \]  

\[ X^{\text{cor}} = \sum \text{Carr}^* B_* = \sum \text{Carr}^{\omega_1 \ldots \omega_r} B_{n_r} \ldots B_{n_1} \]  

\[ \Theta^{\text{cor}} = \sum \text{Scarr}^* B_* = \sum \text{Scarr}^{\omega_1 \ldots \omega_r} B_{n_r} \ldots B_{n_1} \]  

The key ingredient here is a mould \(\text{Carr}^*\) inductively defined by:

\[ \text{Carr}^0 = 0 ; \quad \text{Carr}^0 = 1 ; \quad \text{Carr}^{\omega_1} = 0 \text{ if } \omega_1 = 0 \]  

\[ \text{var}_i \text{Carr}^{\omega} = \sum \text{Carr}^{\omega_1 \omega_2 \omega_3} \text{Carr}^{\omega_2} - \sum \text{Carr}^{\omega_1 \omega_2 \omega_3} \text{Carr}^{\omega_2} \]  

with a variation operator \(\text{var}_i\) that acts as follows:

\[ \text{var}_i M^{\omega_1 \ldots \omega_r} := \omega_i M^{\omega_1 \ldots \omega_r} + M^{\omega_1 \ldots \omega_{i-1} + \omega_{i+1} \ldots \omega_r} - M^{\omega_1 \ldots \omega_{i-1} + \omega_{i+1} \ldots \omega_r} \]  

We have analogous formulas for \(\text{Scarr}^*\). Two points must be emphasised here. The first is that the above induction leaves us sufficient latitude (through the choice of the index \(i\)) to prevent the occurrence of supermultiple small denominators. The second point is that it takes \textit{arborification} to make the expansions (117) and (118) convergent.

4.7 Application 7: Floquet theory.

Floquet theory concerns itself with differential equations with quasi-periodic coefficients. A test case is the system:

\[ \partial_t X(t) = U(t) X(t) \quad \text{with} \]  

\[ U(t) := l A + \sum_{\omega \in \Omega} e^{i \omega t} U_\omega \quad (A, U_\omega \text{ const } ; l, t \in \mathbb{R}, l \gg 1) \]  

\[ \omega \in \Omega := \lambda_1 \mathbb{Z} + \cdots + \lambda_\nu \mathbb{Z} \quad (\lambda_1, \lambda_2 \ldots \text{non-resonant}) \]  

In order to reduce (122) to an elementary, ‘self-solving’ equation:

\[ \partial_t Y(t) = V Y(t) \quad \text{with } V = \text{const} \]  

by means of a change of unknown \(X(t) = \Theta(t) Y(t)\), we must solve:

\[ V + \Theta^{-1}(t) \partial_t \Theta(t) = \Theta^{-1}(t) U(t) \Theta(t) \]  

\[ ^{25}\text{There exists a parallel theory for diffeos. For details, see [EV1],[EV2].} \]
with a constant matrix $V$ whose spectrum $(iv_1, \ldots, iv_\nu)$ can be read off the asymptotic behaviour of the solution of (122). The next steps are broadly parallel to those in the preceding section, except that now multiplication or division by the frequencies $\omega_i$ must be replaced respectively by the action of the operators:

$$V := (\partial_t - ad(V)) \quad ; \quad V := (\partial_t - ad(V))^{-1} \quad (127)$$

The elementary identities:

$$\frac{1}{\omega_1 (\omega_1 + \omega_2)} + \frac{1}{\omega_2 (\omega_1 + \omega_2)} \equiv \frac{1}{\omega_1 \omega_2} \quad (128)$$

upon whose repeated use the induction (120) rests, give way to the identities:

$$V((V B_1^*) B_2^*) + V(B_1^* (V B_2^*)) \equiv (VB_1^*) (VB_2^*) \quad (129)$$

with $B_\omega := e^{i\omega t} B_\omega$ and $B_\omega = \text{const} \quad (130)$

The last step – arborification – is not required in all cases: whether it is or not depends on the group we work in.\footnote{For some details, see [E10].}

### 4.8 Application 8: KAM theory and the survival of invariant tori.

Working under the classical (analytic) KAM assumptions, we perturb an integrable hamiltonian $h$:

$$h(y) = \langle \lambda, y \rangle + \langle y, Q, y \rangle = \sum \lambda_i y_i + \sum Q_{i,j} y_i y_j \quad (131)$$

(with $Q$-independent basic frequencies $\lambda_i$) into a non-integrable $H$:

$$H(x, y) = h(y) + \epsilon b(x, y) \quad (x \in T^\nu, y \in \mathbb{R}^\nu_0) \quad (132)$$

$$= \langle \lambda, y \rangle + \sum_{m,n} H_{m,n}(x, y) \quad (133)$$

The whole point is to start from Bryuno’s (not Siegel’s) diophantine assumptions on the $\lambda_i$’s and to prove the convergence, for $y = 0$ and a small enough perturbation parameter $\epsilon$, of the uncorrected Lindstedt series:

$$\sum H_{m,n}(x, y) = \sum c_{m,n}(\epsilon) e^{2\pi i \langle x, m \rangle} y^n = \sum c_{m,n}(\epsilon) e^{2\pi i \langle \lambda, m \rangle t} y^n \quad (134)$$

$$\omega := < m, \lambda > = \text{‘frequency’} \quad (m \in \mathbb{Z}^\nu)$$

$$\eta := -1 + \|n\| = -1 + \sum n_i = \text{‘grade’} \quad (n \in \mathbb{N}^\nu, \eta \geq -1)$$

\footnotetext{26}{For some details, see [E10].}
Going over from the potential $H$ to the vector field $X^H$, we must partially correct and partially normalise our field $X^H$:

$$X^H - X^\text{cor} \xrightarrow{\text{conj}} X^\text{lin} + X^\text{nor}$$

(135)

frequency($X^\text{cor}$) = 0,  \quad \text{frequency}(X^\text{nor}) = 0  

(136)

grade($X^\text{cor}$) = 0,  \quad \text{grade}(X^\text{nor}) \neq 0  

(137)

by allowing only terms of zero (resp non-zero) grade on the left- (resp right-hand) side of (135). Like in §4.3 the correction still possesses a mould expansion of type:

$$X^\text{cor} = \sum_{r \geq 1} \sum Bicarr^{(\omega_1, \ldots, \omega_r)} X^{H_{m_r, n_r}} \ldots X^{H_{m_2, n_2}} X^{H_{m_1, n_1}}$$

(138)

with frequencies $\omega_i := <m_i, \lambda>$ and grades $\eta_i := -1 + \|n_i\|$. The normal part $X^\text{nor}$ also has a similar mould expansion, but we need not worry about it, since it vanishes for $y = 0$ and so does not contribute to the Lindstedt series.

The alternal mould $Bicarr^\bullet$ is more complex than, but essentially similar to, the mould $Carr^\bullet$ of §4.3. In fact, $Bicarr^\bullet$ reduces to $Carr^\bullet$ when all the grades $\eta_i$ are 0 or, more generally, when to each vanishing partial sum $\omega_i + \ldots + \omega_j = 0$ there corresponds a vanishing partial sum $\eta_i + \ldots + \eta_j = 0$.

We can duplicate in this case all the steps of §4.6 and prove, once again, the non-occurrence of supermultiple small denominators, except that now the formal multiplicity of a divisor is exactly twice what it was in §4.6. That apart, precious little changes. We still must arborify to get the convergence of $X^\text{cor}$. This establishes, for a small enough perturbation parameter $\epsilon$, the convergence of the Lindstedt series for the corrected hamiltonian. Then a standard argument going back to Poincaré (known as “killing the constants” and using the possibility of changing the integration constants) readily yields the convergence of the Lindstedt series for the given hamiltonian itself.$^{27}$

4.9 Application 9: Well-behaved alien derivations.

Roughly speaking, a system $\Delta = \{\Delta_\omega, \omega \in \mathbb{R}^+\}$ of alien derivations is said to be well-behaved if, getting them to act on natural resurgent functions $\varphi$, we get exponential bounds of type $\|\Delta_\omega \varphi\| \leq c_\omega e^{c_\omega \omega}$. This condition, which is useful in certain (not all) applications, is not fulfilled by the simplest and oldest system – that of standard alien derivations. Now, a system $\Delta$ is completely characterised by a system of weights $d^{(\epsilon_1, \ldots, \epsilon_t)}$ with $\epsilon_i \in \{+, -\}$

$^{27}$For some details, see [E10].
and \( \omega_i \in \mathbb{R}^+ \). Further, due to so-called self-consistency constraints, knowing these weights reduces to knowing any one of the three following moulds\(^{28}\):

\[
\text{red}^{\omega_1, \ldots, \omega_r} := (-1)^r \mathbf{d}^{\omega_1^+, \ldots, \omega_r^+} \quad \text{("right-lateral mould") (139)}
\]

\[
\text{led}^{\omega_1, \ldots, \omega_r} := (-1)^r \mathbf{d}^{\omega_1^-, \ldots, \omega_r^-} \quad \text{("left-lateral mould") (140)}
\]

\[
\text{nad}^{t_1, \ldots, t_r} := \epsilon_1 \ldots \epsilon_r \mathbf{d}^{\omega_1^*, \ldots, \omega_r^*} \quad \text{("neutral mould") (141)}
\]

with \( \epsilon_i := \text{sign}(t_i - t_{i-1}) \) \((\forall i < r)\) and \( \epsilon_r := \text{sign}(t_r - t_s) \)

and we have this very useful criterion: the system \( \Delta \) is well-behaved iff, after arborification, one of these moulds (and therefore all three) admit exponential bounds.\(^{29}\)

4.10 Application 10: Well-behaved uniformising averages.

For uniformising convolution averages\(^{30}\) \( \mathbf{m} \) the requirement of being well-behaved is even more essential than for alien derivations. These averages were first devised to overcome the vexing phenomenon of faster-than-exponential growth in the Borel plane along singularity-carrying axes. Like alien derivations, averages admit a description in terms of weights \( \mathbf{m}^{(\omega_1^+, \ldots, \omega_r^+)} \) that are subject to severe self-consistency constraints, and all the information can be compressed into either of three moulds\(^{31}\):

\[
\text{rem}^{\omega_1, \ldots, \omega_r} := (-1)^r \mathbf{m}^{(\omega_1^+, \ldots, \omega_r^+)} \quad \text{("right-lateral mould") (142)}
\]

\[
\text{lem}^{\omega_1, \ldots, \omega_r} := (-1)^r \mathbf{m}^{(\omega_1^-, \ldots, \omega_r^-)} \quad \text{("left-lateral mould") (143)}
\]

\[
\text{nun}^{t_1, \ldots, t_r} := \epsilon_1 \ldots \epsilon_r \mathbf{m}^{(\omega_1^*, \ldots, \omega_r^*)} \quad \text{("neutral mould") (144)}
\]

with \( \epsilon_i := \text{sign}(t_i - t_{i-1}) \) \((\forall i < r)\) and \( \epsilon_r := \text{sign}(t_r - t_s) \)

Here again, well-behavedness has a simple characterisation: the uniformising average \( \mathbf{m} \) is well-behaved iff, after arborification, one of these moulds (and therefore all three) admit exponential bounds.\(^{32}\)

---

\(^{28}\)the first two are alternel; the last one is alternal.

\(^{29}\)For details, see [E11].

\(^{30}\)they turn multivalued functions \( \hat{\varphi} \) over \( \mathbb{R}^+ \) into uniform ones and respect convolution:

\[ \mathbf{m}(\hat{\varphi}_1 \ast \hat{\varphi}_2) \equiv \mathbf{m}(\hat{\varphi}_1) \ast \mathbf{m}(\hat{\varphi}_2) \]

\(^{31}\)the first two are symmetrel; the last one is symmetral.

\(^{32}\)For details, see [Me1],[EM],[E11].
4.11 Application 11: ‘Display’ of a resurgent function.

The *display* of a resurgent function \( f \) is defined by:

\[
f \mapsto \text{display}(f) := f + \sum_{r \geq 1} \sum_{\omega_1} \cdots \sum_{\omega_r} \Delta_{\omega_1} \cdots \Delta_{\omega_r} f
\]  

(145)

It encapsulates in user-friendly form all the information about \( f \). It involves all (successive) alien derivatives of \( f \), along with dual objects, the so-called *pseudovariables*, which multiply according to the shuffle product, behave predictably under alien derivation, and remain inert under natural derivation:

\[
\begin{align*}
Z^{\omega_1} Z^{\omega_2} &= \sum_{\omega \in \text{sha}(\omega_1, \omega_2)} Z^{\omega} \\
\Delta_{\omega_0} Z^{\omega_1} \cdots \Delta_{\omega_r} &= Z^{\omega_1} \cdots \Delta_{\omega_r} (\text{resp } 0) \text{ if } \omega_0 = \omega_1 \text{ (resp } \omega_0 \neq \omega_1) \\
\partial_z Z^{\omega} &= 0
\end{align*}
\]

(146) (147) (148)

These rules ensure that the *display* commutes with all operations (addition, multiplication, ordinary and alien derivation) and makes it an extremely useful tool for:

(a) writing down in compact form all the obstructions to convergence\(^{33}\)
(b) proving transcendence results\(^{34}\).

There are precautions to take, however: although the display may be written down in any dual bases of *ALIEN* and *PSEUDO*, if we want the expansion (145) to be convergent\(^{35}\) we must

(a) work with a well-behaved basis of *ALIEN* and *PSEUDO*
(b) *arborify* the expansion (145).\(^{36}\)

4.12 Application 12: Canonical-spherical Object Synthesis.

*Object Analysis* is concerned with finding the analytic invariants \( \{ A_\omega \} \) of *local analytic objects Ob.*\(^{37}\) *Object Synthesis*, conversely, starts from some

\(^{33}\)i.e. all the Stokes constants, whose non-vanishing prevents \( f \) from being convergent.

\(^{34}\)since any relation \( R(f_1, f_2, \ldots) = 0 \) immediately translates into a corresponding relation between the *displays*, whose impossibility is often conspicuous, in view of the huge mass of constraints which it implies.

\(^{35}\)relative to the natural topology of *RESUR \otimes PSEUDO*

\(^{36}\)For some details, see [E10].

\(^{37}\)these are mostly, but not only, vector fields or diffeomorphisms.
(admissible!) system of invariants $\{A_\omega\}$ and endeavours to produce an object $\text{Ob}$ with precisely those prescribed invariants. The beauty is that there exists:

(a) a canonical solution $\text{Ob}^{\text{can}}$

(b) an entirely explicit, easy-to-handle expression of $\text{Ob}^{\text{can}}$ in terms of mould-comould expansions which involve (on the comould side) the invariants $\{A_\omega\}$ and (on the mould side) a special system of resurgence monomials, the so-called ‘spherical’ or ‘twisted’ monomials.

Here again, the mould-comould expansions always can, and often must be arborified to achieve convergence.\footnote{For details, see [E10].}

\subsection*{4.13 Application 13: Non-linear $q$-equations (F.Menous).}

The technique of arborification has recently been used to great effect by F. Menous\footnote{For details, see [Me2].} to prove that the $q$-difference equation:

\[ x \sigma_q y = y + b(x, y) \quad (b(0, 0) = \partial_q b(0, 0) = 0) \quad (149) \]

with analytic right-hand side and $(\sigma_q f)(x) := f(q x)$, is \textit{analytically} conjugate to one of the following normal forms:

\[ x \sigma_q y = y, \quad x \sigma_q y = y + x \quad (150) \]

\subsection*{4.14 Application 14: The “sandwich equation”.
}

The “sandwich equation” of unknown $f$:

\[ f^{n_1} \circ g_1 \circ f^{n_2} \circ g_2 \circ \ldots \circ f^{n_r} \circ g_r = \text{id} \quad \text{with} \quad n_i \in \mathbb{Z} \quad (151) \]

is clearly the most general equation that may be considered on an unspecified group $G$. If we now take $G$ to be the group of local diffeos of $\mathbb{C}$ and assume the data $g_i$ to be quasirotations, i.e. of the form $x \mapsto c_i x + o(x)$ with $|c_i| = 1$, then, barring global resonance and quasiresonance and assuming $\sum n_i \neq 0$, the unique formal solution of (151) is also \textit{analytic}. To establish this fact, massive arborification of the ‘template-preserving’ sort\footnote{see §1.3} is required.\footnote{For some details, see [EV3].}
5 Algebraic aspects of arborification-coarborification. Haukian moulds and haukian $\sigma$-functions.

5.1 Quadratic coarborification and quadratic fission: induced matrices, induced $\sigma$-functions, induced moulds.

We recall that the general fusion-fission transform:

$$SS = \sum \bullet B\bullet A\bullet \longmapsto SS = \sum # B# A#$$ (152)

involves a fusion rule and a dual fission constraint. The latter leaves considerable latitude. In a differential operator context, there is a natural way of satisfying it.\textsuperscript{42} In a free-associative context, there exists another natural answer, which is the quadratic fission rule.\textsuperscript{43} In matrix notations:

Fusion rule: $A# := F# A\bullet$ (153)

Fission constraint: $B\bullet := B# F\#$ (154)

Quadratic fission rule:

$B# := B\bullet (F\bullet F\#)^{-1} F\bullet$ (155)

$A\bullet := \text{column matrix of type} (r!, 1)$ (156)

$A# := \text{column matrix of type} (r!!, 1)$ (157)

$B\bullet := \text{row matrix of type} (r!, 1)$ (158)

$B# := \text{row matrix of type} (r!!, 1)$ (159)

$F\# := \text{rectangular matrix of type} (r!!, r!)$ (160)

$F\bullet := \text{rectangular matrix of type} (r!, r!!) := tr(F\#)$ (161)

$H\bullet := \text{square matrix of type} (r!, r!) := F\bullet F\#$ (162)

$K\bullet := \text{square matrix of type} (r!, r!) := (F\# F\bullet)^{-1}$ (163)

For definiteness, we concentrate on the case when all $r$ indices inside $\bullet$ are distinct. Then $r!!$ denotes some integer larger than $r!$ that only depends on the chosen type of order. For the arborescent order, there exist exactly $r!$ arborescent $\#$ compatible with a given $\bullet$ and so $r! < r!! < r!^2$.

Among all the fission rules compatible with the fission constraints, quadratic fission stands out as the only one that admits a simple matrix expression.

\textsuperscript{42}the so-called standard coarborification rule, studied at length in §3.

\textsuperscript{43}it minimizes the quadratic fission norm $\| B\bullet \|_\text{fission}^2 := \sum # \leq \bullet < B\#, B\# >$. 
Likewise, among all the fusion-fission transforms of type (152), the special case of arborification-coarborification stands out in at least three respects:

(i) it gives rise, in the group algebra $\mathbb{A}(\mathbb{S}_r)$ of the symmetric group $\mathbb{S}_r$, to a pair of elements $(\text{has}, \text{kas})$ which, despite being mutually inverse, are both expressible by simple, totally explicit formulae.

(ii) after normalisation to $(\text{has}, \text{kas})$ under the condition $\sum_{\sigma \in \mathbb{S}_r} \text{has}(\sigma) = \sum_{\sigma \in \mathbb{S}_r} \text{kas}(\sigma) = 1$, these elements in turn give rise to a pair of moulds $(\text{has}^*, \text{kas}^*)$ which are unexpectedly simple, extend to the whole of $\mathbb{N}$ and even $\mathbb{Z}$, and are of symmetrical type.

(iii) both as moulds and $\sigma$-functions, the above objects extend naturally to a two-parameter family, the haukian objects, which possess a wealth of rather improbable properties, all the more remarkable for completely disappearing when we substitute for the arborescent order any other type of order.

5.2 The symmetric group algebras and $\sigma$-functions.

Throughout, $\mathbb{S}_r$ shall denote the group of all permutations $\sigma$ of $\{1, \ldots, r\}$ and $\mathbb{A}(\mathbb{S}_r)$ shall be the corresponding group algebra, relative to the standard convolution product $\ast$. As for the $\sigma$-functions, they are functions $\sigma \mapsto h(\sigma)$ that are defined simultaneously and uniformly on all groups $\mathbb{S}_r$. Most of the $\sigma$-functions $h, k$ we shall encounter will stand in natural relation to integer-indexed moulds $h^*, k^*$. They will also possess simple invariance properties under a finite group of order 8 that acts on all $\mathbb{S}_r$. This “octo-group” consists of the following operations $\{o_0, o_1, \ldots, o_r\}$:

\[
\begin{align*}
o_0 : & \quad \sigma \mapsto o_0 \sigma := \sigma \\
o_1 : & \quad \sigma \mapsto o_1 \sigma := \sigma^{-1} \\
o_2 : & \quad \sigma \mapsto o_2 \sigma := \text{rev} \sigma \text{ rev} \\
o_3 : & \quad \sigma \mapsto o_3 \sigma := \text{rev} \sigma^{-1} \text{ rev} \\
o_4 : & \quad \sigma \mapsto o_4 \sigma := \text{rev} \sigma \\
o_5 : & \quad \sigma \mapsto o_5 \sigma := \sigma \text{ rev} \\
o_6 : & \quad \sigma \mapsto o_6 \sigma := \text{rev} \sigma^{-1} \\
o_7 : & \quad \sigma \mapsto o_7 \sigma := \sigma^{-1} \text{ rev}
\end{align*}
\]

with $\text{rev} = \text{rev}_r \in \mathbb{S}_r$ denoting the particular permutation (“reversion”) such that $\text{rev}(i) + i \equiv r + 1$. We shall refer to $o_1 \sigma$ and $o_2 \sigma$ as the inverse and reverse of $\sigma$. Apart from the unit element $o_0$, the octo-group comprises five involutions $o_1, \ldots, o_5$ and two elements $o_6, o_7$ of order 4.
5.3 Quadratic coarborification and the fully explicit $\sigma$-functions $\mathsf{has}, \mathsf{kas}$.

$\sigma$-functions ($\mathsf{has}, \mathsf{kas}$) induced by the matrices ($H^*_n, K^*_n$):

For any fusion-fission transform, the matrices $H^*_n$ and $K^*_n$ are clearly invertible, symmetric, and of the form:

$$H^n_{n_1, \ldots, n_r} = h(\sigma), \quad K^n_{n_1, \ldots, n_r} = k(\sigma) \quad \text{with} \quad n_i \equiv n_{\sigma(i)} \quad \sigma \in S_r \quad (172)$$

So, knowing ($H^*_n, K^*_n$) reduces to knowing the induced $\sigma$-functions ($h, k$).

Moreover, since $H^*_n$ and $K^*_n$ are symmetric and mutually inverse, we have:

$$h(\sigma^{-1}) \equiv h(\sigma), \quad k(\sigma^{-1}) \equiv k(\sigma), \quad h \ast k = 1_{A(S_r)} \quad (173)$$

For a general fusion-fission transform, this is about all there is to say. But for the arborification-coarborification transform, ($h, k$) specialises to a highly remarkable pair ($\mathsf{has}, \mathsf{kas}$), which becomes easier to handle when normalised to ($\mathsf{has}, \mathsf{kas}$) under the condition:

$$\sum_{\sigma \in S_r} \mathsf{has}(\sigma) = \sum_{\sigma \in S_r} \mathsf{kas}(\sigma) = 1 \quad (174)$$

We shall now succinctly describe these two objects and their teeming progeny.

Direct expression of $\mathsf{has}(\sigma)$:

$$\mathsf{has}(\sigma) := \prod_{1 \leq j \leq r} \beta_j(\sigma) \quad \in \mathbb{N} \quad (\forall \sigma \in S_r) \quad (175)$$

$$\mathsf{has}(\sigma) := \prod_{1 \leq j \leq r} \frac{2 \beta_j(\sigma)}{j(j+1)} = \frac{1}{h_r} \mathsf{has}(\sigma) \quad \in \mathbb{Q}^+ \quad (\forall \sigma \in S_r) \quad (176)$$

with $\beta_j(\sigma) := \# \{ i : 1 \leq i \leq j , \sigma(i) \leq \sigma(j) \}$

and $h_r := 2^{-r} r! (r+1)! \quad \in \mathbb{N} \quad (177)$

These formulas easily follow from the interpretation of $\mathsf{has}(\sigma)$ as the number of arborescent sequences that are order-compatible with both $\{1, \ldots, r\}$ and $\{\sigma(1), \ldots, \sigma(r)\}$. More unexpected is the existence of a closed expression for the convolution inverse $\mathsf{kas}(\sigma)$.

$^{44}\text{because of their interpretation in terms of norm minimisation. See §5.1}$

40
Direct expression of \( \text{kas}(\sigma) \). We have:

\[
\text{kas}(\sigma) := \frac{1}{h_r} \sum_{\mathcal{P}(0,\sigma) \in \text{Coher}(0,\sigma)} \text{sign}(\mathcal{P}(0,\sigma)). \mathcal{P}(0,\sigma)! \quad \in \mathbb{Q} \tag{179}
\]

\[
= \frac{1}{h_r} \xi(\sigma) \sum_{\mathcal{P}(0,\sigma) \in \text{Coher}(0,\sigma)} \mathcal{P}(0,\sigma)! \tag{180}
\]

\[
\text{kas}(\sigma) := \xi(\sigma) \sum_{\mathcal{P}(0,\sigma) \in \text{Coher}(0,\sigma)} \mathcal{P}(0,\sigma)! \quad \in \mathbb{Z} \tag{181}
\]

with elementary summands defined by:

\[
\mathcal{P}(0,\sigma)! := \frac{r! \: (r-1)!}{\prod_i [p_i(p_i-1)]^* [q_i(q_i-1)]^*} \tag{182}
\]

with \([x]^* := x \ (\text{resp} \ 1) \text{ if } x > 0 \ (\text{resp} \text{ if } x = 0)\) \tag{183}

or equivalently:

\[
\mathcal{P}(0,\sigma)! := \prod_i \text{ca}_p, q_i \tag{184}
\]

with \( \text{ca}_p, q_i := \frac{(p + q - 1)! \: (p + q - 2)!}{p!(p-1)!q!(q-1)!} \) \tag{185}

The sums extend to all maximal coherent binary bracketings\(^4\) of the sequence \(0, \sigma(1), \ldots, \sigma(r)\). Maximal binary bracketings are systems of nested pairs of brackets. They correspond one-to-one to binary trees. The coherence condition means that the integers within each bracket should be some permutation of consecutive integers \(s, s+1, \ldots\). Thus, ‘holes’ are prohibited. As for the products (182),(184), they extend to all pairs \(i\) of nested brackets or, equivalently, to all nodes \(i\) in the associated binary tree. Each of these pairs (or nodes) involves a sequence \(p_i\) of length \(p_i\) in the left bracket and a sequence \(q_i\) of length \(q_i\) in the right bracket, and gives rise to two factors:

(i) the integer factor \(\text{ca}_p, q_i\) defined above

(ii) a sign factor which is 1 (resp -1) if \(p_i < q_i\) (resp. \(p_i > q_i\)), meaning of course that each element of \(p_i\) is less (resp. greater) than each element of \(q_i\).

Multiplied together, the factors \(\text{ca}_p, q_i\) yield the “factorial” \(\mathcal{P}(0,\sigma)!\) and the sign factors yield the global \(\text{sign}(\mathcal{P}(0,\sigma))\). This global sign is actually independent of the bracketing \(\mathcal{P}\). It depends solely on the permutation \(\sigma\). So it

\(^4\)when no such bracketings exist (which becomes possible for \(r \geq 4\), and tends to occur with a probability approaching 1 as \(r\) increases), then of course the right-hand side of (179) should be taken as 0.
may be denoted as $\xi(\sigma)$ and factored out of the sum on the right-hand side of (180). Beware that $\xi(\sigma)$ is \textit{not} the permutation’s \textit{signature} $\epsilon(\sigma)$.

Let us show on two examples how the above rules work. First, let $r = 4$ and $(0, \sigma) = (0, 2, 1, 4, 3)$. We find only two coherent brackets. Here they are, along with the attached factors:

\[
(0) \left( (2)(1) \right) \left( (4)(3) \right) \implies c_{a^*}^{1, 4} c_{a^*}^{2, 2} c_{a^*}^{1, 1} c_{a^*}^{1, 1} = 3 \tag{186}
\]

\[
(0) \left( (2)(1) \right) \left( (4)(3) \right) \implies c_{a^*}^{3, 2} c_{a^*}^{1, 2} c_{a^*}^{1, 1} c_{a^*}^{1, 1} = 6 \tag{187}
\]

The global sign factor being $(-1) \times (-1) = 1$, we find $\text{kas}(\sigma) = 9$.

Now, consider the case $r = 4$ and $(0, \sigma) = (0, 3, 1, 4, 2)$. It is easy to check that there exits no coherent bracketing here. Therefore $\text{kas}(\sigma) = 0$.

Normalisation:
The reason for normalising $(\text{has}, \text{has})$ to $(\text{has}, \text{kas})$ is that the latter form alone leads to an interesting mould extension. In this context, let us record the two parallel formulas:

\[
\sum_{\sigma \in S_r} \text{has}(\sigma) = \left( \sum_{\sigma \in S_r} \text{kas}(\sigma) \right)^{-1} = 2^{-r} r! (r + 1)! \tag{188}
\]

\[
\sum_{\sigma \in S_r} \epsilon(\sigma) \text{has}(\sigma) = \left( \sum_{\sigma \in S_r} \epsilon(\sigma) \text{kas}(\sigma) \right)^{-1} = \text{ent} \left( \frac{r}{2} \right)! \text{ent} \left( \frac{r + 1}{2} \right)! \tag{189}
\]

with $\epsilon(\sigma) := \text{signature of } \sigma$ and $\text{ent}(x) := \text{integer part of } x$.

5.4 The associated moulds $\text{has}^\bullet, \text{kas}^\bullet$.

Definition of $\text{has}^n$ and $\text{kas}^n$ for arbitrary positive sequences $n$:
The relations

\[
\text{has}^{\sigma(1), \ldots, \sigma(r)} := \text{has}(\sigma) \quad \text{kas}^{\sigma(1), \ldots, \sigma(r)} := \text{kas}(\sigma) \tag{190}
\]

define $\text{has}^n, \text{kas}^n$ for any standard sequence $n$ of length $r$, i.e. for any permutation of $\{1, \ldots, r\}$. Now, any sequence of positive integers $n$, of length $r$, coherent or not, but without repetitions, may, for $r^*$ large enough, be embedded in a standard sequence $n^*$ of length $r^*$. Surprisingly, the following two sums:

\[
\text{has}^n := \sum_{n^* \in \text{Standard}(r^*), n^* \supseteq n} \text{has}^{n^*} \quad \text{(independent of } r^*) \tag{191}
\]

\[
\text{kas}^n := \sum_{n^* \in \text{Standard}(r^*), n^* \supseteq n} \text{has}^{n^*} \quad \text{(independent of } r^*) \tag{192}
\]
which range through all \( r*/(r* \cdot r)! \) standard sequences \( n^* \) containing \( n \), do not depend on the choice of \( r* \). Thus \( has^* \) and \( kas^* \) possess a natural extension to all positive, repetition-free sequences \( n \).

**Symmetry of \( has^* \) (conditional) and \( kas^* \) (unconditional):**

The two moulds so defined are symmetric:

\[
\sum_{n \in \text{sha}(n^1, n^2)} has^n \equiv has^{n^1} has^{n^2} \quad \text{if } n^1, n^2, n^1n^2 \in \text{coherent} \quad (193)
\]

\[
\sum_{n \in \text{sha}(n^1, n^2)} kas^n \equiv kas^{n^1} kas^{n^2} \quad \forall n^1 \neq \emptyset, \forall n^2 \neq \emptyset \quad (194)
\]

but whereas the first identity is conditional on all three sequences \( n^1, n^2, n^1n^2 \) being coherent\(^{46} \), the second identity holds in all cases, at least whenever it makes sense, i.e. for any repetition-free sequences \( n^1, n^2, n^1n^2 \).

**Form preservation under arborification:**

The direct expressions for \( has(\sigma), kas(\sigma) \) carry over trivially to \( has^n, kas^n \), at least for standard \( n \), but they also carry over, almost unchanged, to the arborified variants \( has^{\prec}, kas^{\prec} \). For instance, (181) remains in force, with maximal binary bracketings as in (181), with the very same Catalan factors and sign rule, and a “coherence” condition which demands that each parenthesis should contain

(i) some coherent subsequence

(ii) some connected portion of the original tree \( n^\times \)

**Factorisation properties of \( kas^n \):**

Any sequence \( n \) of positive integers factors uniquely into a product of maximal coherent sequences \( n^1n^2\ldots n^k \) and so too does the mould \( kas^* \):

\[
kas^n \equiv kas^{n^1} kas^{n^2} \ldots kas^{n^k} \quad \text{if } n^1 < n^2 \ldots < n^k \quad (195)
\]

\[
kas^n \equiv 0 \quad \text{otherwise} \quad (196)
\]

No such rule holds for \( has^n \), but this is immaterial, as the direct definition is so simple.

**Shift parameter of \( has^n \) and \( kas^n \):**

For any sequence \( n = (n_1, \ldots, n_r) \) and any shift parameter \( s \in \mathbb{N} \) let us set \( ^s n := (s + n_1, \ldots, s + n_r) \). The shift-dependence of \( has^n \) and \( kas^n \) turns out to be remarkably simple. It is:

\(^{46}\)i.e. permutations of unbroken integer sequences.
(i) rational\footnote{with simple poles at the points $s = -2, -3, \ldots, -r - 1$.} of degree at most $2r$ for the former,
(ii) polynomial of degree at most $r$ for the latter.

**Extension of $has^n$ and $kas^n$ to arbitrary integer sequences $n$:**
Simply write any (repetition-free) sequence $n$ as $^s m$ for some positive $m$ and negative $s$, and using rational (resp. polynomial) shift-continuation, set:

$$
has^n := has^s m, \quad kas^n := kas^s m
$$

The result won’t depend on the pair $(s, m)$, but on $n$ alone. *Symmetrality* also is guaranteed by construction, and so too is the persistence of the factorisation (195). The only hurdle, namely the occurrence of $s$-poles which may render $has^n$ (but not $kas^n$) infinite for certain sequences $n$ of *mixed signs*, will be removed by the introduction of a ‘twist’ parameter $t$. See below.

### 5.5 The twist parameter $t$ and the shift parameter $s$.

**Introduction of a ‘twist’ parameter $t$ and survival of all essential properties of $has^n, kas^n$:**
Fixing a real or complex parameter $t$, we first define $has_t$ and its normalised variant $\hat{has}_t$ by formulae closely patterned on (175) and (176):

$$
\begin{align*}
has_t(\sigma) &:= \prod_{1 \leq j \leq r} \left( t + \frac{\beta_j(\sigma)}{2} \right) \quad (\sum_{\sigma} has_t(\sigma) \neq 1) \quad (198) \\
\hat{has}_t(\sigma) &:= \frac{1}{r!} \prod_{1 \leq j \leq r} \frac{t + 2 \beta_j(\sigma)}{t + j + 1} \quad (\sum_{\sigma} has_t(\sigma) = 1) \quad (199)
\end{align*}
$$

with $\beta_j(\sigma) := \# \{ i : 1 \leq i \leq j , \sigma(i) \leq \sigma(j) \}$

Next, we derive $kas_t$ and $\hat{kas}_t$ by straightforward inversion in the group algebra $A(S_r)$. We then construct the moulds $has^* t$ and $kas^* t$ exactly as before, successively for sequences $n$ of *standard*, then *positive*, then *arbitrary* type. For this last step, we use the same trick as before, introducing a shift-parameter $s$ and setting:

$$
\begin{align*}
has^n_t := has^s m &=: has^m_{t,s}, \quad kas^n_t := kas^s m &=: kas^m_{t,s}
\end{align*}
$$

As before, we get the bonus:
(i) of *conditional symmetrality* for $has^* t$ and $has^* t,s$,
(ii) of *unconditional symmetrality* for $kas^* t$ and $kas^* t,s$. 
Broadly speaking, all known properties of \( \text{has}^* \) and \( \text{kas}^* \) seem to survive the introduction of the ‘twist’ parameter \( t \). The \( t \)-dependence itself closely resembles the \( s \)-dependence: rational for \( \text{has}_t^* \) and polynomial for \( \text{kas}_t^* \). Actually, the shift and twist\(^{48}\) parameters coexist and commingle amicably, and the \( t \)-dependence even turns out to be the simpler of the two.

**Twist- and shift-dependence of \( \text{has}^* \).**

\( \text{has}_{t,s}^n \) is a rational function of \( t, s \), of total degree no larger than \( 2r' \), and with at most \( r' \) simple poles of the form \( t + s + 1 + k, \; \inf(n) \leq k \leq \sup(n) \). Note that here \( r' \) is not the length \( r \) of \( n \), but its span \( := 1+\sup(n)−\inf(n) \).

**Twist- and shift-dependence of \( \text{kas}^* \).**

\( \text{kas}_{t,s}^n \) is a polynomial in \((t, s)\), of \( t \)-degree at most \( r−1 \), of \( s \)-degree at most \( 2r−2 \), and of total \((t, s)\)-degree also no larger than \( 2r−2 \). The main thing, however, is the existence of a closed expression for \( \text{kas}_{t,s}^n \). First, we set\(^{49}\):

\[
ca_{p,q}^*(t) := \frac{(p + q - 2)! (p + q - 1)!}{(p - 1)! (q - 1)! (p)! q!} \in \mathbb{Z}[t] 
\]

Next, we define mappings \( P_{t,s} \) by the following induction:

\[
P_{t,s} : \quad n \rightarrow P_{t,s}(n) \in \mathbb{Z}[t, s] \quad (204) \\
P_{t,s}(n) := 1 \quad \text{if} \quad n \text{ has length one. Otherwise} : \quad (205) \\
P_{t,s}(n) := \sum_{n_1,n_2=n} ca_{r_1,r_2}^*(s,t) \xi(n^1,n^2) P_{t,s}(n^1) P_{0,0}(n^2) \quad (206)
\]

with a sum extending to all factorisations of \( n \) into non-empty sequences \( n^1,n^2 \) of length \( r_1,r_2 \); and with sign coefficients defined in this way:

\[
\xi(n^1,n^2) := +1 \quad \text{if} \quad \max(n^1) < \min(n^2) \quad (207) \\
:= -1 \quad \text{if} \quad \max(n^2) < \min(n^1) \quad (208) \\
:= 0 \quad \text{otherwise} \quad (209)
\]

We should pay attention to the highly dissymmetric role assigned to \( n^1 \) and \( n^2 \) on the right-hand side of (206). Now, with all the ingredients in place,

\(^{48}\)This is a mere label, of course: the twist attached to \( \text{has}^* \) and \( \text{kas}^* \) bears no relation to the one attached to the resurgence monomials.

\(^{49}\)Of course, for \( x \not\in \mathbb{N} \), \( x! \) means \( \Gamma(x+1) \).
we may write down the required formulas for any \( \sigma \in S_r \). They read:

\[
\begin{align*}
\text{kas}(\sigma) &= k\text{as}^n := P_{0,0}(-n) \\
\text{kas}_t(\sigma) &= k\text{as}^n_t := P_{t,0}(-n) \\
\text{kas}_{t,s}(\sigma) &= k\text{as}^{n}_{t,s} := P_{t,s}(-n)
\end{align*}
\]

with \( n := (\sigma(1), \ldots, \sigma(r)) \), \( -n := (0, \sigma(1), \ldots, \sigma(r)) \).

For future use let us also define a related, parameter-free \( \sigma \)-function \( k\text{a} \):

\[
\text{ka}(\sigma) = k\text{a}^n := P_{0,0}(n) \quad (n \text{ here, not } -n!)
\]

The corresponding mould \( k\text{a}^* \) turns out to be alternating.

Let us point out, lastly, that \( \text{has}_{t,s}, \text{kas}_{t,s} \) are mutually inverse in \( A(S_r) \) only for \( s = 0 \). For other values of \( s \), the inverse of \( \text{has}_{t,s} \) is unremarkable, and that of \( \text{kas}_{t,s} \) is remarkable (i.e. factorisable and explicitable) only for \( s \in \{0, -1, \ldots, -r\} \).

### 5.6 Basic symmetries for \( \text{has}, \text{kas} \).

These \( \sigma \)-functions present a large number of symmetries, which involve the ‘octo-group’ (see §5.2) and become easier to write down after suitable parameter changes \((t, s) \rightarrow (t', s') \) or \((t'', s'')\) that mix up twist, shift, and length.

First, we have the parity relations in \( \sigma \) (or \( o_1 \)-invariance):

\[
\begin{align*}
\text{has}_t(\sigma) &\equiv \text{has}_t(\sigma^{-1}) \quad \forall \sigma \in S_r \\
\text{kas}_t(\sigma) &\equiv \text{kas}_t(\sigma^{-1}) \quad \forall \sigma \in S_r \\
\text{has}_{t,s}(\sigma) &\neq \text{has}_{t,s}(\sigma^{-1}) \quad \text{generally} \\
\text{kas}_{t,s}(\sigma) &\equiv \text{kas}_{t,s}(\sigma^{-1}) \quad \forall \sigma \in S_r
\end{align*}
\]

Now to the symmetries proper. It is convenient to set:

\[
\begin{align*}
\text{has}_{t',s'}(\sigma) := \text{has}_{t'-1,s'-1-\frac{r}{2}}(\sigma) \quad \forall \sigma \in S_r \\
\text{kas}_{t'',s''}(\sigma) := \text{kas}_{t'-1,s''-1-\frac{r}{2}}(\sigma) \quad \forall \sigma \in S_r
\end{align*}
\]

The \( \sigma \)-function \( \text{has} \) is invariant under one involution only:

\[
\{t', s', \sigma\} \longrightarrow \{-t', -s', o_4 \sigma\} \quad (\text{recall that } o_4 \sigma := \text{rev} \cdot \sigma)
\]

46
but the $\sigma$-function $\text{kas}$ is invariant under 11 involutions (4 independent):

\[
\begin{align*}
\{t'', s'', \sigma\} &\rightarrow \{t'', s'', o_1 \sigma\} &\text{(recall that } o_1 \sigma := \sigma^{-1}) &\quad (221) \\
\{t'', s'', \sigma\} &\rightarrow \{-t'', s'', o_0 \sigma\} &\text{(recall that } o_0 \sigma := \sigma) &\quad (222) \\
\{t'', s'', \sigma\} &\rightarrow \{-t'', s'', o_1 \sigma\} &\quad (223) \\
\{t'', s'', \sigma\} &\rightarrow \{t'', -s'', o_2 \sigma\} &\text{(recall that } o_2 \sigma := \text{rev } \sigma^{-1} \text{rev}) &\quad (224) \\
\{t'', s'', \sigma\} &\rightarrow \{t'', -s'', o_3 \sigma\} &\quad (225) \\
\{t'', s'', \sigma\} &\rightarrow \{-t'', -s'', o_2 \sigma\} &\quad (226) \\
\{t'', s'', \sigma\} &\rightarrow \{-t'', -s'', o_3 \sigma\} &\quad (227) \\
\{t'', s'', \sigma\} &\rightarrow \{s'', t'', o_0 \sigma\} &\text{if } \xi(\sigma) = - \text{ and with factor } (-1)^{r-1} &\quad (228) \\
\{t'', s'', \sigma\} &\rightarrow \{s'', t'', o_1 \sigma\} &\text{if } \xi(\sigma) = - \text{ and with factor } (-1)^{r-1} &\quad (229) \\
\{t'', s'', \sigma\} &\rightarrow \{s'', t'', o_2 \sigma\} &\text{if } \xi(\sigma) = - \text{ and with factor } (-1)^{r-1} &\quad (230) \\
\{t'', s'', \sigma\} &\rightarrow \{s'', t'', o_3 \sigma\} &\text{if } \xi(\sigma) = - \text{ and with factor } (-1)^{r-1} &\quad (231)
\end{align*}
\]

The next symmetries involve a $\sigma$-function $\text{lokas}$ derived from $\text{kas}$ by taking the (mould) logarithm of the corresponding moulds, but after reversion to the original $(s, t)$ parameters, like this:

<table>
<thead>
<tr>
<th>$\sigma$-functions</th>
<th>moulds</th>
<th>moulds</th>
<th>$\sigma$-functions</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\text{kas}_{t,s}$</td>
<td>$\text{kas}_{t,s}$</td>
<td>$\text{lokas}_{t,s}$</td>
<td>$\text{lokas}_{t,s}$</td>
</tr>
<tr>
<td>$\downarrow$</td>
<td>(symmetrical)</td>
<td>(alternal)</td>
<td>$\downarrow$</td>
</tr>
<tr>
<td>$\text{kas}_{t',s'}$</td>
<td>$\text{lokas}_{t',s'}$</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

This $\sigma$-function $\text{lokas}$ is invariant under 7 involutions (3 independent):

\[
\begin{align*}
\{t'', s'', \sigma\} &\rightarrow \{t'', s'', o_5 \sigma\} \text{ with factor } (-1)^{r-1} &\quad (232) \\
\{t'', s'', \sigma\} &\rightarrow \{-t'', s'', o_0 \sigma\} &\quad (233) \\
\{t'', s'', \sigma\} &\rightarrow \{-t'', s'', o_5 \sigma\} \text{ with factor } (-1)^{r-1} &\quad (234) \\
\{t'', s'', \sigma\} &\rightarrow \{t'', -s'', o_2 \sigma\} &\quad (235) \\
\{t'', s'', \sigma\} &\rightarrow \{t'', -s'', o_4 \sigma\} \text{ with factor } (-1)^{r-1} &\quad (236) \\
\{t'', s'', \sigma\} &\rightarrow \{-t'', -s'', o_0 \sigma\} &\quad (237) \\
\{t'', s'', \sigma\} &\rightarrow \{-t'', -s'', o_5 \sigma\} \text{ with factor } (-1)^{r-1} &\quad (238)
\end{align*}
\]

These new symmetries are as unexpected as the previous ones. In particular, they are no direct consequences of the symmetries for $\text{kas}$\textsuperscript{50}.

\textsuperscript{50}Indeed, due to the non-linearity of the taking of mould logarithms, the $(t, s) \leftrightarrow (t'', s'')$ shuttle has the effect of mixing up quite distinct sequence lengths.
5.7 Factorisation properties for has, kas.

The factorisation property for kas already encountered in §5.4 survives the introduction of the twist and shift parameters. For any repetition-free integer sequence \( n \) with its decomposition \( n^1 \ldots n^k \) into a product of coherent factor sequences, we still have:

\[
kas_{t,s}^n \equiv \begin{cases} 
kas_{t,s}^{n^1} kas_{t,s}^{n^2} \ldots kas_{t,s}^{n^k} & \text{if } n^1 < n^2 \ldots < n^k \\
0 & \text{otherwise}
\end{cases}
\] (239)

In combination with the formula (212), which already settles the case of coherent sequences \( n \), the rule (239) covers all possible cases.

Moreover, if a sequence \( n \) contains indices \( n_i \) of both signs, we have a further factorisation result:

\[
kas_{t,s}^n \equiv \begin{cases} 
kas_{t,s}^{n^1} kas_{t,s}^{n^2} & \text{if } n = n^1 \cdot n^2 \text{ with } n^1 \leq 0 < n^2 \\
0 & \text{otherwise}
\end{cases}
\] (240)

5.8 Proofs: main steps.

Catalan numbers and polynomials.

\[
ca_n := \frac{(2n)!}{n(n+1)!} \quad ca_{p,q} := \frac{(p+q)!}{p! q! (p+1)! (q+1)!} \\
ca_n(t) := \frac{(2n+t)!}{n(n+1+t)!} \quad ca_{p,q}(t) := \frac{(p+q+t)!}{p! q! (p+1+t)! (q+1)!} \\
ca_n(t,s) := \frac{(2n+t+s)!}{n(n+1+t+s)!} \quad ca_{p,q}(t,s) := \frac{(p+q+s)!}{p! q! (p+1+t+s)! (q+1)!}
\]

They relate under \( ca_n = ca_{n+1}^* \) and \( ca_{p,q} = ca_{p+1,q+1}^* \) to the earlier coefficients and polynomials, but are sometimes more convenient. Useful identities:

\[
ca_n = \sum_{p+q = n-1, p \geq 0, q \geq 0} ca_{p,q} \quad ; \quad ca_n(t) = \sum_{p+q = n-1, p \geq 0, q \geq 0} ca_{p,q}(t)
\] (243)

**Induction for has** and **has**.

It is implicit in the factorisation rule

**Induction for kas** and **kas**.

Thanks to the factorisation property (239) we may limit ourselves to coherent sequences \( n \), and by playing on the shift parameter \( s \), we may even assume \( n \) to be some permutation of the basic sequence \( (1, \ldots, r) \). That leaves the distinction between normal and antinormal sequences, depending on whether the smallest element 1 precedes or follows the largest element \( r \). The simpler
induction rules apply for antinormal sequences. As usual, we have the choice between two (non-trivially equivalent) variants, the one privileging the smallest element, the other the largest. They go like this:

For antinormal sequences \( n = (r_1, \ldots, r_n) = (a, 1, b) = (c, r, d) \) of length \( r \):

\[
\begin{align*}
\kast_{t,s}^{a,b} &:= ca_{s,r-1}(t) \frac{a}{-1} \kast_{t-1}^{b} & (244) \\
\kast_{t,s}^{c,d} &:= ca_{s,r-1}(t) \frac{d}{-1} \kast_{t,r}^{c} & (245)
\end{align*}
\]

with \( \sim (n_1, \ldots, n_r) := (n_r, \ldots, n_1) \) and \( \tilde{\sim} (n_1, \ldots, n_r) := (-n_1, \ldots, -n_r) \).

For normal sequences \( n = (1, \ldots, r) \), \( \tilde{n} \) is antinormal and the rule is:

\[
\kast_n^{\tilde{n}} := \frac{(-1)^{r-1}}{2} \kast_{t,s}^{c,d} + \sum_{2 \leq k \leq r(n)} (-1)^{r-k} \sum_{n^1n^2\ldots n^k=n} \kast_{t,s}^{c,n^1} \kast_{t,s}^{c,n^2} \ldots \kast_{t,s}^{c,n^k} & (246)
\]

**Main steps:** One checks that the elementary induction for \( \textbf{has} \) and \( \textbf{has}_t \) translates into the above induction for \( \kast \) and \( \kast_t \). Then one shows that the latter agrees (is equivalent) with the direct expressions (181) for \( \kast \) and (211) for \( \kast_t \).

### 5.9 Factorisation properties for the connecting functions \( h\text{ak}, \, h\text{äk} \).

Fix \( t_1, t_2 \). Since \( \textbf{has}_{t_1}, \, \kast_{t_2} \) are even \( \sigma \)-functions, it is readily seen that all \( 2 \times 8 \times 8 \) convolution products of the form \((o_i \textbf{has}_{t_1}) \star (o_j \kast_{t_2})\) and \((o_j \kast_{t_2}) \star (o_i \textbf{has}_{t_1})\), with \( 0 \leq i, j \leq 7 \), actually reduce, modulo the \( o \)-action of the octo-group, to just two of them, e.g. \( h\text{ak}_{t_1,t_2} \) and \( h\text{äk}_{t_1,t_2} \):

\[
\begin{align*}
\hak_{t_1,t_2}(\sigma) &:= \textbf{has}_{t_1} \star \kast_{t_2}(\sigma) = \sum_{\sigma_1,\sigma_2=\sigma} \textbf{has}_{t_1}(\sigma_1) \kast_{t_2}(\sigma_2) & (247) \\
\hak_{t_1,t_2}(\sigma) &:= \textbf{has}_{t_1} \star (o_j \kast_{t_2})(\sigma) = (o_j \textbf{has}_{t_1}) \star \kast_{t_2}(\sigma) \sum_{\sigma_1, \text{rev} \sigma_2=\sigma} \textbf{has}_{t_1}(\sigma_1) \kast_{t_2}(\sigma_2) & (248)
\end{align*}
\]

But the real surprise is that both these “connecting” \( \sigma \)-functions should enjoy the property of maximal factorisation, which \( \textbf{has}_{t_1} \) already possesses, but not

---

\(^{51}\)So the first term in (246) may be calculated according to the rule (244) or (245).

\(^{52}\)I.e. invariant under the change \( \sigma \mapsto \sigma^{-1} \)
Indeed, we have:

\[
\text{hak}_{t_1,t_2}(\sigma) := \frac{1}{r!} \prod_{1 \leq j \leq r} \frac{t_1 + \gamma_j(\sigma) t_2 + \delta_j(\sigma)}{t_1 + j + 1} \quad \forall \sigma \in S_r \tag{250}
\]

\[
\text{h\text{"a}k}_{t_1,t_2}(\sigma) := \frac{1}{r!} \prod_{1 \leq j \leq r} \frac{t_1 + \gamma_j^*(\sigma) t_2 + \delta_j^*(\sigma)}{t_1 + j + 1} \quad \forall \sigma \in S_r \tag{251}
\]

with coefficients \(\gamma_j, \delta_j\) given by:

- if \(\sigma(j-1) < \sigma(j) < \sigma(j+1)\): \(\gamma_j(\sigma) := j - 1\), \(\delta_j(\sigma) := 2 \beta_j(\sigma) + j^2 - j\)
- if \(\sigma(j-1) < \sigma(j) > \sigma(j+1)\): \(\gamma_j(\sigma) := -1\), \(\delta_j(\sigma) := 2 \beta_j(\sigma) - 2j\)
- if \(\sigma(j-1) > \sigma(j) < \sigma(j+1)\): \(\gamma_j(\sigma) := 0\), \(\delta_j(\sigma) := 2 \beta_j(\sigma)\)
- if \(\sigma(j-1) > \sigma(j) > \sigma(j+1)\): \(\gamma_j(\sigma) := -j\), \(\delta_j(\sigma) := 2 \beta_j(\sigma) - j^2 - j\)

and with the same \(\beta_j(\sigma)\) as in the definition (175), (177) of has.

For \(j = 1\) or \(r\), the above inequalities involve numbers \(\sigma(0)\) or \(\sigma(r+1)\) which are not defined, since \(\sigma \in S_r\), but even then one does get the correct answer by setting \(\sigma(0) := 0\) or \(\sigma(r+1) := r+1\). We may also note that there is always a factor\(^{54}\) \(t_1 + 2\) on the numerator of (250), which cancels the \(t_1 + 2\) on the denominator. Similarly, unless \(\sigma = id\), there always has to be at least one factor\(^{55}\) \(t_1 - t_2\) on the numerator of (250) since \(\text{hak}_{t_1,t_2}(\sigma) \equiv 0\) when \(\sigma \neq id\).

Analogous formulas hold for the coefficients \(\gamma_j^*, \delta_j^*\). In fact:

\[
\gamma_j^*(\sigma) \equiv -\gamma_j(\sigma) \quad \forall \sigma \in S_r, \forall j \in \{1,\ldots,r\} \tag{252}
\]

\[
\delta_j^*(\sigma) \equiv -\delta_j(\sigma) + 2 + 2 \sigma(j) \quad \forall \sigma \in S_r, \forall j \in \{1,\ldots,r\} \tag{253}
\]

Here again, there is always a factor\(^{56}\) \(t_1 + 2\) on the numerator of (251), which cancels the one on the denominator. But since generally \(\text{h\text{"a}k}_{t_1,t_2}(\sigma) \neq 0\) there is no ‘permanent’ factor \(t_1 - t_2\) on the numerator of (251).

**Proofs:** These factorisation properties haven’t been proved yet in all cases, but they have been systematically checked on a computer up to \(r = 9\).

\(^{53}\)at least not in that sense. Its own factorisation properties (239) are of a markedly different nature.

\(^{54}\)it corresponds to the largest value of \(j\) such that \(\sigma(1) > \sigma(2) \cdots > \sigma(j)\).

\(^{55}\)it corresponds to the value of \(j\) such that \(\sigma(1) < \sigma(2) < \cdots < \sigma(j) > \sigma(j+1)\) (\(\sigma \neq id\)).

\(^{56}\)it corresponds to the largest value of \(j\) such that \(\sigma(1) > \sigma(2) \cdots > \sigma(j)\).
Moreover, for a large proportion of permutations $\sigma$, they result from the three, clearly equivalent identities that follow:

\begin{align}
\text{kas}_t \ast \text{ha}(\sigma) &= \lambda_r(t) \text{ ka}(\sigma) \quad \forall \sigma \in \mathbb{S}_r \\
\text{has}_t \ast \text{ka}(\sigma) &= \lambda_r^{-1}(t) \text{ ha}(\sigma) \quad \forall \sigma \in \mathbb{S}_r \\
\text{hak}_{t_1,t_2} \ast \text{ha}(\sigma) &= \frac{\lambda_r(t_2)}{\lambda_r(t_1)} \text{ ha}(\sigma) \quad \forall \sigma \in \mathbb{S}_r
\end{align}

with

\begin{align}
\lambda_r(t) := \frac{\Gamma(t + r + 2)}{\Gamma(r)\Gamma(t + 3)} = \frac{(t + 3) \ldots (t + r + 1)}{(r - 1)!}
\end{align}

These identities involve new $\sigma$-functions $\text{ha}, \text{ka}$. The first is elementary, and can be read off the defining identity:

\begin{align}
\sum_{\sigma \in \mathbb{S}_r} \text{ha}(\sigma)e_{\sigma(1)} \ldots e_{\sigma(r)} := \begin{bmatrix} \ldots e_1, e_2 \ldots e_r \end{bmatrix}
\end{align}

The other one, $\text{ka}$, has already received a direct definition in (213). It is closely related to the leading $t$-terms in $\text{kas}_t$ and $\text{kas}_{t,s}$. Indeed:

\begin{align}
\text{ka}(\sigma) &\equiv r \left( \frac{d}{dt} \right)^{r-1} \text{kas}_t(\sigma) \quad \forall t \\
&\equiv \frac{r \Gamma(s + r) \left( \frac{d}{dt} \right)^{r-1} \text{kas}_{t,s}(\sigma)}{\Gamma(r)\Gamma(t + 3)} \quad \forall t, \forall s
\end{align}

It displays maximal symmetry under the action of the octo-group:

\begin{align}
\text{ka}(\sigma) &= \text{ka}(o_i \sigma) \quad \forall \sigma \in \mathbb{S}_r, \forall i \in \{0, 1, 2, 3\} \\
&\equiv (-1)^{r-1} \text{ka}(o_i \sigma) \quad \forall \sigma \in \mathbb{S}_r, \forall i \in \{4, 5, 6, 7\}
\end{align}

The corresponding moulds $\text{ha}^*, \text{ka}^*$ are clearly alternal.$^{57}$

**Convolution group. Link with the ‘organic’ family.**

From the construction of the connecting functions there follow the identities:

\begin{align}
\text{hak}_{t_1,t_2} \ast \text{hak}_{t_2,t_3} &= \text{hak}_{t_1,t_2} \ast \text{hak}_{t_2,t_3} = \text{hak}_{t_1,t_3} \quad \forall t_1, t_2, t_3 \\
\text{hak}_{t_1,t_2} \ast \text{hak}_{t_2,t_3} &= \text{hak}_{t_1,t_2} \ast \text{hak}_{t_2,t_3} = \text{hak}_{t_1,t_3} \quad \forall t_1, t_2, t_3 \\
\text{hak}_{t,t} &= \text{hak}_{t,t} = 1_{h(S_r)} \quad \forall t
\end{align}

To derive from these a true convolution group we must take the limits:

\begin{align}
\text{hok}_t &:= \lim_{t_1 \to \infty} \text{hak}_{t_1,t} \\
\text{hök}_t &:= \lim_{t_1 \to \infty} \text{hak}_{t_1,t}
\end{align}

$^{57}\text{ha}(\sigma) := h^\sigma(1) \ldots \sigma(r), \text{ka}(\sigma) := k^\sigma(1) \ldots \sigma(r)$.
We end up with much simpler $\sigma$-functions:

$$h_{\sigma}(\tau) \equiv h_{\bar{\sigma}}(\tau) \equiv \frac{1}{r!} \prod_{1 \leq j \leq r} (1 + \gamma_j(\sigma) \tau)$$

(268)

with automatic stability under convolution:

$$h_{\sigma_1} \ast h_{\sigma_2} \equiv h_{\sigma_1 \ast \sigma_2} \quad \forall \tau_1, \tau_2$$

(269)

and an unexpected connection with the organic mould family: see §5.19.

5.10 Yet more factorisation properties.

Two ‘dual objects’, namely the scalar products $\sum_{\sigma \in S_r} h_{\sigma_1}(\sigma) h_{\sigma_2}(\sigma)$ and the convolution products $\sum_{\sigma_1 \sigma_2 = \sigma} h_{\sigma_1}(\sigma_1) h_{\sigma_2}(\sigma_2)$ evaluated at $\sigma = id$ also display, as functions of the twist parameters $\tau_1, \tau_2$, quite unexpected factorisation properties. Actually, this holds for all $k$-linear sums:

$$\sum_{\sigma \in S_r} h_{\sigma_1}(\sigma) h_{\sigma_2}(\sigma) \cdots h_{\sigma_k}(\sigma) = \frac{N_{r,k}}{D_{r,k}}$$

(270)

$$\sum_{\sigma \in S_r} \epsilon(\sigma) h_{\sigma_1}(\sigma) h_{\sigma_2}(\sigma) \cdots h_{\sigma_k}(\sigma) = \frac{N'_{r,k}}{D'_{r,k}}$$

(271)

and also for convolutions evaluated at more general permutations $\sigma \in S_r$, like for instance those acting like $rev$ on $\{1, \ldots, j_0\}$ and like $id$ on $\{1+j_0, \ldots, r\}$:

$$(h_{\sigma_1} \ast h_{\sigma_2})(\sigma) = \frac{N'_{r,j_0}}{D'_{r,j_0}}$$

(272)

Indeed, the numerators $N_{r,k}$ and $N'_{r,k}$ factor into products of $r$ polynomials, each of total degree $k$, and the numerator $N'_{r,j_0}$ factors into a product of $r$ quadratic polynomials.

There is no point in either writing down or proving the above factorisations, since they will turn out to be special cases of a more general result. Indeed, the factorisations (270),(271) will reduce to (275),(276) infra with $t_{ij} := \frac{1}{2} t_i + j$, and the factorisation (272) will reduce to (277) infra with $a_j := \frac{1}{2} t_1 + j$ and $b_j := \frac{1}{2} t_2 + j$.

58 The denominators $D_{r,k}$ and $D'_{r,j_0}$ also break down into simple factors, but this was entirely predictable, since all terms in the sums being considered already share the same elementary factors.
Though not nearly as deep as the factorisations of the previous sections, in particular those for the mould $\text{kas}^*$ or the $\sigma$-functions $\text{hak}, \text{håk}$, the unexpected splitting phenomenon occurring in (270),(271),(272) has one merit: when looking for the underlying mechanism, one is led quite naturally to a generalisation of the $\sigma$-functions $\text{has}_t, \text{kas}_t$ under with the twist parameter $t$ is replaced by a parameter sequence $T = \{t_j\}$. The next section shall be devoted to this extension, and the subsequent sections to a search for those particular sequences $T$ that yield the most interesting $\sigma$-functions.

5.11 Extending $\text{has}, \text{kas}$ to $\text{haus}, \text{kaus}$.

Starting from any sequence $T = \{t_1, t_2, t_3 \ldots \}$ we set:

$$\text{haus}_r(\sigma) := \prod_{1 \leq j \leq r} \frac{t_{\beta_j(\sigma)}}{t_1 + t_2 \cdots + t_j} \quad (\forall \sigma \in S_r) \quad (273)$$

with $\beta_j(\sigma)$ as in (177). Normalisation is non-trivial but automatic:

$$\sum_{\sigma \in S_r} \text{haus}_r(\sigma) \equiv 1 \quad \forall r \quad (274)$$

and the ‘superficial’ factorisations of the last section have exact analogues:

$$\sum_{\sigma \in S_r} \text{haus}_{r_1}(\sigma) \cdots \text{haus}_{r_k}(\sigma) = \prod_{1 \leq p \leq r} \sum_{1 \leq j \leq p} \frac{t_{ij}}{\sum_{1 \leq j \leq k} t_{ij}} \quad (275)$$

$$\sum_{\sigma \in S_r} \text{haus}_{r_1}(\sigma) \cdots \text{haus}_{r_k}(\sigma) \epsilon(\sigma) = (-1)^r \prod_{1 \leq p \leq r} \frac{\sum_{1 \leq j \leq p} (-1)^{1+j} \prod_{1 \leq i \leq k} t_{ij}}{\prod_{1 \leq j \leq k} \sum_{1 \leq j \leq p} t_{ij}} \quad \text{with } T_i = \{t_{ij}\} = \{t_{i1}, t_{i2}, t_{i3} \ldots \} \quad \text{and } r = 2r' \text{ or } 2r'+1 \quad (276)$$

The factorisation (272) for convolution products also has an analogue:

$$\sum_{\sigma \in S_r} (\text{haus}_A * \text{haus}_B)(\sigma) = \prod_{1 \leq p \leq r} \frac{\sum_{1 \leq j \leq p} a_j b_{\sigma(j)}}{(\sum_{1 \leq j \leq p} a_j)(\sum_{1 \leq j \leq p} b_j)} \quad (277)$$

which holds for all sequences $A = \{a_1, a_2 \ldots \}, B = \{b_1, b_2 \ldots \}$ and all permutations $\sigma$ of the form $\sigma_{j_0}$:

$$\sigma_{j_0}(j) = 1 + j_0 - j \quad (\text{resp } j) \quad \text{if } j \leq j_0 \quad (\text{resp } j > j_0) \quad (278)$$

But we would also like the deeper properties of $\text{has}, \text{kas}$ to survive. In other words, we would like to come up with pairs $\text{haus}_r, \text{kaus}_r$ of mutually
inverse σ-functions such that:
(i) \( \text{kau}_x(\sigma) \) has low degree denominators and is expressible in closed, transparent form
(ii) \( \text{kau}_x(\sigma) \) vanishes for most \(^{59}\) permutations \( \sigma \)
(iii) \( \text{haus}_x, \text{kau}_x \) admit natural mould extensions \( \text{haus}^*_x, \text{kau}^*_x \) with nice properties such as symmetrality.
(iv) there exist simple connecting σ-functions \(^{60}\) with maximum factorisation.
(v) \( \text{haus}_x, \text{kau}_x \) possess simple images under most linear representations of the symmetric groups \( S_r \).

As it turns out, there are three types of sequences \( T \), and only three, which answer this long wish list. They are:

\[
\text{Ta}_t := \left[ \frac{t}{2} + n \right]_{n=1}^{n=+\infty} \quad \text{‘arithmetic sequence’ (279)}
\]

\[
\text{Tu}_x := \left[ x^n \right]_{n=1}^{n=+\infty} \quad \text{‘geometric sequence’ (280)}
\]

\[
\text{Tu}_{x,t} := \left[ \frac{t^n - x^n}{t} \right]_{n=1}^{n=+\infty} \quad \text{‘bigeometric sequence’ (281)}
\]

Moreover, since \( (\text{haus}_x, \text{kau}_x) \) depend, not on the sequence \( T \) as such, but on its class \( \tilde{T} \) up to homotheties \( \{t_1, t_2, \ldots\} \mapsto \{ct_1, ct_2, \ldots\} \), these three classes \( \tilde{T}_a, \tilde{T}_u, \tilde{T}_{u,t} \) constitute a two-dimensional connected manifold. Indeed:

\[
\tilde{T}_a = \lim_{\epsilon \to 0} \tilde{T}_{1+2\epsilon, 1-\epsilon}
\]

\[
\tilde{T}_u = \tilde{T}_{u,0} := \lim_{t \to 0} \tilde{T}_{u,t}
\]

Arithmetic sequences yield the familiar pair \( (\text{has}_t, \text{kas}_t) = (\text{haus}_{\text{Ta}_t}, \text{kau}_{\text{Ta}_t}) \).

So let us turn successively to the geometric and bigeometric sequences.

### 5.12 Restricting \( \text{haus}, \text{kau} \) to \( \text{hus}, \text{kus} \).

**σ-functions hus**\(_x\) and \( \text{kus}_x \). Setting:

\[
(\text{hus}_x, \text{kus}_x) := (\text{haus}_x, \text{kau}_x) \quad \text{with} \quad T = \text{Tu}_x := \{x, x^2, x^3, \ldots\} \quad (284)
\]

\(^{59}\)more precisely, for all permutations that admit no maximal coherent binary bracketing; see §5.3

\(^{60}\)i.e. σ-functions \( \text{hau}_{r_1, r_2} \) such that \( \text{haus}_{r_1} \equiv \text{hau}_{r_1, r_2} \ast \text{haus}_{r_2} \).
we get:

\[
\text{hus}_x(\sigma) =: \frac{\text{hus}^*_x(\sigma)}{\text{DH}_r(x)} = \frac{x^{\beta^*(\sigma)}}{\text{DH}_r(x)}
\] (285)

\[
\text{kus}_x(\sigma) =: \frac{\text{kus}^*_x(\sigma)}{\text{DK}_r(x)} \quad \text{with} \quad \text{kus}^*_x(\sigma) \in \mathbb{Z}[x^2] \text{ or } x \mathbb{Z}[x^2]
\] (286)

with simple, cyclotomic denominators:

\[
\text{DH}_r(x) := \prod_{1 \leq k \leq r} \frac{1 - x^k}{1 - x}
\] (287)

\[
\text{DK}_r(x) := \prod_{2 \leq k \leq r} \frac{(1 - x)(1 - x^{k-1})}{(1 - x^k)} \quad \text{if } \ r \geq 2 \quad \text{(DK}_1(x) := 1)
\] (288)

and with simple numerators. Those of \( \text{hus}_x \) are monomials of exponent:

\[
\beta^*(\sigma) := -r + \sum_{1 \leq j \leq r} j \beta_j(\sigma)
\] (289)

\[
\equiv \# \{ (i, j) : 1 \leq i < j \leq r, \ \sigma(i) < \sigma(j) \}
\] (289)

and those of \( \text{kus}_x \) are even\(^{61} \) polynomials of low degree.

**Numerators of \( \text{hus}_x \) and \( \text{kus}_x \).**

Unexpected as the simplicity of the denominators \( \text{DK}_r(x) \) may be, the truly interesting part is the numerators \( \text{kus}^*_x \). Like with \( \text{kas}_t \), they depend on the maximal coherent binary bracketings of the sequence \( \{\sigma(1),...,\sigma(r)\} \):

- when no such bracketings exist, the numerator vanishes
- when there is only one bracketing, we have maximal factorisation into cyclotomic factors
- when there are several bracketings, we get very peculiar superpositions of such products, with many residual aspects of ‘cyclotomicity’.

All cases are covered by a completely explicit generalisation of formula (211) which involves the so-called Gaussian polynomials which are the \( q \)-analogues of the binomial coefficients so abundantly present in the definition of the operators \( P_{t,s} \) of (211).

**Symmetries of \( \text{hus}_x \) and \( \text{kus}_x \).**

With \( \xi(\sigma) \) as in (180), we have:

\[
\text{kus}_x(\sigma, \sigma) \equiv \text{kus}_x(\sigma) \quad \forall \sigma \in S_r, \quad \forall i \in \{0, 1, 2, 3\}
\] (290)

\[
\text{kus}_x(\sigma, \sigma) \equiv \text{kus}^1_x(\sigma) \quad \forall \sigma \in S_r, \quad \forall i \in \{4, 5, 6, 7\}
\] (291)

\[
\text{kus}^1_x(\sigma) \equiv (-1)^{r-1} x^{-\xi(\sigma)r(r-1)/2} \text{kus}_x(\sigma)
\] (292)

\[
\text{kus}^*_x(\sigma) \equiv \epsilon_r \epsilon(\sigma) \text{kus}^*_x(\sigma)
\] (293)

\(^{61}\)up to an occasional factor \( x \), present whenever \( \xi(\sigma) = -1 \).
with $\epsilon_r := 1$ if $r = 0$ or $1 \mod 4$ and $\epsilon_r := -1$ if $r = 2$ or $3 \mod 4$.

Connections between $\text{kus}_x$ and $\text{kas}_t$:

$$\text{kas}_0(\sigma) \equiv \text{kus}_1^*(\sigma^+) \quad \forall \sigma \in \mathbb{S}_r \ , \ \sigma^+ \in \mathbb{S}_{r+1} \quad (294)$$

where $\sigma^+$ stands for the natural extension of $\sigma$ to $\mathbb{S}_{r+1}$. \footnote{62}{ie $\sigma^+(j) := \sigma(j)$ for $j = 1, \ldots, r$ and $\sigma^+(r + 1) := r + 1$}

### 5.13 Endowing $\text{hus}, \text{kus}$ with a twist parameter $t$.

Turning now to the bigeometric sequences, we set:

$$(\text{hus}_{x,t}, \text{kus}_{x,t}) := (\text{haus}_T, \text{kaus}_T) \quad \text{with} \quad T = T_{u,x,t} := \left[\frac{x^n}{t} - \frac{t}{x^n}\right]_1^+ \quad (295)$$

As usual, the ‘direct’ $\sigma$-function $\text{hus}_{x,t}$ holds no mysteries. Its numerator is elementary, and its denominator breaks up into simple factors that are immediately obtainable from the general formula (273) for $\text{haus}_T$ after the substitution $t_n \leadsto \frac{x^n}{t}$.

More remarkable are the simplifications that occur with the $\sigma$-function $\text{kus}_{x,t}$. Its denominator $DK_r(x, t)$ also breaks up into simple factors: we have on the one hand the cyclotomic factors of $x$ alone, already present in the denominators $DK_r(x)$ of $\text{kus}_x$, and on the other hand, in equal number, elementary factors that depend on both $x$ and $t$. Explicitly:

$$\text{hus}_{x,t}(\sigma) = \frac{\text{hus}_{x,t}(\sigma)}{DH_r(x, t)} = \frac{\text{hus}_{x,t}(\sigma)}{DH_r(x) DH^*_r(x, t)} \quad \forall \sigma \in \mathbb{S}_r$$

$$\text{kus}_{x,t}(\sigma) = \frac{\text{kus}_{x,t}(\sigma)}{DK_r(x, t)} = \frac{\text{kus}_{x,t}(\sigma)}{DK_r(x) DK^*_r(x, t)} \quad \forall \sigma \in \mathbb{S}_r$$

$$DH^*_r(x, t) := \prod_{1 \leq k \leq r} (t^2 - x^{k+1}) \quad (296)$$

$$DK^*_r(x, t) := \prod_{1 \leq k \leq r} \frac{(t^{2k} - x^{k+1})}{(t^2 - x^{k+1})} \quad (297)$$

The really non-trivial part of $\text{kus}_{x,t}$, however, is its numerator. Like with $\text{kas}_t$ and $\text{kus}_{x,t}$, the new numerator $\text{kus}^*_r(\sigma)$ depends on the maximal coherent binary bracketings of the sequence $\{\sigma(1), \ldots, \sigma(r)\}$:

- when no such bracketings exist, the numerator vanishes
- when there is only one bracketing, the numerator breaks up completely into simple factors
- when there exist several bracketings, we get a superposition of such terms.

All cases are covered by a suitable generalisation of formula (212).
5.14 Factorisation properties for the connecting functions $\text{huk}, \text{hük}$.

Their construction runs parallel to that of $\text{hak}, \text{hák}$. We set:

$$
\text{huk}_{x,t_1,t_2}(\sigma) := \text{hus}_{x,t_1} \ast \text{kus}_{x,t_2}\left(\sigma\right) = \sum_{\sigma_1 \sigma_2 = \sigma} \text{hus}_{x,t_1}(\sigma_1) \text{kus}_{x,t_2}(\sigma_2) \quad (298)
$$

$$
\text{hük}_{x,t_1,t_2}(\sigma) := \text{hus}_{x,t_1} \ast (\sigma \text{kus}_{x,t_2}\left(\sigma\right)) = (\sigma \text{hus}_{x,t_1}) \ast \text{kus}_{x,t_2}\left(\sigma\right) \quad (299)
$$

= \sum_{\sigma_1, \text{rev. } \sigma_2 = \sigma} \text{hus}_{x,t_1}(\sigma_1) \text{kus}_{x,t_2}(\sigma_2) \quad (300)

and we encounter once again the miracle of maximal factorisation, for both numerators and denominators:

$$
\text{huk}_{x,t_1,t_2}(\sigma) = \frac{\text{huk}^*_{x,t_1,t_2}(\sigma)}{\text{DHK}^*_r(x, t_1, t_2)} \quad ; \quad \text{hük}_{x,t}(\sigma) = \frac{\text{hük}^*_{x,t_1,t_2}(\sigma)}{\text{DHK}^*_r(x, t_1, t_2)} \quad \forall \sigma \in \mathbb{S}_r
$$

$$
\text{DHK}^*_r(x; t_1, t_2) := \text{DHK}^*_r(x, t_1) \text{DK}^*_r(x, t_2) = \prod_{1 \leq k \leq r} \left( t_1^k - x^{k+1}\right) \prod_{1 \leq k \leq r} \left( \frac{t_2^k - x^{k+1}}{t_2^k - x^k}\right)
$$

$$
\text{huk}^*_{x,t_1,t_2} = t_2^{\gamma}(\sigma) x^{\delta}(\sigma) \prod_{1 \leq j \leq r} \left( t_1 t_2^{\gamma_j}(\sigma) + x^{\delta_j}(\sigma)/2\right) \prod_{1 \leq j \leq r} \left( t_1 t_2^{\gamma_j}(\sigma) - x^{\delta_j}(\sigma)/2\right)
$$

$$
\text{hük}^*_{x,t_1,t_2} = t_2^{\gamma}(\sigma) x^{\delta}(\sigma) \prod_{1 \leq j \leq r} \left( t_1 t_2^{\gamma_j}(\sigma) + x^{\delta_j}(\sigma)/2\right) \prod_{1 \leq j \leq r} \left( t_1 t_2^{\gamma_j}(\sigma) - x^{\delta_j}(\sigma)/2\right)
$$

with the very same $\gamma_j, \delta_j, \gamma^*, \delta^*$ as in §5.9 63 and with elementary corrective factors $t_2^{\gamma}(\sigma)x^{\delta}(\sigma)$ or $t_2^{\gamma}(\sigma)x^{\delta}(\sigma)$ which account for the global invariance under the change $(x, t_1, t_2) \rightarrow (x^{-1}, t_1^{-1}, t_2^{-1})$. To highlight this invariance, we may also write down our connecting functions as follows:

$$
\text{huk}_{x,t_1,t_2} = \prod_{1 \leq j \leq r} \frac{\text{bigeo}(x^{\frac{j+1}{2}}, t_2)}{\text{bigeo}(x^{\frac{j+1}{2}}, t_1)} \frac{\text{bigeo}(x^{\frac{j+1}{2}}, t_2)}{\text{bigeo}(x^{\frac{j+1}{2}}, t_1)} \quad (301)
$$

$$
\text{hük}_{x,t_1,t_2} = \prod_{1 \leq j \leq r} \frac{\text{bigeo}(x^{\frac{j+1}{2}}, t_2)}{\text{bigeo}(x^{\frac{j+1}{2}}, t_1)} \frac{\text{bigeo}(x^{\frac{j+1}{2}}, t_2)}{\text{bigeo}(x^{\frac{j+1}{2}}, t_1)} \quad (302)
$$

$$
\text{bigeo}(x, t) := \frac{x}{t} = \frac{t}{x} \quad (303)
$$

63Note in passing that $\delta_j(\sigma)$ and $\delta^*_j(\sigma)$ always being even integers, the above products amount to entire factorisations.
When the parameters \(x, t_1, t_2\) go to 1 simultaneously and with all three numbers \(x - 1, t_1 - 1, t_2 - 1\) in fixed ratios, we clearly retrieve as a special case the factorisations (250),(251) for the ‘arithmetic’ case:

\[
\prod_{1 \leq j \leq r} \frac{(t_1 + \gamma_j(\sigma)t_2 + \delta_j(\sigma))}{j (j + 1)}
\]

\[
\prod_{1 \leq j \leq r} \frac{(t_1 + \gamma_j^*(\sigma)t_2 + \delta_j^*(\sigma))}{j (j + 1)}
\]

5.15 The pair \(\text{hus}, \text{kus}\) as a \(q\)-analogue of \(\text{has}, \text{kas}\). The ‘haukian’ family of \(\sigma\)-functions.

The pairs \((\text{hus}_x, \text{kus}_x)\) and \((\text{hus}_{x,t}, \text{kus}_{x,t})\) may be looked upon as \(q\)-analogues of \((\text{has}_T, \text{kas}_T)\) and \((\text{has}_t, \text{kas}_t)\) respectively, with \(x\) functioning as \(q\)-parameter. The associated moulds \((\text{hus}_x^*, \text{kus}_x^*)\) and \((\text{hus}_{x,t}^*, \text{kus}_{x,t}^*)\) even display a symmetry \(sui\ generis\), which resembles symmetrality and might be called \(q\)-symmetrality. But we cannot afford to go into these matters here. Be it enough to say that the three pairs:

\[
(\text{has}_t, \text{kas}_t) \quad \text{‘arithmetic’}
\]

\[
(\text{hus}_x, \text{kus}_x) \quad \text{‘geometric’}
\]

\[
(\text{hus}_{x,t}, \text{kus}_{x,t}) \quad \text{‘bigeometric.’}
\]

which due to (282), (283) constitute a connected manifold, seem to enjoy a unique status, not only among all pairs \((\text{haus}_T, \text{kaus}_T)\), but even among all pairs \((h, k)\) of mutually inverse \(\sigma\)-functions. They fully deserve a name of their own: let us call them \textit{haukian functions}.\textsuperscript{64}

5.16 Representation theory of finite groups and ‘haukian’ \(\sigma\)-functions.

The existence of simple images \(\sum_{\sigma} h(\sigma)\rho(\sigma), \sum_{\sigma} k(\sigma)\rho(\sigma)\) under the elementary, one-dimensional representations \(\rho(\sigma) := 1\) (trivial) or \(\rho(\sigma) := \epsilon(\sigma)\) (signature) is guaranteed for all pairs \((\text{haus}_T, \text{kaus}_T)\) by the formulas (275), (276). But if we move on to general, higher-dimensional representations \(\rho\) of the symmetric groups \(S_r\), the haukian family once again stands out for the simplicity of its behaviour, in particular for the distribution pattern of its standard factors inside the determinants of the representations. Results

\textsuperscript{64}The \(h\) stands for the direct function; the \(k\) for its convolution inverse; and the diphthong \(au\) refers to the \(a\) and \(u\) of the arithmetic and (bi)geometric cases.
here are still incomplete, so we mention just two formulae, relative to the 
r-dimensional representations:

\[ \rho_r(\sigma) \cdot e_i := e_{\sigma(i)} \quad \forall \sigma \in S_r, \ i \in \{1, \ldots, r\} \] (304)

Typically, we get the familiar factors but with altered multiplicities:

\[
\begin{align*}
\det \left( \sum_{\sigma \in S_r} \text{has}_x(\sigma) \rho_r(\sigma) \right) &= \prod_{1 \leq k \leq r-1} (1 + k)^{-1+2r-3k} \prod_{1 \leq k \leq r-1} (t + 2 + k)^{3k} \\
\det \left( \sum_{\sigma \in S_r} \text{hus}_x(\sigma) \rho_r(\sigma) \right) &= \prod_{1 \leq k \leq r-1} (x^k - 1)^{1+2r-3k}
\end{align*}
\] (305) (306)

5.17 \ σ-functions originating in uniform lamination.

We now take leave of the haukian family and consider a few other \( \sigma \)-functions that arise in the context of our fusion-fission transforms. The first is connected with the uniform lamination-colamination described in §1.8. It involves the alternal mould \( \text{lad}^* \) (of flat, difference-type; see §2.4) which also occurs in the construction of the standard alien derivations. The closely related mould \( \text{sad}^* \) will resurface in §5.19.

5.18 \ σ-functions originating in quadratic lamination.

The quadratic lamination-colamination described in §1.9 also gives rise to interesting \( \sigma \)-functions \( \text{hes}, \text{kes}, \text{ke} \). The first two are mutually inverse and all three are simple. Let \( \mathbb{B} \) be the associative algebra freely generated by \( e_1, e_2, \ldots \) and let \( \overline{1} \mathbb{B} \) be the corresponding Lie algebra, with its natural embedding in \( \mathbb{B} \). The projection \( \text{proj}_1 : \mathbb{B} \to \overline{1} \mathbb{B} \) characterised in §1.9 involves a \( \sigma \)-function \( \text{ke} \) which, though not invertible, is closely related to an invertible one, \( \text{kes} \), whose inverse \( \text{hes} \) is unexpectedly simple: it assumes only zeros or powers of 2 as its values.

**Projection** \( \text{proj}_1 : \mathbb{B} \to \overline{1} \mathbb{B} \): We have five equivalent expressions:
\[ praj_1(e_1...e_r) = \sum_{\sigma \in S_r} ke(\sigma) e_{\sigma(1)}...e_{\sigma(r)} \]

(307)

\[ = \sum_{\sigma \in S_r} \frac{1}{r} ke(\sigma) [..[e_{\sigma(1)}, e_{\sigma(2)}]...e_{\sigma(r)}] \]

(308)

\[ = \sum_{\sigma \in S_r} \frac{1}{r} ke(\sigma) [e_{\sigma(1)}...[e_{\sigma(r-1)}, e_{\sigma(r)}]..] \]

(309)

\[ = \sum_{\tau \in S_{r-1}} kes(\tau) [..[e_1, e_{\tau(2)}]...e_{\tau(r)}] \]

(310)

\[ = \sum_{\tau \in S_{r-1}} kes(\tau) [e_{\tau(1)}...[e_{\tau(r-1)}, A_r]..] \]

(311)

which make manifest the one-to-one correspondance that exists between \(kes(\tau)\) as defined on \(S_{r-1} = S_l\) and \(ke(\sigma)\) as defined on \(S_r\):

\[ kes(\tau) \overset{\text{def}}{=} ke(\sigma) \quad \text{with} \quad \sigma(1) := \tau(1), \ldots, \sigma(l) := \tau(l), \sigma(r) := r \]

(312)

\[ ke^{n_1, r, n^2} \overset{\text{def}}{=} (-1)^{r_2} \sum_{\tau \in sha(n_1, \tilde{n}^2)} kes^n \]

(313)

Only the second relation calls for comments. For convenience it is written in mould form, and the sum ranges over all shuffles \(n\) of \(n^1\) and of the reverse \(\tilde{n}^2\) of \(n^2\). The integer \(r_2\) is of course the length of \(n^2\).

Properties of \(kes\). Here are the main ones:

\[ kes(id_l) = \frac{1}{1 + l} = \frac{1}{r} \]

(314)

\[ kes(\tau) \text{ is maximal for } \tau = id_l \]

(315)

\[ kes(\tau) = kes(\tau^{-1}) \quad \text{(parity)} \]

(316)

\[ kes(\tau) = kes(\tau^*) \quad \text{with } \tau^* = rev_1 \circ \tau \circ rev_1 \quad \text{(symmetry)} \]

(317)

\[ \sum_{\tau \in S_l} kes(\tau) = \frac{l! l!}{(2l)!} \]

(318)

\[ \sum_{\tau \in S_l} e_{\tau} kes(\tau) = \frac{(l/2)! (l/2)!}{l!} \quad \text{for } l \text{ even} \]

(319)

\[ \sum_{\tau \in S_l} e_{\tau} kes(\tau) = \frac{((l-1)/2)! ((l-1)/2)!}{2 (l-1)!} \quad \text{for } l \text{ odd} \]

(320)

\[ kes \text{ has an inverse } hes \text{ in the group algebra } A(S_l) \]

(321)
Properties of $hes$. We have:
\[
hes(\tau) \in \{0, 2^1, \ldots, 2^l\} \text{ if } \tau \in S_l
\] (322)
with the actual values given by a simple rule. That rule is best described by deriving $hes$ from a more general, real-indexed and flat (i.e. locally constant) mould $hes^\ast$. The link is simply:
\[
hes(\tau) = hes^\ast(\tau(1), \ldots, \tau(l)) \text{ if } \tau \in S_l
\] (323)
and $hes^\ast$ is defined by the following induction:
\[
hes^\omega = cohes^{\omega^1} hes^{\omega^2} cohes^{\omega^3}
\] (324)
Here $\omega = (\omega_1, \ldots, \omega_l)$ is any sequence of $l$ distinct real number. The sequence $\omega^1$ (resp $\omega^3$) is obtained from $\omega$ by retaining only the terms $\omega_i$ such that $\omega_i < \omega_1$ (resp $\omega_i > \omega_l$). The mid-sequence $\omega^2$ is obtained from $\omega$ by retaining only the terms $\omega_i$ such that $\omega_1 < \omega_i < \omega_l$ as well as the term $\omega_1^−$ immediately inferior to $\omega_1$ (if it exists) and the term $\omega_l^+\ast$ immediately superior to $\omega_l$ (if it exists). Some of the factor sequences $\omega^i$ may reduce to the empty sequence $\emptyset$, but the above relations amount to a true induction since in all cases $\text{length}(\omega^2) \leq \text{length}(\omega) - 2$.

To complete the induction rules we must set:
\[
hes^\emptyset = 1 \quad \text{(325)}
\]
\[
cohes^\emptyset = 2 \quad \text{and for } \omega \neq \emptyset \quad \text{(326)}
\]
\[
cohes^\omega = 1 \quad \text{if } \omega \text{ is an increasing sequence} \quad \text{(327)}
\]
\[
cohes^\omega = 0 \quad \text{otherwise} \quad \text{(328)}
\]

Remarks about the proofs: Though less than two page long, the proof has to be skipped in this expository paper. Let us just point out the reason for the occurrence of powers of 2 in $hes$. They stem from the standard scalar products of Lie elements of the form $[\ldots [e_{\sigma(1)}, e_{\sigma(2)}], \ldots, e_{\sigma(k)}]$ which happen to be exact powers of 2.

5.19 $\sigma$-functions with treble stability.

Stability under $\ast, \times, \circ$.

To conclude this unashamedly ‘botanical’ chapter in character, we give two instances of $\sigma$-function that display a treble stability:

(i) stability under the convolution product $\ast$.

(ii) stability of the associated mould under mould multiplication $\times$.  

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(iii) stability of the associated mould under mould composition \( \circ \).
Of course, all three stabilities are completely independent.\(^{65}\)

**The ‘uniform’ mould family.**

The following moulds are associated with the so-called *uniform average* of resurgent theory. Setting \( \text{remu}_a^* = tu_a^* \) as in §2.3 and \( \text{namu}_a^* = \text{sad}_a^* \) as in §2.4 we have:

\[
\begin{align*}
\text{remu}_a^* \times \text{remu}_b^* &\equiv \text{remu}_{a+b}^* & \forall a, b \in \mathbb{C} \\
\text{remu}_a^* \circ \text{remu}_b^* &\equiv \text{remu}_{ab}^* & \forall a, b \in \mathbb{C} \\
\text{namu}_a^* \ast \text{namu}_b^* &\equiv \text{namu}_{ab}^* & \forall a, b \in \mathbb{C}
\end{align*}
\]

The proofs are quite short. Far more interesting is the next example.

**The ‘organic’ mould family.**

The mould \( \text{remo}_a^* \) and the closely related mould \( \text{romo}_a^* \) were defined in §2.7. They are essentially the ‘lateral moulds’ (see §4.10) associated with the important ‘organic average’ which is central to resurgence theory. Built from these one-parameter moulds, we have the two-parameter mould \( \text{somo}_{a,b}^* \), also defined in §2.7, and its unexpected closure properties under mould multiplication and mould composition (see §2.7). But on top of these, we have also stability under convolution. Indeed, along with these ‘lateral’ moulds there goes a ‘neutral’ mould \( \text{namo}_a^* \), whose associated \( \sigma \)-function \( \text{namo}_a^* \) turns out to essentially coincide with the \( \sigma \)-function \( \text{hok}^* \) already encountered in connection with the family \( \{ \text{has}, \text{kas}, \text{hak} \} \). Indeed, it can be shown that:

\[
\text{namo}_a^*(\sigma) \equiv a^r \text{hok}^{\frac{1}{2}}(\sigma) & \forall \sigma \in S_r, \forall a \in \mathbb{C}
\]

The closure under convolution follows at once:

\[
\text{namo}_a^* \ast \text{namo}_b^* \equiv \text{namo}_{ab}^* & \forall a, b \in \mathbb{C}
\]

6 Conclusion and complements.

6.1 Unique status of arborification-coarborification among all fusion-fission transforms.

In the introduction, we pointed out the effectiveness of the arborification-coarborification transform in *analysis*. In §4 we backed up this claim with a

\(^{65}\)the first one is at constant length \( r \), the others mix up various lengths.
string of applications. In §2 and §3 we examined the combinatorial mechanisms behind the method’s success, and the reasons for its superiority over other, *a priori* equally attractive fusion-fission transforms. In the last section, §5, this unique status received a further boost, and that too from an unexpected quarter: *algebra*.

To take stock, let us briefly retrace our main steps in §5. Starting from a series \(^66\) of mutually inverse matrices \((H^∗, K^∗)\), which arise naturally when investigating arborification in a free associative context, we have successively encountered all the objects which grace the following table:

\[
\begin{array}{ccc}
(H^∗, K^∗) & (\text{has}^∗_t, \text{kas}^∗_t) & (\text{has}^∗_{t,s}, \text{kas}^∗_{t,s}) \\
\downarrow & \downarrow & \downarrow \\
(\text{has}, \text{kas}) & (\text{has}_t, \text{kas}_t) & (\text{hak}_t, \text{hak}_t) \\
\downarrow & \downarrow & \downarrow \\
\text{q-analogue} & (\text{haus}_T, \text{kaus}_T) & \text{q-analogue} \\
\downarrow & \downarrow & \downarrow \\
(\text{hus}_x, \text{kus}_x) & (\text{hus}_{x,t}, \text{kus}_{x,t}) & (\text{huk}_{x,t}, \text{huk}_{x,t}) \\
\downarrow & \downarrow & \downarrow \\
\sigma-\text{functions} & (\text{hus}^*_{x,t}, \text{kus}^*_{x,t}) & (\text{hus}^*_{x,t,s}, \text{kus}^*_{x,t,s})
\end{array}
\]

These *haukian* objects, some of them moulds, the others σ-functions, turned out to possess no end of unexpected properties:

a) the σ-functions go in pairs of mutually inverse \(^67\) elements, with both terms admitting numerous symmetries, possessing quite explicit expressions, notably simple denominators, and also presenting a tendency towards maximal factorisation – all of which is quite uncommon for mutually inverse σ-functions.\(^68\)

b) unlike σ-functions ‘picked at haphazard’, ours possess natural extensions to integer-indexed, rational valued moulds, the only restriction being that the indices have to be pair-wise distinct.

---

\(^66\)these square matrices of order \(r!\) are defined for all \(r\).

\(^67\)in the convolution algebras \(A(S_r)\) of the symmetric groups \(S_r\).

\(^68\)indeed, inversion in the algebras \(A(S_r)\) tends to produce huge denominators.
c) the moulds so produced, in turn, display precise symmetries (either symmetry or, more rarely, alternality), which may be common enough in "natural moulds", but rather surprising in the present instance\textsuperscript{69}.

d) there is a tantalising connection between these "haukian" moulds and the moulds of the "organic family", which have a quite distinct origin.

But now comes the crux: although the entire construction, starting from the matrix pair \((H^*, K^*)\) down to the whole set of characters in the above Table, can be duplicated for any other fusion-fission transform, relative to any type of order (all partial orders, laminescent orders, arborescent orders of binary, or ternary type, etc etc) none of these parallel constructions\textsuperscript{70} retains any of the rich structure or endearing simplicity which is the hallmark of the haukian family. Although, at the moment, these curious haukian properties seem to have no direct relevance to arborification-coarborification as a tool for convergence-restoration in analysis, they certainly enhance its uniqueness status. Even if devoid of deeper meaning, this 'agreement' between analysis and algebra\textsuperscript{71} which we observe here is very gratifying.

### 6.2 Local-analyticity, free-analyticity, alien-analyticity.

\(\mathbb{C}[x_1, \ldots, x_\nu]\) resp. \(\mathbb{C}\{x_1, \ldots, x_\nu\}\) are well-established notations for the ring of all formal, resp. local-analytic\textsuperscript{72} power series in the \(\nu\) commuting indeterminates \(x_i\) and with coefficients in \(\mathbb{C}\). Going over to non-commuting indeterminates \(X_i\), the question arises: What could be the natural counterpart \(\mathbb{C}\{X_1, \ldots, X_\nu\}\) of \(\mathbb{C}\{x_1, \ldots, x_\nu\}\)? And how could we characterise its elements:

\[
SS = \sum_{0 \leq r \leq \infty} \sum_{i_k \in \{1, \ldots, \nu\}} A^{i_1, i_2, \ldots i_r} X_{i_1} \ldots X_{i_2} X_{i_1} \quad (A^* \in \mathbb{C}) \quad (334)
\]

preferably in terms of bounds on \(A^*\)? That of course will depend on which future 'specialisations' we have in mind for our indeterminates \(X_i\).

\(S_1\) : finite-dimensional specialisations, e.g. in spaces \(\text{End}(V)\) of endomorphisms of \(\mu\)-dimensional vector spaces \(V\), with \(\mu\) finite but otherwise unre-

\textsuperscript{69}at any rate, these mould symmetries are not a simple rephrasing, nor even a consequence, of the \(\sigma\)-function symmetries.

\textsuperscript{70}as far as we could see. We did explore quite a few options.

\textsuperscript{71}a similar 'convergence' is also a feature of resurgence theory which, despite having its moorings in analysis, often tastes like pure algebra.

\textsuperscript{72}i.e. with non-zero convergence radius.
lated to $\nu$.

$S_2$: infinite-dimensional specialisations, e.g. in the spaces $\text{Der}(\mathbb{C}\{x_1, \ldots, x_\mu\})$ of ordinary derivations of the ring of convergent power series of $\mu$ variables.\(^73\)

For definiteness, let us restrict ourselves to specialisations $X_i \mapsto \text{spe}(X_i)$ that are homogeneous and degree-increasing:

$$\text{spe}(X_i) : \quad x^m \mathbb{C} \longrightarrow x^{m+d_i} \mathbb{C} \quad (d_i \in \mathbb{N}^\mu, \forall m \in \mathbb{N}^\mu)$$  \quad (335)

$S_3$: alien specialisations, i.e. incarnations in the space $\text{ALIEN}$ of alien derivations of some space of resurgent functions. Here again, assume for definiteness that $\text{spe}(X_i)$ specialises to homogeneous alien derivations.\(^74\)

So, against this backdrop of possible specialisations, let us weigh the merits of the three types of majorisations on $A^\bullet$ which naturally spring to mind. They are:

- $M_1 : \quad |A^{i_1, \ldots, i_r}| \leq c_0 c^r_1$ \quad (336)
- $M_2 : \quad |A^{i_1, \ldots, i_r}| \leq c_0 c^r_1 \frac{1}{r!}$ \quad (337)
- $M_3 : \quad |A^{(i_1, \ldots, i_r)}^\prec| \leq c_0 c^r_1 \quad \text{with} \quad A^\prec := \sum_{\prec \leq \bullet} A^\bullet$ \quad (338)

for some finite positive constants $c_0 = c_0(\text{SS}), c_1 = c_1(\text{SS})$ and with, on the third line, the usual convention of straight arborification (see §1.2).

Condition $M_1$ is adequate for specialisations of type $S_1$, but clearly not for those of type $S_2$, even in the case of a single $X_1$, and much less for type $S_3$.

Condition $M_2$, on the other hand, ensures the convergence of specialisations $S_2$ and $S_3$, but is unnecessarily stringent.

The “proper” condition would seem to be the one involving arborification, namely $M_3$. As we saw, it implies the convergence of all specialisations $S_2$, and it does so at a much lesser cost\(^75\) in fact, at a minimal cost. Moreover, the space $\mathbb{C}\{\{X_1, \ldots, X_\nu\}\}$ of all $SS$ subject to $M_3$\(^76\) enjoys all the stability

\(^73\)here again, $\mu$ is unrelated to $\nu$, and can be any finite number.

\(^74\)i.e. alien derivations of a given frequency $\omega$, like $\Delta_\omega$ or $[\ldots[\Delta_{\omega_1}, \Delta_{\omega_2}] \ldots \Delta_{\omega_r}]$ with $\sum \omega_i = \omega$, but no superpositions corresponding to different $\omega$’s.

\(^75\)in the uninteresting case of a single variable $X_1$, where non-commutativity doesn’t come into play, $M_3$ is readily seen to coincide with $M_2$, but for several variables it is considerably weaker.

\(^76\)with constants that depend on $SS$. 

65
properties that one may wish for, e.g. under multiplication and substitution. We then speak of free-analyticity.

Condition \( M_3 \) also happens to be the weakest condition that guarantees the convergence of specialisations of type \( S_3 \). Dually, it is the strongest condition to be verified by the displayed and restricted forms of natural resurgent functions. We speak in that context of alien-analyticity.

7 Tables.

7.1 The \( \sigma \)-functions has, kas .
To handle integers only, we set: \( \text{has}^*(\sigma) := \frac{r(1+r)!}{2^r} \) \text{has}(\sigma) \ \forall \sigma \in \mathbb{S}_r .

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7.2 The $\sigma$-functions has, kas with a twist parameter.

\[
\begin{align*}
\text{has}_r^*(\sigma) & := r! (t+2)(t+3) \ldots (t+r+1) \text{ has}_t(\sigma) \quad \forall \sigma \in S_r \quad (339) \\
\text{kas}_r^*(\sigma) & := r! \text{ has}_t(\sigma) \quad \forall \sigma \in S_r \quad (340)
\end{align*}
\]
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<th>$\text{kas}_t^*$</th>
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<td>$(t + 4)$</td>
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7.3 The $\sigma$-functions has, kas with twist and shift.

We set:

\[
\text{has}^*_t,s(\sigma) := \frac{(2r)!}{2^r} (t + s + 2) \ldots (t + s + r + 1) \text{has}_{t,s}(\sigma) \quad \forall \sigma \in S_r \tag{341}
\]

\[
\text{kas}^*_t,s(\sigma) := r! \text{has}_{t,s}(\sigma) \quad \forall \sigma \in S_r \tag{342}
\]

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<td>$3 t^2 + 6 t s + 3 s^2 + 12 t + 13 s + 12$</td>
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<td>[1, 2, 3]</td>
<td>$15 (t + s + 4) (t^2 + 2 t s + s^2 + 8 t + 7 s + 12)$</td>
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<td>[1, 3, 2]</td>
<td>$15 t^3 + 45 t^2 s + 45 t s^2 + 15 s^3 + 150 t^2 + 300 t s + 147 s^2 + 480 t + 468 s + 480$</td>
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<td>$15 (t + s + 2) (t^2 + 2 t s + s^2 + 4 t + 5 s + 4)$</td>
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<td>$(t s + s^2 + t + 4 s + 6) (t s + s^2 + 2 t + 4 s + 7)$</td>
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<tr>
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7.4 The $\sigma$-functions hak, hok.

\[
\text{hak}_{a,b}^*(\sigma) := r! (a + 2)(a + 3) \ldots (a + r + 1) \text{hak}_{a,b}(\sigma) \quad \forall \sigma \in S_r \tag{343}
\]

\[
\text{hok}_b^*(\sigma) := r! \text{hok}_b(\sigma) \quad \forall \sigma \in S_r \tag{344}
\]
\[\begin{array}{ccc}
\sigma & \text{hak}_{a,b} & \text{hok}_{b}^{*} \\
1 & (a + 2) & 1 \\
1, 2 & (a + 2)(a + b + 6) & (1 + b) \\
2, 1 & (a - b)(a + 2) & (1 - b) \\
1, 2, 3 & (a + 2)(a + b + 6)(a + b + 12) & (1 + b)(1 + 2b) \\
1, 3, 2 & (a + 2)(a - b)(a + 4) & (1 - b) \\
2, 1, 3 & (a - b)(a + 2)(a + 2b + 12) & (1 - b)(1 + 2b) \\
2, 3, 1 & (a + 2)(a - b)(a + 2) & (1 - b) \\
3, 1, 2 & (a - b)(a + 2)(a + 2b + 10) & (1 - b)(1 + 2b) \\
3, 2, 1 & (a - b)(a - 2b - 4)(a + 2) & (1 - b)(1 - 2b) \\
1, 2, 3, 4 & (a + 2)(a + b + 6)(a + 2b + 12)(a + 3b + 20) & (1 + b)(1 + 2b)(1 + 3b) \\
1, 2, 4, 3 & (a + 2)(a + b + 6)(a - b)(a + 6) & (1 + b)(1 - b) \\
1, 3, 2, 4 & (a + 2)(a - b)(a + 4)(a + 3b + 20) & (1 - b)(1 + 3b) \\
1, 3, 4, 2 & (a + 2)(a + b + 6)(a - b)(a + 4) & (1 + b)(1 - b) \\
1, 4, 2, 3 & (a + 2)(a - b)(a + 4)(a + 3b + 18) & (1 - b)(1 + 3b) \\
1, 4, 3, 2 & (a + 2)(a - b)(a - 3b - 8)(a + 4) & (1 - b)(1 - 3b) \\
2, 1, 3, 4 & (a - b)(a + 2)(a + 2b + 12)(a + 3b + 20) & (1 - b)(1 + 2b)(1 + 3b) \\
2, 1, 4, 3 & (a - b)(a + 2)(a - b)(a + 6) & (1 - b)(1 - b) \\
2, 3, 1, 4 & (a + 2)(a - b)(a + 2)(a + 3b + 20) & (1 - b)(1 + 3b) \\
2, 3, 4, 1 & (a + 2)(a + b + 6)(a - b)(a + 2) & (1 + b)(1 - b) \\
2, 4, 1, 3 & (a + 2)(a - b)(a + 2)(a + 3b + 18) & (1 - b)(1 + 3b) \\
2, 4, 3, 1 & (a + 2)(a - b)(a - 3b - 8)(a + 2) & (1 - b)(1 - 3b) \\
3, 1, 2, 4 & (a - b)(a + 2)(a + 2b + 10)(a + 3b + 20) & (1 - b)(1 + 2b)(1 + 3b) \\
3, 1, 4, 2 & (a - b)(a + 2)(a - b)(a + 4) & (1 - b)(1 - b) \\
3, 2, 1, 4 & (a - b)(a - 2b - 4)(a + 2)(a + 3b + 20) & (1 - b)(1 - 2b)(1 + 3b) \\
3, 2, 4, 1 & (a - b)(a + 2)(a - b)(a + 2) & (1 - b)(1 - b) \\
3, 4, 1, 2 & (a + 2)(a - b)(a + 2)(a + 3b + 16) & (1 - b)(1 + 3b) \\
3, 4, 2, 1 & (a + 2)(a - b)(a - 3b - 10)(a + 2) & (1 - b)(1 - 3b) \\
4, 1, 2, 3 & (a - b)(a + 2)(a + 2b + 10)(a + 3b + 18) & (1 - b)(1 + 2b)(1 + 3b) \\
4, 1, 3, 2 & (a - b)(a + 2)(a - b - 2)(a + 4) & (1 - b)(1 - b) \\
4, 2, 1, 3 & (a - b)(a - 2b - 4)(a + 2)(a + 3b + 18) & (1 - b)(1 - 2b)(1 + 3b) \\
4, 2, 3, 1 & (a - b)(a + 2)(a - b - 2)(a + 2) & (1 - b)(1 - b) \\
4, 3, 1, 2 & (a - b)(a - 2b - 4)(a + 2)(a + 3b + 16) & (1 - b)(1 - 2b)(1 + 3b) \\
4, 3, 2, 1 & (a - b)(a - 2b - 4)(a - 3b - 10)(a + 2) & (1 - b)(1 - 2b)(1 - 3b) \\
\end{array}\]
7.5 The $\sigma$-functions häk, hök.

\[
häk^\ast_{a,b}(\sigma) := r!(a+2)(a+3) \ldots (a+r+1) \text{ häk}_{a,b}(\sigma) \quad \forall \sigma \in S_r \quad (345)
\]

\[
hök^\ast_b(\sigma) := r! \text{ hök}_b(\sigma) \quad \forall \sigma \in S_r \quad (346)
\]

\[
\begin{array}{ccc}
\sigma & \text{häk}^\ast_{a,b} & \text{hök}^\ast_b \\
[1] & (a+2) & 1 \\
[1, 2] & (a+2)(a-b) & (1-b) \\
[2, 1] & (a+b+6)(a+2) & (1+b) \\
[1, 2, 3] & (a+2)(a-b)(a-2b-4) & (1-b)(1-2b) \\
[1, 3, 2] & (a+2)(a+b+8)(a+2) & (1+b)(1-3b) \\
[2, 1, 3] & (a+b+6)(a+2)(a-2b-4) & (1+b)(1-2b) \\
[2, 3, 1] & (a+4)(a+b+8)(a+2) & (1+b) \\
[3, 1, 2] & (a+b+8)(a+2)(a-2b-4) & (1+b)(1-2b) \\
[3, 2, 1] & (a+b+8)(a+2b+10)(a+2) & (1+b)(1+2b) \\
[1, 2, 3, 4] & (a+b+6)(a+2)(a-2b-4)(a-3b-10) & (1+b)(1-2b)(1-3b) \\
[1, 2, 4, 3] & (a+2)(a-b)(a+b+10)(a+2) & (1-b)(1+b) \\
[1, 3, 2, 4] & (a+2)(a+b+8)(a+2)(a-3b-10) & (1+b)(1-3b) \\
[1, 3, 4, 2] & (a+2)(a-b+2)(a+b+10)(a+2) & (1-b)(1+b) \\
[1, 4, 2, 3] & (a+2)(a+b+10)(a+2)(a-3b-10) & (1+b)(1-3b) \\
[1, 4, 3, 2] & (a+2)(a+b+10)(a+3b+16)(a+2) & (1+b)(1+3b) \\
[2, 1, 3, 4] & (a+b+6)(a+2)(a-2b-4)(a-3b-10) & (1+b)(1-2b)(1-3b) \\
[2, 1, 4, 3] & (a+b+6)(a+2)(a+b+10)(a+2) & (1+b)(1+b) \\
[2, 3, 1, 4] & (a+4)(a+b+8)(a+2)(a-3b-10) & (1+b)(1-3b) \\
[2, 3, 4, 1] & (a+4)(a-b+2)(a+b+10)(a+2) & (1-b)(1+b) \\
[2, 4, 1, 3] & (a+4)(a+b+10)(a+2)(a-3b-10) & (1+b)(1-3b) \\
[2, 4, 3, 1] & (a+4)(a+b+10)(a+3b+16)(a+2) & (1+b)(1+3b) \\
[3, 1, 2, 4] & (a+b+8)(a+2)(a-2b-4)(a-3b-10) & (1+b)(1-2b)(1-3b) \\
[3, 1, 4, 2] & (a+b+8)(a+2)(a+b+10)(a+2) & (1+b)(1+b) \\
[3, 2, 1, 4] & (a+b+8)(a+2b+10)(a+2)(a-3b-10) & (1+b)(1+2b)(1-3b) \\
[3, 2, 4, 1] & (a+b+8)(a+4)(a+b+10)(a+2) & (1+b)(1+b) \\
[3, 4, 1, 2] & (a+6)(a+b+10)(a+2)(a-3b-10) & (1+b)(1-3b) \\
[3, 4, 2, 1] & (a+6)(a+b+10)(a+3b+16)(a+2) & (1+b)(1+3b) \\
[4, 1, 2, 3] & (a+b+10)(a+2)(a-2b-4)(a-3b-10) & (1+b)(1-2b)(1-3b) \\
[4, 1, 3, 2] & (a+b+10)(a+2)(a+b+10)(a+2) & (1+b)(1+b) \\
[4, 2, 1, 3] & (a+b+10)(a+2b+10)(a+2)(a-3b-10) & (1+b)(1+2b)(1-3b) \\
[4, 2, 3, 1] & (a+b+10)(a+10)(a+b+10)(a+2) & (1+b)(1+b) \\
[4, 3, 1, 2] & (a+b+10)(a+2b+12)(a+2)(a-3b-10) & (1+b)(1+2b)(1-3b) \\
[4, 3, 2, 1] & (a+b+10)(a+2b+12)(a+3b+16)(a+2) & (1+b)(1+2b)(1+3b)
\end{array}
\]
7.6 The $\sigma$-functions haus, kaus.

We set $T := [t_1, t_2, t_3, \ldots]$ and

$$\text{haus}^*_T(\sigma) := t_1(t_1 + t_2) \cdots (t_1 + \cdots + t_r) \text{haus}_T(\sigma) \quad \forall \sigma \in S_r$$

$$\begin{array}{cccc}
\sigma & \text{haus}^*_T & \sigma & \text{haus}^*_T \\
[1] & t_1 & [1, 2, 3] & t_1 t_2 t_3 & [2, 3, 1] & t^3_1 t_2 \\
[1, 2] & t_1 t_2 & [1, 3, 2] & t_1 t^2_2 & [3, 1, 2] & t_1^3 t_2 \\
[2, 1] & t^2_1 & [2, 1, 3] & t^2_1 t_3 & [3, 2, 1] & t_1^3 \\
[1, 2, 3, 4] & t_1 t_2 t_3 t_4 & [2, 3, 1, 4] & t^2_1 t_2 t_4 & [3, 4, 1, 2] & t^2_1 t_2 \\
[1, 2, 4, 3] & t_1 t_2 t^2_3 & [2, 3, 4, 1] & t^2_1 t_2 t_3 & [3, 4, 2, 1] & t^2_1 t_2 \\
[1, 3, 2, 4] & t_1 t^2_2 t_4 & [2, 4, 1, 3] & t^2_1 t_2 t_3 & [4, 1, 2, 3] & t_1^3 t_2 t_3 \\
[1, 3, 4, 2] & t_1 t^2_2 t_3 & [2, 4, 3, 1] & t^2_1 t^2_2 & [4, 1, 3, 2] & t_1^3 t_2 \\
[1, 4, 2, 3] & t_1 t^2_2 t_3 & [3, 1, 2, 4] & t^2_1 t_2 t_4 & [4, 2, 1, 3] & t_1^3 t_3 \\
[1, 4, 3, 2] & t_1 t^2_2 & [3, 1, 4, 2] & t^2_1 t_2 t_3 & [4, 2, 3, 1] & t_1^3 t_3 \\
[2, 1, 3, 4] & t^2_1 t_3 t_4 & [3, 2, 1, 4] & t^3_1 t_4 & [4, 3, 1, 2] & t_1^4 t_2 \\
[2, 1, 4, 3] & t^2_1 t^2_3 & [3, 2, 4, 1] & t^3_1 t^2_3 & [4, 3, 2, 1] & t_1^4 \\
\end{array}$$

7.7 The $\sigma$-functions hus, kus.

Reverting to the simple cyclotomic polynomials of §5.12, we set:

$$\text{hus}^*_x(\sigma) := \text{hus}_x(\sigma) \prod_{1 \leq k \leq r} \frac{(1 - x^k)}{(1 - x)} \quad \forall \sigma \in S_r$$

$$\text{kus}^*_x(\sigma) := \text{kus}_x(\sigma) \prod_{1 \leq k \leq r} \frac{(1 - x)(1 - x^k(k-1))}{(1 - x^k)} \quad \forall \sigma \in S_r$$
<table>
<thead>
<tr>
<th>$\sigma$</th>
<th>$\text{hus}_x^*$</th>
<th>$\text{kus}_x^*$</th>
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<tr>
<td>[1, 2]</td>
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<td>$x$</td>
</tr>
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<td>[2, 1]</td>
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<td>$-1$</td>
</tr>
<tr>
<td>[1, 2, 3]</td>
<td>$x^3$</td>
<td>$x^3 (x^2 + 1)$</td>
</tr>
<tr>
<td>[1, 3, 2]</td>
<td>$x^2$</td>
<td>$-x^4$</td>
</tr>
<tr>
<td>[2, 1, 3]</td>
<td>$x^2$</td>
<td>$-x^4$</td>
</tr>
<tr>
<td>[2, 3, 1]</td>
<td>$x$</td>
<td>$-x$</td>
</tr>
<tr>
<td>[3, 1, 2]</td>
<td>$x$</td>
<td>$-x$</td>
</tr>
<tr>
<td>[3, 2, 1]</td>
<td>1</td>
<td>$x^2 + 1$</td>
</tr>
<tr>
<td>[1, 2, 3, 4]</td>
<td>$x^6$</td>
<td>$x^6 (x^8 + 2 x^6 + x^4 + 2 x^2 + 1)$</td>
</tr>
<tr>
<td>[1, 2, 4, 3]</td>
<td>$x^5$</td>
<td>$-x^7 (x^2 + 1) (x^4 + 1)$</td>
</tr>
<tr>
<td>[1, 3, 2, 4]</td>
<td>$x^5$</td>
<td>$-x^7 (x^2 + 1) (x^4 - x^2 + 1)$</td>
</tr>
<tr>
<td>[1, 3, 4, 2]</td>
<td>$x^4$</td>
<td>$-x^{10}$</td>
</tr>
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<td>$-x^{10}$</td>
</tr>
<tr>
<td>[1, 4, 3, 2]</td>
<td>$x^3$</td>
<td>$x^9 (x^2 + 1)$</td>
</tr>
<tr>
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<td>$x^5$</td>
<td>$-x^7 (x^2 + 1) (x^4 + 1)$</td>
</tr>
<tr>
<td>[2, 1, 4, 3]</td>
<td>$x^4$</td>
<td>$x^8 (x^2 - x + 1) (x^2 + x + 1)$</td>
</tr>
<tr>
<td>[2, 3, 1, 4]</td>
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<td>$-x^{10}$</td>
</tr>
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<td>[2, 3, 4, 1]</td>
<td>$x^3$</td>
<td>$-x^3 (x^2 + 1)$</td>
</tr>
<tr>
<td>[2, 4, 1, 3]</td>
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<td>0</td>
</tr>
<tr>
<td>[2, 4, 3, 1]</td>
<td>$x^2$</td>
<td>$x^4$</td>
</tr>
<tr>
<td>[3, 1, 2, 4]</td>
<td>$x^4$</td>
<td>$-x^{10}$</td>
</tr>
<tr>
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<td>$x^3$</td>
<td>0</td>
</tr>
<tr>
<td>[3, 2, 1, 4]</td>
<td>$x^3$</td>
<td>$x^9 (x^2 + 1)$</td>
</tr>
<tr>
<td>[3, 2, 4, 1]</td>
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<td>$x^4$</td>
</tr>
<tr>
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<td>$-x^2 (x^2 - x + 1) (x^2 + x + 1)$</td>
</tr>
<tr>
<td>[3, 4, 2, 1]</td>
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<td>$x (x^2 + 1) (x^4 + 1)$</td>
</tr>
<tr>
<td>[4, 1, 2, 3]</td>
<td>$x^3$</td>
<td>$-x^3 (x^2 + 1)$</td>
</tr>
<tr>
<td>[4, 1, 3, 2]</td>
<td>$x^2$</td>
<td>$x^4$</td>
</tr>
<tr>
<td>[4, 2, 1, 3]</td>
<td>$x^2$</td>
<td>$x^4$</td>
</tr>
<tr>
<td>[4, 2, 3, 1]</td>
<td>$x$</td>
<td>$x (x^2 + 1) (x^4 - x^2 + 1)$</td>
</tr>
<tr>
<td>[4, 3, 1, 2]</td>
<td>$x$</td>
<td>$x (x^2 + 1) (x^4 + 1)$</td>
</tr>
<tr>
<td>[4, 3, 2, 1]</td>
<td>1</td>
<td>$-(x^8 + 2 x^6 + x^4 + 2 x^2 + 1)$</td>
</tr>
</tbody>
</table>
7.8 The $\sigma$-functions $\text{hus, kus}$ with a twist parameter.

We set:

\[
\begin{align*}
\text{hus}^*_x(t)(\sigma) &:= DH_r(x, t) \text{hus}_x(t)(\sigma) &\forall \sigma \in S_r \\
\text{kus}^*_x(t)(\sigma) &:= DK_r(x, t) \text{kus}_x(t)(\sigma) &\forall \sigma \in S_r \\

DH_r(x, t) &:= DH_r(x) \text{DH}^*_r(x, t) = \prod_{1 \leq k \leq r} \frac{(1 - x^k)}{(1 - x)} \prod_{1 \leq k \leq r} \frac{(t^2 - x^{k+1})}{(t^2 - x^k)} \\

DK_r(x, t) &:= DK_r(x) \text{DK}^*_r(x, t) = \prod_{2 \leq k \leq r} \frac{(1 - x)(1 - x^{k(k-1)})}{(1 - x^k)} \prod_{1 \leq k \leq r} \frac{(t^{2k} - x^{k(k+1)})}{(t^2 - x^k)} \\

\sigma &\begin{array}{c|cc}
\text{hus}^*_x(t) \\ \hline
1 & -(x^2 - t^2) \\
[1, 2] & (x^4 - t^2)(x^2 - t^2) \\
[2, 1] & x(x^2 - t^2)^2 & -(x^2 - t^2) \\
[1, 2, 3] & -(x^2 - t^2)(x^4 - t^2)(x^6 - t^2) & (x^6 - t^2) K_{123} \\
[1, 3, 2] & -x(x^2 - t^2)(x^4 - t^2)^2 & -(x^2 - t^2)(x^6 - t^4) \\
[2, 1, 3] & -x(x^2 - t^2)^2(x^6 - t^2) & -(x^2 - t^2)(x^{12} - t^4) \\
[2, 3, 1] & -x^2(x^2 - t^2)^2(x^4 - t^2) & -(x^2 - t^2)(x^6 - t^4) \\
[3, 1, 2] & -x^2(x^2 - t^2)^2(x^4 - t^2) & -(x^2 - t^2)(x^6 - t^4) \\
[3, 2, 1] & -x^3(x^2 - t^2)^3 & x^3(x^2 + 1)(x^2 - t^2)(x^6 - t^4) \\
\end{array}
\end{align*}
\]

with $K_{123} := x^{10} + x^8 - x^6 t^2 + x^4 t^2 - x^2 t^4 - t^4$.

7.9 The $\sigma$-functions $\text{huk, h"uk}$.

We set:

\[
\begin{align*}
\text{huk}^*_{x, a, b}(\sigma) &:= D\text{HK}_r(x; a, b) \text{huk}_x(t)(\sigma) &\forall \sigma \in S_r \\
\text{h"uk}^*_{x, a, b}(\sigma) &:= D\text{HK}_r(x; a, b) \text{h"uk}_x(t)(\sigma) &\forall \sigma \in S_r \\

D\text{HK}_r(x; a, b) &:= DH^*_r(x, a) \text{DK}^*_r(x, b) = \prod_{1 \leq k \leq r} (a^2 - x^{k+1}) \frac{(b^{2k} - x^{k(k+1)})}{(b^{2} - x^{k+1})} \\
\end{align*}
\]

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\[\sigma \quad \text{huk}^*_{x; a, b} \quad \text{huk}^*_{x; a, b}\]

\[
\begin{array}{ccc}
[1] & (a^2-x^2) & (a^2-x^2) \\
[1, 2] & (a^2-x^2)(a^2b^2-x^6) & b^2x^3(a^2-x^2)(a^2b^2-1) \\
[2, 1] & b^2x^3(a^2-x^2)(a^2b^2-1) & (a^2-x^2)(a^2b^2-x^6) \\
[1, 2, 3] & (a^2-x^2)(a^2b^2-x^6)(a^2b^4-x^{12}) & b^6x^{11}(a^2-x^2)(a^2b^2-1)(a^2b^4-x^4) \\
[1, 3, 2] & b^4x^7(a^2-x^2)(a^2b^2-1)(a^2-x^4) & b^2x^4(a^2b^2-x^6)(a^2b^4-x^4) \\
[2, 1, 3] & b^2x^3(a^2-x^2)(a^2b^2-1)(a^2b^4-x^{12}) & b^4x^8(a^2-x^2)(a^2b^2-x^6)(a^2b^4-x^4) \\
[2, 3, 1] & b^4x^3(a^2-x^2)(a^2b^2-1) & b^2x^3(a^2-x^2)(a^2b^2-1) \\
[3, 1, 2] & b^2x^4(a^2-x^2)(a^2b^2-1)(a^2b^4-x^{10}) & b^4x^7(a^2-x^2)(a^2b^2-x^6)(a^2b^4-x^4) \\
[3, 2, 1] & b^6x^{11}(a^2-x^2)(a^2b^2-1)(a^2b^4-x^4) & (a^2-x^2)(a^2b^2-x^6)(a^2b^4-x^{10}) \\
\end{array}
\]

7.10 The \(\sigma\)-functions ke and hes, kes.

We set:

\[
\text{kes}^*(\sigma) := \delta_r \text{kes}(\sigma) \quad \text{but} \quad \text{ke}^*(\sigma) := \delta_{r-1} \text{ke}(\sigma) \quad \forall \sigma \in S_r
\]

with \(\delta_r := \frac{(2r)!}{r! r!} \delta^*_r\)

and \(\delta_1 = \delta_2 = \delta_3 = 1, \quad \delta_4 = 3^2, \quad \delta_5 = 2^5 \cdot 3, \quad \delta_6 = 2^6 \cdot 3^2 \cdot 5^2 \cdot 41\)

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<th>(\text{ke}^*)</th>
<th>(\sigma)</th>
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<td>kes*</td>
<td>( \sigma )</td>
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