Combinatorial tidbits from resurgence theory and mould calculus.

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Abstract.

- §1 shows that the flexion structure with all its wealth of bimould operations is already contained *in nuce* in mould calculus.
- §2 shows on a striking example how even the simplest identities of Analysis may acquire unsuspected depth after mouldification.
- §3 shows how to decompose any mould, naturally and explicitely, into a sum of d-alternal moulds.
- §4 concerns itself with the algebraic-combinatorial side of *singularity composition* as a tool for characterising/describing/constructing *minimal convolution domains.*
- §5 recalls the notion of *iso-differential operators* – a sort of hyper-Schwarzians uniquely suited to functional composition – and asserts the existence on them of a remarkable positive cone spanned by extremal basis elements with startling combinatorial properties.

Contents

1 From moulds to bimoulds, and back.

1.1 The mould-bimould nexus.

Our aim is twofold:
• to remove the stigma of artificiality that seems to attach to the construction of *bimoulds* and the underlying *flexion structure.*
• to show that *bimoulds* arise from *moulds* with the same inevitability as *complex* from *real* numbers.

$$
\begin{array}{lcll}
\textit{moulds} & \| & \textit{bimoulds} & (\textit{flexion structure}) \\
(\mathrm{mu},\mathrm{lu}) & \| & & (\mathrm{gami},\mathrm{ami}) \\
 & \nwarrow\!\nwarrow & & \nearrow \\
 & & (\mathrm{gaxi},\mathrm{axi}) & \to (\mathrm{gari},\mathrm{ari}),(\mathrm{gali},\mathrm{ali}),\dots,(\mathrm{gawa},\mathrm{awa}),(\mathrm{giwi},\mathrm{iwi}) \\
 & \nearrow\!\nearrow & & \searrow \\
(\mathrm{ko},\mathrm{lo}) & \| & & (\mathrm{gani},\mathrm{ani})
\end{array}
$$

1.2 Moulds and their uses.

• **Mould product** $mu = \times$**. The corresponding Lie bracket** $lu = [\dots]$**.**

$$C^\bullet = \mathrm{mu}(A^\bullet, B^\bullet) = A^\bullet \times B^\bullet \iff C^{\boldsymbol{\omega}} = \sum_{\boldsymbol{\omega^1}.\boldsymbol{\omega^2} = \boldsymbol{\omega}} A^{\boldsymbol{\omega^1}} B^{\boldsymbol{\omega^2}} \tag{1}$$

$$C^\bullet = \mathrm{lu}(A^\bullet, B^\bullet) = [A^\bullet\,,\,B^\bullet] \iff C^{\boldsymbol{\omega}} = A^\bullet \times B^\bullet - B^\bullet \times A^\bullet \tag{2}$$

N.B. $\boldsymbol{\omega^1}$ or $\boldsymbol{\omega^2}$ in (1) may be $\emptyset$.

• **Mould composition** $ko = \circ$**. The corresponding Lie bracket** lo**.**

$$C^\bullet = \mathrm{ko}(A^\bullet, B^\bullet) = A^\bullet \circ B^\bullet \Longleftrightarrow C^{\boldsymbol{\omega}} = \sum_{\boldsymbol{\omega^1}\dots\boldsymbol{\omega^s} = \boldsymbol{\omega}}^{1\leq s \leq r(\boldsymbol{\omega})} A^{|\boldsymbol{\omega^1}|,\dots,|\boldsymbol{\omega^s}|} \prod_{1\leq i\leq s} B^{\boldsymbol{\omega^i}} \tag{3}$$

$$C^\bullet = \mathrm{lo}(A^\bullet, B^\bullet) \Longleftrightarrow C^{\boldsymbol{\omega}} = \sum_{\boldsymbol{\omega^1}\boldsymbol{\omega^2}\boldsymbol{\omega^3} = \boldsymbol{\omega}} \left(A^{\boldsymbol{\omega^1},|\boldsymbol{\omega^2}|,\boldsymbol{\omega^3}} B^{\boldsymbol{\omega^2}} - B^{\boldsymbol{\omega^1},|\boldsymbol{\omega^2}|,\boldsymbol{\omega^3}} A^{\boldsymbol{\omega^2}}\right) \tag{4}$$

N.B. Each sequence $\boldsymbol{\omega^i}$ in (3) as well as $\boldsymbol{\omega^2}$ in (4) has to be $\neq \emptyset$ but $\boldsymbol{\omega^1}$ or $\boldsymbol{\omega^3}$ in (4) may be $\emptyset$, separately or simultaneously.

• **Properties and uses.**
Mould multiplication naturally arises whenever we multiply two or several

sums of type $\sum_\bullet A^\bullet B_\bullet$ – i.e. sums resulting from the contraction of scalar-valued moulds $A^\bullet$ and co-moulds $B_\bullet$ with values in associative, non commutative algebras. Plus in scores of other contexts.

Mould composition naturally arises whenever we change bases in graded algebras. Plus in scores of other contexts.

Moreover, most natural moulds happen to possess a definite *symmetry type*, which the main mould operations (to wit *mu, lu, ko, lo*, plus *logmu*, *expmu*, plus *logko*, *expko*, plus a whole array of *mould derivations*[1]) either *conserve* or *transmute* into other symmetry types. The main, but by no means only, symmetry types are: *symmetr***a***l, symmetr***e***l, altern***a***l, altern***e***l.*

1.3 Bimoulds and their uses. The flection structure.

• The double $u_i//v_i$ indexation and the four flexion symbols $\rceil, \lceil, \rfloor, \lfloor$. The action of the four *flectors* is always relative to a specified sequence factorisation $\mathbf{w} = \mathbf{w}^1\mathbf{w}^2\dots\mathbf{w}^s$. The rules, which involve only u_i-sums,[2] pair-wise v_i-differences and preserve $\sum u_i\, v_i$, are apparent on the following example. Given the factorisation

$$\mathbf{w} = \dots\mathbf{a}.\mathbf{b}\dots = \dots(\begin{smallmatrix} u_3 & , & u_4 & , & u_5 \\ v_3 & , & v_4 & , & v_5 \end{smallmatrix})(\begin{smallmatrix} u_6 & , & u_7 & , & u_8 & , & u_9 \\ v_6 & , & v_7 & , & v_8 & , & v_9 \end{smallmatrix})\dots$$

the flectors $\rceil$, $\lceil$, $\rfloor$, $\lfloor$ act as follows:

$$\mathbf{a}\rceil := (\begin{smallmatrix} u_3 & , & u_4 & , & u_{5,6,7,8,9} \\ v_3 & , & v_4 & , & v_5 \end{smallmatrix}) \qquad \lceil\mathbf{b} := (\begin{smallmatrix} u_{3,4,5,6} & , & u_7 & , & u_8 & , & u_9 \\ v_6 & , & v_7 & , & v_8 & , & v_9 \end{smallmatrix}) \tag{5}$$

$$\mathbf{a}\rfloor := (\begin{smallmatrix} u_3 & , & u_4 & , & u_5 \\ v_{3:6} & , & v_{4:6} & , & v_{5:6} \end{smallmatrix}) \qquad \lfloor\mathbf{b} := (\begin{smallmatrix} u_6 & , & u_7 & , & u_8 & , & u_9 \\ v_{6:5} & , & v_{7:5} & , & v_{8:5} & , & v_{9:5} \end{smallmatrix}) \tag{6}$$

with the usual shorthand: $u_{i,j,\dots} := u_i + u_j + \dots\ ,\ v_{i:j} := v_i - v_j$.

• The $\boldsymbol{u}$-pattern determines the $\boldsymbol{v}$-pattern and *vice versa.*

• The core involution *swap.*

$$B^\bullet = \mathrm{swap}(A^\bullet) \quad\Longleftrightarrow\quad B^{(\begin{smallmatrix} u_1 & , & \dots & , & u_r \\ v_1 & , & \dots & , & v_r \end{smallmatrix})} = A^{(\begin{smallmatrix} v_r & , & v_{r-1:r} & , & \dots & , & v_{2:3} & , & v_{1:2} \\ u_{1\dots r} & , & u_{1\dots r-1} & , & \dots & , & u_{12} & , & u_1 \end{smallmatrix})} \tag{7}$$

Anything definable in terms of the four flexions retains this property even after undergoing the *swap*. The most interesting bimoulds are *dimorphic* bimoulds, i.e. bimoulds such that $A^\bullet$ and *swap*$A^\bullet$ both belong to a definite symmetry type – often the same one, but not necessarily so. On top of the

[1]derivations, that is, relative to mould multiplication *mu*.

[2]More precisely: sums of *consecutive* u_i's. The construction of *ARRI//GARRI* in §1.7 *infra* shows how essential this restriction is.

four symmetry types defined on moulds, bimoulds often display symmetry types *sui generis*, of a more exotic nature.

• **The overarching structure** $AXI//GAXI$ **and its seven sub-structures induced by involutions** $\mathcal{A}_R^\bullet := involA_L$**.**
Both the group $GAXI$ and its algebra AXI consist of pairs $\mathcal{A}^\bullet = (\mathcal{A}_L^\bullet, \mathcal{A}_R^\bullet)$. The left and right components $\mathcal{A}_L^\bullet$ and $\mathcal{A}_L^\bullet$ are w_i-indexed bimoulds, where $w_i = \binom{u_i}{v_i}$. The group operation *gaxi*, the bracket *axi* and the corresponding exponential-logarithm *expaxi, logaxi*, are all expressible in terms of the four flectors. Of special interest are the sub-structures of $AXI//GAXI$ consisting of pairs of the form $\mathcal{A}_R^\bullet := involA_L$, with an involution *invol* so chosen as to permit stability under *gaxi* or *axi*. Up to isomorphism, there exist exactly seven such sub-structures,[3] to which we must add the two *lateral* sub-structures $AMI//GAMI$ and $ANI//GANI$, where the pair $(\mathcal{A}_L^\bullet, \mathcal{A}_R^\bullet)$ reduces to either $\mathcal{A}_L^\bullet$ or $\mathcal{A}_R^\bullet$. Moreover, we have full closure under the involution *swap*, which transmutes any one of the $4 \times 7 + 4 \times 2 = 32$ bimould operations into finite sequences of those same 32 operations.

• **The core structure** $ARI//GARI$**.**
Of those magic seven sub-structures, four (and especially two[4]) stand out – those namely that preserve *dimorphy*, e.g. bi-alternality or bi-symmetrality. Moreover, when restricted to dimorphic objects, those sub-structures actually coincide, thus sparing us the agony of choosing between them. In practice, one works most of the time with the sub-structure $ARI//GARI$, induced by the involution $invol.A^\bullet := -A^\bullet$ for the algebra ARI, and $invol.A^\bullet := invmu.A^\bullet$ for the group $GARI$.

• **'Accidental' origin of the flexion structure:**
Said structure was first encountered in the early 90ies while investigating the quite specific resurgence pattern of the divergent solutions of *singular* and *singularly perturbed* differential equations expanded in power series of the singular parameter. The hybrid make-up of their singularities in the Borel plane found its natural reflection in a double $(u_i\,, v_i)$-indexation. The basic notion in this context was the so-called *scramble* transform, which already foreshadowed the full-fledged flexion structure.

[3] They rejoice in the mellifluous names of $ARI//GARI$, $ALI//GALI$, $ALA//GALA$, $ILI//GILI$, $AWI//GAWI$, $AWA//GAWA$, $IWI//GIWI$.
[4] Namely $ARI//GARI$ and $ALI//GALI$.

• **Other domains of application:**
Chief amongst them is the investigation of *arithmetical dimorphy*, which is the dominant feature of many $\mathbb{Q}$-rings of transcendental numbers: multizetas, hyperlogs, the so-called *naturals* etc etc.

But even classical elementary *moulds* tend to go in pairs and, when viewed as *bimoulds* (under the suitable adjunction of a second row of mute indices – i.e. indices on which they do not effectively depend), the two terms of those pairs get exchanged by the *swap*.

The *swap*, in fact, is as central to mould/bimould calculus as the Fourier transform is to Analysis. The two notions, incidentally, are not altogether unrelated.

1.4 From moulds to bimoulds.

Under mould composition, any two moulds $\mathcal{A}^\bullet$ and $\mathcal{B}^\bullet$, u_i-indexed for convenience, and of the form[5]

$$\mathcal{A}^\bullet := \mathcal{A}_L^\bullet \times Id^\bullet \times \mathcal{A}_R^\bullet \qquad , \qquad \mathcal{B}^\bullet := \mathcal{B}_L^\bullet \times Id^\bullet \times \mathcal{B}_R^\bullet \tag{8}$$

yield a mould $\mathcal{C}^\bullet := \mathcal{A}^\bullet \circ \mathcal{B}^\bullet$ also of the form $\mathcal{C}^\bullet := \mathcal{C}_L^\bullet \times Id^\bullet \times \mathcal{C}_R^\bullet$ with

$$\mathcal{C}_L^\bullet \;:=\; \Big(\mathcal{A}_L^\bullet \circ (\mathcal{B}_L^\bullet \times Id^\bullet \times \mathcal{B}_R^\bullet)\Big) \times \mathcal{B}_L^\bullet \tag{9}$$

$$\mathcal{C}_R^\bullet \;:=\; \mathcal{B}_R^\bullet \times \Big(\mathcal{A}_R^\bullet \circ (\mathcal{B}_L^\bullet \times Id^\bullet \times \mathcal{B}_R^\bullet)\Big) \tag{10}$$

and – *lo and behold!* – the right-hand terms involve only *'inflectable'* summands, leading straightaway to the *GAXI*-structure.

Thus, at depth $r = 5$, the left component $\mathcal{C}_L^{u_1,u_2,u_3,u_4,u_5}$, once expanded, takes the form of a sum with exactly 89 summands, each one of which can be *inflected* (i.e. turned into a bimould component under proper introduction of v_i-indices) and that too in a unique way. Here are five such u_i-indexed summands, fairly reprentative of all possible combinations, and their $\binom{u_i}{v_i}$-indexed counterparts, with the usual abbreviations for index sums

[5] L for *left*, R for *right*.

and differences:

$$\begin{array}{rcl}
\mathcal{A}_L^{u_{1,2,3}}\,\mathcal{B}_R^{u_2,u_3}\,\mathcal{B}_L^{u_4,u_5} & \mapsto & \mathcal{A}_L^{\binom{u_{1,2,3}}{v_1}}\,\mathcal{B}_R^{\binom{u_2\ ,\ u_3}{v_{2:1}\ ,\ v_{3:1}}}\,\mathcal{B}_L^{\binom{u_4\ ,\ u_5}{v_4\ ,\ v_5}} \\
\mathcal{A}_L^{u_{1,2},u_3}\,\mathcal{B}_L^{u_1}\,\mathcal{B}_L^{u_4,u_5} & \mapsto & \mathcal{A}_L^{\binom{u_{1,2}\ u_3}{v_2\ \ v_3}}\,\mathcal{B}_L^{\binom{u_1}{v_{1:2}}}\,\mathcal{B}_L^{\binom{u_4\ ,\ u_5}{v_4\ ,\ v_5}} \\
\mathcal{A}_L^{u_{1,2},u_3,u_{4,5}}\,\mathcal{B}_R^{u_2}\,\mathcal{B}_R^{u_5} & \mapsto & \mathcal{A}_L^{\binom{u_{1,2}\ ,\ u_3\ ,\ u_{4,5}}{v_1\ ,\ v_3\ ,\ v_4}}\,\mathcal{B}_R^{\binom{u_2}{v_{2:1}}}\,\mathcal{B}_R^{\binom{u_5}{v_{5:4}}} \\
\mathcal{A}_L^{u_{1,2},u_{3,4}}\,\mathcal{B}_L^{u_1}\,\mathcal{B}_L^{u_3}\,\mathcal{B}_L^{u_5} & \mapsto & \mathcal{A}_L^{\binom{u_{1,2}\ ,\ u_{3,4}}{v_2\ ,\ v_4}}\,\mathcal{B}_L^{\binom{u_1}{v_{1:2}}}\,\mathcal{B}_L^{\binom{u_3}{v_{3:4}}}\,\mathcal{B}_L^{\binom{u_5}{v_5}} \\
\mathcal{A}_L^{u_1,u_{2,3,4}}\,\mathcal{B}_L^{u_2,u_3}\,\mathcal{B}_L^{u_5} & \mapsto & \mathcal{A}_L^{\binom{u_1\ ,\ u_{2,3,4}}{v_1\ ,\ v_4}}\,\mathcal{B}_L^{\binom{u_2\ ,\ u_3}{v_{2:4}\ ,\ v_{3:4}}}\,\mathcal{B}_L^{\binom{u_5}{v_5}}
\end{array}$$

Another way of putting it would be to say that the mould composition *ko*, when restricted to u_i-indexed elements of the form (8), yields the subgroup of *GAXI* whose elements reduce to $Id^\bullet$ unless all v_i vanish or, what amounts to the same, the subgroup of *GAXI* whose elements do not depend on the v_i-indices:

$$\big(moulds^\bullet \times Id^\bullet \times moulds^\bullet\big)_{\mathrm{ko}} \sim GAXI\|_{\boldsymbol{v}=0} \sim GAXI\|_{\boldsymbol{v}\text{-}const.}$$

The same applies for the *lo*-bracket and the algebra *AXI*:

$$\big(moulds^\bullet \times Id^\bullet \times moulds^\bullet\big)_{\mathrm{lo}} \sim AXI\|_{\boldsymbol{v}=0} \sim AXI\|_{\boldsymbol{v}\text{-}const.}$$

1.5 From bimoulds to moulds.

A bimould $A^\bullet$ is said to be *internal* if, for all r, it verifies two dual properties, which in *short* notation read:

$$\{u_1 + \ldots u_r \neq 0\} \implies \{A^{\binom{u_1\ ,\ldots,\ u_r}{v_1\ ,\ldots,\ v_r}} \equiv 0\} \tag{11}$$

$$\{v_i - v_i' = \text{const}\,;\ \forall i\} \implies \{A^{\binom{u_1\ ,\ldots,\ u_r}{v_1\ ,\ldots,\ v_r}} \equiv A^{\binom{u_1\ ,\ldots,\ u_r}{v_1'\ ,\ldots,\ v_r'}}\} \tag{12}$$

and in *long* notation assume the more natural form:

$$\{u_0 \neq 0\} \implies \{A^{\left(\left[\begin{smallmatrix}u_0\\ v_0\end{smallmatrix}\right]\begin{smallmatrix},\ u_1\ ,\ldots,\ u_r\\ ,\ v_1\ ,\ldots,\ v_r\end{smallmatrix}\right)} \equiv 0\} \tag{13}$$

$$\{\forall\, v_0\,,\ \forall\, v_0'\} \implies \{A^{\left(\left(\left(\left[\begin{smallmatrix}u_0\\ v_0\end{smallmatrix}\right]\begin{smallmatrix},\ u_1\ ,\ldots,\ u_r\\ ,\ v_1\ ,\ldots,\ v_r\end{smallmatrix}\right)\right.\right.} \equiv A^{\left(\left(\left(\left[\begin{smallmatrix}u_0\\ v_0'\end{smallmatrix}\right]\begin{smallmatrix},\ u_1\ ,\ldots,\ u_r\\ ,\ v_1\ ,\ldots,\ v_r\end{smallmatrix}\right)\right.\right.}\} \tag{14}$$

Internals constitute an ideal ARI_{intern} of *ARI* resp. a normal subgroup $GARI_{intern}$ of *GARI*. The elements of the corresponding quotients are referred to as *externals*:

$$\mathrm{ARI}_{\mathrm{extern}} := \mathrm{ARI}/\mathrm{ARI}_{\mathrm{intern}} \tag{15}$$

$$\mathrm{GARI}_{\mathrm{extern}} := \mathrm{GARI}/\mathrm{GARI}_{\mathrm{intern}} \tag{16}$$

Moreover, when restricted to internals, the *ari* bracket reduces, up to order, to the simpler *lu* bracket, and the *gari* product, again up to order, reduces to the *mu* product:

$$\mathrm{ari}(A^\bullet, B^\bullet) \equiv \mathrm{lu}(B^\bullet, A^\bullet) \quad , \quad \forall A^\bullet, B^\bullet \in \mathrm{ARI}_{\mathrm{intern}} \tag{17}$$

$$\mathrm{gari}(A^\bullet, B^\bullet) \equiv \mathrm{mu}(B^\bullet, A^\bullet) \quad , \quad \forall A^\bullet, B^\bullet \in \mathrm{GARI}_{\mathrm{intern}} \tag{18}$$

On the face of it, the identity (18) is highly surprising, since $\mathrm{lu}(B^\bullet, A^\bullet)$ is bilinear in $A^\bullet$ and $B^\bullet$, whereas $\mathrm{gari}(A^\bullet, B^\bullet)$ is violently non-linear in $B^\bullet$ — at least when the factors are *general* moulds instead of *internal* ones, as assumed in (18).

• **Subspace of bimoulds:**

The above applies in particular to those *internal* bimoulds that are either $\boldsymbol{u}$- or $\boldsymbol{v}$-constant.

Thus, the highly complex *bimould calculus*, with its exquisite intricacy of structure and plethora of inflected operations, and the more elementary, un-inflected *mould calculus* reveal themselves to be, in a sense, subcases of each other!

1.6 How to retrieve the bimoulds' two-tier indexation $\binom{u_1 \,...\, u_r}{v_1 \,...\, v_r}$ along with the core involution *swap*.

As already pointed out, as far as the flexion operations are concerned, the u_i-pattern wholly determines the complete $\binom{u_i}{v_i}$-pattern. But are there natural incentives (other than the absolutely compelling phenomenon of *dimorphy*) for introducing the v_i-indices in the first place? Yes, there are – rooted in simple considerations of *notational economy.* The *swap*, too, is an inescapable feature of the landscape.

As for the idea of applying mould composition to elements of the form $\mathcal{A}^\bullet := \mathcal{A}_L^\bullet \times Id^\bullet \times \mathcal{A}_R^\bullet$, that too is a very natural step – especially when $\mathcal{A}_{\mathcal{L}}^\bullet \times \mathcal{A}_{\mathcal{R}}^\bullet \equiv \mathbf{1}^\bullet$ with $\mathcal{A}_{\mathcal{L}}^\bullet$, $\mathcal{A}_{\mathcal{R}}^\bullet$ symmetral and, as a consequence, $\mathcal{A}^\bullet$ alternal.

A case in point is provided by the hyperlogarithmic monics $U^\bullet$ and $V^\bullet$, associated respectively with the Δ- and ∂-friendly canonical resurgence monomials $\mathcal{U}^\bullet(z)$ and $\mathcal{V}^\bullet(z)$. Not only are these monics $U^\bullet$ and $V^\bullet$ reciprocal under mould composition (see (22)), but they are globally homogeneous in their indices:

$$U^{c\,\omega_1,\dots,c\,\omega_r} \equiv U^{\omega_1,\dots,\omega_r} \quad ; \quad V^{c\,\omega_1,\dots,c\,\omega_r} \equiv V^{\omega_1,\dots,\omega_r} \quad (\forall c)$$

This homogeneousness is an invitation to express each monic $U^{\boldsymbol{\omega}}$ or $V^{\boldsymbol{\omega}}$ of length r in terms of simpler monics, of strictly lesser depth. The actual decomposition takes the form

$$U^\bullet \;=\; \Big(U_L^\bullet \times Id^\bullet \times U_R^\bullet\Big) \circ Rot^\bullet \tag{19}$$

$$V^\bullet \;=\; Ret^\bullet \circ \Big(V_L^\bullet \times Id^\bullet \times V_R^\bullet\Big) \tag{20}$$

with symmetral, *mu*-inverse, non-homogeneous monics $(U_L^\bullet, U_R^\bullet)$ and $(V_L^\bullet, V_R^\bullet)$ and elementary, locally constant, *ko*-reciprocal moulds $Rot^\bullet$, $Ret^\bullet$, which incidentally reduce to $Id^\bullet$ when all indices ω_i are real positive. Altogether, we get the relations:

$$\begin{aligned}
Rot^\bullet \circ Ret^\bullet &= Id^\bullet = \textit{composition-unit} &(21)\\
U^\bullet \circ V^\bullet &= Id^\bullet = \textit{composition-unit} &(22)\\
V_L^\bullet \times V_R^\bullet &= \mathbf{1}^\bullet \;= \textit{multiplication-unit} &(23)\\
U_L^\bullet \times U_R^\bullet &= \mathbf{1}^\bullet \;= \textit{multiplication-unit} &(24)\\
\mathrm{gari}(U_L^\bullet, V_L^\bullet) &= \mathbf{1}^\bullet \;= \mathrm{GARI}\text{-}\textit{unit} &(25)
\end{aligned}$$

The bottom line is that the purely *mould-like* monics $U_L^\bullet$ and $V_L^\bullet$ are mutually inverse under what is, essentially, the *bimould* group product *gari*.

1.7 Cautionary remark: *ARRI//GARRI*, a deceptive look-alike of *ARI//GARI*.

Let *alex* ("alternal extensor") be the mould-to-mould operator so defined:

$$(\mathrm{alex}\, A)^{\boldsymbol{u}} = \sum_{\boldsymbol{u} = (\boldsymbol{u'}, u_i, \boldsymbol{u''})}^{1 \le i \le r(\boldsymbol{u})} (-1)^{r(\boldsymbol{u''})} \sum_{\boldsymbol{u^*} \in \,\mathbf{shuffle}(\boldsymbol{u'}, \widetilde{\boldsymbol{u''}})} A^{\boldsymbol{u^*}} \tag{26}$$

Here, $r = r(\boldsymbol{u})$ is the length of $\boldsymbol{u}$ and $r'' = r - i = r(\boldsymbol{u''})$ that of $\boldsymbol{u''}$. Further, $\widetilde{\boldsymbol{u''}}$ denotes the order-reversed sequence $\boldsymbol{u''}$. In the two extreme cases, when $i = 1$ resp. $i = r$, the second sum reduces to a single term $A^{\boldsymbol{u^*}}$, with $\boldsymbol{u^*} = \widetilde{\boldsymbol{u''}}$ resp. $\boldsymbol{u^*} = \boldsymbol{u'}$. As for the operator *alex*, it owes its name to the fact that *alex* $A^\bullet$ is automatically alternal (whatever the nature of $A^\bullet$) and carries depth-r components that are defined as superpositions of the depth-$(r-1)$ components of $A^\bullet$.

Now, one readily checks that mould composition, whether it be $ko \sim \circ$ or the infinitesimal variant *lo*, preserves moulds of the form *alex* $A^\bullet$. To be precise, the relations:

$$ko(\mathit{alex}\, A^\bullet, \mathit{alex}\, B^\bullet) \;=\; \mathit{alex}\, C^\bullet \tag{27}$$

$$lo(\mathit{alex}\, A_*^\bullet, \mathit{alex}\, B_*^\bullet) \;=\; \mathit{alex}\, C_*^\bullet \tag{28}$$

unambiguously define a group operation *garri* and a Lie bracket *arri*:

$$(A^\bullet, B^\bullet) \quad \mapsto \quad C^\bullet = garri(A^\bullet, B^\bullet) \tag{29}$$
$$(A^\bullet_*, B^\bullet_*) \quad \mapsto \quad C^\bullet_* = arri(A^\bullet_*, B^\bullet_*) \tag{30}$$

What's more, when expanding $C^\bullet$ in terms of $A^\bullet, B^\bullet$ or $C^\bullet_*$ in terms of $A^\bullet_*, B^\bullet_*$, we get a bunch of terms that look deceptively like 'flexions': they can indeed, in a unique way, be supplemented by v_i-indices so as to preserve the scalar product $\sum u_i\, v_i$ and involve only pair-wise v_i differences and partial u_i-sums. However – and here lies the crux – those u_i-sums don't always consist of *consecutive* u_i's. So we aren't dealing with genuine flexions here, and this becomes immediately apparent the moment we apply the involution *swap*: unlike *swap gari*$(A^\bullet, B^\bullet)$, which involves only *bona fide* flexions, *swap garri*$(A^\bullet, B^\bullet)$ throws up numerous pathological expressions of type $v_i - v_j + v_k$ and worse.

However, when we restrict ourselves to symmetral (resp. alternal) bimoulds, the non-flexion object *GARRI* (resp. *ARRI*) can be shown to coincide with the flexion object *GARI* (resp. *ARI*), which in turn, *again in that case*, happens to coincide with other flexion objects such as *GALI*, *GAWI* etc (resp. *ALI*, *AWI* etc). This applies even more when dealing with *dimorphic* bimoulds. Thus, in the situations that really matter we have but *one* pair group/algebra to consider, and we choose to take it in its simplest and most natural incarnation, namely the *ARI//GARI* incarnation.

2 Mould extensions of classical functions.

2.1 The right way to proceed.

The temptation to resist is to pick classical functions or identities at random and say *"Let us mouldify them; let us come up with some multivariate extension"*. The thing is that there usually exist scores and scores of *a priori* possible "multivariate extensions", most of them useless and highly forgettable – mere mathematical refuse. So the wise attitude is to let Analysis be our guide, and introduce only those moulds that genuine problems require for their solution. Such moulds are abundant enough. Unlike the gratuitous ones, they are almost guaranteed to be interesting, property-rich, and long-lived. And oftener than not, they shed unexpected light on the one-variable starting point, as the following example shows.

2.2 The Gamma function and Euler's reflection formula.

$*$ **Definition of** $V^{\bullet}$ (altern**a**l).
The σ-dependent monics $V^{\bullet}$ are derived from the ∂-friendly, σ-dependent resurgence monomials $\mathcal{V}(z)^{\bullet}$

$$(\partial_z + |\boldsymbol{\omega}|)\, \mathcal{V}^{\binom{\sigma_1\ ,\dots,\ \sigma_r}{\omega_1\ ,\dots,\ \omega_r}}(z) = -\mathcal{V}^{\binom{\sigma_1\ ,\dots,\ \sigma_{r-1}}{\omega_1\ ,\dots,\ \omega_{r-1}}}(z)\ \ z^{\sigma_r - 1} \tag{31}$$

under alien differentiation

$$\Delta_{\omega_0}\, \mathcal{V}^{\binom{\sigma_1\ ,\dots,\ \sigma_r}{\omega_1\ ,\dots,\ \omega_r}}(z) = \sum_{\omega_1+\cdots+\omega_i=\omega_0} V^{\binom{\sigma_1\ ,\dots,\ \sigma_r}{\omega_1\ ,\dots,\ \omega_i}}\, \mathcal{V}^{\binom{\sigma_1\ \ ,\dots,\ \sigma_r}{\omega_{i+1}\ ,\dots,\ \omega_r}}(z) \tag{32}$$

$*$ **Definition of** $W^{\bullet}$ (symmetr**a**l).
The associated monics $W^{\bullet}$ admit a direct definition via the multiple integral:

$$\mathcal{W}^{\binom{\sigma_1\ ,\dots,\ \sigma_r}{\omega_1\ ,\dots,\ \omega_r}} = \int_{0<y_r<\cdots<y_1<+\infty)} \prod_{1\le i\le r} \left(e^{-\omega_i\, y_i}\ y_i^{\sigma_i-1} dy_i\right) \tag{33}$$

that converges on $\{\Re\,\omega_i > 0\ ,\ \Re\,\widehat{\sigma}_i > 0\}$ (with $\widehat{\sigma}_i := \sigma_i + \cdots + \sigma_r$) and admits a meromorphic extension to $\mathbb{C}^r_{\bullet} \times \mathbb{C}^r$.

$*$ **Definition of** $Qe^{\bullet}$ (altern**e**l).

$$Qe^{\sigma_1,\dots,\sigma_r} := \frac{1}{2\pi i}\frac{(-1)^{r-1}}{r!} \sum_{\epsilon_1,\dots,\epsilon_r\in\{+,-\}} \epsilon_1\, p!\, q!\ \exp\big(\pi i(\epsilon_1\sigma_1 + \cdots + \epsilon_r\sigma_r)\big) \tag{34}$$

where p (resp. q) is the number of $+$ (resp. $-$) signs in $(\epsilon_2,\dots,\epsilon_r)$. Notice that ϵ_1 is omitted, so that $p+q=r-1$.

$*$ **A multivariate extenstion of Euler's reflection formula.**
Our three moulds happen to be related under mould composition:

$$V^{\bullet} = Qe^{\bullet} \circ W^{\bullet} \tag{35}$$

with $V^{\bullet}$ altern**a**l, $V^{\bullet}$ symmetr**a**l, $Qe^{\bullet}$ altern**e**l. At depth $r=1$ we have

$$V^{\binom{\sigma_1}{\omega_1}} = \frac{\omega_1^{-\sigma_1}}{\Gamma(1-\sigma_1)} \quad , \quad W^{\binom{\sigma_1}{\omega_1}} = \omega_1^{-\sigma_1}\Gamma(\sigma_1) \quad , \quad Qe^{\sigma_1} = \frac{\sin(\pi\sigma_1)}{\pi} \tag{36}$$

so that in this case (35) reduces to Euler's reflection formula

$$\frac{1}{\Gamma(1-\sigma_1)} \equiv \frac{\sin(\pi\sigma_1)}{\pi}\,\Gamma(\sigma_1) \tag{37}$$

Even the trivial re-ordering of the reflection formula:

$$\frac{\pi}{\sin(\pi\sigma_1)}\,\frac{1}{\Gamma(1-\sigma_1)} \equiv \Gamma(\sigma_1) \tag{38}$$

has a *non-trivial* mould equivalent, namely

$$Qo^\bullet \circ V^\bullet = W^\bullet \qquad \big(\textit{where}\;\; Qo^\bullet \circ Qe^\bullet = Id^\bullet\big) \tag{39}$$

with a *symmetr***al** mould $Qo^\bullet$, composition-reciprocal to the *altern***e***l* $Qe^\bullet$ and expressible by a formula markedly different from, and much more complex than, formula (34).

The functional equation $\Gamma(\sigma+1) = \Gamma(\sigma)\,\sigma$ also has its mould extension – two in fact, one for $V^\bullet$ and another for $W^\bullet$.

2.3 Other examples.

They include: multizetas, multigammas, inverse multigammas, multi-Bernoullis, multifactorials, etc etc etc.

It should be noted that one and the same classical function or notion may admit *several, equally natural* mould extensions[6](to say nothing of the unatural ones).

Then the pervasive phenomenon of dimorphy, which is mould-specific (it manifests only after mouldification) often enters the picture, contributing new structure, posing new problems, and spicing up everything.

3 Natural projectors.

3.1 Notion of d-projectors. Dual formulation.

Projectors operating on free associative algebras.
Let $\mathcal{L}$ be a free Lie algebra and $\mathcal{E}$ its enveloping algebra.
Question: Is there a privileged way of turning the *filtration*

$$\mathcal{L} = \mathcal{E}_{[1]} \subset \mathcal{E}_{[2]} \subset \mathcal{E}_{[3]} \cdots \subset \mathcal{E} \tag{40}$$

into a natural *gradation*

$$\mathcal{E}_1 \cup \mathcal{E}_2 \cup \mathcal{E}_3 \cup \cdots = \mathcal{E} \tag{41}$$

[6]usually two, but sometimes more, as with the Bernoulli numbers and polynomials.

More to the point, we ask for a complete system of natural projectors $proj_d$ with explicit formulae for the structure coefficients $h_d(\sigma)$.

$$\mathrm{proj}_d : \quad \Delta_{\omega_1}\,\Delta_{\omega_2}\ldots\Delta_{\omega_r} \mapsto h_d(\sigma)\,\Delta_{\omega_{\sigma(1)}}\,\Delta_{\omega_{\sigma(2)}}\ldots\Delta_{\omega_{\sigma(r)}} \tag{42}$$

and the natural conditions

$$\mathrm{proj}_d(\Delta_{\omega_1}\,\Delta_{\omega_2}\ldots\Delta_{\omega_r}) \quad \in \quad \mathcal{E}_d \subset \mathcal{E}_{[d]} \tag{43}$$

$$\sum_{1\le d\le r} \mathrm{proj}_d(\Delta_{\omega_1}\,\Delta_{\omega_2}\ldots\Delta_{\omega_r}) \quad = \quad \Delta_{\omega_1}\,\Delta_{\omega_2}\ldots\Delta_{\omega_r} \tag{44}$$

Projectors operating on moulds.
Our main concern is actually with the dual problem: how to decompose any given mould into a sum of d-alternal components

$$\mathrm{proj}_d \,.\, A^{\omega_1,\ldots,\omega_r} \quad = \quad \sum_{\sigma\in\mathfrak{S}_r} h_d(\sigma)\,A^{\sigma(\omega_1,\ldots,\omega_r)} \quad \in \quad \mathcal{A}lt_d \tag{45}$$

the definition of d-alternal being

$$\{A^\bullet \;\; d\text{-alternal}\} \;\;\Leftrightarrow\;\; \{ \sum_{\boldsymbol{\omega}\in \mathrm{shuffle}(\boldsymbol{\omega^1},\ldots,\boldsymbol{\omega^{d+1}})} A^{\boldsymbol{\omega}} \equiv 0 \quad , \quad \forall \boldsymbol{\omega^i} \neq \emptyset \;\; \} \tag{46}$$

The two notions – degree-d and d-alternal – are indeed dual, since

$$\{A^\bullet \;\; d\text{-alternal}\} \;\;\Leftrightarrow\;\; \{ \; \sum_\bullet A^\bullet\,\Delta_\bullet \in \mathcal{E}_d(\Delta) \;\; \} \tag{47}$$

3.2 Dyn'kin projectors / natural projectors.

Dyn'kin projectors:

They rely on repeated bracketing:

$$\Delta_{\omega_1}\,\Delta_{\omega_2}\ldots\Delta_{\omega_r} \mapsto \frac{1}{r}\;[[\ldots[\Delta_{\omega_1}\,\Delta_{\omega_2}]\ldots\ldots]\Delta_{\omega_r}] \tag{48}$$

They are economical (they involve 2^{r-1} permutations rather than $r!$) but unsatisfactory, to the extent that they priviledge multi-bracketing from one side (here, the left side), and above all because they possess no natural extension to the higher degrees. The notion of *symmetral* mould shows a way out.

Notion of symmetral mould:

$$\{A^\bullet \text{ symmetral}\} \;\Leftrightarrow\; \{ \sum_{\boldsymbol{\omega}\in \text{shuffle}(\boldsymbol{\omega^1},\boldsymbol{\omega^2})} A^{\boldsymbol{\omega}} \equiv A^{\boldsymbol{\omega^1}}\, A^{\boldsymbol{\omega^2}} \quad , \quad \forall \boldsymbol{\omega^1}\,,\, \boldsymbol{\omega^2}\ \} \tag{49}$$

$$\{A^\bullet \text{ symmetral}\} \;\Leftrightarrow\; \{ \sum_\bullet A^\bullet\, \Delta_\bullet \ \textit{ is a formal automorphism}\} \tag{50}$$

Natural projectors for symmetral moulds:

The construction is staightforward. It rests entirely on the *mould logarithm*, the *mould powers* of that logarithm, and their *symmetral linearisations* according to formula (49) :

$$\text{proj}_1\, A^\bullet \;:=\; \text{logmu}\, A^\bullet \;\in\; (1\text{-}\textit{alternal}) \tag{51}$$

$$= \sum_{1\leq n} \frac{(-1)^{n-1}}{n} \overbrace{(A^\bullet - 1^\bullet)\times\cdots\times(A^\bullet - 1^\bullet)}^{n\ \textit{times}} \tag{52}$$

$$(\textit{after symmetral linearisation})$$

$$\text{proj}_d\, A^\bullet \;:=\; \frac{1}{d!} \overbrace{\text{logmu}\, A^\bullet \times\cdots\times \text{logmu}\, A^\bullet}^{d\ \textit{times}} \tag{53}$$

$$(d\text{-}\textit{alternal}\ ,\ \textit{after symmetral linearisation})$$

The construction automatically fulfills conditions (43)-(44).

Natural projectors for general moulds:

We extend the above projectors to *all* moulds $A^\bullet$, irrespective of their symmetry type, by treating them *as if* they were symmetral.

In a way, this completely solves our problem, since it provides an effective construction for our natural projectors $proj_d$. But we ask for more — we want explicit formulae for the the structure coefficients $h_d(\sigma)$ attached to those projectors. That too can be had, but we must first introduce two auxiliary constructions.

3.3 Permutation algebra and permutation convolution.

Permutation shuffling:

If $\boldsymbol{\sigma'} \in \mathfrak{S}_{r'}$, $\boldsymbol{\sigma''} \in \mathfrak{S}_{r''}$, and $\boldsymbol{\sigma} \in \mathfrak{S}_{r'+r''}$, the relation

$$\boldsymbol{\sigma} \in \text{sha}(\boldsymbol{\sigma'},\boldsymbol{\sigma''}) \qquad \Big(\text{read : } < \boldsymbol{\sigma} \textit{ is a shuffling of } \boldsymbol{\sigma'} \textit{ and } \boldsymbol{\sigma''} > \Big) \tag{54}$$

will be taken to mean

$$[\boldsymbol{\sigma}(1),...,\boldsymbol{\sigma}(r'+r'')] \in \mathrm{sha}\Big(\underline{[\boldsymbol{\sigma'}(1),\ldots,\boldsymbol{\sigma'}(r')]},\overline{[r'+\boldsymbol{\sigma''}(1),\ldots,r'+\boldsymbol{\sigma''}(r'')]}\Big) \tag{55}$$

This leads to defining two dual products: a product $\overline{\times}$ acting on the symbols $Z^{\boldsymbol{\sigma}}$ and a convolution product $\underline{\times}$ acting on scalar functions of $\boldsymbol{\sigma}$:

$$Z^{\boldsymbol{\sigma'}}\,\overline{\times}\,Z^{\boldsymbol{\sigma''}} \;:=\; \sum_{\boldsymbol{\sigma}\in\mathrm{sha}(\boldsymbol{\sigma'},\boldsymbol{\sigma''})} Z^{\boldsymbol{\sigma}} \tag{56}$$

$$(f\,\underline{\times}\,g)(\sigma) \;:=\; \sum_{\boldsymbol{\sigma}\in\mathrm{sha}(\boldsymbol{\sigma'},\boldsymbol{\sigma''})} f(\boldsymbol{\sigma'})\,g(\boldsymbol{\sigma''}) \tag{57}$$

The $\overline{\times} \leftrightarrow \underline{\times}$ duality finds its reflection in the identity:

$$\Big(\sum_{\boldsymbol{\sigma'}} f(\boldsymbol{\sigma'})\;Z^{\boldsymbol{\sigma'}}\Big)\,\overline{\times}\,\Big(\sum_{\boldsymbol{\sigma''}} f(\boldsymbol{\sigma''})\;Z^{\boldsymbol{\sigma''}}\Big) \;\equiv\; \sum_{\boldsymbol{\sigma}}(f\,\underline{\times}\,g)(\boldsymbol{\sigma})\;Z^{\boldsymbol{\sigma}} \tag{58}$$

$\overline{\times}$ and $\underline{\times}$ are clearly associative and non-commutative.

3.4 Structure coefficients of the natural projectors.

If we set

$$\mathcal{Z} := 1 + \sum_{1\leq r} Z^{\mathrm{id}_r} \qquad \textit{with} \qquad \mathrm{id_r} \in \mathfrak{S}_r \tag{59}$$

then the above definition of the natural projectors translates to

$$\sum_{1\leq r}\sum_{\sigma\in\mathfrak{S}_r} \mathrm{h}_d(\sigma)\,Z^{\sigma} = \frac{1}{d!}\;(\log\mathcal{Z})^d \tag{60}$$

$\mathcal{Z}$ is a fairly trivial object, but *log*.$\mathcal{Z}$ and $(\textit{log}.\mathcal{Z})^d$ are not. Fortunately:

Fact: *The structure coefficients $h_d(\sigma)$ depend only on the degree d and the type (p,q) of σ^{-1}.*

Here, a permutation $\tau \in \mathbb{S}_r$ is said to be of *type* (p,q) if there are exactly p increases and q decreases in the sequence $(\tau(1),\ldots,\tau(r))$. Thus, $p{+}q = r{-}1$.

3.5 Typal algebra and typal convolution.

Since those symbols Z^{σ} (resp. functions $f(\sigma)$) that depend only on the *type* of σ^{-1} *do not* retain that property under $\overline{\times}$ (resp. $\underline{\times}$), there is no automatic extension of these products to type-dependent objects. However, the consideration, for each type (p,q), of the particular substitution $\sigma^+_{p,q}$ (resp. $\sigma^-_{p,q}$) whose inverse is of type (p,q) and first steadily increases, then steadily decreases (resp. does the opposite) suggests a satisfactory definition for the product of typal symbols $\mathbf{Z}^{p,q}$ and the convolution of typal functions $f(p,q)$. These definitions read:

$$\mathbf{Z}^{p_1,q_1} \,\overline{*}\, Z^{p_2,q_2} \;:=\; \delta(p_1)\,\mathbf{Z}^{p_1+p_2,1+q_1+q_2} + \mathbf{Z}^{1+p_1+p_2,q_1+q_2}\,\delta(q_2) \tag{61}$$

$$(f \,\underline{*}\, g)(p,q) \;:=\; \sum_{q_1+q_2=p-1}^{0\le q_1,0\le q_2} f(0,q_1)\,g(p,q_2) + \sum_{p_1+p_2=p-1}^{0\le p_1,0\le p_2} f(p_1,q)\,g(p_2,0) \tag{62}$$

with δ denoting the discrete dirac: $\delta(0)=1$ and $\delta(t)=0$ if $t\neq 0$.

Once again, both products $\overline{*}$ and $\underline{*}$ are associative and non-commutative, though this was not a foregone conclusion, in view in the latitude in the passage $(\overline{\times},\underline{\times}) \to (\overline{*},\underline{*})$.

Here are a few useful identities:

$$\overbrace{\mathbf{Z}^{0,0}\,\overline{*}\,\mathbf{Z}^{0,0}\,\overline{*}\,\ldots\mathbf{Z}^{0,0}}^{r\ times} \equiv \sum_{r_1+r_2=r-1}^{0\le r_1,0\le r_2} \frac{(r_1+r_2)!}{r_1!\,r_2!}\;\mathbf{Z}^{r_1,r_2} \tag{63}$$

$$\overbrace{f\,\underline{*}\,f\,\underline{*}\,\ldots f}^{p+q+1\ times} \equiv \Big(f(0,0)\Big)^{p+q+1} \tag{64}$$

$$\overbrace{f\,\underline{*}\,f\,\underline{*}\,\ldots f}^{k\ times} \equiv\; 0 \qquad \textit{if} \qquad k > 1+p+q \tag{65}$$

The upshot is that we can replace the *permutational convolution* (57) by the far simpler *typal convolution* (62) in the defining identities of the structure coefficients. Concretely, (59)-(60) gets replaced by (66)-(67):

$$\mathcal{Z} := 1 + \sum_{1\le r} \mathbf{Z}^{r-1,0} \tag{66}$$

$$\sum_{1\le r}\sum_{0\le p,0\le q} \mathrm{h}_d(p,q)\;\mathbf{Z}^{p,q} = \frac{1}{d!}\,(\log\mathcal{Z})^d \qquad \Big(h_d(\sigma) = h_d(p(\sigma^{-1}),q(\sigma^{-1}))\Big) \tag{67}$$

and the relative ease of calculation in the typal algebra shall lead us to very explicit, very convenient formulae for our structure coefficients $h_d(p,q)$.

3.6 The structure coefficients $h_d(p,q)$. Main statements.

For each pair $(p,q) \geq (0,0)$, consider the Laurent series

$$\mathcal{H}_{p,q}(t) := t\, e^{(p-q)\,t/2} \left(e^{t/2} - e^{-t/2}\right)^{-p-q-2} = \sum_{-p-q-1 \leq d < +\infty} h_d^{p,q}\, t^n \tag{68}$$

Their constant terms vanish: $h_0^{p,q} \equiv 0$. Their power series part (i.e. the positive powers of t) is interesting in its own right, but of no direct relevance to our problem. It is their Laurent part that matters to us, because it carries the sought-after structure coefficients $h_d(p,q)$. One can indeed show that:

$$h_d(p,q) \equiv \frac{1}{d!}\, h_{-d}^{p,q} \quad (1 \leq d \leq p+q+1) \tag{69}$$

In particular

$$h_1(p,q) = (-1)^q\, \frac{p!\; q!}{(p+q+1)!} \qquad , \qquad h_{p+q+1}(p,q) = \frac{1}{(p+q+1)!} \tag{70}$$

For the $p \leftrightarrow q$ exchange, the obvious symmetry relation holds

$$h_d(p,q) \equiv (-1)^{1+p+q-d}\, h_d(q,p) \tag{71}$$

3.7 Further properties.

For $1 \leq d \leq r$ let $h_d^r := [h_d(r-1,0), \ldots, h_d(r-1-i,i), \ldots, h_d(0,r-1)]$ and let $\mathbb{H}_r$ be the $r \times r$ matrix made up from the the rows h_d^r. Then the above symmetry relation becomes

$$\mathbb{H}_r||^{i,j} = (-1)^{r-i}\, \mathbb{H}_r||^{i,1+r-j} \tag{72}$$

More remarkably:

$$\det(\mathbb{H}_r) = \frac{1}{1!\,2!\,3!\ldots r!} \tag{73}$$

Furthermore, the inverse matrix $K_r := (\mathbb{H}_r)^{-1}$ has only integer entries, so-called *Eulerian numbers* in fact, which are explicitly given by

$$\mathbb{K}_r||^{i,j} = \sum_{1 \leq s \leq r} (-1)^{i-s}\, \frac{(r+s)!}{(i-s)!(1+r+s-i)!}\, s^j \tag{74}$$

We have a new symmetry relation

$$\mathbb{K}_r||^{i,j} \equiv \mathbb{K}_r||^{1+r-j,j} \tag{75}$$

that follows from the corresponding relation for $\mathbb{H}_r$ but isn't at all obvious from the formula (74) for $\mathbb{K}_r\|^{i,j}$.

Here are our two matrices for $r = 7$.

$$7!\,\mathbb{H}_7 \quad = \quad \begin{bmatrix} 720 & -120 & 48 & -36 & 48 & -120 & 720 \\ 1764 & -154 & 28 & 0 & -28 & 158 & -1764 \\ 1624 & 49 & -56 & 49 & -56 & 49 & 1624 \\ 735 & 140 & -35 & 0 & 35 & -140 & -735 \\ 175 & 70 & 7 & -14 & 7 & 70 & 175 \\ 21 & 14 & 7 & 0 & -7 & -14 & -21 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

$$\mathbb{K}_7 \quad = \quad \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ -6 & -4 & 0 & 8 & 24 & 56 & 120 \\ 15 & 5 & -9 & -19 & 15 & 245 & 1191 \\ -20 & 0 & 16 & 0 & -80 & 0 & 2416 \\ 15 & -5 & -9 & 19 & 15 & -245 & 1191 \\ -6 & 4 & 0 & -8 & 24 & -56 & 120 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

Decomposition into pure degree parts.

The tests performed so far suggest that remarkable moulds often (not always) tend to possess remarkable d-alternal natural projections.

Behaviour of the pure-degree parts under the brackets *lu* and *lo*.

The following parity-driven stability properties hold:

$$\mathrm{lu}(\mathrm{Alt}_{d_1}, \mathrm{Alt}_{d_2}) \quad \in \quad \mathrm{Alt}_{d_1+d_2-1} \oplus \mathrm{Alt}_{d_1+d_2-3} \oplus \mathrm{Alt}_{d_1+d_2-5} \oplus \dots \qquad (76)$$
$$\mathrm{lo}(\mathrm{Alt}_{d_1}, \mathrm{Alt}_{d_2}) \quad \in \quad \mathrm{Alt}_{d_1+d_2-1} \oplus \mathrm{Alt}_{d_1+d_2-3} \oplus \mathrm{Alt}_{d_1+d_2-5} \oplus \dots \qquad (77)$$

When either d_1 or d_2 (or both) is $= 1$, the above inclusions simplify to:

$$\mathrm{lu}(\mathrm{Alt}_{d_1}, \mathrm{Alt}_{d_2}) \quad \in \quad \mathrm{Alt}_{d_1+d_2-1} \qquad (78)$$
$$\mathrm{lo}(\mathrm{Alt}_{d_1}, \mathrm{Alt}_{d_2}) \quad \in \quad \mathrm{Alt}_{d_1+d_2-1} \qquad (79)$$

3.8 When does a permutation subalgebra reduce to a typal subalgebra?

If we ask for subalgebras whose elements $f(\sigma)$ depend only on the type of σ, then the answer is both simple and uninteresting: $f(\sigma)$ must depend on the sole length $r := p + q + 1$ of the permutation.

If on the other hand we ask for subalgebras whose elements $f(\sigma)$ depend only on the type of σ^{-1}, then the answer is that all maximal, type-dependent, permutation subalgebras depend on a single parameter c and their elements are all of the form:[7]

$$f(\sigma) = a_1\, h^{\underline{\times} 1}(\sigma) + a_2\, h^{\underline{\times} 2}(\sigma) + a_3\, h^{\underline{\times} 3}(\sigma) + \dots \quad (\forall a_n \in \mathbb{C}) \tag{80}$$

with $h(\sigma) := (-1)^q\, c^{p+q+1}\, p!\, q!/(p+q+1)!$ and $h^{\underline{\times} n}$ standing for the n^{th}-convolution power of h. Thus, the homogeneousness parameter c aside, we fall back on the typal algebras of §3.5.

3.9 Remarks by J.-Y. Thibon.

J.-Y. Thibon pointed out to us that, in the free algebra setting, the notion of *natural projectors* was already known, as well as the indirect analytical expression of their structure coefficients $h_d(\sigma) = h_d(p, q)$ in terms of the $\mathbb{K}_r$ matrices (made up of Eulerian numbers).

However, the direct expression of $h_d(\sigma)$ in terms of the matrices $\mathbb{H}_r = (\mathbb{K}_r)^{-1}$ (made up of 'pseudo-Bernoulli' numbers) appears to be new, and so too the expression of the $\mathbb{H}_r$-entries $h_d(p, q)$ in terms of the generating function (68)-(69). In our approach, moreover, the emphasis is on the moulds and their d-alternal components, rather than on the free enveloping algebras and their d-degree components.

Regarding the *typal algebra* and *typal convolution* of §3.5, J.-Y. Thibon drew our attention to the existence of a more general construction, based not on the pairs (p, q), but on a finer analysis of the permutations σ that takes into account the lenghts $(p_1, q_1, p_2, q_2, ...)$ of the alternating increasing-decreasing subsequences inside $(\sigma(1), ..., \sigma(r))$.

4 Minimal convolution domains.

4.1 Motivation. Characterisation and construction of convolution domains.

If two germs $\widehat{\varphi}_1$ and $\widehat{\varphi}_2$ at $0_\bullet$ (ramified origin of $\mathbb{C}_\bullet = \widetilde{\mathbb{C} - \{0\}}$) have the property of *endless analytic continuation* over $\mathbb{C}_\bullet$, then so does the convolution product $\widehat{\varphi}$:

$$\widehat{\varphi} = \widehat{\varphi}_1 * \widehat{\varphi}_2 \qquad \Leftrightarrow \qquad \widehat{\varphi}(\zeta) \overset{locally}{:=} \int_{0_\bullet}^{\zeta} \widehat{\varphi}(\zeta_1)\, \widehat{\varphi}(\zeta - \zeta_1)\, d\zeta_1 \tag{81}$$

[7]No convergence problem in (80), thanks to (65) or rather the analogue of (65) in the permutation algebra.

Moreover, two singularities somewhere *over* ω_1 and ω_2 respectively for $\widehat{\varphi}_1$ and $\widehat{\varphi}_2$ will necessarily result in singularities *over* $\omega_1 + \omega_2$ for $\widehat{\varphi}$. But where exactly? On which Riemann sheets?

In theory, the notion of SSS-path (*self-symmetrical and self-symmetrically shrinkable* path, along which the convolution integral (81) has to be calculated), provides an answer, but an impractical one, since when a path Γ grows in length, the shortest SSS-path Γ_* equivalent to Γ tends to grow incredibly faster. So an alternative machinery — a transparent algebraic-combinatorial machinery — is called for.

4.2 Atomic singularities and atomic alien operators.

Alongside the usual alien operators, *standard* and *lateral*, which possess convenient co-products but only measure *averages* of singularities, we must consider *atomic alien operators*, which measure isolated singularities but possess less transparent co-products. We then introduce three parallel systems of dual notions – the so-called *pseudo-variables* or *symbolic singularities* – endowed with a *product* or *multiplication* (dual to the alien operators' co-product) that will adequately reflect, in the atomic case, the composition of singularities under convolution. So we have to juggle these six systems:

	Standard basis	*Lateral basis*	*Atomic system*
Alien operators	$\{\Delta_{\omega_1,\ldots,\omega_r}\}$	$\{\Delta^+_{\omega_1,\ldots,\omega_r}\}$	$\{D_{\omega_1,\ldots,\omega_r}\}$
Singularity symbols	$\{Z^{\omega_1,\ldots,\omega_r}\}$	$\{Z_+^{\omega_1,\ldots,\omega_r}\}$	$\{S^{\omega_1,\ldots,\omega_r}\}$

with $\omega_i \in \mathbb{C}_\bullet := \widetilde{\mathbb{C} - \{0\}}$, $\epsilon_i \in \{+,-\}$. Mark that in the general case, the indices ω_i are in $\mathbb{C}_\bullet$, not $\mathbb{C}^*$. In the atomic system, the sequence $(\omega_1, \ldots, \omega_r)$ represents the *taut* (i.e. shortest) *broken line*[8] Γ connecting a given ramification point with the origin $0_\bullet \in \mathbb{C}_\bullet$. Moreover, when two consecutive ω_i and ω_{i+1} are *co-axial* or *'aligned'* (again, on $\mathbb{C}_\bullet$, not $\mathbb{C}$), one must interpose a sign $\epsilon \in \{+,-\}$ between them to specify *how* the *taut broken line* Γ bypasses the singularity — whether to the right or to the left.

So we have two very distinct difficulties here — those that arise from winding broken lines and their possible self-crossings, and those that stem

[8] If $\frac{1}{2\pi}(\arg\omega_{i+1} - \arg\omega_i) \in [n-\frac{1}{2}, n+\frac{1}{2}[$ resp. $]-n-\frac{1}{2}, -n+\frac{1}{2}]$ with $n \in \mathbb{N}^*$, that should be taken to mean that our broken line Γ turns n times, in the positive resp. negative direction, round the relevant turning point located somewhere over $\omega_{i,*} := \dot{\omega}_1 + \ldots + \dot{\omega}_i$. If $\frac{1}{2\pi}(\arg\omega_{i+1} - \arg\omega_i) \in]-\frac{1}{2}, 0[\cup]0, \frac{1}{2}[$, there is no local self-crossing of Γ at $\omega_{i,*}$. Lastly, if $\arg\omega_{i+1} = \arg\omega_i$, we must interpose a sign ϵ_i between $\arg\omega_i$ and $\arg\omega_{i+1}$ to specify whether Γ circumvents $\omega_{i,*}$ to the right or to the left.

from the presence of subsequences made up of co-axial ω_i's. The two difficulties neatly separate, and, somewhat counter-intuitively, the first one turns out to be the more tractable of the two. So here we shall address the main difficulty, arising from co-axial singularities.

4.3 The axial case.

For convenience, we assume our axis to be $\mathbb{R}^+$. Our six systems become:

	Standard basis	*Lateral basis*	*Atomic system*
Alien operators	$\{\Delta_{\omega_1,\dots,\omega_r}\}$	$\{\Delta^+_{\omega_1,\dots,\omega_r}\}$	$\{D_{\omega_1,\epsilon_1,\omega_2,\dots,\epsilon_{r-1},\omega_r}\}$
Singularity symbols	$\{Z^{\omega_1,\dots,\omega_r}\}$	$\{Z_+^{\omega_1,\dots,\omega_r}\}$	$\{S^{\omega_1,\epsilon_1,\omega_2,\dots,\epsilon_{r-1},\omega_r}\}$

with $\omega_i \in \mathbb{R}^+$, $\epsilon_i \in \{+,-\}$. We require all six conversion rules. Those between *standard* and *lateral* readily follow from the identities:

$$\sum_{0<\omega} \Delta_\omega = \log\big(1 + \sum_{0<\omega} \Delta^+_\omega\big) \tag{82}$$

$$1 + \sum_{0<\omega} \Delta^+_\omega = \exp\big(\sum_{0<\omega} \Delta_\omega\big) \tag{83}$$

The conversion rules between *lateral* to *atomic* are quite elementary.

That leaves only the case *standard* $\leftrightarrow$ *atomic.*

• **The conversion rule from** *atomic* **to** *standard***, expressed for convenience in terms of the** *singularity symbols***, reads:**

$$S^{\binom{\epsilon}{\omega}} = \sum_{\binom{\epsilon}{\omega}=\binom{\boldsymbol{\epsilon}^{s_1}}{\boldsymbol{\omega}^{s_1}}\epsilon_{s_1}\binom{\boldsymbol{\epsilon}^{s_2}}{\boldsymbol{\omega}^{s_2}}\epsilon_{s_2}\dots\binom{\boldsymbol{\epsilon}^{s_k}}{\boldsymbol{\omega}^{s_k}}}^{0\le k\le r} \epsilon_{s_1}\dots\epsilon_{s_k}\, \Lambda^{\boldsymbol{\epsilon}^1}\dots\Lambda^{\boldsymbol{\epsilon}^k}\;\; Z^{|\boldsymbol{\omega}^1|,\dots,|\boldsymbol{\omega}^k|} \tag{84}$$

$$\textit{with}\quad \Lambda^{\epsilon_1,\dots,\epsilon_n} = \lambda^{p,q} = \frac{p!\;q!}{(p+q+1)!} \quad \Big(p := \sum_{\epsilon_i=+} 1 \;\;,\;\; q := \sum_{\epsilon_i=-} 1\Big) \tag{85}$$

In formula (84), each $\binom{\boldsymbol{\epsilon}^{s_k}}{\boldsymbol{\omega}^{s_k}}$ represents an alternate scalar/sign sequence that necessarily begins with some scalar ω_* and ends with some other scalar ω_{**}. The lower-indexed, intermediary ϵ_{s_k}, on the other hand, are *single signs*, whereas the upper-indexed, bold-face $\boldsymbol{\epsilon}^{s_k}$ are *sign sequences.*

• **The reverse conversion, from *standard* to *atomic*, reads:**

$$Z^{\omega_1,\dots,\omega_r} = \sum_{\epsilon_i\in\{+,-\}} \frac{1}{r!}\, \Xi^{\epsilon_1,\dots,\epsilon_{r-1}}\; S^{\omega_1,\epsilon_1,\omega_2,\epsilon_2,\dots,\epsilon_{r-1},\omega_r} \tag{86}$$

with sign-indexed coefficients $\Xi^\bullet$ that reduce to integer-indexed, integer-valued coefficients $\xi^\bullet$:

$$\Xi^{\epsilon_1,\dots,\epsilon_n} = \epsilon_1\dots\epsilon_n\,|\Xi^{\epsilon_1,\dots,\epsilon_n}| \tag{87}$$

$$|\Xi^{\epsilon_1,\dots,\epsilon_n}| = \xi^{n_1,\dots,n_s} \quad \textit{if} \quad (\epsilon_1,\dots,\epsilon_n) = (\pm)^{n_1}(\mp)^{n_2}\dots(\#)^{n_s}\,\Big(n_i>0\Big)$$

The coefficients $\xi^\bullet$ in turn are defined by the induction

$$\xi^{n_1} = 1 \tag{88}$$

$$\xi^{n_1,\dots,n_s} = \sum_{1\le j\le s} \widehat{\xi}^{\,n_1,\dots,n_j-1,\dots,n_r} \tag{89}$$

with the convention

$$\widehat{\xi}^{\,n_1,\dots,n_j-1,\dots,n_r} := \xi^{\,n_1,\dots,n_j-1,\dots,n_r} \quad \textit{if} \quad n_j\ge 2 \tag{90}$$

$$:= \xi^{\,n_1,\dots,n_{j-1}+n_{j+1},\dots,n_r} \quad \textit{if} \quad n_j = 1 \tag{91}$$

Thus we get $\xi^{n_1}\equiv 1$, $\xi^{n_1,n_2}\equiv\frac{(n_1+n_2)!}{n_1!\,n_2!}$, $\xi^{n_1,\dots,n_r}\equiv\xi^{n_r,\dots,n_1}$ and:

$$\begin{array}{lllll}
\xi^{2,1,1}=9 & \xi^{1,2,1}=11 & \dots & \dots & \xi^{1,1,1,1}=16\\
\xi^{3,1,1}=14 & \xi^{2,2,1}=26 & \xi^{2,1,2}=19 & \dots & \xi^{1,1,1,1,1}=61\\
\xi^{4,1,1}=20 & \xi^{3,2,1}=50 & \xi^{2,2,2}=70 & \dots & \xi^{1,1,1,1,1,1}=272\\
\xi^{\{1\}^7}=1385 & \xi^{\{1\}^8}=7936 & \xi^{\{1\}^9}=50520 & \dots &
\end{array}$$

4.4 Multiplication of axial singularities.

The algebra *Singax* of axial singularities is spanned by the symbols

$$S^{\binom{\epsilon}{\omega}} = S^{\omega_1\,,\,\epsilon_1\,,\,\omega_2\,,\,\epsilon_2\,,\,\omega_3,\dots,\omega_{r-1}\,,\,\epsilon_{r-1}\,,\,\omega_r} \qquad \Big(\omega_i\in\mathbb{R}^+\ ,\ \epsilon_i\in\{+,-\}\Big) \tag{92}$$

modulo the equivalence relations

$$S^{\dots,\omega_i\,,\,+\,,\,\omega_{i+1},\dots} + S^{\dots,\omega_i\,,\,-\,,\,\omega_{i+1},\dots} = S^{\dots,\omega_i+\omega_{i+1},\dots} \tag{93}$$

whose geometric interpretation is self-evident.

Lemma: *The following multiplication rule on the singularity symbols $S^\bullet$*

$$S^{\binom{\epsilon'}{\omega'}} . S^{\binom{\epsilon''}{\omega''}} \quad = \quad \sum_{\boldsymbol{\omega} \,\in\, \text{shuffle}(\boldsymbol{\omega'},\boldsymbol{\omega''})} \big(\prod \epsilon'_i \prod \epsilon''_i \prod \epsilon_i\big) \quad S^{\binom{\epsilon}{\omega}} \tag{94}$$

$$\begin{aligned}
\epsilon_i &:= \epsilon'_j \quad \textit{if} \quad (\omega_i, \omega_{i+1}) = (\omega'_j, \omega'_{j+1})\\
\epsilon_i &:= \epsilon''_j \quad \textit{if} \quad (\omega_i, \omega_{i+1}) = (\omega''_j, \omega''_{j+1})\\
\epsilon_i &:= + \quad \textit{if} \quad (\omega_i, \omega_{i+1}) = (\omega'_j, \omega''_k)\\
\epsilon_i &:= - \quad \textit{if} \quad (\omega_i, \omega_{i+1}) = (\omega''_k, \omega'_j)
\end{aligned}$$

is compatible[9] *with the equivalence relation (93) and faithfully reflects the behaviour of atomic singularities under convolution.*

4.5 Three options, none perfect.

Singularity composition is clearly three things:
(i) associative
(ii) commutative
(iii) entire

The last point means that, under convolution, a simple pole with residue 1 somewhere over ω_1 composed with a simple pole with residue 1 somewhere over ω_2 can only produce simple poles with residues $n(\omega) \in \mathbb{Z}$ at various points ω over $\omega_1 + \omega_2$, with $n(\omega)$ effectively depending on ω.

The product on *Singax*, of course, reflects these three properties, but only *modulo the equivalence relations* (93). The funny thing is that there is no way to define the *Singax*-product so as to make all three properties *simultaneously manifest*: turn it or twist it, at least one of the three must remain hidden. In fact, the choice is between three main options:

(i) **Option 1**: We define the *Singax*-product as in (94). Commutativity is hidden, associativity and entireness manifest.

(ii) **Op[tion 2**: We define the *Singax*-product as the half-sum

$$\frac{1}{2}\left(S^{\binom{\epsilon'}{\omega'}} . S^{\binom{\epsilon''}{\omega''}} + S^{\binom{\epsilon''}{\omega''}} . S^{\binom{\epsilon'}{\omega'}}\right) \tag{95}$$

[9] i.e. the subspace spanned by the elements of the form

$$S^{\dots,\omega_i,+,\omega_{i+1},\dots} + S^{\dots,\omega_i,-,\omega_{i+1},\dots} - S^{\dots,\omega_i+\omega_{i+1},\dots}$$

is an ideal relative to the product (94).

with each product calculated as in (94). That (almost) preserves the entire character and restores manifest commutativity, but at the cost of associativity, which on balance matters more.

(ii) **Op[tion 3**: We define the *Singax*-product based on the conversion rule from *atomic* to *standard*: see *infra.* That ensures manifest associativity and manifest communitativity (and this is indeed *the only way of getting both*), but at the cost of entireness and simplicity: not only are the new formulae ridden with complicated rational coefficients, but they also involve an absurdly large number of summands.

To sum up the three trade-offs:

	associativity	commutativity	entireness
option 1	*manifest*	*hidden*	*manifest*
option 2	*hidden*	*manifest*	*semi-manifest*
option 3	*manifest*	*manifest*	*hidden*

Proofs. Establishing the rules for the *Singax*-product corresponding to Option 1 (resp. Option 3) is a three-stepped affair:
(i) changing from the *atomic* to the *lateral* (resp. *standard*) system.
(ii) expressing the product of the singularity symbols in that system.
(iii) reverting to the atomic system.

Option 2, lastly, is entirely derivative on Option 1.

Analytical expression of singlarity composition.
It is encapsulated in a simple matrix identity

$$S^{\binom{\epsilon^1}{\omega^1}}.S^{\binom{\epsilon^2}{\omega^2}} = \overline{\overline{S}}_{r_1+r_2}.M^{\,\boldsymbol{\epsilon^1};\boldsymbol{\epsilon^2}}.\underline{\underline{S}}^{\,\boldsymbol{\omega^1};\boldsymbol{\omega^2}} \tag{96}$$

whose interpretation is as follows:

• $S^{\binom{\epsilon^1}{\omega^1}}$ and $S^{\binom{\epsilon^2}{\omega^2}}$ stand for singularities of depth r_1 and r_2, with sequences $\boldsymbol{\omega^1}$, $\boldsymbol{\omega^2}$ of length r_1, r_2 and sign sequences $\boldsymbol{\epsilon^1}$, $\boldsymbol{\epsilon^2}$ of length r_1-1, r_2-1.

• $M^{\,\boldsymbol{\epsilon^1};\boldsymbol{\epsilon^2}}$ is a matrix with $2^{r_1+r_2-1}$ rows and $\frac{(r_1+r_2)!}{r_1!r_2!}$ columns

• The matrices $M_I^{\,\boldsymbol{\epsilon^1};\boldsymbol{\epsilon^2}}$, $M_{II}^{\,\boldsymbol{\epsilon^1};\boldsymbol{\epsilon^2}}$, $M_{III}^{\,\boldsymbol{\epsilon^1};\boldsymbol{\epsilon^2}}$ relative to option I, II, III have entries respectively in $\mathbb{Z}$, $(1/2)\,\mathbb{Z}$, $(1/(r_1+r_2)!)\,\mathbb{Z}$, but the entries in each row always add up to an integer – the same in all three cases (see the red columns in the examples below).

• $\overline{\overline{S}}_{r_1+r_2}$ denotes the row with $2^{\,r_1+r_2-1}$ entries of the form $\overline{S}^{\,\boldsymbol{\epsilon}} = \overline{S}^{\,\epsilon_1,\dots,\epsilon_{r_1+r_2-1}}$, arranged in lexicographic order, with the sign sequences $\boldsymbol{\epsilon}$ assuming all possible values.

• $\underline{\underline{S}}^{\,\boldsymbol{\omega^1};\boldsymbol{\omega^2}}$ denotes the column with $\frac{(r_1+r_2)!}{r_1!r_2!}$ entries $\underline{S}^{\boldsymbol{\omega}}$, where ω runs through all shufflings of $\boldsymbol{\omega^1}$ and $\boldsymbol{\omega^2}$, taken in lexicographic order.

• When expanding the right-hand side of (96), each product $\overline{S}^{\,\boldsymbol{\epsilon}}.\underline{S}^{\,\boldsymbol{\omega}}$ must be re-interpreted as the singularity symbol $S^{\binom{\boldsymbol{\epsilon}}{\boldsymbol{\omega}}} = S^{\omega_1,\epsilon_1,\omega_2,\epsilon_2,\dots,\omega_{r_1+r_2}}$.

Here are two examples, corresponding to the case $(r_1, r_2) = (2, 2)$, first for $(\boldsymbol{\epsilon^1};\boldsymbol{\epsilon^2}) = ((+);(+))$, then $(\boldsymbol{\epsilon^1};\boldsymbol{\epsilon^2}) = ((+);(-))$. All three matrices $M_{I}^{\,\boldsymbol{\epsilon^1};\boldsymbol{\epsilon^2}}$, $M_{II}^{\,\boldsymbol{\epsilon^1};\boldsymbol{\epsilon^2}}$, $M_{III}^{\,\boldsymbol{\epsilon^1};\boldsymbol{\epsilon^2}}$ are listed side by side for comparison,

$$M_{I}^{(+);(+)} = \left|\begin{array}{cccccc|c} 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 & 0 & 0 & -1 \\ 0 & -1 & 0 & 0 & -1 & 0 & -2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right.$$

$$M_{II}^{(+);(+)} = \frac{1}{2} \left|\begin{array}{rrrrrr|r} 1 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & -1 & -1 & 0 & 0 & -1 \\ -1 & -1 & 0 & 0 & -2 & 0 & -2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & -1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right.$$

$$M_{III}^{(+);(+)} = \frac{1}{24} \left|\begin{array}{rrrrrr|r} 8 & 1 & 3 & 3 & 1 & 8 & 1 \\ -2 & -3 & -7 & -7 & -3 & -2 & -1 \\ -16 & -5 & -3 & -3 & -5 & -16 & -2 \\ -2 & 3 & -1 & -1 & 3 & -2 & 0 \\ -2 & -3 & -7 & -7 & -3 & -2 & -1 \\ 0 & 5 & 7 & 7 & 5 & 0 & 1 \\ -2 & 3 & -1 & -1 & 3 & -2 & 0 \\ 0 & -1 & 1 & 1 & -1 & 0 & 0 \end{array}\right.$$

$$M_{I}^{(+);(-)} = \left|\begin{array}{cccccc|c} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & -1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right.$$

$$M_{II}^{(+);(-)} = \frac{1}{2} \left|\begin{array}{cccccc|c} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 \\ -1 & 0 & -1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & -1 & 0 & 0 & -1 & 0 & -1 \\ 0 & 0 & -1 & 0 & 0 & -1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right.$$

$$M_{III}^{(+);(-)} = \frac{1}{24} \left|\begin{array}{rrrrrr|r} 2 & -1 & 1 & -3 & -1 & 2 & 0 \\ 12 & 3 & -1 & 7 & 3 & 0 & 1 \\ 2 & 5 & 7 & 3 & 5 & 2 & 1 \\ -12 & -3 & -7 & 1 & -3 & 0 & -1 \\ 0 & 3 & -1 & 7 & 3 & 12 & 1 \\ -2 & -5 & -3 & -7 & -5 & -2 & -1 \\ 0 & -3 & -7 & 1 & -3 & -12 & -1 \\ -2 & 1 & 3 & -1 & 1 & -2 & 0 \end{array}\right.$$

4.6 Mathematical status of the algebra *Singax* .

(i) Doesn't appear to have been ever considered as a tool for modeling the behaviour of singularities under convolution or for characterising the *minimal convolution domains.*
(ii) However, the rule for singularity composition, especially in the simple form (94) corresponding to Option 1, seems too basic an algebraic object to have altogether escaped notice. Our hunch is that it must have already cropped up in *other* mathematical contexts. But which ones? Suggestive of what unsuspected linkages?

4.7 The algebra *Sing* of general atomic singularities above $\mathbb{C}_\bullet$

As pointed out, it is but a short step from understanding the composition of *axial atomic singularities* to mastering that of *general atomic singularities.* So we shall be content here with a few hints. The symbols for general atomic singularities are of the form:

$$S^{\omega_1,\ldots,\omega_r} \quad , \quad D_{\omega_1,\ldots,\omega_r} \quad , \quad \big(\omega_i \in \mathbb{C}_\bullet := \widetilde{\mathbb{C} - \{0\}}\big) \tag{97}$$

Consecutive indices ω_i, ω_{i+1} are separated by an $\epsilon_i \in \{+,-\}$ *only* if they are aligned, i.e. if $\arg \omega_i$ and $\arg \omega_{i+1}$ are equal — as elements of $\mathbb{R}$, not $\mathbb{R}/(2\pi\mathbb{Z})$.

Key to the discussion is the decomposition of the atomic operators in blocks of standard, higher-order alien derivations *and* of the rotation[10] R:

$$D_{\boldsymbol{\omega}} = \sum_{n_1 \le n \le n_2} R^n \sum_{*} c_{n,\boldsymbol{\omega}}\, \Delta_{\boldsymbol{\omega}} = \sum_{n_1 \le n \le n_2} R^n \sum_{*} c^{+}_{n,\boldsymbol{\omega}}\, \Delta^{+}_{\boldsymbol{\omega}} \quad \big(c_{n,\boldsymbol{\omega}} \in \mathbb{Q}\big) \tag{98}$$

The bounds (n_1, n_2) are the same in both sums, and $n_1 \le 0 \le n_2$. As for the number $n_2 - n_1 - 1$ of distinct secondary $\sum_*$-blocks in (98) that are preceded by effective rotations R^n $(n \ne 0)$, that number depends in a rather interesting way on the self-crossing pattern of the broken line Γ encoded by the sequence $\boldsymbol{\omega}$.

4.8 Minimal convolution domains.

The above rules for the composition of symbolic atomic singularities are all we need to characterise/describe/construct all minimal convolution domains, and this seems to be the only workable approach (except in the relatively simple case when all singularities lie *over* a lattice $\tau_1\mathbb{Z}$ or $\tau_1\mathbb{Z} + \tau_2\mathbb{Z}$).

4.9 Example: the 'arithmetical axis' $\overline{\mathcal{R}_{\ln \mathbb{P}}}$.

Let $\mathbb{P}$ be the set of all prime integers $\ne 1$ and $\ln \mathbb{P}$ its image under $\ln = \log$. Let us call *'arithmetical axis'* and denote by $\overline{\mathcal{R}_{\ln \mathbb{P}}}$ the minimal convolution domain *generated*[11] or *spawned*[12] by $\mathcal{R}_{\ln \mathbb{P}} := \widetilde{\mathbb{C} - \ln \mathbb{P}}$.

This 'arithmetical axis' $\overline{\mathcal{R}_{\ln \mathbb{P}}}$ is a highly non-trivial *'sub-surface'* of $\mathcal{R}_{\ln \mathbb{N}} := \widetilde{\mathbb{C} - \ln \mathbb{N}}$. The fact is that $\overline{\mathcal{R}_{\ln \mathbb{P}}}$ has far more Riemann sheets than $\mathcal{R}_{\ln \mathbb{P}}$,

[10] R stands for a one-turn local rotation (round $0_\bullet$) in the Borel plane.
[11] a somewhat improper term in this context.
[12] a mathematically less loaded word.

but far fewer than $\mathcal{R}_{\ln\mathbb{N}} \equiv \overline{\mathcal{R}_{\ln\mathbb{N}}}$. The 'arithmetical axis' has indeed singularities *over* each $\ln n \in \ln\mathbb{N}$ but, especially when n is highly divisible, of the total number 2^{n-1} of forward paths that connect $0_\bullet$ with $\ln n$ on $\mathbb{C} - \ln\mathbb{N}$, only slighthly more than half have an actual singularity of $\overline{\mathcal{R}_{\ln\mathbb{P}}}$ as their end-point.

For definiteness, let us sort out the case of a quadrat-frei integer n with prime factoration $n = p_1 \dots p_r$. To that end, let us compose the elementary singularities $S^{\ln p_1} \dots S^{\ln p_1}$ according to the procedure of Option 1 first, then of Option 3. We find:

Option 1

$$S^{\ln p_1} \dots S^{\ln p_r} = \sum_{\sigma\in\mathfrak{S}_r} \epsilon_1^\sigma \dots \epsilon_{r-1}^\sigma \;\; S^{\ln p_{\sigma(1)},\epsilon_1^\sigma,\ln p_{\sigma(2)},\epsilon_2^\sigma,\dots,\ln p_{\sigma(r)}} \tag{99}$$

with

$$\epsilon_i^\sigma \;:=\; + \quad \textit{if} \quad \sigma(i) < \sigma(i+1) \tag{100}$$

$$\epsilon_i^\sigma \;:=\; - \quad \textit{if} \quad \sigma(i) > \sigma(i+1) \tag{101}$$

Option 3

$$S^{\ln p_1} \dots S^{\ln p_r} = \frac{1}{r!} \sum_{\sigma\in\mathfrak{S}_r} \sum_{\epsilon_i\in\{+,-\}} \Xi^{\epsilon_1,\dots,\epsilon_{r-1}} S^{\ln p_{\sigma(1)},\epsilon_1 \ln p_{\sigma(2)},\epsilon_2,\dots,\ln p_{\sigma(r)}} \tag{102}$$

with the integer-valued coefficients $\Xi^\bullet$ defined as in (87).

Next, to accommodate and compare all these summands and register the effect of possible – in fact, very numerous – cancellations, we must re-write the sums (99) and (102) in terms of the *position-based* singularity symbols $S^{\overline{\bullet}}$, which relate to the *interval-based* symbols $S^\bullet$ as follows[13]:

$$S^{\omega_1,\epsilon_1,\omega_2,\epsilon_2,\omega_3,\epsilon_3,\dots} \;=\; S^{\overline{\omega_1,\epsilon_1,\omega_1+\omega_2,\epsilon_2,\omega_1+\omega_2+\omega_3,\epsilon_3,\dots}} \tag{103}$$

$$S^{\overline{\eta_1,\epsilon_1,\eta_2,\epsilon_2,\eta_3,\epsilon_3,\dots}} \;=\; S^{\eta_1,\epsilon_1,\eta_2-\eta_1,\epsilon_2,\eta_3-\eta_2,\epsilon_3,\dots} \tag{104}$$

The total configuration of singularity symbols now assumes the form:

$$S^{\ln p_1} \dots S^{\ln p_r} = \sum_{\epsilon_{d_1},\dots,\epsilon_{d_\rho}\in\{+,-\}}^{d_i|n \;\; (d_i\neq 1,n)} \Theta^{\epsilon_{d_1},\dots,\epsilon_{d_\rho}} \;\; S^{\overline{\ln d_1,\epsilon_{d_1},\dots,\ln d_\rho,\epsilon_{d_\rho},\ln n}} \tag{105}$$

[13]It is only in the last stages of singularity analysis, for convenient comparison and to arrive at the total picture, that one should resort to the position-based symbols $S^{\overline{\bullet}}$. By themselves these $S^{\overline{\bullet}}$ obey no simple multiplication rule, so that in the early stages one has to work with the interval-based symbols $S^\bullet$,

with all $\rho := 2^r - 2$ strict divisors d_i of n simultaneously coming into play, so that no further cancellations can take place.[14] Here are the values of $\Theta^\bullet$, calculated successively by the first and third method.

Option 1

$$\Theta^{\epsilon_{d_1},\ldots,\epsilon_{d_\rho}} = \sum_{\sigma\in\mathfrak{S}_r} \gamma_1(\sigma,\boldsymbol{\epsilon_d})\,\gamma_2(\sigma,\boldsymbol{\epsilon_d})\ \ldots\ \gamma_{r-1}(\sigma,\boldsymbol{\epsilon_d}) \tag{106}$$

with

$$\gamma_i(\sigma,\boldsymbol{\epsilon_d}) \quad = \quad \begin{cases} +1 & \textit{if} \quad \epsilon_{p_{\sigma(1)}\cdots p_{\sigma(i)}} = + \quad \textit{and} \quad \sigma(i) < \sigma(i{+}1) \\ -1 & \textit{if} \quad \epsilon_{p_{\sigma(1)}\cdots p_{\sigma(i)}} = - \quad \textit{and} \quad \sigma(i) > \sigma(i{+}1) \\ 0 & \textit{otherwise} \end{cases}$$

Option 3

$$\Theta^{\epsilon_{d_1},\ldots,\epsilon_{d_\rho}} = \frac{1}{r!}\sum_{\sigma\in\mathfrak{S}_r}\ \sum_{\epsilon_{d_i}\in\{+,-\}} \Xi^{\epsilon_{p_{\sigma(1)}},\epsilon_{p_{\sigma(1)}p_{\sigma(2)}},\ldots,\epsilon_{p_{\sigma(1)}\cdots p_{\sigma(i-1)}}} \tag{107}$$

Both formulae (106) and (107) yield the same (integer) value for $\Theta^\bullet$. Though (106) is by far the simpler of the two, formula (107) is not entirely without merit either. It shows in particular that $|\Theta^{\epsilon_{d_1},\ldots,\epsilon_{d_\rho}}|$ reaches its maximum[15] when each sign ϵ_d is $+$ or $-$ depending on the parity of the number of primes p_i involved in the factorisation of the divisor d.

[14]Inside $\overline{S^\bullet}$, the logatithms $\ln d_i$ of the various divisors must of course appear in their proper order, which order depends on n, but this is immaterial, since neither the crucial coefficients $\Theta^\bullet$ nor the underlying combinatorics depend on that order.

[15]that maximum is none other than $\xi^{1,\ldots,1}$, with 1 repeated $r{-}1$ times: see towards the end of §4.3.

5 Isodifferential operators and their natural basis.

5.1 The bialgebra *ISO* of iso-differential operators.

An *iso-differential operator* or *iso-differentiation* of iso-degree n is an operator of the form:

$$Df := \sum_{1\leq r\leq n} \sum_{1\leq n_i}^{n_1+\ldots n_r=n} a_{n_1,\ldots,n_r}\, H^{(n_1)} \ldots H^{(n_r)} \quad \textit{with} \quad H = \log(1/f') \tag{108}$$

$$:= \sum_{1\leq r\leq n} \sum_{1\leq n_i}^{n_1+\ldots n_r=n} b_{n_1,\ldots,n_r} \frac{f^{(1+n_1)}}{f'} \ldots \frac{f^{(1+n_r)}}{f'} \tag{109}$$

These operators are uniquely adapted to *functional composition* in general, and more particularly to the description of the so-called *'universal asymptotics'* since they always produce the same asymptotic series when made to act on ultra-slow germs.

Due to their double homogeneousness (– the *iso* part of their name alludes to that –) they are essentially invariant under pre- and post-composition by simulitudes S:

$$D(S\circ f) \equiv Df \qquad ; \qquad D(f\circ S) \equiv \alpha^n \qquad (S(z) = \alpha z + \beta) \tag{110}$$

They also generate an interesting bialgebra, since they possess
(i) a commutative product $\times$, *distinct* from the non-commutative operator composition and additive with respect to the iso-degree:

$$(D_1 \times D_2)\, f \;:=\; (D_1\, f).(D_2\, f) \tag{111}$$

$$\mathrm{ideg}(D_1\times D_2) \;=\; \mathrm{ideg}(D_1) + \mathrm{ideg}(D_2) \tag{112}$$

(ii) a non-commutative coproduct $D \mapsto \sigma(D)$:

$$\sigma(D) := \sum_{\deg D = \deg D_1 + \deg D_2} a_D^{D_1,D_2}\, D_1 \otimes D_2 = D_1 \otimes \mathbf{1} + \mathbf{1}\otimes D_2 + \ldots \tag{113}$$

that reflects the action of iso-differentiations on composition products:

$$D(f_2\circ f_1) := \sum_{\mathrm{ideg} D = \mathrm{ideg} D_1 + \mathrm{ideg} D_2} a_D^{D_1,D_2}\, (D_1 f_1)\, (D_2 f_2)\circ f_1\, .\, (f_1')^{n_2} \quad (n_2 := \mathrm{ideg} D_2) \tag{114}$$

(iii) an involution $D \mapsto \widetilde{D}$:

$$Dg \;\equiv\; (\widetilde{D} f)\circ g\, .\, (g')^n \qquad (n = \mathrm{ideg} D\, ,\, f\circ g = id) \tag{115}$$

that reflects the action of iso-differentiations on functional inverses.

But the most arresting feature of ISO is the existence of a doubly stable positive cone ISO^+ with a remarkable system $\{\mathrm{Da}_n, n \in \mathbb{N}\}$ of extremal generators.

5.2 The stable and co-stable positive cone $ISO^+ \subset ISO$.

For reasons that will become clear after a page or two, we must consider the formal[16] expansions $\ell e^{\langle \bullet \rangle}(x)$ defined in this way:

$$\ell e^{\langle n_1,\dots,n_r \rangle}(x) \;=\; \sum_{1 \le p_1 < p_2 < \dots < p_r < +\infty} (L'_{p_1}(x))^{n_1} \dots (L'_{p_r}(x))^{n_r} \tag{116}$$

where L_p denotes the p^{th} iterate of $L := \log$. The series $\ell e^{\langle \bullet \rangle}(x)$ consist of monomials of the form:

$$\lambda_\sigma \;=\; L'_{q_1} \dots L'_{q_r} \quad (1 \le q_1 \le q_2 \le \dots \le q_n \;,\; n = n_1 + \dots + n_r) \tag{117}$$

with an alternative indexation by transfinite ordinals

$$\tau \;=\; \omega^{n-1} q_1 + \omega^{n-2}(q_2 - q_1) + \omega^{n-3}(q_3 - q_2) + \dots + (q_n - q_{n-1}) \tag{118}$$

that reflects the natural ordering of the monomials: the larger τ as an ordinal, the faster the rate of decrease of λ_σ as a germ.

Next, again for good reasons, we must change from the system $\ell e^{\langle \bullet \rangle}$ to a new system $\ell a^{\langle \bullet \rangle}$, via mould post-composition by two composition-reciprocal moulds $sa^\bullet$ and $cosa^\bullet$:

$$sa^{n_1,\dots,n_r} \;:=\; \frac{1}{(n_1 + \dots + n_r)(n_2 + \dots + n_r)\dots n_r} \tag{119}$$

$$cosa^{n_1,\dots,n_r} \;:=\; (-1)^{r-1}\, n_1 \tag{120}$$

$$sa^\bullet \circ cosa^\bullet \;:=\; Id^\bullet \qquad (sa^\bullet \textit{ symmetr}\boldsymbol{al}\,,\; cosa^\bullet \textit{ altern}\boldsymbol{el}) \tag{121}$$

The conversion formulae $\ell e^{\langle \bullet \rangle} \leftrightarrow \ell a^{\langle \bullet \rangle}$ read:

$$\ell a^{\langle \bullet \rangle}(x) = \ell e^{\langle \bullet \rangle}(x) \circ sa^\bullet \qquad ; \qquad \ell e^{\langle \bullet \rangle}(x) = \ell a^{\langle \bullet \rangle}(x) \circ cosa^\bullet \tag{122}$$

The product rules:

$$\ell e^{\langle \boldsymbol{n'} \rangle} . \ell e^{\langle \boldsymbol{n''} \rangle} \;=\; \sum_{\boldsymbol{n} \in \boldsymbol{she}(\boldsymbol{n'},\boldsymbol{n''})} \ell e^{\langle \boldsymbol{n} \rangle} \qquad \big(\boldsymbol{she} = \textit{contracting shuffle}\big) \tag{123}$$

$$\ell a^{\langle \boldsymbol{n'} \rangle} . \ell a^{\langle \boldsymbol{n''} \rangle} \;=\; \sum_{\boldsymbol{n} \in \boldsymbol{sha}(\boldsymbol{n'},\boldsymbol{n''})} \ell a^{\langle \boldsymbol{n} \rangle} \qquad \big(\boldsymbol{sha} = \textit{plain shuffle}\big) \tag{124}$$

[16] *formal* indeed since there exists no common interval $[c, +\infty[$ on which all $L_p(x)$ are simultaneously defined.

simply mean that $\ell e^{\langle\bullet\rangle}$ (resp. $\ell e^{\langle\bullet\rangle}$) is symmetr**e**l (resp. symmetr**a**l).

The ordinary derivation $\partial := d/dx$ acts slightly differently on the two systems $\ell e^{\langle\bullet\rangle}$ and $\ell a^{\langle\bullet\rangle}$:

$$\partial\, \ell e^{\langle \boldsymbol{n}\rangle} \;=\; -\sum_{\boldsymbol{n'} n_j \boldsymbol{n''}=\boldsymbol{n}} (n_j+|\boldsymbol{n''}|)\left(\ell e^{\langle \boldsymbol{n'},1,n_j,\boldsymbol{n''}\rangle} + \ell e^{\langle \boldsymbol{n'},1+n_j,\boldsymbol{n''}\rangle}\right) \tag{125}$$

$$\partial\, \ell a^{\langle \boldsymbol{n}\rangle} \;=\; -\sum_{\boldsymbol{n'} n_j \boldsymbol{n''}=\boldsymbol{n}} (n_j+|\boldsymbol{n''}|)\ell a^{\langle \boldsymbol{n'},1,n_j,\boldsymbol{n''}\rangle} - \sum_{\boldsymbol{n'} n_j \boldsymbol{n''}=\boldsymbol{n}} n_j\, \ell a^{\langle \boldsymbol{n'},1+n_j,\boldsymbol{n''}\rangle} \tag{126}$$

Roughly speaking, the reason for considering $\ell e^{\langle\bullet\rangle}$ and $\ell a^{\langle\bullet\rangle}$ is that any iso-differential operator D, acting on any smooth *germ* (i.e. any real germ f on $[\ldots,+\infty]$ that grows more slowly than any finite iterate of *log*), produces a (trans)-asymptotic series which *does not depend* on f and can be analysed in terms of these $\ell e^{\langle\bullet\rangle}$ or $\ell a^{\langle\bullet\rangle}$. However, not all $\ell e^{\langle\bullet\rangle}$ or $\ell a^{\langle\bullet\rangle}$ can be obtained in this way, but only relatively few of them. This is the starting point of a fascinating theory — the *universal asymptotics of slow germs*. But our perspective here is different: we are more interested in the iso-differential operators *per se*, and we use $\ell e^{\langle\bullet\rangle}$ and $\ell a^{\langle\bullet\rangle}$ merely to define, by duality, symbolic operators $\mathbb{D}\mathrm{e}^{\langle\bullet\rangle}$ and $\mathbb{D}\mathrm{a}^{\langle\bullet\rangle}$ that will span a bialgebra $\overline{ISO}$ *larger*, and in many ways *more convenient*, than *ISO* — even though in the last analysis the structure that matters remains *ISO*.

Our symbolic operators $\mathbb{D}\mathrm{e}^{\langle\bullet\rangle}$ and $\mathbb{D}\mathrm{a}^{\langle\bullet\rangle}$ immediately inherit a simple co-product dual to the products (123)-(124), plus a natural product, plus an action on them of $\partial := d/dx$ patterned on (125)-(126):

$$-\partial\, \mathbb{D}\mathrm{e}^{\langle \boldsymbol{n}\rangle} \;=\; \sum_{\boldsymbol{n'} n_j \boldsymbol{n''}=\boldsymbol{n}} (n_j+|\boldsymbol{n''}|)\left(\mathbb{D}\mathrm{e}^{\langle \boldsymbol{n'},1,n_j,\boldsymbol{n''}\rangle} + \mathbb{D}\mathrm{e}^{\langle \boldsymbol{n'},1+n_j,\boldsymbol{n''}\rangle}\right) \tag{127}$$

$$-\partial\, \mathbb{D}\mathrm{a}^{\langle \boldsymbol{n}\rangle} \;=\; \sum_{\boldsymbol{n'} n_j \boldsymbol{n''}=\boldsymbol{n}} (n_j+|\boldsymbol{n''}|)\mathbb{D}\mathrm{a}^{\langle \boldsymbol{n'},1,n_j,\boldsymbol{n''}\rangle} + \sum_{\boldsymbol{n'} n_j \boldsymbol{n''}=\boldsymbol{n}} n_j\, \mathbb{D}\mathrm{a}^{\langle \boldsymbol{n'},1+n_j,\boldsymbol{n''}\rangle} \tag{128}$$

The positive cone $\overline{ISO}^{+}$ of $\overline{ISO}$ generated by the system $\mathbb{D}\mathrm{a}^{\langle\bullet\rangle}$ is doubly stable – under the product and co-product – and the same holds for the positive cone ISO^{+} that it induces on ISO. For more information on ISO^{+}, see §6 of *The Natural Growth Scale*[17]. But let us here proceed forthwith to the exciting part – the extremal basis $\{\mathrm{Da}_n, n \in \mathbb{N}\}$ of ISO^{+}.

[17] Accessible on our Webpage: < *http : //www.math.u-psud.fr/ ∼ ecalle/* >

5.3 The extremal basis $\{\mathrm{Da}_n, n \in \mathbb{N}\}$ of ISO^+ and its combinatorial aspects.

Before stating (some of) the main results, let us get a few definitions and notations out of the way. For any non-ordered sequence of the form

$$\{\boldsymbol{n}\} = \{n_1, n_2, \ldots, n_r\} = \{m_1^{(r_1)}, m_2^{(r_2)}, \ldots, m_s^{(r_s)}\} \tag{129}$$

$$\textit{with} \qquad n_1 \leq n_2 \leq \cdots \leq n_r \quad \textit{and} \quad m_1 < m_2 < \cdots < m_s \tag{130}$$

the *multiplicity correction* $\mu^{\{\boldsymbol{n}\}}$ is defined as

$$\mu^{\{\boldsymbol{n}\}} = \prod_{1 \leq j \leq s} \frac{1}{(1+r_j)!} \tag{131}$$

and we denote $\overrightarrow{\boldsymbol{n}}$ resp. $\overleftarrow{\boldsymbol{n}}$ the ordered sequence obtained by arranging the elements of $\{\boldsymbol{n}\}$ in increasing resp. decreasing order. If $\boldsymbol{n}$ is an ordered sequence, $\widetilde{\boldsymbol{n}}$ denotes the same sequence with its order reversed. Lastly, for any $t \in \mathbb{R}$ we set:

$$(t)^+ := |t| \quad \textit{if} \quad t > 0 \qquad \textit{and} \qquad (t)^+ := 0 \quad \textit{if} \quad t \leq 0 \tag{132}$$

$$(t)^- := |t| \quad \textit{if} \quad t < 0 \qquad \textit{and} \qquad (t)^- := 0 \quad \textit{if} \quad t \geq 0 \tag{133}$$

Alonside $\mathrm{Da}^{\{\boldsymbol{n}\}}$ and $\mathbb{D}\mathrm{a}^{\langle\boldsymbol{n}\rangle}$ we also require the variants:

$$\underline{\mathrm{Da}}^{\{n_1,\ldots,n_r\}} := \frac{\mathrm{Da}^{\{n_1,\ldots,n_r\}}}{\prod(1+n_j)!} \quad ; \quad \underline{\mathbb{D}\mathrm{a}}^{\langle n_1,\ldots,n_r\rangle} := \frac{\mathbb{D}\mathrm{a}^{\langle n_1,\ldots,n_r\rangle}}{\prod n_j\,(1+n_j)} \tag{134}$$

Proposition: The extremal basis.
For each $n_1 \geq 1$ *there exists a unique iso-operator* $\mathrm{Da}^{\{n_1\}} = (n_1+1)!\,\underline{\mathrm{Da}}^{\{n_1\}}$ *in the positive cone* $\mathrm{Da}^+ \subset \mathbb{D}\mathrm{a}^+$ *verifying the normalisation condition*

$$\mathrm{Da}^{\{n_1\}} = (n_1-1)!\,\mathbb{D}\mathrm{a}^{\langle n_1\rangle} + \ldots \quad \Leftrightarrow \quad \underline{\mathrm{Da}}^{\{n_1\}} = \underline{\mathbb{D}\mathrm{a}}^{\langle n_1\rangle} + \ldots \tag{135}$$

and characterised by either of the following properties:
(i) among all iso-operators so normalised, $\mathrm{Da}^{\{n_1\}}$ *and* $\underline{\mathrm{Da}}^{\{n_1\}}$ *are* least elements *in the cone* Da^+
(ii) the expression of $\mathrm{Da}^{\{n_1\}}$ *resp.* $\underline{\mathrm{Da}}^{\{n_1\}}$ *in the basis* $\mathbb{D}\mathrm{a}^{\langle\boldsymbol{n}\rangle}$ *resp.* $\underline{\mathbb{D}\mathrm{a}}^{\langle\boldsymbol{n}\rangle}$ *involves no weakly decreasing sequences* $\boldsymbol{n} = (n_1, ..., n_r)$ *of length* $r \geq 2$.

The system $\mathrm{Da}^{\{\bullet\}}$ or $\underline{\mathrm{Da}}^{\{\bullet\}}$ constitutes the so-called extremal basis of ISO.

Proposition: Analytical properties of the extremal basis.
The elements of the positive basis are given by $\mathrm{Da}^{\{1\}} = 2\,\underline{\mathrm{Da}}^{\{1\}} := \mathrm{Dn}^{\{1\}}$ *and by an induction rule*

$$\mathrm{Da}^{\{n\}} \;=\; -\partial\,\mathrm{Da}^{\{n-1\}} - \sum_{|\boldsymbol{n}|=n} \mathrm{ka}_{\{\boldsymbol{n}\}} \quad \mu^{\{\boldsymbol{n}\}}\,\mathrm{Da}^{\{\boldsymbol{n}\}} \tag{136}$$

$$(1{+}n)\,\underline{\mathrm{Da}}^{\{n\}} \;=\; -\partial\,\underline{\mathrm{Da}}^{\{n-1\}} - \sum_{|\boldsymbol{n}|=n} \underline{\mathrm{ka}}_{\{\boldsymbol{n}\}} \quad \mu^{\{\boldsymbol{n}\}}\,\underline{\mathrm{Da}}^{\{\boldsymbol{n}\}} \tag{137}$$

involving non-negative coefficients $\underline{\mathrm{ka}}_{\{\boldsymbol{n}\}}$ *with a clear multiplicative structure:*

$$\underline{\mathrm{ka}}_{\{\boldsymbol{n}\}} = (r_1{-}1)\,(-1{+}n_1)^{-}\,(-1{+}n_1{-}n_2)^{-}\prod_{2\le j\le r}(-1{+}n_1{+}\ldots{+}n_{j-1}{-}n_j)^{+} \tag{138}$$

Here r_1 *denotes the multiplicity of the smallest element in the non-ordered sequence* $\boldsymbol{n}$*. As a consequence,* $\underline{\mathrm{ka}}_{\{\boldsymbol{n}\}}$ *is* >0 *if and only if*

$$n_1 = n_2 \ge 2 \quad \textit{and} \quad n_1 + n_2 + \cdots + n_{j-1} \ge 2 + n_j \tag{139}$$

The expansion of the $\mathrm{Da}^{\{n\}}, \underline{\mathrm{Da}}^{\{n\}}$ *in the* $\mathbb{D}\mathrm{a}^{\langle\boldsymbol{n}\rangle}$ *basis:*

$$\mathrm{Da}^{\{n\}} \;=\; \sum_{1\le r}\;\sum_{n=n_1+\ldots+n_r} \mathrm{ta}_{n_1,\ldots,n_r}\;\mathbb{D}\mathrm{a}^{\langle n_1,\ldots,n_r\rangle} \tag{140}$$

$$\underline{\mathrm{Da}}^{\{n\}} \;=\; \sum_{1\le r}\;\sum_{n=n_1+\ldots+n_r} \underline{\mathrm{ta}}_{n_1,\ldots,n_r}\;\underline{\mathbb{D}\mathrm{a}}^{\langle n_1,\ldots,n_r\rangle} \tag{141}$$

as well as the expression of the involution $\sim :\mathrm{Da}^{\{n\}} \mapsto \widetilde{\mathrm{Da}}^{\{n\}}$

$$\widetilde{\mathrm{Da}}^{\{n\}} \;=\; \mathrm{fa}_{\{\boldsymbol{n}\}} \quad \mu^{\{\boldsymbol{n}\}}\,\mathrm{Da}^{\{\boldsymbol{n}\}} \tag{142}$$

$$\widetilde{\underline{\mathrm{Da}}}^{\{n\}} \;=\; \underline{\mathrm{fa}}_{\{\boldsymbol{n}\}} \quad \mu^{\{\boldsymbol{n}\}}\,\underline{\mathrm{Da}}^{\{\boldsymbol{n}\}} \qquad (\textit{with} \quad \underline{\mathrm{fa}}_{\{\boldsymbol{n}\}} = (-1)^{r(\boldsymbol{n})}\underline{\mathrm{ta}}_{\vec{\boldsymbol{n}}}) \tag{143}$$

or that the co-product[18] $\sigma :\ \mathrm{Da}^{\{n\}} \mapsto \sigma(\mathrm{Da}^{\{n\}})$

$$\sigma(\mathrm{Da}^{\{n\}}) \;=\; \sum_{|\boldsymbol{p}|+|\boldsymbol{q}|=n} \mathrm{ha}_{\{\boldsymbol{p}\},\{\boldsymbol{q}\}}\;\mu^{\{\boldsymbol{p}\}}\,\mathrm{Da}^{\{\boldsymbol{p}\}} \otimes \mu^{\{\boldsymbol{q}\}}\,\mathrm{Da}^{\{\boldsymbol{q}\}} \tag{144}$$

$$\sigma(\underline{\mathrm{Da}}^{\{n\}}) \;=\; \sum_{|\boldsymbol{p}|+|\boldsymbol{q}|=n} \underline{\mathrm{ha}}_{\{\boldsymbol{p}\},\{\boldsymbol{q}\}}\;\mu^{\{\boldsymbol{p}\}}\,\underline{\mathrm{Da}}^{\{\boldsymbol{p}\}} \otimes \mu^{\{\boldsymbol{q}\}}\,\underline{\mathrm{Da}}^{\{\boldsymbol{q}\}} \qquad (\textit{with}\ \ \underline{\mathrm{ha}}_{\{\boldsymbol{p}\},\{\boldsymbol{q}\}} = \underline{\mathrm{ta}}_{\overleftarrow{\boldsymbol{p}},\overleftarrow{\boldsymbol{q}}})$$

[18]Recall that σ is co-associative, but not co-commutative

also involve non-negative integers[19] $\underline{\mathrm{ta}}_{\boldsymbol{n}}$, $\underline{\mathrm{fa}}_{\{\boldsymbol{n}\}}$, $\underline{\mathrm{ha}}_{\{\boldsymbol{p}\},\{\boldsymbol{q}\}}$, *but the only structure constants with a transparent factorization are those coefficients* $\underline{\mathrm{ha}}_{\{\boldsymbol{p}\},\{\boldsymbol{q}\}}$ *for which one of the sequences* $\{\boldsymbol{p}\} = \{p_1 \leq p_2 \leq ...\}$ *or* $\{\boldsymbol{q}\} = \{q_1 \leq q_2 \leq ...\}$ *is of length one:*

$$\underline{\mathrm{ha}}_{\{\boldsymbol{p}\},\{q_1\}} = (q_1 - p_1)^+ \prod_{2 \leq j \leq r} (q_1 + p_1 + p_2 + ... + p_{j-1} - p_j)^+ \tag{145}$$

$$\underline{\mathrm{ha}}_{\{p_1\},\{\boldsymbol{q}\}} = (p_1 - q_1)^- \prod_{2 \leq j \leq r} (p_1 + q_1 + q_2 + ... + q_{j-1} - q_j)^+ \tag{146}$$

More information on the iso-differential operators Da_n and their combinatorial aspects may be found in §7 of *The Natural Growth Scale.*[20] Numerous related tables are also available there, in §14.

[19]except $\underline{\mathrm{fa}}_{\{\boldsymbol{n}\}}$ whose sign is that of $(-1)^{r(\boldsymbol{n})}$

[20]Accessible on our Webpage: < *http : //www.math.u-psud.fr/ ∼ ecalle/* >