Finitary flexion algebras.

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N.B. This is only the last chapter of a long, still incomplete paper titled "Finitary Flexion Algebras"*. We are posting it ahead of the rest because of its actualness.*

Contents

8 Addendum: singulators and bisingulators.

Unlike the seven preceding sections, which merely re-hash old unpublished material, the present Addendum was written quite recently and in some haste, following a series of exchanges with Leila Schneps, who apprised us of an – apparently quite brilliant – thesis, yet to be defended, by Samuel Baumard, a PhD student of hers. Though we are not, by any stretch, cognizant of the full substance of the thesis[1], what little we learnt of it caught our attention and prompted us to write down a series of remarks – partly to put the student's investigation (about *bisingulators*) into perspective by relating it to earlier work of ours (on *singulators*); partly to prove a conjecture by L. Schneps who, on the strength of numerical evidence, correctly surmised that the *carma* bialternals could be expressed as sums of some special *ari*-brackets; and partly, in fact chiefly, to suggest further and to our mind quite promising lines of investigation.[2]

8.1 A simple but useful lemma about *push*-invariants.

We have it from L.Schneps that her student proved the following lemma:

Lemma 8.1.1 *Bialternal bimoulds of the form*

$$M^{w_1,\dots,w_r} = N^{w_1,\dots,w_r}\, P(u_0)\, P(u_1) \dots P(u_r) \qquad \big(u_0 := -(u_1 + \dots + u_r)\big) \quad (1)$$

with $N^\bullet$ $\boldsymbol{u}$*-entire and* $\boldsymbol{v}$*-constant, are stable under* ari.

In other words, for such bimoulds the *ari*-bracketing, contrary to expectations, produces no "bad" poles of type $P(u_i + \dots u_j)$ with $1 < j-i < r-1$.

We have not seen the proof in question[3] but we wish to point out that the lemma actually results from another statement which, though stronger, is actually simpler to establish:

Lemma 8.1.2 Push-*invariant bimoulds of the form*

$$M^{w_1,\dots,w_r} = N^{w_1,\dots,w_r}\, P(u_0)\, P(u_1) \dots P(u_r) \qquad \big(u_0 := -(u_1 + \dots + u_r)\big) \quad (2)$$

with $N^\bullet$ $\boldsymbol{u}$*-entire and* $\boldsymbol{v}$*-constant, are stable under* ari.

Since bialternality classically implies *push*-invariance, the latter Lemma implies the former. Moreover, since in (2) the *push*-invariance of $M^\bullet$ is equivalent to that of $N^\bullet$, it suffices to check that for any pair $S_1^\bullet, S_2^\bullet$ of arbitrary bimoulds of lengths r_1, r_2, the *ari*-product

$$M^\bullet := \mathrm{ari}(M_1^\bullet, M_2^\bullet) \qquad \textit{with} \qquad N_i^\bullet := \mathrm{pushinvar}.S_i^\bullet \quad (\forall i \in \{1,2\}) \quad (3)$$

[1] We have to say that, though S.Baumard was apparently put to work on subjects close to ours, and that too with methods due to us (flexion algebra etc), we were not at any stage informed of his progress. In fact, we initially learnt of this PhD project by pure happenstance, from a chance remark by our esteemed Orsay colleague, Pierre Pansu.

[2] To underscore the kinship and mutual convertibility between the earlier operators ("singulators") introduced by us in the early 2000s and those recently used by S.Baumard, we shall refer to the latter as "bisingulators" and adhere throughout to a terminology and notations probably alien to S.Baumard but as close as possible to the ones already in use for the "singulators".

[3] though L. Schneps gave us a sketch of the argument leading to it.

has no "bad" poles. The verification is staightforward:

(i) one first deals with the case when the length-1 components of $M_1^\bullet$, $M_2^\bullet$ have no poles at $u_1 = 0$.
(ii) one calculates $M^{\boldsymbol{w}}$ while keeping P as P
(iii) for any (i,j) with $1 \leq i < j \leq r := r_1 + r_2$ (due to the *push*-invariance of $M^\bullet$ one may in fact assume i to be 1) one calculates the coefficients $H_{i,j}^+$ and $H_{i,j}^-$ respectively of $P(u_i + ... + u_j)$ and $P(-(u_i + ... + u_j))$ in $M^{\boldsymbol{w}}$
(iv) one sets $u_j := -(u_1 + ... + u_{j-1})$ in $H_{i,j}^+$ and $H_{i,j}^-$, which thus become $h_{i,j}^+$ and $h_{i,j}^-$
(v) one then checks (preferably using the *"long notation"*, i.e. with u_0) that due to pairwise cancellations $h_{i,j}^+$ and $h_{i,j}^-$, and not just the difference $h_{i,j}^+ - h_{i,j}^-$, separately vanish.
(vi) lastly, one easily removes the restriction (i). To do this, it is in fact enough to deal with the case when $M_1^{w_1} = P(u_1).P(u_0) = -P(u_1)^2$ and $M_2^{w_1}$ is regular at $u_1 = 0$.

Casual remark: As just mentioned, the proof makes no use of the actual form of P, not even its imparity: it would work just as well with P changed to an arbitrary meromorphic function with a simple pole at the origin.[4]

Important remark: Lemma 8.1.2 no longer holds if we remove the assumption of $\boldsymbol{v}$-constancy. Or rather, it still holds, but only if we impose special constraints on the $\boldsymbol{v}$-dependency. Now, it so happens that in the case of *n-coloured multizetas*, we are led to consider bimoulds that do depend on the v_i-variables[5] but which automatically verify these additional constraints.

8.2 (Bi)singulators, (bi)singulands, (bi)singulates.

The *(bi)singulators* are bimould operators that turn regular inputs of the right parity – the *(bi)singulands* – into singular and bialternal outputs – the *(bi)singulates*. All three come in two forms: *simple* or *composite*.

Let us start with the *simple* (bi)singulators $(bi)slank_r$. These operators turn (bi)singulands $(bi)s\o nd_r$ from $BIMU_1$ into (bi)singulates $(bi)s\o t_r$ in

[4]For instance, we might change P to Q with $Q(t) := c/\tan(c\,t)$.
[5]they range over a discrete set $\mathbb{Z}/\frac{1}{n}\mathbb{Z}$.

$BIMU_r$:

$$\mathrm{slank}_r.\mathrm{s\o nd}^\bullet_r \;:=\; \mathrm{senk}_r(\mathrm{pal}^\bullet).\mathrm{s\o nd}^\bullet_r \qquad (cf.[E3], \S 5) \tag{4}$$

$$\mathrm{bislank}_r.\mathrm{bis\o nd}^\bullet_r \;:=\; \vec{\mathrm{ari}}(\mathrm{bis\o nd}^\bullet_r, \overbrace{\kappa\alpha^\bullet_{-2}, \ldots, \kappa\alpha^\bullet_{-2}}^{(r\text{–}1)\ times}) \tag{5}$$

Throughout, the length-1 bimoulds $\kappa\alpha^\bullet_d$ are defined by $\kappa\alpha^{w_1}_d := u_1^d$.

The operator $slank_r$ produces a bialternal singulate if and only if the singuland $s\o nd_r^{w_1}$ has the right parity in w_1, namely *odd* when r is *even*, and *even* when r is *odd*. The operator $bislank_r$, on the other hand, produces bialternal bisingulates if and only if the bisinguland $bis\o nd_r^{w_1}$ is an *even* function of w_1, whatever the value of r.

Under $slank_r$ (resp. $bislank_r$), the homogeneous degree drops by $r-1$ (resp. $2\,r-2$) units.

Lastly, it was stated in [E3] (§5.7, p 107) that the singulates $s\o t^\bullet_r :=$ $slank_r.s\o nd^\bullet_r$ have only "good" poles, that is to say poles of the form $P(u_i)$ or $P(u_0) := P(-u_1...-u_r)$, to the exclusion of "bad" poles of type $P(u_i+\cdots+u_j)$ with $1 < |i\text{–}j| < r\text{–}1$, which the flexion operations might have been expected to produce, and which they do indeed produce when the singuland does not possess the right parity. Moreover, we are told by L.Schneps that her student has proved an analogous statement for the bisingulates, assuming of course the bisinguland to be even. As we shall see, due to the equivalence of singulates and bisingulates and on the strength of the above Lemmas in §8.1, both statements corroborate each other. The remarkable thing, however, which we may note in passing, is that in both cases the parity condition guaranteeing bialternality coincides with the one that keeps "bad" poles at bay!

Let us now define the *composite* (bi)singulators $(bi)slank_{r_1,\ldots,r_n}$. These operators turn (bi)singulands $(bi)s\o nd^\bullet_{r_1,\ldots,r_n}$ from $BIMU_n$ into (bi)singulates $(bi)s\o t^\bullet_{r_1,\ldots,r_n}$ in $BIMU_{r_1+\ldots r_n}$. They are characterised by the following straightforward multilinearity property, valid for all degrees d_i:

$$\mathrm{slank}_{r_1,\ldots,r_n}.\mathrm{mu}(\kappa\alpha^\bullet_{d_1},...,\kappa\alpha^\bullet_{d_n}) = \vec{\mathrm{ari}}(\mathrm{slank}_{r_1}.\kappa\alpha^\bullet_{d_1},...,\mathrm{slank}_{r_n}.\kappa\alpha^\bullet_{d_n}) \tag{6}$$

$$\mathrm{bislank}_{r_1,\ldots,r_n}.\mathrm{mu}(\kappa\alpha^\bullet_{d_1},...,\kappa\alpha^\bullet_{d_n}) = \vec{\mathrm{ari}}(\mathrm{bislank}_{r_1}.\kappa\alpha^\bullet_{d_1},...,\mathrm{bislank}_{r_n}.\kappa\alpha^\bullet_{d_n}) \tag{7}$$

The notation $\vec{ari}$ signals that the bracketing goes from left to right.[6]

Although the *simple* (bi)singulators $(bi)slank_r$ make direct sense only for $r > 1$, it is convenient to set:

$$\mathrm{slank}_1 = \mathrm{bislank}_1 := \mathrm{id} : \qquad \mathrm{BIMU}_1 \to \mathrm{BIMU}_1 \tag{8}$$

[6] In [E3], we had adopted the opposite convention.

so as to be able to handle *composite* (bi)singulators $(bi)slank_{r_1,\ldots,r_n}$ with partial indices $r_i \geq 1$.

Another convenient tool has to be mentioned in this context: namely the operators $preslank_{r_1,\ldots,r_n}$ and $prebislank_{r_1,\ldots,r_n}$, which are defined exactly as in (6) and (7) but with $\vec{ari}$ changed to $pre\vec{ari}$. Taken in isolation, they fail to produce bialternals (they do so only collectively, in the right combinations[7]) but they have one great merit: they relieve us of the necessity of choosing (necessarily arbitrary) bases in the spaces spanned by all $(bi)slank_{r_1,\ldots,r_n}$ of a given length $r = \sum r_i$.

8.3 Mutual convertibilty of singulates and bisingulates. The algebra $ALAL_{sing}$.

The monomial pilot formula behind convertibility.

For any even d, consider the simple "atomic" (bi)singulates:

$$\mathrm{sat}^{\bullet}_{r,d} \;:=\; \mathrm{slank}_r.\kappa\alpha^{\bullet}_{d+r-1} \tag{9}$$

$$\mathrm{bisat}^{\bullet}_{r,d} \;:=\; \mathrm{bislank}_r.\kappa\alpha^{\bullet}_{d+2\,r-2} \;=\; \vec{ari}(\kappa\alpha^{\bullet}_{d+2\,r-2}, \overbrace{\kappa\alpha^{\bullet}_{-2},\ldots,\kappa\alpha^{\bullet}_{-2}}^{(r-1)\ times}) \tag{10}$$

Both have the same length r; the same homogeneous degree d; and the same *effective polarity*[8] of order $r-1$. Now, a careful calculation shows that:

$$\mathrm{bisat}^{\boldsymbol{w}}_{r,d} \equiv \frac{(r-1)!\,(d+2\,r-2)!}{(d+r-1)!}\,\mathrm{sat}^{\boldsymbol{w}}_{r,d} \qquad mod\ \ \mathrm{Polar}_{r-2} \tag{11}$$

Incidentally, if we introduce the "weight" $s := r+d$, the conversion factor $\frac{(r-1)!\,(d+2\,r-2)!}{(d+r-1)!}$ assumes the slightly more pleasant shape $\frac{(r-1)!\,(s+r-2)!}{(s-1)!}$

The general pilot formula, and what it tells us.

Consider now general (bi)singulands subject to no other restriction than regularity at $u_1 = 0$:

$$\mathrm{s\o t}^{\bullet}_{r} \;:=\; \mathrm{slank}_r.\mathrm{s\o nd}^{\bullet}_{r} \qquad with\ \ \mathrm{s\o nd}^{w_1}_{r} := A(u_1) \tag{12}$$

$$\mathrm{bis\o t}^{\bullet}_{r} \;:=\; \mathrm{bislank}_r.\mathrm{bis\o nd}^{\bullet}_{r} \qquad with\ \ \mathrm{bis\o nd}^{w_1}_{r} := B(u_1) \tag{13}$$

[7]i.e. in sums of type $\sum c^{r_1,\ldots,r_n}\, pre(bi)slank_{r_1,\ldots,r_n}$, with alternal coefficients $c^{\bullet}$.

[8]The term *effective polarity* really explains itself. It differs of course from the *apparent polarity* (i.e. the degree of the denominator you get after brutally factoring everything), which in the present instance would be $r{+}1$. For composite (bi)singulands, the discrepancy between effective and apparent polarity is even larger.

The corresponding (bi)singulates verify the following identities, which neatly isolate the terms of highest polar order (in the present instance, of order $r-1$):

$$\text{sø}\mathrm{t}_r^{\boldsymbol{w}} = \sum_{0\leq i<j\leq r} \frac{(-1)^{j-i-1}(r-1)!}{(j-i-1)!(r-j+i)!}.\frac{A(u_i)}{(r-1)!} \prod_{0\leq k\leq r}^{k\notin\{i,j\}} P(u_k) \;\; mod \; \mathrm{Polar}_{r-2} \quad (14)$$

$$= \sum_{0\leq i<j\leq r} \frac{(-1)^{j-i+r}(r-1)!}{(j-i-1)!(r-j+i)!}.\frac{A(u_j)}{(r-1)!} \prod_{0\leq k\leq r}^{k\notin\{i,j\}} P(u_k) \;\; mod \; \mathrm{Polar}_{r-2} \quad (15)$$

$$\text{bisø}\mathrm{t}_r^{\boldsymbol{w}} = \sum_{0\leq i<j\leq r} \frac{(-1)^{j-i-1}(r-1)!}{(j-i-1)!(r-j+i)!}.B^{(r-1)}(u_i) \prod_{0\leq k\leq r}^{k\notin\{i,j\}} P(u_k) \;\; mod \; \mathrm{Polar}_{r-2} \quad (16)$$

$$= \sum_{0\leq i<j\leq r} \frac{(-1)^{j-i+r}(r-1)!}{(j-i-1)!(r-j+i)!}.B^{(r-1)}(u_j) \prod_{0\leq k\leq r}^{k\notin\{i,j\}} P(u_k) \;\; mod \; \mathrm{Polar}_{r-2} \quad (17)$$

Remark 8.3.1. Non-trivialness of desingularisation.
Due to the parity conditions laid upon the singuland A (resp. bisinguland B), the right-hand sides of (12) and (13) (resp. (14) and (15)) clearly coincide modulo $Polar_{r-2}$ and define a bimould $Søt_r^\bullet$ (resp. $Bisøt_r^\bullet$) that is automatically *bisymmetral*, but again only modulo $Polar_{r-2}$. By suitably modifying these right-hand sides, one could easily ensure the *exact* alternality of $Søt_r^\bullet$ and $Bisøt_r^\bullet$ and even their *exact* alternality in combination with their *exact push*-invariance.[9] What no elementary trick can achieve, though, is *exact* bialternality. Were that possible, we would be spared many a headache: we could simply subtract from $søt_r^\bullet - bisøt_r^\bullet$ the *exactly* bialternal part $Søt_r^\bullet - Bisøt_r^\bullet$ that carries all the polarity of order $r-1$, and by so doing kick-start a simple and effective *conversion algorithm.* Unfortunately, no such short-cut can work: the only way to remove polar parts of a given order while respecting the *double symmetries* is the arduous process, essentially perinomal in nature, that is explained in [E1], [E2], [E3] and that shall be applied here, in §8.8 to the special task of converting singulates and bisingulates into one another.

Remark 8.3.2. The apparent differentiation of bisingulands: a strange phenomenon halfway between fact and artefact.
A striking difference between the pilot formula for singulates and binsingulates is this: the highest polar part as given by (12) or (13) involves the

[9]Ultimately, this is due to the *finitariness* of the combination alternality+*push*-invariance, in sharp contrast to the *non-finitariness* of bialternality. See §1 *supra*.

singuland A in unadulterated form, whereas (14) and (15) involve the derivative $B^{(r-1)}$ of the bisinguland. This is not to say, of course, that bisingulators, which as linear operators resolve into a sequence of purely algebraic manipulations, actually *differentiate* the singulands they act upon. They do no such thing: if $bis\o nd_r^{w_1} := B(u_1)$ has only simple poles away from zero[10], the bisinguland $bis\o t_r^{w_1,\dots,w_r}$ also will exhibit only simple poles[11] away from zero. There is no danger of poles of order r popping out of thin air – as would be the case if actual differentiation had taken place. Still, we should not dismiss this apparent "differentiation" as a mere optical illusion, for it has two momentous consequences.

First consequence: When converting bisingulates into singulates (see §8.8 *infra*) we shall have to subtract from the bisingulate $bis\o t^\bullet$ (produced from B) a singulate $s\o t^\bullet$ (produced from $A := (r-1)!\,B^{(r-1)}$). This subtraction, to which there is no alternative, shall lead to *actual* differentiation and to a very tangible proliferation of multiple poles.

Second consequence: If we were to use bisingulates instead of singulates in the perinomal construction of $luma^\bullet/lumi^\bullet$ and $ruma^\bullet/rumi^\bullet$ (see [E2], [E3]), the difficulty would be exactly the reverse (integration rather than differentiation), only ten times worse. Indeed, to offset singulands $A(u_1)$ carrying simple poles $c.(u_1{-}n_1)^{-1}$ we would have no choice but to rope in bisingulands $B(u_1)$ carrying logarithmic terms of the form $c.\frac{1}{(r-1)!}\,\partial_{u_1}^{-r+1}.(u_1-n_1)^{-1} = c.\frac{(u_1-n_1)^{r-2}}{(r-1)!\,(r-2)!}\log(u_1{-}n_1)+(\ldots)$, and this would at once land us into an almighty mess. This fact alone, from the very start, disqualifies bisingulates as a vehicle for perinomal calculus.

Conversion rules at length $r = 2$.

This is actually the only conversion rule that we shall require for tackling the *carma* bialternals. So let us deal with it with some care. Let us first recall the three polynomials that go into the making of the *doma* bialternals:[12]

$$\mathrm{fa}(u_1,u_2) := u_1\,u_2\,(u_1+u_2)\,(u_1-u_2)\,(u_1+2\,u_2)\,(u_2+2\,u_1) \quad (18)$$
$$\mathrm{ha}(u_1,u_2) := u_1^2+u_1\,u_2+u_2^2 \quad (19)$$
$$\mathrm{ga}(u_1,u_2) := (u_1+u_2)^2\,u_1^2\,u_2^2 \quad (20)$$

[10]By definition, (bi)singulands are regular at the origin.

[11]respective to any *given* variable u_i or to any *given* partial sum $u_i+\ldots+u_j$.

[12]For details, see [E3], §7.2. The bimoulds *doma* constitute a basis for the space of all length-2 bialternals. Here, we may adopt the simpler indexation $doma_{m,n}^{w_1.w_2} := fa(u_1,u_2)\,\big(ha(u_1,u_2)\big)^m\,\big(ga(u_1,u_2)\big)^n$.

In the *"long notation"* (with $u_0 := -u_1 - u_2$), they assume the simpler form:

$$\mathrm{fa}(u_1,u_2) \;:=\; u_0\,u_1\,u_2\,(u_0-u_1)\,(u_1-u_2)\,(u_2-u_0) \tag{21}$$
$$\mathrm{ha}(u_1,u_2) \;:=\; -u_0\,u_1 - u_1\,u_2 - u_2\,u_0 \tag{22}$$
$$\mathrm{ga}(u_1,u_2) \;:=\; u_0^2\,u_1^2\,u_2^2 \tag{23}$$

Let us now collect all our (bi)singulands inside generating power series of t:

$$\mathrm{sund}_2^{w_1}(t) \;:=\; \frac{2\,u_1}{(1-u_1^2\,t^2)^2} \tag{24}$$
$$\mathrm{bisund}_2^{w_1}(t) \;:=\; \frac{u_1^2}{(1-u_1^2\,t^2)} \tag{25}$$

In the *long notation* we get:

$$(\mathrm{bislank}_2.\mathrm{bisund}_2)^{w_1,w_2} - (\mathrm{slank}_2.\mathrm{sund}_2)^{w_1,w_2} =$$
$$\frac{u_0\,u_1\,u_2\,(u_0-u_1)\,(u_1-u_2)\,(u_2-u_0)\,\big(3+(u_0\,u_1+u_1\,u_2+u_2\,u_0)\,t^2\big)\,t^6}{(1-u_0^2\,t^2)^2\,(1-u_1^2\,t^2)^2\,(1-u_2^2\,t^2)^2} \tag{26}$$

which readily translates into:

$$(\mathrm{bislank}_2.\mathrm{bisund}_2)^{w_1,w_2} - (\mathrm{slank}_2.\mathrm{sund}_2)^{w_1,w_2} =$$
$$\frac{(\mathrm{fa}).(3-\mathrm{ha}\,t^2)\,t^6}{1-4\,\mathrm{ha}\,t^2+6\,\mathrm{ha}^2\,t^4-(2\,\mathrm{ga}+4\,\mathrm{ha}^3)\,t^6+(\mathrm{ha}^4+4\,\mathrm{ha}\,\mathrm{ga})\,t^8-2\,\mathrm{ha}^2\mathrm{ga}\,t^{10}+\mathrm{ga}^2\,t^{12}} \tag{27}$$

Expanding both sides of (27) as power series of t and equating the coefficients of t^d (d even), we immediately get the singulates and bisingulates of same degree translated into each other plus a string of *doma* bialternals.

Conversion rules at higher lengths ($r \geq 3$).

Proposition 8.3.1. *The twin processes of singulation and bisingulation are globally equivalent: they generate exactly the same space of singular bisymmetrals with "good poles". The mutual convertibility of singulates and bisingulates is guaranteed by the existence, for all pairs* $(\mathrm{s\o nd}_r^{\bullet}, \mathrm{bis\o nd}_r^{\bullet})$ *such that*

$$\mathrm{s\o nd}_r^{w_1} = (r-1)!\;\partial_{u_1}^{r-1}\,\mathrm{bis\o nd}_r^{w_1} \qquad \big(\mathrm{s\o nd}_r^{\bullet}, \mathrm{bis\o nd}_r^{\bullet} \in \mathrm{BIMU}_1^{\mathrm{ent}}\big) \tag{28}$$

of identities of the form:

$$\mathrm{bislank}_r.\mathrm{bis\o nd_r}^{\bullet} \;=\; \mathrm{slank}_r.\mathrm{s\o nd_r}^{\bullet} + \sum_{2\leq s\leq r}^{\sum r_i = r} \mathrm{slank}_{r_1,\dots,r_s}.\mathrm{s\o nd}^{\bullet}_{r_1,\dots,r_s} \tag{29}$$

$$\mathrm{slank}_r.\mathrm{s\o nd_r}^{\bullet} \;=\; \mathrm{bislank}_r.\mathrm{bis\o nd_r}^{\bullet} + \sum_{2\leq s\leq r}^{\sum r_i = r} \mathrm{bislank}_{r_1,\dots,r_s}.\mathrm{bis\o nd}^{\bullet}_{r_1,\dots,r_s} \tag{30}$$

(i) with all singulands $sønd^\bullet_{r_1,\dots,r_s}$ $\boldsymbol{v}$-constant and in $BIMU^{ent}_s$
(ii) with all proper singulates $søt^\bullet_{r_1,\dots,r_s} = slank_{r_1,\dots,r_s}.sønd^\bullet_{r_1,\dots,r_s}$ in $BIMU^{sing}_r$
(iii) with one last 'improper' singulate $søt^\bullet_{1,\dots,1}$ in $BIMU^{ent}_r$.

Thus, at lengths 3 and 4 we have:

$$\mathrm{bislank}_3.\mathrm{bisønd}_3{}^\bullet = \mathrm{slank}_3.\mathrm{sønd}_3{}^\bullet + \mathrm{slank}_{1,2}.\mathrm{sønd}^\bullet_{1,2} + \mathrm{slank}_{1,1,1}.\mathrm{sønd}^\bullet_{1,1,1} \tag{31}$$

$$\begin{aligned}\mathrm{bislank}_4.\mathrm{bisønd}_4{}^\bullet &= \mathrm{slank}_4.\mathrm{sønd}_4{}^\bullet + \mathrm{slank}_{3,1}.\mathrm{sønd}^\bullet_{3,1} + \mathrm{slank}_{2,2}.\mathrm{sønd}^\bullet_{2,2}\\ &\quad +\mathrm{slank}_{2,1,1}.\mathrm{sønd}^\bullet_{2,1,1} + \mathrm{slank}_{1,1,1,1}.\mathrm{sønd}^\bullet_{1,1,1,1}\end{aligned} \tag{32}$$

Important remark: By itself, our pilot formula (11) *does not* tell us that the difference $bislank_r.bisønd_r{}^\bullet - slank_r.sønd_r{}^\bullet$ can be expanded into pure sums of singulates or pure sums of bisingulates. It just tells us that, if such expansions exist, they can involve only strictly *composite* singulates or bisingulates, i.e. terms of lesser polarity. The *existence* itself of expansions of type (29) and (30) flows from other considerations – namely, from the formula (67) in §8.8 *infra* which is an exact analogue of (29) in the perinomal setting.[13]

8.4 Reminder about the elements of $ALIL_{\mathrm{ent}}$ and their representation.

We recall that the systems $\{løma^\bullet_s; s\ odd\}$ constructed in [E3], §6 through an inductive shuttle of singularisation-desingularisation are of the form:

$$\mathrm{løma}^\bullet_s = \mathrm{adari}(\mathrm{pal}^\bullet).\Big(\mathrm{søt}^\bullet_{s;1} + \mathrm{søt}^\bullet_{s;3} + \mathrm{søt}^\bullet_{s;5} + \mathrm{søt}^\bullet_{s;7} + \dots\Big) \tag{33}$$

with $\boldsymbol{v}$-constant singulates $søt^\bullet_{s;r}$ in $ALAL_{sing}$
(i) of odd length r,
(ii) of homogeneous $\boldsymbol{u}$-degree $d = s-r$,
(iii) of formal $\boldsymbol{u}$-polarity[14] ranging between 1 and $r-2$
(iv) with each $søt^\bullet_{s;r}$ (save of course the initial and trivial $søt^\bullet_{s;1}$) given as a

[13]Remarkably, (30) has no perinomal analogue: if has to be derived directly from (29) by inversion.

[14]The ban on terms of formal polarity 0 means that we exclude from $søt_{s;r}$ any trivial contribution of the form $slank_{1,\dots,1}.sønd^\bullet_{1,\dots,1}$. These are the *"naive bialternals"*. They necessarily occur, for instance, in the conversion formulae (29) and (30). They are absent from (35), (37) etc — simply because we *banished them*! However, distinct realisations of $løma^\bullet_s$, like for instance the three canonical realisations $lama^\bullet_s$, $loma^\bullet_s$, $luma^\bullet_s$ may, and for large enough values of s always do, differ pairwise by *quite special* – and rather rare – terms of the form $slank_{1,\dots,1}.sønd^\bullet_{1,\dots,1}$. These are the so-called *"wandering bialternals"*.

sum of strictly *composite* elementary singulates $slank_{r_1,\dots,r_n}\, s\o nd^{\bullet}_{r_1,\dots,r_n}$ with $0 < n < r$. Thus:

$$\begin{aligned}
\mathrm{s\o t}^{\bullet}_{s;1} &= \kappa\alpha^{\bullet}_{s-1} \qquad\qquad (\,\kappa\alpha^{w_1}_{d} := u_1^d\,) && (34)\\
\mathrm{s\o t}^{\bullet}_{s;3} &= \mathrm{slank}_{2,1}.\mathrm{s\o nd}^{\bullet}_{2,1} && (35)\\
\mathrm{s\o t}^{\bullet}_{s;5} &= \mathrm{slank}_{4,1}.\mathrm{s\o nd}^{\bullet}_{4,1} + \mathrm{slank}_{3,2}.\mathrm{s\o d}^{\bullet}_{3,2} + \mathrm{slank}_{3,1,1}.\mathrm{s\o nd}^{\bullet}_{3,1,1}\\
&\quad +\mathrm{slank}_{2,1,2}.\mathrm{s\o nd}^{\bullet}_{2,1,2} + \mathrm{slank}_{2,1,1,1}.\mathrm{s\o nd}^{\bullet}_{2,1,1,1} && (36)
\end{aligned}$$

For instance, the singuland $s\o nd^{w_1,w_2}_{2,1}$, which is the only one that we shall require in the next paragraph, is a homogeneous polynomial in (u_1, u_2) of degree $s-2$. When acted upon by the singulator $slank_{2,1}$, it produces a singulate $s\o t^{w_1,w_2,w_3}_{2,1}$ that is a homogeneous rational fraction in (u_1, u_2, u_3) of degree $s-3$.

8.5 All *carma* bialternals can be expressed as sums of singulates or bisingulates, at one's choice.

For any fixed system $\{l\o ma^{\bullet}_s; s = 1, 3, 5 \dots\}$, the *carma* bialternals of degree d are constructed (see E1] §17 , [E3] §7) from *precarma* polynomials of degree $d+2$, i.e. from alternal polynomials of the form:

$$\mathrm{precar}^{x_1,x_2} = \sum_{d_1+d_2=d+2}^{d_i\,even\geq 2} c_{d_1,d_2}\, x_1^{d_1}\, x_2^{d_2} \qquad (37)$$

$$\textit{with} \qquad 0^{\bullet} \equiv \sum_{d_1+d_2=d+2}^{d_i\,even\geq 2} c_{d_1,d_2}\, \mathrm{ari}(\kappa\alpha^{\bullet}_{d_1}, \kappa\alpha^{\bullet}_{d_2}) \qquad (38)$$

The corresponding *carma* bialternal $\mathrm{c\o rma}^{\bullet} \in BIMU_4$ is then defined by the following identity, taken at length $r = 4$:

$$\mathrm{c\o rma}^{\bullet} = \frac{1}{2} \sum_{d_1+d_2=d+2}^{d_i\,even\geq 2} c_{d_1,d_2}\, \mathrm{ari}(\mathrm{l\o ma}^{\bullet}_{1+d_1}, \mathrm{l\o ma}^{\bullet}_{1+d_2}) \qquad (39)$$

In view of (33), (34), (35), this yields:

$$\begin{aligned}
\mathrm{c\o rma}^{\bullet} &= \frac{1}{2}\,\mathrm{adari}(\mathrm{pal}^{\bullet}) \sum_{d_1+d_2=d+2}^{d_i\,even\geq 2} c_{d_1,d_2}\ \mathrm{ari}(\mathrm{s\o t}^{\bullet}_{1+d_1;1}, \mathrm{s\o t}^{\bullet}_{1+d_2;1}) && (40)\\
&+ \frac{1}{2}\,\mathrm{adari}(\mathrm{pal}^{\bullet}) \sum_{d_1+d_2=d+2}^{d_i\,even\geq 2} c_{d_1,d_2}\ \mathrm{ari}(\mathrm{s\o t}^{\bullet}_{1+d_1;1}, \mathrm{s\o t}^{\bullet}_{1+d_2;3}) && (41)\\
&+ \frac{1}{2}\,\mathrm{adari}(\mathrm{pal}^{\bullet}) \sum_{d_1+d_2=d+2}^{d_i\,even\geq 2} c_{d_1,d_2}\ \mathrm{ari}(\mathrm{s\o t}^{\bullet}_{1+d_1;3}, \mathrm{s\o t}^{\bullet}_{1+d_2;1}) && (42)
\end{aligned}$$

Due to (34) and (38), the right-hand side of (40) vanishes. So we are left with the contributions (41) and (42), from which the operator $adari(pal^\bullet)$ may be removed, since it acts on bimoulds which are already in $BIMU_4$. Thus:

$$\text{cørma}^\bullet \;=\; \frac{1}{2}\sum_{d_1+d_2=d+2}^{d_i\,even\geq 2} c_{d_1,d_2}\;\text{ari}(\text{søt}^\bullet_{1+d_1;1},\text{søt}^\bullet_{1+d_2;3}) \qquad (43)$$

$$+\;\frac{1}{2}\sum_{d_1+d_2=d+2}^{d_i\,even\geq 2} c_{d_1,d_2}\;\text{ari}(\text{søt}^\bullet_{1+d_1;3},\text{søt}^\bullet_{1+d_2;1}) \qquad (44)$$

Or again, due to the alternality of *precar*:

$$\text{cørma}^\bullet \;=\; \sum_{d_1+d_2=d+2}^{d_i\,even\geq 2} c_{d_1,d_2}\;\text{ari}(\text{søt}^\bullet_{1+d_1;3},\text{søt}^\bullet_{1+d_2;1}) \qquad (45)$$

In view of (34) and (35), this means, quite simply, that $cørma^\bullet$ is a *composite* singulate of type $søt^\bullet_{2,1,1}$, which as such immediately converts into a *composite* bisingulate of type $bisøt^\bullet_{2,1,1}$ (plus of course a harmless string of trivial terms of the form $søt^\bullet_{1,1,1,1} = bisøt^\bullet_{1,1,1,1}$) under the simple rule of §8.3 that exchanges *simple* singulates of type $søt^\bullet_2$ and *simple* bisingulates of type $bisøt^\bullet_2$. Resorting to formula (27) and choosing for example the canonical ("perinomal") realisation $luma^\bullet$ of $løma^\bullet$ with the corresponding *singulands* $sut^\bullet_{2,1}$ given by the formula (6.28) in [E3] (set $(r_1, r_2) = (1, 2)$ rather than $(2, 1)$ due to the change of ordering convention), we arrive at the perinomal realisation *curma* of our *cørma* bialternals, with a *totally* explicit expansion in terms of *bisingulates* – an expansion which, however, does not compare too favourably with the original expression in terms of *singulates*.

8.6 The messy structure of $ALAL_{sing}$. "Wandering" bialternals at all polar heights.

Polynomial (bi)singulands $(bi)sønd^{w_1,\ldots,w_s}_{r_1,\ldots,r_s}$, that is to say (bi)singulands with values in $\mathbb{C}[[u_1, \ldots, u_s]]$, often produce (bi)singulates of *effective* polarity strictly less than their formal polarity $\sum(r_i-1) = r-s$. At the lowest end of the polarity chain, they produce singularity-free (i.e polynomial) bialternals – the proper "(bi)wandering bialternals" – which, while rather thin on the ground (they are incomparably less numerous than the general polynomial bialternals), are still responsible for the residual indeterminacy that mars the so-called semi-canonical realisations of $løma^\bullet$, $røma^\bullet$ etc.

This indeterminacy only disappears, at the cost of much hard work, in the fully canonical realisationss, of which three types are known:

(i) the perinomal type $luma^\bullet$, $ruma^\bullet$ etc, which maximises functional smoothness
(ii) the arithmetical type $lama^\bullet$, $rama^\bullet$ etc, which maximises arithmetical smoothness
(iii) the mixed type $loma^\bullet$, $roma^\bullet$ etc, which makes the best of both worlds by taking advantage of the symmetries observed in the perinomal construction while retaining a measure of arithmetical smoothness.

The bottom-line is that these wandering bialternals are a very real nuisance. They make a whole mess of the structure of $ALAL_{sing}$ – a hard fact that cannot be papered over or done away with by a simple sleight of hand.[15]

One of the most glaring manifestations of the wandering bialternals with their nuisance value is this: the perinomal decompositions (see (50), (51) *infra*), which have the merit of uniqueness, sometimes express Taylor-rational singulates[16] in terms of Taylor-irrational singulands.[17] No contadiction here – this is just the wanderers at their tricks! For instance, it is still a moot question (which we haven't found the time to address) whether the perinomal conversion formula (67) expresses the Taylor-rational bisingulate on the left-hand side in terms of Taylor-rational or Taylor-irrational singulands on the right-hand side.

8.7 The tidy structure of $ALAL_{eumero}$. Uniqueness of decomposition.

Perinomal calculus.[18]
Consider the elementary singulands

$$\mathrm{sund}^\bullet_{\left[{n_1 \atop r_1}\right]} \quad := \quad P(u_1{-}n_1) + (-1)^{r_1}.P(u_1{+}n_1) \tag{46}$$

They are in $BIMU_1^{even}$ if r_1 is odd and in $BIMU_1^{odd}$ if r_1 is even. They can therefore be subjected to the singulators $slank_{r_1}$ to produce simple bialternal singulates

$$\mathrm{sut}^\bullet_{\left[{n_1 \atop r_1}\right]} \quad := \quad \mathrm{slank}_{r_1}.\mathrm{sund}^\bullet_{\left[{n_1 \atop r_1}\right]} \quad \textit{for} \ \ r_1 > 1 \tag{47}$$

[15]We may be wrong, but we suspect that little would be gained by equating $ALAL_{sing}$ with other structures, for behind the changed appearance these stuctures would perforce be of equal messiness.

[16]i.e. singulates which, after multiplication by $u_0u_1..u_r$, produce power series with only rational Taylor coefficients.

[17]i.e. singulands carrying at least some irrational Taylor coefficients.

[18]What follows is a very sketchy account of perinomal calculus. For a more detailed introduction, see [E3] and also [E2].

For $r_1 = 1$, we must set $sut^\bullet_{\left[{n_1 \atop 1}\right]} := sund^\bullet_{\left[{n_1 \atop 1}\right]}$. Of course, the singulands $sund^\bullet_{\left[{n_1 \atop r_1}\right]}$ are in BIMU_1 while the same-indexed singulates $sut^\bullet_{\left[{n_1 \atop r_1}\right]}$ are in BIMU_{r_1}. This is one of those minor notational inconsistencies that cannot be helped and that we must learn to take in our stride.

We can then proceed to form the composite bialternal singulates

$$\mathrm{sut}^\bullet_{\left[{n_1 \atop r_1} {,\dots, \atop ,\dots,} {n_s \atop r_s}\right]} := \vec{\mathrm{ari}}(\mathrm{sut}^\bullet_{\left[{n_1 \atop r_1}\right]}, \dots, \mathrm{sut}^\bullet_{\left[{n_s \atop r_s}\right]}) \tag{48}$$

Each such length-r singulate, whether simple or composite, posssesses three outstanding properties:

$\mathbf{P_1}$: it is *bialternal* (hence an even function of $\boldsymbol{w}$, which here reduces to $\boldsymbol{u}$)

$\mathbf{P_2}$: it is a meromorphic function of $\boldsymbol{u}$, with all its singularities located at multi-integers $\boldsymbol{n} := (n_1, \dots, n_r)$ and of *eupolar* type, i.e. of the form

$$\mathrm{Const}\, \mathcal{P}^{\left({u_1-n_1 \atop v_1} {,\dots, \atop ,\dots,} {u_r-n_r \atop v_r}\right)} \qquad \textit{with} \qquad \mathcal{P}^{\left({u_1 \atop v_1} {,\dots, \atop ,\dots,} {u_r \atop v_r}\right)} \in \mathrm{Flex}_r(\mathrm{Pa}) \tag{49}$$

$\mathbf{P_3}$: it has only *good* poles at the origin, i.e. poles of the form $P(u_i)$ with $0 \le i \le r$ and the usual convention $P(u_0) := P(-(u_1 + \dots + u_r))$.

It is therefore tempting to consider the space $ALAL^{\#}_{eumero}$ of all formal combinations $A^\bullet$, finite or infinite, of eupolar multipoles (49) that verify[19] the conditions $\mathbf{P_1}$, $\mathbf{P_2}$, $\mathbf{P_3}$.

Rather unsurprisingly, $\mathrm{ALAL}^{\#}_{eumero}$ *turns out to be stable under* ari[20]*. But the real beauty is that each length-r element* $\mathrm{A}^\bullet_r$ *in the algebra* $\mathrm{ALAL}^{\#}_{eumero}$ *can be expressed, in a unique way, as a perinomal expansion of the form*

$$\mathrm{A}^\bullet_r = \sum_{s \ge 1} \sum_{r_i \ge 1}^{\sum r_i = r} \sum_{n_i \ge 1} \theta^{\left({n_1 \atop r_1} \middle| {\dots \atop \dots} \middle| {n_s \atop r_s}\right)} \mathrm{presut}^\bullet_{\left[{n_1 \atop r_1} \middle| {\dots \atop \dots} \middle| {n_s \atop r_s}\right]} \tag{50}$$

$$= \sum_{s \ge 1} \sum_{r_i \ge 1}^{\sum r_i = r} \sum_{n_i \ge 1} \theta^{\left({n_1 \atop r_1} \middle| {\dots \atop \dots} \middle| {n_s \atop r_s}\right)} \frac{1}{s}\, \mathrm{sut}^\bullet_{\left[{n_1 \atop r_1} \middle| {\dots \atop \dots} \middle| {n_s \atop r_s}\right]} \tag{51}$$

with well-defined alternal coefficients $\theta^\bullet$ *.*

Remark 8.7.1. Of the two expansions (50) and (51), the former is the more important from a theoretical viewpoint, since it automatically ensures the

[19]globally, i.e. when regrouped inside $A^\bullet$.

[20]Contrary to appearances, the calculation of $C^\bullet = ari(A^\bullet, B^\bullet)$ offers no difficulty even when $A^\bullet$ and $B^\bullet$ are both infinite combinations of eupolar multipoles, since for any given multi-integer $\boldsymbol{n}$, only a finite number of multipoles in $A^\bullet$ and $B^\bullet$ are liable to contribute to the multipoles of $C^\bullet$ over $\boldsymbol{n}$.

alternality of $\theta^\bullet$. Its only drawback is the presence in it of *preari*-brackets

$$\mathrm{presut}^\bullet_{\left[{n_1 \,,\ldots,\, n_s \atop r_1 \,,\ldots,\, r_s}\right]} := \vec{\mathrm{preari}}(\mathrm{sut}^\bullet_{\left[{n_1 \atop r_1}\right]}, ..., \mathrm{sut}^\bullet_{\left[{n_s \atop r_s}\right]}) \tag{52}$$

which, taken in isolation, do not belong to $\mathrm{ALAL}^{\#}_{eumero}$. The expansion (51) is of course to be preferred in practical calculations, since it involves fewer individual summands[21] and these, moreover, are all in $\mathrm{ALAL}^{\#}_{eumero}$. The drawback here is that (51), on its own, does not imply the alternality of $\theta^\bullet$. However, that can be easily remedied: if from the start we *impose* alternality on $\theta^\bullet$ and choose a basis - *any basis* - in the space spanned by the $\mathrm{sut}^\bullet_{\left[{n_1 \atop r_1}\middle|{\ldots \atop \ldots}\middle|{n_s \atop r_s}\right]}$, the corresponding coefficients $\theta^\bullet$ will be uniquely determined and will, by alternality, unambiguously determine the whole system.

Remark 8.7.2. A remarkable feature of the expansions (50) or (51) is that they often involve an *infinite* number of summands even when $A^\bullet$ itself is only a *finite* combination of eupolar multipoles. We shall encounter too striking instances of this phenomenon – first in the conversion formulae (67), then in the Exercise at the end of the subsection §8.8. (see (74)).

Remark 8.7.3. In all natural instances, the so-called *perinomal coefficients* $\theta^\bullet$ featuring in (50) and (51) are calculable from definite induction rules (that crucially involve the action of $Sl_s(\mathbb{Z})$) and, oftener than not, expressible in terms of the entries of continuous fractions (when $s = 2$) or of higher-dimensional analogues (when $s \geq 3$).

Remark 8.7.4. The decompositions (50) and (51) associate to each $A^\bullet$ in $\mathrm{ALAL}^{\#}_{eumero}$ well-defined singulands

$$\mathrm{Poten}_{r_1,\ldots,r_s}.A^\bullet := \sum_{n_i \geq 1} \theta^{\left({n_1 \atop r_1}\middle|{\ldots \atop \ldots}\middle|{n_s \atop r_s}\right)} \mathrm{mu}(\mathrm{sund}^\bullet_{\left[{n_1 \atop r_1}\right]}, \ldots, \mathrm{sund}^\bullet_{\left[{n_s \atop r_s}\right]}) \tag{53}$$

$$\mathrm{Poten}_{[r_1,\ldots,r_s]}.A^\bullet := \sum_{n_i \geq 1} \theta^{\left({n_1 \atop r_1}\middle|{\ldots \atop \ldots}\middle|{n_s \atop r_s}\right)} \frac{1}{s} \vec{\mathrm{lu}}(\mathrm{sund}^\bullet_{\left[{n_1 \atop r_1}\right]}, \ldots, \mathrm{sund}^\bullet_{\left[{n_s \atop r_s}\right]}) \tag{54}$$

which may collectively be viewed as some sort of "potential" from which $A^\bullet$ is "derived".

Remark 8.7.5. Nearly all elements of $ALAL^{\#}_{eumero}$ encountered in actual life possess nice convergence properties, and so do their canonical expansions

[21] due to the *a priori* relation between multiple Lie brackets.

(50) and (51) – although the two statements are by no means equivalent (see Remark 8.7.2 *supra*). These properties amount
(i) either to the absolute convergence of all summands (though this is rarely the case)
(ii) or to the absolute convergence of the 'corrected summands', i.e. after subtraction from each summand of a suitable constant or of a simple polyomial of fixed degree
(iii) or again to blockwise convergence, for some natural choice of blocks.

We won't bother with these distinctions here, only state that there exists a nice subalgebra $ALAL_{eumero}$ of $ALAL^{\#}_{eumero}$ whose elements $A^\bullet$ are not just formal combinations of eupolar multipoles but *bona fide* meromorphic functions – *eumeromorphic* functions, for short.

Remark 8.7.6. For any such $A^\bullet_r$ of given length r, the question naturally arises of resumming their "potentials" (53), (54) , i.e. the singulands implicit in the canonical expansions (50), (51). Since the basic mono-singulands possess these elementary Taylor expansions

$$\mathrm{sund}^{w_1}_{\left[{n_1 \atop r_1}\right]} := -2 \sum_{\sigma_1 \geq 1}^{\sigma_1+r_1 \text{ odd}} n_1^{-\sigma_1}\ u_1^{\sigma_1-1} \tag{55}$$

our "potentials", if at all summable, ought to possess expansions of the form

$$(\mathrm{Poten}_{r_1,\dots,r_s}.A)^{w_1,\dots,w_s} := \sum_{\sigma_i \geq 1}^{\sigma_i+r_i\ odd} \rho^{\left({\sigma_1 \atop r_1}\middle|{\dots \atop \dots}\middle|{\sigma_s \atop r_s}\right)}\ u_1^{\sigma_1-1}\dots u_s^{\sigma_s-1} \tag{56}$$

with Taylor coefficients given by

$$\rho^{\left({\sigma_1 \atop r_1}\middle|{\dots \atop \dots}\middle|{\sigma_s \atop r_s}\right)} := (-2)^s \sum_{n_i \geq 1}\ \theta^{\left({n_1 \atop r_1}\middle|{\dots \atop \dots}\middle|{n_s \atop r_s}\right)} n_1^{-\sigma_1}\dots n_s^{-\sigma_s} \tag{57}$$

Now, in all instances investigated so far, the series (57) converge absolutely for all large enough σ_i's, and "semi-converge" for the remaining small values. In either case, summation or resummation is straightforward. It yields the so-called *perinomal numbers* $\rho^\bullet$, whose arithmetical nature (rational, algebraic, multizetaic, general-transcendental) is clearly of prime theoretical importance. It is closely tied up with the shape of the perinomal induction that defines the corresponding *perinomal coefficient* $\theta^\bullet$.

These scanty indications should suffice to show how much we gain by changing from $ALAL_{sing}$ to $ALAL_{eumero}$. All the obnoxious irregularity, redundancy and indeterminacy inherent in the first structure dissipate at one magic stroke, like the famed Mists of Avalon, when we move to $ALAL_{eumero}$. New problems inevitably arise, but exhilarating ones, and of a totally new order. They revolve around two main themes: the *perinomal coefficients* $\theta^\bullet$ with their defining inductions; and the *perinomal numbers* $\rho^\bullet$ with their arithmetical properties. Perinomal equations and the group action of $Sl_r(\mathbb{Z})$ dominate the whole field. *Perinomal calculus* is truly the beating heart of *flexion theory.* It opens such rich vistas that one could be forgiven for calling it a wonder within a wonder.

8.8 The perinomal conversion formula.

For the reasons laid down in Remark 8.3.2 (*"First consequence"*), if we are to succeed in our attempt at expressing bisingulates in terms of singulates in the perinomal context, we cannot avoid the introduction of higher-order poles. So we must introduce new parameters π_i to measure that "excess polarity".

For the singulates, this leads to the following definitions:

$$\mathrm{sund}^{\bullet}\begin{bmatrix} n_1 \\ \pi_1 \\ r_1 \end{bmatrix} := (u_1 - n_1)^{-1-\pi_1} + (-1)^{r_1+\pi_1}.(u_1 + n_1)^{-1-\pi_1} \tag{58}$$

$$\mathrm{sut}^{\bullet}\begin{bmatrix} n_1 \\ \pi_1 \\ r_1 \end{bmatrix} := \mathrm{slank}_r.\mathrm{sund}^{\bullet}\begin{bmatrix} n_1 \\ \pi_1 \\ r_1 \end{bmatrix} \quad \in \mathrm{BIMU}_r \tag{59}$$

$$\mathrm{sut}^{\bullet}\begin{bmatrix} n_1 & ,\dots, & n_s \\ \pi_1 & ;\dots; & \pi_s \\ r_1 & ;\dots; & r_s \end{bmatrix} := \vec{\mathrm{ari}}\,(\mathrm{sut}^{\bullet}\begin{bmatrix} n_1 \\ \pi_1 \\ r_1 \end{bmatrix}, \dots, \mathrm{sut}^{\bullet}\begin{bmatrix} n_s \\ \pi_s \\ r_s \end{bmatrix}) \tag{60}$$

$$\mathrm{presut}^{\bullet}\begin{bmatrix} n_1 & ,\dots, & n_s \\ \pi_1 & ;\dots; & \pi_s \\ r_1 & ;\dots; & r_s \end{bmatrix} := \vec{\mathrm{preari}}\,(\mathrm{sut}^{\bullet}\begin{bmatrix} n_1 \\ \pi_1 \\ r_1 \end{bmatrix}, \dots, \mathrm{sut}^{\bullet}\begin{bmatrix} n_s \\ \pi_s \\ r_s \end{bmatrix}) \tag{61}$$

For the bisingulates, we have a parallel set of definitions

$$\mathrm{bisund}^{\bullet}\begin{bmatrix} n_1 \\ \pi_1 \\ r_1 \end{bmatrix} := (u_1 - n_1)^{-1-\pi_1} - (-1)^{\pi_1}.(u_1 + n_1)^{-1-\pi_1} \tag{62}$$

$$\mathrm{bisut}^{\bullet}\begin{bmatrix} n_1 \\ \pi_1 \\ r_1 \end{bmatrix} := \mathrm{slank}_r.\mathrm{bisund}^{\bullet}\begin{bmatrix} n_1 \\ \pi_1 \\ r_1 \end{bmatrix} \quad \in \mathrm{BIMU}_r \tag{63}$$

$$\mathrm{bisut}^{\bullet}\begin{bmatrix} n_1 & ,\dots, & n_s \\ \pi_1 & ;\dots; & \pi_s \\ r_1 & ;\dots; & r_s \end{bmatrix} := \vec{\mathrm{ari}}\,(\mathrm{bisut}^{\bullet}\begin{bmatrix} n_1 \\ \pi_1 \\ r_1 \end{bmatrix}, \dots, \mathrm{bisut}^{\bullet}\begin{bmatrix} n_s \\ \pi_s \\ r_s \end{bmatrix}) \tag{64}$$

$$\mathrm{prebisut}^{\bullet}\begin{bmatrix} n_1 & ,\dots, & n_s \\ \pi_1 & ;\dots; & \pi_s \\ r_1 & ;\dots; & r_s \end{bmatrix} := \vec{\mathrm{preari}}\,(\mathrm{bisut}^{\bullet}\begin{bmatrix} n_1 \\ \pi_1 \\ r_1 \end{bmatrix}, \dots, \mathrm{bisut}^{\bullet}\begin{bmatrix} n_s \\ \pi_s \\ r_s \end{bmatrix}) \tag{65}$$

However, as we shall see, for bisingulates the only case of real relevance is $s = 1$ and $\pi_1 = 0$, since perinomal conversion will prove feasible in one direction only – from bisingulates to singulates.

The general conversion formula in perinomal form.

Proposition 8.7.1: Canonical perinomal conversion of bisingulates into singulates. *With the above notations there exists a unique, well-defined, three-tiered*[22]*, rational valued, alternal bimould* $\theta^\bullet$ *such that for any* r *the identity holds*

$$bisut^{\bullet}_{\left[\begin{smallmatrix}1\\0\\r\end{smallmatrix}\right]} = \sum_{s\geq 1} \sum_{r_i\geq 1}^{\sum r_i=r} \sum_{\pi_i\geq 0}^{\sum \pi_i=r-1} \sum_{n_i\geq 1} \theta^{\left(\begin{smallmatrix}n_1\\\pi_1\\r_1\end{smallmatrix}\middle|\begin{smallmatrix}\dots\\\vdots\end{smallmatrix}\middle|\begin{smallmatrix}n_s\\\pi_s\\r_s\end{smallmatrix}\right)} presut^{\bullet}_{\left[\begin{smallmatrix}n_1\\\pi_1\\r_1\end{smallmatrix}\middle|\begin{smallmatrix}\dots\\\vdots\end{smallmatrix}\middle|\begin{smallmatrix}n_s\\\pi_s\\r_s\end{smallmatrix}\right]} \tag{66}$$

$$= \sum_{s\geq 1} \sum_{r_i\geq 1}^{\sum r_i=r} \sum_{\pi_i\geq 0}^{\sum \pi_i=r-1} \sum_{n_i\geq 1} \theta^{\left(\begin{smallmatrix}n_1\\\pi_1\\r_1\end{smallmatrix}\middle|\begin{smallmatrix}\dots\\\vdots\end{smallmatrix}\middle|\begin{smallmatrix}n_s\\\pi_s\\r_s\end{smallmatrix}\right)} \frac{1}{s}\, sut^{\bullet}_{\left[\begin{smallmatrix}n_1\\\pi_1\\r_1\end{smallmatrix}\middle|\begin{smallmatrix}\dots\\\vdots\end{smallmatrix}\middle|\begin{smallmatrix}n_s\\\pi_s\\r_s\end{smallmatrix}\right]} \tag{67}$$

with an elementary first sum that reduces to a single term

$$\sum_{n_1\geq 1} \theta^{\left(\begin{smallmatrix}n_1\\r-1\\r_1\end{smallmatrix}\right)} \mathrm{presut}^{\bullet}_{\left[\begin{smallmatrix}n_1\\r-1\\r_1\end{smallmatrix}\right]} \equiv \theta^{\left(\begin{smallmatrix}1\\r-1\\r_1\end{smallmatrix}\right)} \mathrm{sut}^{\bullet}_{\left[\begin{smallmatrix}1\\r-1\\r_1\end{smallmatrix}\right]} \equiv (-1)^{r-1} \left((r-1)!\right)^2 \mathrm{sut}^{\bullet}_{\left[\begin{smallmatrix}1\\r-1\\r_1\end{smallmatrix}\right]} \tag{68}$$

The existence and uniqueness of the above expansions, while far from trivial, result from an adaptation of the two central lemmas of perinomal calculus – despite the presence, unusual in the perinomal context, of higher-order poles.

A word about convergence: to get absolute convergence in the expansions (74), (67), certain precautions have to be taken, such as subtracting suitable constants (or simple polynomials of fixed degree) from each individual summand or, alternatively, taking care of performing the summation block-wise, with suitable blocks. This is standard perinomal practice. In any case, when calculating the Taylor coefficients, absolute convergence is automatic for all but a finite number of them.

From perinomal to polynomial. Reverse conversion.

The singulands $sund^{\bullet}_{\left[\begin{smallmatrix}n_1\\\pi_1\\r_1\end{smallmatrix}\middle|\begin{smallmatrix}\dots\\\vdots\end{smallmatrix}\middle|\begin{smallmatrix}n_s\\\pi_s\\r_s\end{smallmatrix}\right]} := mu\left(sund^{\bullet}_{\left[\begin{smallmatrix}n_1\\\pi_1\\r_1\end{smallmatrix}\right]}, \dots, sund^{\bullet}_{\left[\begin{smallmatrix}n_s\\\pi_s\\r_s\end{smallmatrix}\right]}\right)$ implicitely involved in the singulates $sut^{\bullet}_{\left[\begin{smallmatrix}n_1\\\pi_1\\r_1\end{smallmatrix}\middle|\begin{smallmatrix}\dots\\\vdots\end{smallmatrix}\middle|\begin{smallmatrix}n_s\\\pi_s\\r_s\end{smallmatrix}\right]}$ of (67) carry only multipoles of the form $\prod_{i=1}^{i=s} P(u_i - n_i)$ with $n_i \in \mathbb{Z}^*$. They can therefore be expanded into power series of the u_i's. All these power series can be regrouped into one (about convergence issues, see the above remarks) and then subjected to the corresponding singulator $\mathrm{slank}_{r_1,\dots,r_s}$. Fixing a degree d and collecting all terms of total homogeneous degree d on the right-hand side of (67) and

[22] i.e. with three tiers of indices (n_i, π_i, r_i).

on the left-hand side of (74), we clearly arrive at a conversion formula of type (29), for any chosen monomial $\mathit{bis\o nd}^{\bullet}_{r}$. This system of formulae (29), in turn, is easily invertible to the system of formulae (30). The remarkable thing, however, is that the original perinomal conversion formulae (38) or (39) *cannot*, try as we may, be inverted in perinomal form, for the reasons given in Remark 8.3.2 (*"Second consequence"*): any attempt to do so would immediately unleash an epidemic of logarithmic singularities.

The case $r = 2$.

The general formula yields

$$\begin{aligned}
\mathrm{bisut}^{\bullet}_{\left[\begin{smallmatrix}1\\0\\2\end{smallmatrix}\right]} &= \theta^{\left(\begin{smallmatrix}1\\1\\2\end{smallmatrix}\right)}\,\mathrm{sut}^{\bullet}_{\left[\begin{smallmatrix}1\\1\\2\end{smallmatrix}\right]} + \sum_{n_i\geq 1}\theta^{\left(\begin{smallmatrix}n_1\\0\\1\end{smallmatrix}\middle|\begin{smallmatrix}n_2\\1\\1\end{smallmatrix}\right)}\mathrm{presut}^{\bullet}_{\left[\begin{smallmatrix}n_1\\0\\1\end{smallmatrix}\middle|\begin{smallmatrix}n_s\\1\\1\end{smallmatrix}\right]} + \sum_{n_i\geq 1}\theta^{\left(\begin{smallmatrix}n_1\\0\\1\end{smallmatrix}\middle|\begin{smallmatrix}n_2\\1\\1\end{smallmatrix}\right)}\mathrm{presut}^{\bullet}_{\left[\begin{smallmatrix}n_1\\1\\1\end{smallmatrix}\middle|\begin{smallmatrix}n_s\\0\\1\end{smallmatrix}\right]}\\
&= \theta^{\left(\begin{smallmatrix}1\\1\\2\end{smallmatrix}\right]}\,\mathrm{sut}^{\bullet}_{\left[\begin{smallmatrix}1\\1\\2\end{smallmatrix}\right)} + \sum_{n_i\geq 1}\theta^{\left(\begin{smallmatrix}n_1\\0\\1\end{smallmatrix}\middle|\begin{smallmatrix}n_2\\1\\1\end{smallmatrix}\right)}\mathrm{sut}^{\bullet}_{\left[\begin{smallmatrix}n_1\\0\\1\end{smallmatrix}\middle|\begin{smallmatrix}n_s\\1\\1\end{smallmatrix}\right]} \qquad (69)
\end{aligned}$$

with alternal coefficients $\theta^{\left(\begin{smallmatrix}n_1\\0\\1\end{smallmatrix}\middle|\begin{smallmatrix}n_2\\1\\1\end{smallmatrix}\right)} \equiv -\theta^{\left(\begin{smallmatrix}n_2\\1\\1\end{smallmatrix}\middle|\begin{smallmatrix}n_1\\0\\1\end{smallmatrix}\right)}$ easily calculable by the following induction

$$\begin{aligned}
\theta^{\left(\begin{smallmatrix}n_1\\0\\1\end{smallmatrix}\middle|\begin{smallmatrix}n_2\\1\\1\end{smallmatrix}\right)} &= 0 && \textit{if}\quad (n_1,n_2)\ \textit{not co-prime}\\
&= \tfrac{1}{2} && \textit{if}\quad (n_1,n_2) = (1,1)\\
&= \theta^{\left(\begin{smallmatrix}n_1-n_2\\0\\1\end{smallmatrix}\middle|\begin{smallmatrix}n_2\\1\\1\end{smallmatrix}\right)} && \textit{if}\quad n_1 > n_2\\
&= \theta^{\left(\begin{smallmatrix}n_1\\0\\1\end{smallmatrix}\middle|\begin{smallmatrix}n_2-n_1\\1\\1\end{smallmatrix}\right)} - \theta^{\left(\begin{smallmatrix}n_2-n_1\\0\\1\end{smallmatrix}\middle|\begin{smallmatrix}n_1\\1\\1\end{smallmatrix}\right)} && \textit{if}\quad n_2 > n_1
\end{aligned}$$

The general case $r \geq 3$.

For $r = 3$ when get ten non-trivial perinomal coefficients $\theta^{\bullet}$, which due to alternality reduce to four, for example these four:

$$\theta^{\left(\begin{smallmatrix}n_1\\0\\1\end{smallmatrix}\middle|\begin{smallmatrix}n_2\\2\\2\end{smallmatrix}\right)}\ ,\quad \theta^{\left(\begin{smallmatrix}n_1\\1\\1\end{smallmatrix}\middle|\begin{smallmatrix}n_2\\1\\2\end{smallmatrix}\right)}\ ,\quad \theta^{\left(\begin{smallmatrix}n_1\\0\\1\end{smallmatrix}\middle|\begin{smallmatrix}n_2\\0\\1\end{smallmatrix}\middle|\begin{smallmatrix}n_3\\2\\1\end{smallmatrix}\right)}\ ,\quad \theta^{\left(\begin{smallmatrix}n_1\\0\\1\end{smallmatrix}\middle|\begin{smallmatrix}n_2\\1\\1\end{smallmatrix}\middle|\begin{smallmatrix}n_3\\1\\1\end{smallmatrix}\right)} \qquad (70)$$

All are calculable under simple induction rules similar to the one we just encountered in the case $r = 2$. All vanish when (n_1, n_2) or (n_1, n_2, n_3) are not co-prime. The first two coefficients $\theta^{\bullet}$ (of length 2) are still expressible in terms of the continued fraction of n_1/n_2, while for the last two coefficients $\theta^{\bullet}$ (of length 3) the continuous fraction has to be replaced by a quite interesting analogue for homogeneous integer triplets (n_1, n_2, n_3).

Such 'higher-order continuous fractions' in fact exist for integer sequences $(n_1, ..., n_s)$ of any length and have to be taken into consideration when studying the perinomal coefficients $\theta^\bullet$ of the general conversion formula (67). We intend to soon return to the subject in a much more detailed and explicit paper. Meanwhile, here is a nice little warm-up exercise for readers who might wish to acquaint themselves with perinomal calculus.

Recommended exercise: conversion formulae for *ari* and *oddari*.

Consider the elementary, length-1 bimoulds $Ev_n^\bullet$ and $Od_n^\bullet$

$$\begin{aligned} \mathrm{Ev}_n^{w_1} &:= P(u_1{-}n) - P(u_1{+}n) \qquad \big(\mathrm{Ev}_n^\bullet \in \mathrm{BIMU}_1^{\mathrm{even}}\big) &(71)\\ \mathrm{Od}_n^{w_1} &:= P(u_1{-}n) + P(u_1{+}n) \qquad \big(\mathrm{Od}_n^\bullet \in \mathrm{BIMU}_1^{\mathrm{odd}}\big) &(72) \end{aligned}$$

and recall the definition the *oddari*-bracket from $BIMU_1^{odd} \times BIMU_1^{odd}$ into $BIMU_2^{\underline{\mathrm{al}}/\underline{\mathrm{al}}}$:

$$\begin{aligned} C^\bullet &= \mathrm{oddari}(A^\bullet, B^\bullet) \qquad \Longrightarrow &(73)\\ C^{\binom{u_1\,,\,u_2}{v_1\,,\,v_2}} &:= +A^{\binom{u_1}{v_1}}B^{\binom{u_2}{v_2}} + A^{\binom{-u_1-u_2}{v_2}}B^{\binom{u_1}{v_1-v_2}} + A^{\binom{u_2}{v_2-v_1}}B^{\binom{-u_1-u_2}{-v_1}}\\ &\quad\; -B^{\binom{u_1}{v_1}}A^{\binom{u_2}{v_2}} - B^{\binom{-u_1-u_2}{-v_2}}A^{\binom{u_1}{v_1-v_2}} - B^{\binom{u_2}{v_2-v_1}}A^{\binom{-u_1-u_2}{-v_1}} \end{aligned}$$

Show that for any fixed pair of positive integers (m_1, m_2) (resp. (n_1, n_2)) there exist unique expansions of the form

$$\mathrm{oddari}(\mathrm{Od}_{m_1}^\bullet, \mathrm{Od}_{m_2}^\bullet) = \sum_{0<n_1<n_2} \mathrm{H}_{m_1,m_2}^{n_1,n_2}\, \mathrm{ari}(\mathrm{Ev}_{n_1}^\bullet, \mathrm{Ev}_{n_2}^\bullet) \qquad (74)$$

$$\mathrm{ari}(\mathrm{Ev}_{n_1}^\bullet, \mathrm{Ev}_{n_2}^\bullet) = \sum_{0<m_1<m_2} \mathrm{K}_{n_1,n_2}^{m_1,m_2}\, \mathrm{oddari}(\mathrm{Od}_{m_1}^\bullet, \mathrm{Od}_{m_2}^\bullet) \qquad (75)$$

Show that the structure coefficients $H_{m_1,m_2}^{n_1,n_2}$, $K_{n_1,n_2}^{m_1,m_2}$ automatically vanish unless $gcd(m_1, m_2) = gcd(n_1, n_2)$. We may therefore assume that each pair (m_1, m_2) and (n_1, n_2) is co-prime. Show further that $H_\bullet^\bullet$ and $K_\bullet^\bullet$ always assume their values in the set $\{0, 2, -2\}$ except when the upper and lower indices coincide, in which case $H_{n_1,n_2}^{n_1,n_2} \equiv K_{n_1,n_2}^{n_1,n_2} \equiv -1$. Concentrate on the coefficients $H_\bullet^\bullet$ which are somewhat simpler than the $K_\bullet^\bullet$. Show that for any fixed (m_1, m_2) the vanishing or non-vanishing of $H_{n_1,n_2}^{n_1,n_2}$ only depends on the last entry in the continuous fraction $cofra(n_2/n_1)$ and that, in the non-vanishing case, the sign before 2 depends only on the parity of the length $\#cofra(n_2/n_1)$ of that continued fraction. Thus, for $(m_1, m_2) = (1, 2)$ and $(n_1, n_2) \neq (1, 2)$ and co-prime, $H_{n_1,n_2}^{n_1,n_2}$ never vanishes and is given by $H_{n_1,n_2}^{n_1,n_2} \equiv (-1)^{\#cofra(n_2/n_1)}\, 2$.

8.9 Singulates vs bisingulates. Merits and demerits.

Let us now take stock:

8.9.1. Deceptive simplicity of the bisingulators.
On the face of it, the bisingulators are *much simpler* to define than the singulators. The former require only multiple *ari*-bracketing, while the latter draw on the bisymmetral $pal^\bullet/pil^\bullet$ with all the attendant paraphernalia. Unlike the singulators, bisingulators are also capable of acting on bimoulds of any length, not just of length one. But this appearance turns out to be *deceptive*. The bisingulators' ease of definition is more a liability than an asset – an index of rawness, so to speak. The complexity in the singulators' make, on the other hand, is really a measure of their sophistication – the price to pay for fine-tuning our singularity-generating operators and achieving maximal economy of means in the operands.

8.9.2. Built-in redundancy in the bisingulands.
In the polynomial context, bisingulate expansions are needlessly wasteful in the sense of requiring polynomial singulands of abnormally high degree. Indeed, fix a length r, a degree δ, and a polar order π. The only monomial (bi)singulands capable of generating (bi)singulates of length r, homogeneous degree δ, and effective polarity π, are of the form:

$$\text{sønd}^{w_1,\dots,w_s}_{\left[\substack{d_1\ ,\dots,\ d_s\\ r_1\ ,\dots,\ r_s}\right]} = u_1^{d_1}\dots u_s^{d_s} \quad \textit{with} \ \sum d_i = \delta+\pi \ \ \textit{and} \ \ r_i+d_i \ \ \textit{odd} \tag{76}$$

$$\text{bisønd}^{w_1,\dots,w_s}_{\left[\substack{d_1\ ,\dots,\ d_s\\ r_1\ ,\dots,\ r_s}\right]} = u_1^{d_1}\dots u_s^{d_s} \quad \textit{with} \ \sum d_i = \delta+2\,\pi \ \ \textit{and} \ \ d_i \ \ \textit{even} \geq 2 \tag{77}$$

$$(\textit{and} \qquad s = r-\pi \ \ , \ \sum r_i = r \quad \textit{in both cases})$$

The latter clearly span a larger space than the former:

$$\dim(\text{SAND}_{r,\delta,\pi}) =: \alpha(r,\delta,\pi) \ll \beta(r,\delta,\pi) := \dim(\text{BISAND}_{r,\delta,\pi}) \tag{78}$$

8.9.3. Built-in indeterminacy in the bisingulates.
Let $(BI)SAT_{r,\delta,\pi}$ denote the space of (bi)singulates generated by all the (bi)singulands in $(BI)SAND_{r,\delta,\pi}$. One is a good deal larger than the other, but due to the polynomial conversion formulae (29) and (30), their quotient by $Polar_{r,\delta,\pi-1}$ coincide:

$$\dim\big(\text{SAT}_{r,\delta,\pi}/\text{Polar}_{r,\delta,\pi-1}\big) = \dim\big(\text{BISAT}_{r,\delta,\pi}/\text{Polar}_{r,\delta,\pi-1}\big) =: \gamma(r,\delta,\pi) \tag{79}$$

As a consequence, the mass of "biwandering bialternals" is going to exceed that of "wandering bialternals":

$$\dim\left(\mathrm{WANDER}_{r,\delta,\pi-1}\right) = \alpha(r,\delta,\pi) - \gamma(r,\delta,\pi) \tag{80}$$
$$\dim\left(\mathrm{BIWANDER}_{r,\delta,\pi-1}\right) = \beta(r,\delta,\pi) - \gamma(r,\delta,\pi) \tag{81}$$

This applies for all $\pi > 0$, including $\pi = 1$. This in turn means that the "semi-canonical" realisations of $l\o ma^\bullet / l\o mi^\bullet$ or $r\o ma^\bullet / r\o mi^\bullet$ that rely on bisingulates will be blighted by a greater indeterminacy than the "semi-canonical" realisations based on singulates.

8.9.4. Asymmetry in the singulate-bisingulate conversion rules.
Moving now from the polynomial to the perinomal setting, we note this arresting fact: while there are explicit and beautiful formulae for converting the perinomal bisingulates $bisut^\bullet$ into perinomal singulates $sut^\bullet$, no such formulae exist for the reverse change – at least not in perinomal form (they exist of course in polynomial form (30)). So we have mutual convertibility all right, but of a highly asymmetric sort – smooth and explicit from bisingulates to singulates, complicated and derivative in the reverse direction.

8.9.5. Because they violate 'simple polarity', bisingulators are constitutionally unsuited for perinomal calculus.
Multizeta algebra, when reframed in what is its natural language – flexion theory – makes constant use of eumeromorphic functions *without poles of order higher than* 1 (with respect, that is, to any given variable or combination of variables). While the singulators effortlessly move, breathe and operate within this pre-ordained setting, the bisingulators immediately drag us out of it (as explained in Remark 8.3.2) either by generating (albeit indirectly) poles of higher order[23] or by introducing (again, indirectly) logarithmic singularities.[24] This inadequation to perinomal calculus is perhaps the most damning indictment of bisingulators, and the reason why they could never step into the full spectrum of roles filled by the singulators.

8.9.6. Residual importance of bisingulators.
For all their flaws, bisingulators constitute one of the two big operator families capable of producing *singular bialternality* 'on demand', and it certainly would be rash to dismiss them as undeserving of attention. The fact remains that exploration is still in its early stages and that our knowledge of these objects is scanty at best.[25] Situations may yet emerge, for all we know,

[23]like in the formula (57)

[24]See Remark 8.3.2, *"Second consequence"*.

[25]As far as we are concerned, we originally (about twelve years ago) dabbled with bisin-

where the bisingulators could show themselves at an advantage and manifest greater flexibility than their elders, the well-established singulators. In any case, the bare fact of their existence is of already of some consequence: it has inspired the beautiful, richly structured conversion formula (67), some aspects of which (like the nature – rational or otherwise – of the Taylor coefficients carried by the singulands $sund^{\bullet}$ implicit in (67)) are still unclear.

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gulators only at length 2 and 3; soon noticed that they were outperformed by singulators; and in the sequel completely forgot about them – somewhat prematurely, we now suspect.