Combinatorial aspects of matrix models

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Abstract. We show that under reasonably general assumptions, the first order asymptotics of the free energy of matrix models are generating functions for colored planar maps. This is based on the fact that solutions of the Schwinger-Dyson equations are, by nature, generating functions for enumerating planar maps, a remark which bypasses the use of Gaussian calculus.

1. Introduction

It has long been used in combinatorics and physics that moments of Gaussian matrices have a valuable combinatorial interpretation. The first result in this direction is due to Wigner (1958) who proved that the trace of even moments of a $N \times N$ Hermitian matrix A with i.i.d centered entries with covariance N^{-1} converge as N goes to infinity towards the Catalan numbers which enumerate a large variety of combinatorial objects such as non crossing pair-partitions, rooted trees or Dick paths. If one restricts to Gaussian entries, that is matrices following the law μ_N of the **GUE** which is the probability measure on the set $\mathcal{H}_N$ of $N \times N$ Hermitian matrices with density

$$\mu_N(dA) = \frac{1}{Z_N} e^{-\frac{N}{2}\mathrm{tr}(A^2)} \prod_{1 \leqslant i \leqslant j \leqslant N} d\Re e(A_{ij}) \prod_{1 \leqslant i < j \leqslant N} d\Im m(A_{ij}),$$

it occurs that the corrections to this convergence count graphs which can be embedded on surface of higher genus, a fact which was used by Harer and Zagier (1986). This enumerative property was fully developed after 't Hooft, who considered generating functions of such moments. For instance, c.f. Zvonkin (1997), we have the formal expansion

$$F^N_{tx^4} = \frac{1}{N^2} \log \int e^{-Nt\mathrm{tr}(A^4)} d\mu_N(A) = \sum_{g \geqslant 0} N^{-2g} \sum_{k \geqslant 1} \frac{(-t)^k}{k!} C(k, g)$$

with

$$C(k, g) = \mathrm{Card}\{ \text{ maps with genus } g \text{ with } k \text{ vertices of valence } 4\}$$

Here, maps are connected oriented diagrams which can be embedded into a surface of genus g in such a way that edges do not cross and the faces of the graph

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(which are defined by following the boundary of the graph) are homeomorphic to a disc. The valence of the vertices comes from the quartic potential. The counting is done up to equivalent classes, i.e. up to homeomorphism. Let us stress that the above equality is only formal and should be understood in the sense that all the derivatives at the origin on both sides of the equality match, i.e. for all $k \in \mathbb{N}$,

$$(-1)^k \partial_t^k F_{tx^4}^N|_{t=0} = \sum_{g \geqslant 0} \frac{1}{N^{2g}} C(k,g).$$

This equality can be proved thanks to Wick's formula since $\partial_t^k F_{tx^4}^N|_{t=0}$ depends only on moments of gaussian variables (note above that the sum is in fact finite).

Such expansions can be generalized to arbitrary polynomial functions (to enumerate maps with vertices of different degrees) and to several-matrices integrals to enumerate colored maps. More precisely, let V be a polynomial of m non-commutative variables, $V_{\mathbf{t}} = \sum_{i=1}^n t_i q_i$ with some monomials q_i and complex parameters $\mathbf{t} = (t_i)_{1 \leqslant i \leqslant n}$ such that $\mathrm{tr} V_{\mathbf{t}}(A_1, \cdots, A_m)$ is real for any self adjoint matrices $A_1, \cdots, A_m$. Then, the free energy expands formally into

$$F_{V_{\mathbf{t}}}^N = \frac{1}{N^2} \log \int e^{-N \mathrm{tr}(V_{\mathbf{t}}(A_1, \cdots, A_m))} \prod_{i=1}^n d\mu_N(A_i) = \sum_{g \geqslant 0} \frac{1}{N^{2g}} F_g(t_1, \cdots, t_n)$$

where for $g \in \mathbb{N}$, F_g is a generating function for the enumeration of colored maps of genus g related to the monomials $(q_i)_{1 \leqslant i \leqslant n}$ (see section 3).

The interest in such formal expansions lies in the hope to be able to estimate the free energy F_V^N when N goes to infinity by probability techniques, henceforth finding formulae for the generating functions $(F_g)_{g \geqslant 0}$. Such a strategy can only be validated if the expansion is not only formal, but for reasonable (small but non zero) parameters $\mathbf{t} = (t_1, \cdots, t_n)$, for all $k \in \mathbb{N}$ and for N large enough,

$$F_{V_{\mathbf{t}}}^N = \sum_{g=0}^k \frac{1}{N^{2g}} F_g(t_1, \cdots, t_n) + o(\frac{1}{N^{2k}}).$$

This means that one can invert the limits of t small and N large in the expansion.

Our aim is to look beyond this formal approach and try to justify this inversion of limits.

In the case of one matrix integrals, this problem is quite well understood at any level of the expansion and for any reasonable potentials V (see Albeverio S. and M. (2001) and Ercolani and McLaughlin (2003) for instance).

Several matrix models are much harder. In the physics literature, the focus is mostly on a few specific integrals; we refer the interested reader to the reviews Di Francesco et al. (1995); Gross D. and S. (1991). In the mathematical literature, fewer matrix integrals could be analyzed and only their first order asymptotics could be derived (see Mehta and Mahoux (1991); Mehta (1981) and Guionnet (2003); Gross and Matytsin (1995)). Even for these last integrals, the relation of their first order asymptotics with the related enumeration problem was not yet established rigorously. In combinatorics, another road was opened by Bousquet-Melou and Schaeffer (2002), following the ideas of Tutte (1968), to enumerate colored planar maps; instead of studying matrix models, they used directly bijection between maps and well labeled trees.

To establish such a relation, we shall study an even more interesting quantity than the free energy, namely, the limiting empirical distribution of matrices; for

$A_1, \cdots, A_m \in \mathcal{H}_N^m$, it is defined as the linear form on the set $\mathbb{C}\langle X_1, \cdots, X_m\rangle$ of polynomials of m non-commutative variables so that

$$\hat{\mu}^N_{A_1,\cdots,A_m}(P) = \frac{1}{N}\mathrm{tr}(P(A_1, \cdots, A_m)) \quad \forall P \in \mathbb{C}\langle X_1, \cdots, X_m\rangle.$$

Let μ_V^N be the Gibbs measure on $\mathcal{H}_N^m$ given by

$$\mu_V^N(dA_1, \cdots, dA_m) = e^{-N^2 F_V^N} e^{-N\mathrm{tr}(V(A_1,\cdots,A_m))} \prod_{i=1}^m d\mu_N(A_i)$$

with F_V^N as above. We shall prove that for reasonnable V, for all polynomials P, $\mu_V^N(\hat{\mu}^N_{A_1,\cdots,A_m}(P))$ converges and its limit is a generating function for maps. We prove this in two steps

• First, we study the solution τ in the algebraic dual of $\mathbb{C}\langle X_1, \cdots, X_m\rangle$ of the so-called Schwinger-Dyson equations $\mathbf{SD[V]}$:

$$\tau_V(1) = 1, \forall P \in \mathbb{C}\langle X_1, \cdots, X_m\rangle, \forall i \in \{1, \cdots, m\}$$

$$\tau_V((X_i + D_i V)P) = \tau_V \otimes \tau_V(\partial_i P).$$

Here, ∂_i and D_i are respectively the non-commutative derivative and the cyclic derivative with respect to the i^{th} variable (see paragraph 2.2). We give sufficient conditions on V so that solutions to this equation exist and are unique.

Moreover, we relate solutions to $\mathbf{SD[V]}$ with generating functions of planar maps. Let us describe these planar maps. We associate to $(X_i)_{1\leqslant i \leqslant m}$ m half-edges of different colors, and to a monomial $q(\mathbf{X}) = X_{i_1} \cdots X_{i_p}$ a star with p colored half-edges by ordering clockwise the half-edges corresponding to $X_{i_1}, \cdots, X_{i_p}$. Such a star is said to be of type q. Note that it has a distinguished half-edge, the first one, X_{i_1}, and its half-edges are oriented by the above clockwise order (one should imagine the star to be fat, each half-edge made of two parallel segments which have opposite orientation, the whole orientation being given by the clockwise order). This defines a bijection between non-commutative monomials and stars. Alternatively, a star can be seen as an oriented circle with colored dots and one marked dot. A map is a connected graph with colored stars, each half-edge of each star being glued with exactly one half-edge of the same color and orientation and the edges obtained in this way do not cross each other (see a more precise description of the planar maps we enumerate in subsection 2.5). As edges around a vertex are cyclically ordered, one can find a canonical embedding of this graph on a surface. The map is said to be planar if we obtain a sphere by this construction.

We can now relate Schwinger-Dyson's equation and maps enumeration:

Theorem 1.1. *Let $V_{\mathbf{t}} = \sum_{i=1}^n t_i q_i$ with t_i in $\mathbb{C}$ and q_i monomials. For all $R > 2$, there exists an open neighborhood $U \subset \mathbb{C}^n$ of the origin (a ball of positive radius) such that:*

- *For $\mathbf{t} \in U$, there exists a unique $\tau_{\mathbf{t}} \in \mathbb{C}\langle X_1, \cdots, X_m\rangle^*$ which is a solution to $\mathbf{SD}[V_{\mathbf{t}}]$ and such that for all p, for all $i_1, \cdots, i_p$ in $\{1, \cdots, m\}$, $|\tau_{\mathbf{t}}(X_{i_1} \cdots X_{i_1})| \leqslant R^p$.*
- *For all P monomial in $\mathbb{C}\langle X_1, \cdots, X_m\rangle$, $\mathbf{t} \to \tau_{\mathbf{t}}(P)$ is analytic on U and for all $k_1, \cdots, k_n$ integers, $(-1)^{\Sigma k_i} \partial_{t_1}^{k_1} \cdots \partial_{t_n}^{k_n} \tau_{\mathbf{t}}(P)|_{\mathbf{t}=0}$ is the number of maps with k_i stars of type q_i and one of type P.*

Hence, $(\tau_{V_{\mathbf{t}}})_{|t|\leqslant\varepsilon}$ are generating states for the enumeration of colored planar maps and Schwinger-Dyson's equations can be viewed as the generating differential equations to enumerate colored planar maps. This is due to the fact that the action of the derivatives ∂_i and D_i on monomials, under the above bijection between stars and monomials, produces natural operations on planar maps.

• Then, we shall see (see section 3) that, under some appropriate assumptions on V, $\hat{\mu}^N_{A_1,\cdots,A_m}$ converges almost surely under $\mu^N_V(dA_1,\cdots,dA_m)$ towards a solution τ_V to the Schwinger-Dyson equations $\mathbf{SD}[V]$.

We show under rather general assumptions that the limit points of $\hat{\mu}^N_{A_1,\cdots,A_m}$ will solve a weak form of the Schwinger-Dyson equation (see section 3.1) which turns into its strong form if the limit points are compactly supported, i.e have all the moments of monomial functions of degree d bounded by R^d for some finite constant R. For small t_i's, this proves that $\hat{\mu}^N_{A_1,\cdots,A_m}$ converges almost surely towards the solution of $\mathbf{SD[V]}$ if we know that the limit points of $\hat{\mu}^N_{A_1,\cdots,A_m}$ satisfy such a bound. We then give sufficient conditions to obtain such an a priori estimate.

For instance, if we consider a convex potential V (see section 3.2) we have:

Theorem 1.2. *Let $a\in[0,1]$, let U_a be the set of t_i's for which $V+\frac{a}{2}\sum_i X_i^2 = V_{\mathbf{t}}+\frac{a}{2}\sum_i X_i^2$ is convex, then there exists $\varepsilon>0$ such that for $(t_i)_{1\leqslant i\leqslant n}\in U_a\cap B(0,\varepsilon)$, $\hat{\mu}^N_{A_1,\cdots,A_m}$ converges in $L^1(\mu^N_{V_{\mathbf{t}}})$ and almost surely to the unique solution to $\mathbf{SD}[V_{\mathbf{t}}]$ as described in theorem 1.1*

Remark: For the one matrix model, Ercolani and McLaughlin (2003) assumed that $V_{\mathbf{t}}=\sum_{i=1}^{2D}t_iX_i$ with $t_{2D}>\gamma\sum_{i=1}^{2D-1}|t_i|$ and $t_{2D}<T$ for some $\gamma,T>0$. Note that if T,γ are large enough the hypothesises of Theorem 1.2 are satisfied.

For general potential V, we obtain a similar result provided we add a cut-off (see section 3.3).

Coming back to the free energy of matrix models, we conclude (see Theorem 3.3) that when the empirical distribution of matrices converges towards the solution to Schwinger-Dyson's equations, the free energy is also a generating function of the associated planar maps.

As a consequence, we can apply these results to the study of Voiculescu's microstates entropy (see section 4) and show that the microstates entropy can be estimated at the solutions to $\mathbf{SD[V]}$ when the t_i's are small enough.

Finally, we compare diverse approaches to the enumeration of planar maps by either using matrix models or combinatorics techniques.

The results of this paper are clearly known, at least at a subconscious level, by physicists, but we could not find any proper reference on the subject. However, we want to emphasize that the use of Schwinger-Dyson's equations is well spread in physics. This paper is rather elementary but provides a mathematical framework to the study of matrix models and related map enumeration. We hope it will demystify this interesting field of physics to mathematicians, or at least to probabilists. The generalization of the techniques developed in this paper to higher order expansions is the subject of a forthcoming article.

2. Schwinger-Dyson's equations and combinatorics

2.1. *Tracial states.* Let $\mathbb{C}\langle X_1,\cdots,X_m\rangle$ be the set of polynomial functions in m self-adjoint non-commutative variables. We endow $\mathbb{C}\langle X_1,\cdots,X_m\rangle$ with the involution given for all $z \in \mathbb{C}$, all $i_1, \cdots, i_p \in \{1, \cdots, m\}$ and all $p \in \mathbb{N}$, by

$$(zX_{i_1} \cdots X_{i_p})^* = \bar{z} X_{i_p} \cdots X_{i_1}.$$

We will say that P in $\mathbb{C}\langle X_1, \cdots, X_m\rangle$ is self-adjoint if $P^* = P$.

For any $R > 0$, completing $\mathbb{C}\langle X_1, \cdots, X_m\rangle$ for the norm

$$||P||_R = \sup_{\mathcal{A}\ C^*-\text{algebra}} \sup_{\substack{a_1,\cdot\cdot,a_m \in \mathcal{A}, \\ a_i = a_i^*, \|a_i\|_{\mathcal{A}} \leqslant R}} \|P(a_1, \cdots, a_m)\|_{\mathcal{A}}$$

produces a C^*-algebra $\mathbb{C}\langle X_1, \cdots, X_m\rangle_R = (\mathbb{C}\langle X_1, \cdots, X_m\rangle, ||.||_R, *)$.

We let $\mathbb{C}\langle X_1, \cdots, X_m\rangle^*$ be the set of self-adjoint linear forms on $\mathbb{C}\langle X_1, \cdots, X_m\rangle$ (i.e linear forms such that $\tau(a^*) = \overline{\tau(a)}$), and denote $\mathbb{C}\langle X_1, \cdots, X_m\rangle_R^*$ the subset of $\mathbb{C}\langle X_1, \cdots, X_m\rangle^*$ of continuous forms with respect to the norm $||.||_R$, i.e the topological dual of $\mathbb{C}\langle X_1, \cdots, X_m\rangle_R$.

We let $\mathcal{M}^m$ be the set of laws of m bounded self-adjoint non-commutative variables, that is the subset of elements τ of $\mathbb{C}\langle X_1, \cdots, X_m\rangle^*$ such that

$$\tau(PP^*) \geqslant 0, \quad \tau(PQ) = \tau(QP) \quad \forall P, Q \in \mathbb{C}\langle X_1, \cdots, X_m\rangle, \quad \tau(1) = 1. \tag{2.1}$$

For any $R < \infty$, $\mathcal{M}_R^m = \mathbb{C}\langle X_1, \cdots, X_m\rangle_R^* \cap \mathcal{M}^m$ is a compact metric space for the weak*-topology by Banach-Alaoglu theorem. Elements of $\mathcal{M}^m = \cup_{R \geqslant 0} \mathcal{M}_R^m$ are said to be compactly supported, by analogy with the case $m = 1$ where they are indeed compactly supported probability measures. A family $(\tau_t)_{t \in I}$ of elements of $\mathcal{M}_R^m$ for some $R < \infty$ is said to be uniformly compactly supported.

To deal with variables which do not have all their moments, we will sometimes change the set of test functions and, following Cabanal Duvillard and Guionnet (2001), consider instead of $\mathbb{C}\langle X_1, \cdots, X_m\rangle$ the complex vector space $\mathcal{C}_{st}^m(\mathbb{C})$ generated by the Stieljes functionals

$$ST^m(\mathbb{C}) = \{ \overrightarrow{\prod_{1 \leqslant i \leqslant p}} (z_i - \sum_{k=1}^m \alpha_i^k \mathbf{X}_k)^{-1} | \quad z_i \in \mathbb{C} \backslash \mathbb{R}, \alpha_i^k \in \mathbb{R}, p \in \mathbb{N} \} \tag{2.2}$$

where $\prod^{\rightarrow}$ is the non-commutative product. We can give to $ST^m(\mathbb{C})$ an involution and a norm

$$||F||_\infty = \sup_{\mathcal{A} C^*-\text{algebra}} \sup_{a_i = a_i^* \in \mathcal{A}} ||F(a_1, \cdots, a_m)||_\infty$$

where the supremum is taken on unbounded operators affiliated with $\mathcal{A}$, which turns it into a C^*-algebra. We denote $\mathcal{C}_{st}^m(\mathbb{R}) = \{G = F + F^*, F \in \mathcal{C}_{st}^m(\mathbb{C})\}$. We let $\mathcal{M}_{ST}^m$ be the set of linear forms on $\mathcal{C}_{st}^m(\mathbb{C})$ which satisfy (2.1) (but with functions of $\mathcal{C}_{st}^m(\mathbb{C})$ instead of $\mathbb{C}\langle X_1, \cdots, X_m\rangle$). If one equips $\mathcal{M}_{ST}^m$ with its weak topology, then $\mathcal{M}_{ST}^m$ is a compact metric space (see Cabanal Duvillard and Guionnet (2001)).

2.2. *Non-commutative derivatives.* We define the non-commutative derivative with respect to X_i, $\partial_i : \mathbb{C}\langle X_1, \cdots, X_m\rangle \to \mathbb{C}\langle X_1, \cdots, X_m\rangle \otimes \mathbb{C}\langle X_1, \cdots, X_m\rangle$ given by the Leibnitz rule

$$\partial_i(PQ) = \partial_i P \times (1 \otimes Q) + (P \otimes 1) \times \partial_i Q$$

for any $P, Q \in \mathbb{C}\langle X_1, \cdots, X_m\rangle$ and the condition

$$\partial_i X_j = \mathbb{1}_{i=j} 1 \otimes 1.$$

In other words, if P is a non-commutative monomial

$$\partial_i P = \sum_{P=P_1 X_i P_2} P_1 \otimes P_2$$

where the sum runs over over all possible decomposition of P as $P_1 X_i P_2$. This definition can be extended to $\mathcal{C}^m_{st}(\mathbb{C})$ by keeping the above Leibnitz rule (but with P, Q in $\mathcal{C}^m_{st}(\mathbb{C})$) and

$$\partial_i (z_i - \sum_{k=1}^m \alpha_k \mathbf{X}_k)^{-1} = \alpha_i (z_i - \sum_{k=1}^m \alpha_k \mathbf{X}_k)^{-1} \otimes (z_i - \sum_{k=1}^m \alpha_k \mathbf{X}_k)^{-1}.$$

We also define the cyclic derivative D_i as follows. Let $m : \mathbb{C}\langle X_1, \cdots, X_m\rangle^{\otimes 2} \to \mathbb{C}\langle X_1, \cdots, X_m\rangle$ (resp. $\mathcal{C}^m_{st}(\mathbb{C}) \otimes \mathcal{C}^m_{st}(\mathbb{C}) \to \mathcal{C}^m_{st}(\mathbb{C})$) be defined by $m(P \otimes Q) = QP$. Then, we set

$$D_i = m \circ \partial_i.$$

Note that, if P is a non-commutative monomial,

$$D_i P = \sum_{P=P_1 X_i P_2} P_2 P_1.$$

2.3. *Schwinger-Dyson's equation.* Let V be in $\mathbb{C}\langle X_1, \cdots, X_m\rangle$ and consider the following equation on $\mathbb{C}\langle X_1, \cdots, X_m\rangle^*$; we will say that $\tau \in \mathbb{C}\langle X_1, \cdots, X_m\rangle^*$ satisfies the Schwinger-Dyson equation with potential V, denoted in short $\mathbf{SD[V]}$, if and only if for all $i \in \{1, \cdots, m\}$ and $P \in \mathbb{C}\langle X_1, \cdots, X_m\rangle$,

$$\tau(1) = 1, \quad \tau \otimes \tau(\partial_i P) = \tau((D_i V + X_i)P) \qquad \mathbf{SD[V]}$$

These equations are called Schwinger-Dyson's equations in physics, but in free probability, one would rather say that the conjugate variable (or alternatively the non-commutative Hilbert transform) $\partial_i^* 1$ under τ is equal to $X_i + D_i V$ for all $i \in \{1, \cdots, m\}$.

2.4. *Uniqueness of the solutions to* $\mathbf{SD[V]}$ *for small parameters.* Let

$$V(X_1, \cdots, X_m) = V_{\mathbf{t}}(X_1, \cdots, X_m) = \sum_{i=1}^n t_i q_i(X_1, \cdots, X_m)$$

where the q_i's are fixed monomial functions of m non-commutative indeterminates and $\mathbf{t} = (t_1, \cdots, t_n)$ are complex parameters.

In this paragraph, we shall consider solutions to $\mathbf{SD}[V_{\mathbf{t}}]$ which satisfy a compactness condition that we shall discuss in the following subsections. Let $R \in \mathbb{R}^+$ (We will always assume $R \geqslant 1$ without loss of generality).

$\mathbf{(H(R))}$*An element $\tau \in \mathbb{C}\langle X_1, \cdots, X_m\rangle^*$ satisfies* $\mathbf{(H(R))}$ *if and only if for all $k \in \mathbb{N}$,*

$$\max_{1 \leqslant i_1, \cdots, i_k \leqslant m} |\tau(X_{i_1} \cdots X_{i_k})| \leqslant R^k.$$

In the sequel, we denote D the degree of V, that is the maximal degree of the $q_i's$; $q_i(X) = X_{j_1^i} \cdots X_{j_{d_i}^i}$ with, for $1 \leqslant i \leqslant n$, $\deg(q_i) =: d_i \leqslant D$ and equality holds for some i.

The main result of this paragraph is

Theorem 2.1. *If we fix $R > 0$ then, there exists $\eta > 0$ such that for all $\mathbf{t} \in \mathbb{C}^n$ such that $|\mathbf{t}| := max_i |t_i| < \eta$,* $\mathbf{SD}[V_\mathbf{t}]$ *has at most one solution which satsifies* $(\mathbf{H(R)})$.

Remark: Note here that it could be believed at first sight that the solutions to $\mathbf{SD[V]}$ are not unique since they depend on the trace of high moments $\tau(q_j P)$. However, our compactness assumption $(\mathbf{H(R)})$ gives uniqueness because it forces the solution to be in a small neighborhood of the law $\tau_0 = \sigma^m$ of m free semi-circular variables, so that perturbation analysis applies. We shall see in Theorem 2.7 that this solution is actually the one which is related with the enumeration of maps.

Proof.

Let us assume we have two solutions τ and τ'. Then, by the equation $\mathbf{SD[V]}$, for any monomial function P of degree $l-1$, for $i \in \{1, \cdots, m\}$,

$$(\tau - \tau')(X_i P) = ((\tau - \tau') \otimes \tau)(\partial_i P) + (\tau' \otimes (\tau - \tau'))(\partial_i P) - (\tau - \tau')(D_i V P)$$

Hence, if we let for $l \in \mathbb{N}$

$$\Delta_l(\tau, \tau') = \sup_{\text{monomial } Q \text{ of degree l}} |\tau(Q) - \tau'(Q)|$$

we get, since if P is of degree $l-1$,

$$\partial_i P = \sum_{k=0}^{l-2} p_k^1 \otimes p_{l-2-k}^2$$

where p_k^i, $i = 1, 2$ are monomial of degree k or the null monomial, and $D_i V$ is a finite sum of monomials of degree smaller than $D-1$,

$$\begin{aligned}\Delta_l(\tau, \tau') &= \max_{P \text{ of degree } l-1} \max_{1 \leqslant i \leqslant m} \{|\tau(X_i P) - \tau'(X_i P)|\} \\ &\leqslant 2 \sum_{k=0}^{l-2} \Delta_k(\tau, \tau') R^{l-2-k} + C|\mathbf{t}| \sum_{p=0}^{D-1} \Delta_{l+p-1}(\tau, \tau')\end{aligned}$$

with a finite constant C (which depends on n only). For $\gamma > 0$, we set

$$d_\gamma(\tau, \tau') = \sum_{l \geqslant 0} \gamma^l \Delta_l(\tau, \tau').$$

Note that under $(\mathbf{H(R)})$, this sum is finite for $\gamma < (R)^{-1}$. Summing the two sides of the above inequality times γ^l we arrive at

$$d_\gamma(\tau, \tau') \leqslant 2\gamma^2 (1 - \gamma R)^{-1} d_\gamma(\tau, \tau') + C|\mathbf{t}| \sum_{p=0}^{D-1} \gamma^{-p+1} d_\gamma(\tau, \tau').$$

We finally conclude that if $(R, |\mathbf{t}|)$ are small enough so that we can choose $\gamma \in (0, R^{-1})$ so that

$$2\gamma^2 (1 - \gamma R)^{-1} + C|\mathbf{t}| \sum_{p=0}^{D-1} \gamma^{-p+1} < 1$$

then $d_\gamma(\tau, \tau') = 0$ and so $\tau = \tau'$ and we have at most one solution. Taking $\gamma = (2R)^{-1}$ shows that this is possible provided

$$\frac{1}{R^2} + C|\mathbf{t}| \sum_{p=0}^{D-1} (2R)^{p-1} < 1$$

so that when R goes to $+\infty$ we need to take $|\mathbf{t}|$ of order at most R^{2-D}.

$\square$

2.5. *Combinatorics.* In this paragraph we describe the combinatorial objects we are considering. A star is the neighbourhood of a vertex in a planar graphs i.e. it is a vertex with some half-edges coming out of it. Theses half-edges are ordered in the clockwise order starting from a distinguished one. We associate to each $i \in \{1, \cdots, m\}$ a different color. Then, we define a bijection between oriented edges-colored stars with a distinguished half-edge and non-commutative monomials as follows. For any $i \in \{1, \cdots, m\}$, we associate to X_i a half-edge of color i. We shall say that a star is of type $q(X_1, \cdots, X_m) = X_{i_1} \cdots X_{i_l}$ if it is a star with l half-edges which we color clockwise ; the first half-edge will be of color i_1, the second of color i_2 ... etc ... until the l^{th} half-edge is colored with color i_l. Note that this star possesses a distinguished half-edge, the one corresponding to X_{i_1}, and an orientation, corresponding to the clockwise order (see figure 1 for an example). By convention, the star of type $q = 1$ is simply a point.

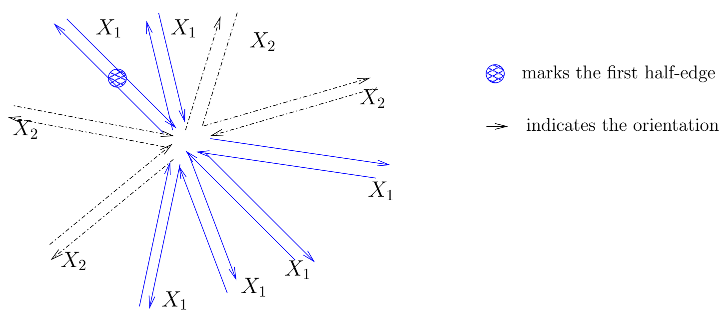

FIGURE 2.1. The star of type $q(X) = X_1^2 X_2^2 X_1^4 X_2^2$

A map is a connected graph whose vertices are colored stars, each half-edge is glued with exactly one half-edge of the same color and orientation. Because the edges coming out of a star are cyclically ordered, we can define the faces of this graph and thus find an embedding of this graph on an orientated surface in such a way that edges do not cross each other (see Zvonkin (1997)). A map is planar if this surface has genus zero, i.e. is the sphere. Planar maps can be thought as graphs embedded on the sphere up to homeomorphism. Maps are only considered up to an homeomorphism of the sphere. Now we will be interested in enumerating maps with a fixed set of stars, we define for q_i the family of monomials which appear in V and $\overline{k} = (k_1, \cdots, k_n)$ a family of integers:

$$\mathcal{M}_{\overline{k}} = \sharp\{\text{planar maps build with } k_i \text{ stars of type } q_i\}.$$

and

$$\mathcal{M}_{\overline{k}}(P) = \sharp\{\text{planar maps build with } k_i \text{ stars of type } q_i \text{ and one of type } P\}.$$

In that set, each star is labeled and has a marked half-edge (which corresponds to its first variable) so that for example if $V = X^4$, $\mathcal{M}_2 = 36$.

Due to the fact that everything is labeled, we enumerate lots of very similar objects. A way to avoid this problem is to look at the maps as they are enumerated by combinatorialists (see Bousquet-Melou and Schaeffer (2002)). The idea is to forget every label and to add a root which is defined as a star and a half-edge of this star. We will say that a map is rooted at a vertex of type P if its root is of type P with the marked half-edge the first one in the above construction of a star from a monomial. We can define for P a monomial, $\overline{k} = (k_1, \cdots, k_n)$,

$$\mathcal{D}_{\overline{k}}(P) = \sharp\{\text{maps with } k_i \text{ stars of type } q_i \text{ rooted at a vertex of type } P\}$$

To go from these rooted maps to the previous one we only have to label each star and be careful about the symmetry of the stars in order to specify a half-edge by star. More precisely, let us define the degree of symmetry $s(q)$ of a monomial q as follows. Let $\omega : \mathbb{C}\langle X_1, \cdots, X_m\rangle \to \mathbb{C}\langle X_1, \cdots, X_m\rangle$ be the linear function so that for all $i_k \in \{1, \cdots, m\}$, $1 \leqslant k \leqslant p$

$$\omega(X_{i_1} X_{i_2} \cdots X_{i_p}) = X_{i_2} \cdots X_{i_p} X_{i_1}$$

and, with $\omega^p = \omega \circ \omega^{p-1}$, define

$$s(q) = \sharp\{0 \leqslant p \leqslant \deg\,(q) - 1 | \omega^p(q) = q\}.$$

We easily see that for all monomial P, distinct monomials q_i (but one of them may be equal to P), and integers k_i:

$$\mathcal{D}_{k_1,\ldots,k_n}(P) = \frac{\mathcal{M}_{k_1,\ldots,k_n}(P)}{\Pi_{i=1}^n k_i! s(q_i)^{k_i}} \tag{2.3}$$

2.6. *Graphical interpretation of Schwinger-Dyson's equations.* We shall now make an assumption on the solutions of Schwinger-Dyson's equation $\mathbf{SD}[V_\mathbf{t}]$ when the parameters belong to an open convex neighborhood of the origin, namely

(H) *There exists a convex neighborhood $U \in \mathbb{C}^n$, a finite real number R and a family $\{\tau_\mathbf{t}, t \in U\}$ of linear forms on $\mathbb{C}\langle X_1, \cdots, X_m\rangle$ so that for all $\mathbf{t}$ in U, $\tau_\mathbf{t}$ is a solution of $\mathbf{SD}[V_\mathbf{t}]$ which satisfies* $\mathbf{H(R)}$.

Note that up to take a smaller set U, we can assume that the conclusions of Theorem 2.1 are valid, i.e for all $\mathbf{t} \in U$ there exists an unique solution to $\mathbf{SD}[V_\mathbf{t}]$ which satisfies $\mathbf{H(R)}$.

The central result of this article is then

Theorem 2.2. *Assume that* **(H)** *is satisfied. Then*

(1) *For any $P \in \mathbb{C}\langle X_1, \cdots, X_m\rangle$, $\mathbf{t} \in U \to \tau_\mathbf{t}(P)$ is $\mathcal{C}^\infty$ at the origin in the sense that for all $\overline{k} = (k_1, k_2, \cdots, k_n) \in \mathbb{N}^n$ there exists $\varepsilon_{\overline{k}} > 0$ so that $\partial_{t_1}^{k_1} \cdots \partial_{t_n}^{k_n} \tau_\mathbf{t}(P)$ exists on $U_\varepsilon = U \cap B(0, \varepsilon)$ with $B(0,\varepsilon) = \{\mathbf{t} \in \mathbb{C}^n : |\mathbf{t}| \leqslant \varepsilon\}$.*

(2) *We let $\tau^{\overline{k}}(P) = (-1)^{k_1+\cdots+k_n} \partial_{t_1}^{k_1} \cdots \partial_{t_n}^{k_n} \tau_\mathbf{t}(P)|_{\mathbf{t}=0}$. Then, we have for all $P \in \mathbb{C}\langle X_1, \cdots, X_m\rangle$ and all $i \in \{1, \cdots, m\}$,*

$$\tau^{\overline{k}}(X_i P) = \sum_{\substack{0 \leqslant p_j \leqslant k_j \\ 1 \leqslant j \leqslant n}} \prod_{j=1}^n C_{k_j}^{p_j} \tau^{\overline{p}} \otimes \tau^{\overline{k} - \overline{p}}(\partial_i P) + \sum_{1 \leqslant j \leqslant n} k_j \tau^{\overline{k} - 1_j}((D_i q_j) P) \tag{2.4}$$

where $1_j(i) = \mathbb{1}_{i=j}$ and $\tau^{\overline{k}}(1) = \mathbb{1}_{\overline{k}=0}$.

(3) *Moreover the $\mathcal{M}_{\overline{k}}(P)$'s satisfy the same family of equations (2.4) than the $\tau_{\overline{k}}(P)$'s. Hence, for any monomial $P \in \mathbb{C}\langle X_1, \cdots, X_m\rangle$, any $k_1, \cdots, k_n \in \mathbb{N}$,*

$$\tau^{\overline{k}}(P) = \mathcal{M}_{\overline{k}}(P).$$

Proof.

• The smoothness of $\mathbf{t} \to \tau_{\mathbf{t}}$ comes as in the proof of Theorem 2.1 from Schwinger-Dyson's equations and induction on the degree of the test polynomial function. Denote $V = V_{\mathbf{t}}$, $\tau = \tau_{\mathbf{t}}$ and take $\mathbf{t} = (t_1, \cdots, t_n), \mathbf{t}' = (t'_1, t'_2, .., t'_n) \in U$. By $\mathbf{SD[V]}$,

$$(\tau_{\mathbf{t}} - \tau_{\mathbf{t}'})[(X_i + D_i V_{\mathbf{t}})P] = (\tau_{\mathbf{t}} - \tau_{\mathbf{t}'}) \otimes \tau_{\mathbf{t}}(\partial_i P) + \tau_{\mathbf{t}'} \otimes (\tau_{\mathbf{t}} - \tau_{\mathbf{t}'})(\partial_i P) \\ + \tau_{\mathbf{t}'}[(D_i V_{\mathbf{t}'} - D_i V_{\mathbf{t}})P]$$

By our finite moment assumption, we deduce that if P is a monomial function of degree $l-1$, for any $i \in \{1, \cdots, m\}$,

$$|\tau_{\mathbf{t}}[(X_i + D_i V_{\mathbf{t}})P] - \tau_{\mathbf{t}'}[(X_i + D_i V_{\mathbf{t}})P]|$$

$$\leqslant 2 \sum_{k=0}^{l-2} \max_{Q \text{ monomial of degree } \leqslant k} |\tau_{\mathbf{t}}[Q] - \tau_{\mathbf{t}'}[Q]| R^{l-2-k} + \sum_{1 \leqslant i \leqslant n} |t_i - t'_i| R^{l+D-1}.$$

Thus we deduce that for any $p \in \mathbb{N}$,

$$\Delta_l(\tau_{\mathbf{t}}, \tau_{\mathbf{t}'}) = \max_i \max_{P \text{ monomial of degree } p-1} |\tau_{\mathbf{t}}(X_i P) - \tau_{\mathbf{t}'}(X_i P)|$$

$$\leqslant 2 \sum_{k=0}^{l-2} \Delta_k(\tau_{\mathbf{t}}, \tau_{\mathbf{t}'}) R^{l-2-k} + \sum_{i=1}^{n} |t_i| \Delta_{l+d_i-1}(\tau_{\mathbf{t}}, \tau_{\mathbf{t}'}) + \sum_{1 \leqslant i \leqslant n} |t_i - t'_i| R^{l+D-1}.$$

Now, let $\gamma \in (0, R^{-1})$ and let's sum both sides of this inequality multiplied by γ^l to obtain, with $d_\gamma(\tau_{\mathbf{t}}, \tau_{\mathbf{t}'}) = \sum_{l \geqslant 0} \gamma^l \Delta_l(\tau_{\mathbf{t}}, \tau_{\mathbf{t}'})$,

$$d_\gamma(\tau_{\mathbf{t}}, \tau_{\mathbf{t}'}) \leqslant 2(1 - \gamma R)^{-1} \gamma^2 d_\gamma(\tau_{\mathbf{t}}, \tau_{\mathbf{t}'}) \\ + \sum_{i=1}^{n} |t_i| \gamma^{-d_i+1} d_\gamma(\tau_{\mathbf{t}}, \tau_{\mathbf{t}'}) + (1 - \gamma R)^{-1} \sum_{1 \leqslant i \leqslant n} |t_i - t'_i| R^{D-1}.$$

Since by definition $\Delta_l(\tau_{\mathbf{t}}, \tau_{\mathbf{t}'}) \leqslant 2R^l$, $d_\gamma(\tau_{\mathbf{t}}, \tau_{\mathbf{t}'})$ is finite for $\gamma R < 1$ we arrive at

$$(1 - 2\gamma^2(1 - R\gamma)^{-1} - \sum_{1 \leqslant i \leqslant n} |t_i| \gamma^{-D+2}) d_\gamma(\tau_{\mathbf{t}}, \tau_{\mathbf{t}'}) \leqslant (1 - R\gamma)^{-1} \sum_{1 \leqslant i \leqslant n} |t_i - t'_i| R^{D-1}.$$

Now, for ε small enough, we can find $\gamma = \gamma(|t|) > 0$ so that

$$1 - 2\gamma^2(1 - R\gamma)^{-1} - \sum_{1 \leqslant i \leqslant n} |t_i| \gamma^{-D+2} > 0$$

and so

$$\sum_{l \geqslant 0} \gamma^l \Delta_l(\tau_{\mathbf{t}}, \tau_{\mathbf{t}'}) \leqslant C(\mathbf{t}) \sum_{1 \leqslant i \leqslant n} |t_i - t'_i|$$

which implies that for all $l \in \mathbb{N}$

$$\Delta_l(\tau_{\mathbf{t}}, \tau_{\mathbf{t}'}) \leqslant C(\mathbf{t}) \gamma^{-l} \sum_{1 \leqslant i \leqslant n} |t_i - t'_i|$$

so that for any monomial function P, $\mathbf{t} \to \tau_{\mathbf{t}}(P)$ is Lipschitz in $U_\varepsilon := U \cap B(0, \varepsilon)$ for ε small enough. Moreover, we have proved that there exists $\eta_0(\varepsilon) = \gamma^{-1} < \infty$, so that

$$\Delta_l(\tau_{\mathbf{t}}, \tau_{\mathbf{t}'}) \leqslant C_0(\varepsilon) \eta_0(\varepsilon)^l |\mathbf{t} - \mathbf{t}'| \text{ with } |\mathbf{t} - \mathbf{t}'| = \max_{1 \leqslant i \leqslant n} |t_i - t'_i|. \tag{2.5}$$

Consequently, $\tau_{\mathbf{t}}$ is almost surely differentiable in U_ε and the derivative satisfies

$$\partial_{t_k}\tau_{\mathbf{t}}[(X_i + D_iV_{\mathbf{t}})P] + \tau_{\mathbf{t}}[D_iq_kP] = \partial_{t_k}\tau_{\mathbf{t}} \otimes \tau_{\mathbf{t}}(\partial_iP) + \tau_{\mathbf{t}} \otimes \partial_{t_k}\tau_{\mathbf{t}}(\partial_iP) \qquad (2.6)$$

for almost all $\mathbf{t} \in U_\varepsilon$. Since $\mathbb{C}\langle X_1, \cdots, X_m\rangle$ is countable, these equalities hold simultaneously for all $P \in \mathbb{C}\langle X_1, \cdots, X_m\rangle$ almost surely, let U'_ε be this subset of U_ε of full probability.

Inequality (2.5) implies that

$$\max_{1\leqslant k\leqslant m}\ \max_{P \text{ monomial of degree } l} |\partial_{t_k}\tau_{\mathbf{t}}(P)| \leqslant C_0(\varepsilon)\eta_0(\varepsilon)^l$$

for all $\mathbf{t} \in U'_\varepsilon$. This bound in turn shows that we can redo the argument as above to see that for $|\mathbf{t}|$ small enough, $\mathbf{t} \to \partial_{t_k}\tau_{\mathbf{t}}(P)$ is Lipschitz. Indeed, if we set

$$\Delta_1(l) = \Delta_l^1(\partial\tau_{\mathbf{t}}, \partial\tau_{\mathbf{t}}) = \max_{1\leqslant k\leqslant m}\ \max_{P \text{ monomial of degree } l} |\partial_{t_k}\tau_{\mathbf{t}}(P) - \partial_{t_k}\tau_{\mathbf{t}'}(P)|$$

we get, for $\mathbf{t}', \mathbf{t} \in U'_\varepsilon$,

$$\Delta_1(l) \leqslant 2\sum_{k=0}^{l-2} \Delta_1(k)R^{l-2-k} + C_0(\varepsilon)|\mathbf{t}-\mathbf{t}'|l\eta_0(\varepsilon)^l + \sum_{i=1}^{n} |t_i|\Delta_1(l + d_i - 1)$$

so that we get that by summation, for $\gamma < \min(R^{-1}, \eta_0(\varepsilon)^{-1})$,

$$(1 - 2(1 - R\gamma)^{-1}\gamma^2 - \sum_{i=1}^{n} |t_i|\gamma^{-d_i+1}) \sum_{l\geqslant 0} \Delta_1(l)\gamma^l \leqslant \gamma^2 C_0(\varepsilon)(1 - \gamma\eta_0(\varepsilon))^{-2}|\mathbf{t} - \mathbf{t}'|.$$

Hence, again, we can choose $\eta_1(\varepsilon) < \infty$ big enough so that there exists $C_1(\varepsilon) < \infty$ so that if ε is small enough

$$\Delta_1(l) \leqslant C_1(\varepsilon)\eta_1(\varepsilon)^l|\mathbf{t} - \mathbf{t}'|.$$

In particular, this shows that we can extend $\mathbf{t} \in U'_\varepsilon \to \partial_{t_k}\tau_{\mathbf{t}}(P)$ for all monomial functions P continuously in U_ε and so the equality (2.6) holds everywhere. Now, we can proceed by induction to see that $\mathbf{t} \to \tau_{\mathbf{t}}(P)$ is $\mathcal{C}^\infty$ differentiable in a neighborhood of the origin. More precisely, for any $\overline{k} = (k_1, \cdots, k_n)$ there exists $\varepsilon = \varepsilon_{\overline{k}} > 0$ so that on U_ε,

$$\tau_{\mathbf{t}}^{\overline{k}}(P) = (-1)^{k_1+\cdots+k_n}\partial_{t_1}^{k_1}\cdots\partial_{t_m}^{k_n}\tau_{\mathbf{t}}(P)$$

exists and furthermore satisfies the equation

$$\tau_{\mathbf{t}}^{\overline{k}}((X_i + D_iV_{\mathbf{t}})P) = \sum_{\substack{0\leqslant p_i\leqslant k_i\\ 1\leqslant i\leqslant n}} \prod_{i=1}^{n} C_{k_i}^{p_i}\tau_{\mathbf{t}}^{\overline{p}} \otimes \tau_{\mathbf{t}}^{\overline{k}-\overline{p}}(\partial_iP) + \sum_{1\leqslant j\leqslant m} k_j\tau_{\mathbf{t}}^{\overline{k}-\mathbb{1}_j}((D_iq_j)P)$$

Applying this result at the origin, we obtain the second point.

• We now show the combinatorial interpretation of (2.4). It is based on the observation that the $\{\tau^{\overline{k}}(P), \overline{k} \in \mathbb{N}^n\}$ and the $\{\mathcal{M}_{\overline{k}}(P), \overline{k} \in \mathbb{N}^n\}$ satisfy the same inductive relations (2.4).

Let us first interpret graphically $\tau^0 = \tau_0$. τ_0 satisfies by definition $\mathbf{SD[0]}$ which is well known to have a unique solution given by the law of m free semi-circular variables (see Voiculescu (1991)). Then, $\tau_0(X_{i_1}\cdots X_{i_k})$ can be computed for instance using cumulants techniques as developed by Speicher (1997); it counts the number of planar maps which can be constructed from the star associated to $X_{i_1}\cdots X_{i_k}$ by gluing together the half-edges of the star colorwise. We prove again this result by induction over the degree of the monomial function. We put $\mathcal{M}_{\overline{k}}(1) = \mathbb{1}_{\overline{k}=0}$ by convention and then we start the induction. Let $i \in \{1, \cdots, m\}$ and $P = X_i Q$. To compute $\mathcal{M}(X_i Q)$, we break the edge between the distinguished half-edge X_i and the other half-edge of Q with which it was glued, then erasing these two half-edges. Since the maps are planar, this decomposes the planar map into two planar maps (see figure 2.2) corresponding respectively to the stars Q_1, Q_2 for any possible choices of Q_1, Q_2 so that $Q = Q_1 X_i Q_2$. Hence

$$\mathcal{M}(X_i Q) = \sum_{Q = Q_1 X_i Q_2} \mathcal{M}(Q_1)\mathcal{M}(Q_2).$$

Thus, if $\mathcal{M}(R) = \tau_0(R)$ for all monomial of degree strictly smaller than P,

$$\mathcal{M}(X_i Q) = \sum_{Q = Q_1 X_i Q_2} \tau_0(Q_1)\tau_0(Q_2) = \tau_0 \otimes \tau_0(\partial_i Q)$$

which completes the argument since the right hand side is exactly $\tau_0(X_i Q)$.

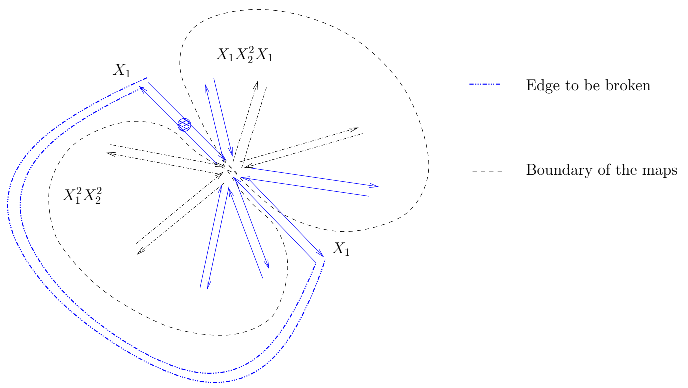

FIGURE 2.2. Decomposition $P(X) = X_1X_2^2X_1^4X_2^2$ into $X_1X_2^2X_1 \otimes X_1^2X_2^2$

We now consider the general case; let us assume that for $|\overline{k}| \leqslant M$, the graphical interpretation has been obtained for all monomial and that for $|\overline{k}| = M + 1$, it has been proved for monomial of degree smaller or equal to L. By the preceding, we can take $M \geqslant 1$ and $L \geqslant 1$ since for all $\overline{k} \neq 0$, $\tau^{\overline{k}}(1) = 0$. Again, we shall show that $\mathcal{M}(P, (q_1, k_1), \cdots, (q_n, k_n))$ satisfies the same induction relation than $\tau^{\overline{k}}(P)$.

Let us consider a star of type $X_i P$ (rooted at the half-edge X_i, with its inner orientation) with P a monomial of degree less than L and $|\overline{k}| = \sum k_i = M+1$. Now, in order to compute $\mathcal{M}(X_i P, (q_1, k_1), \cdots, (q_n, k_n))$, we break the edge between the distinguished half-edge X_i (which has color i) and the other half-edge with which it was glued.

The first possibility is that it was glued with an edge of the star P. Then, since the maps are planar, this decomposes the map in two planar maps. If this half-edge was given by the X_i so that $P = P_1 X_i P_2$, one of this planar map contain the star of type P_1 and the other the star of type P_2, which have also a distinguished half-edge and are oriented. If one of this planar map is glued with k_j stars of type q_j , $0 \leqslant k_j \leqslant n$, the other map is glued with the remaining stars, that is $k_j - p_j$ stars of type q_i. There are $\prod_{j=1}^n C_{k_j}^{p_j}$ ways to choose p_j among k_j stars of type q_j for $1 \leqslant j \leqslant n$ (recall here that stars are labeled). Since we do that for all (P_1, P_2)

so that P have the above decomposition, we obtain the planar maps corresponding actually to the stars associated with the monomials of $\partial_i P$. Note that the case where one of the monomial in $\partial_i P$ is the monomial 1 shows up when $P = X_i Q$ or QX_i for some monomial Q and the weight corresponds then to the case where we glue the first half-edge X_i in $X_i P$ with its left or right neighbor. In this case, none of these two half-edges can be glued with another star, and there is only one possibility to glue these two half-edges otherwise, which corresponds to the weight $\tau^{\overline{k}}(1) = \mathbb{1}_{\overline{k}=0}$.

Hence, the number of planar maps corresponding to this configuration is given by

$$\sum_{\substack{0 \leqslant p_j \leqslant k_j \\ 1 \leqslant j \leqslant n}} \sum_{P = P_1 X_i P_2} \prod_{1 \leqslant j \leqslant n} C_{k_j}^{p_j} \mathcal{M}_{p_1, \cdots, p_n}(P_1) \mathcal{M}_{k_1 - p_1, \cdots, k_n - p_n}(P_2)$$

$$= \sum_{\substack{0 \leqslant p_j \leqslant k_j \\ 1 \leqslant j \leqslant n}} \prod_{1 \leqslant j \leqslant n} C_{k_j}^{p_j} \tau^{\overline{p}} \otimes \tau^{\overline{k} - \overline{p}}(\partial_i P)$$

where we finally used our induction hypothesis.

The other possibility is that this edge is glued with a star of type q_j for some $j \in \{1, \cdots, n\}$. In this case, erasing the edge means that we destroy a star of type q_j and replace the star of type $X_i P$ and the star of type q_j by a single bigger star. If $q_j = Q_1 X_i Q_2$, we replace the two stars of type $X_i P$ and q_j by a single one of type $Q_2 Q_1 P$ (see figure 2.3). Since we do that with all the possible edges of color i in q_j, we find that we can glue all monomials appearing in $D_i q_j$, and so the corresponding weight is given by $\tau^{\overline{k} - \mathbb{1}_j}(D_i q_j P)$ times k_j, the number of ways to choose one star among k_j of type q_j.

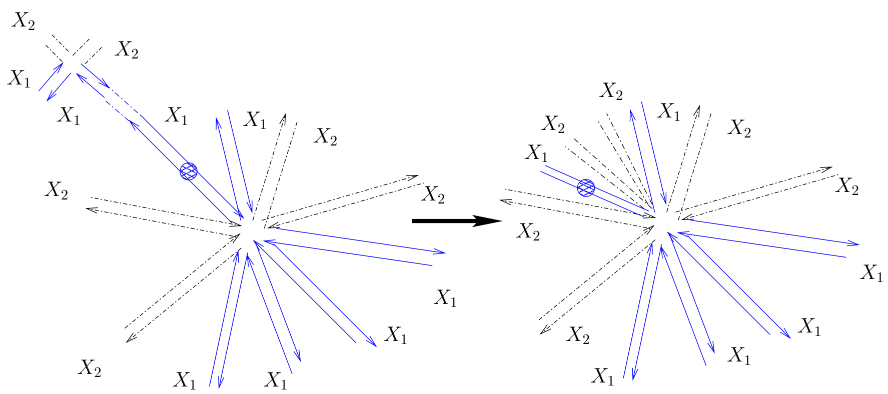

FIGURE 2.3. The merging of $q(X) = X_1^2 X_2^2 = X_1 X_1 X_2^2$ and $X_1 P$ into $X_1 X_2^2 P$

Hence, by induction, we proved that the number of planar maps with k_j stars of type q_j and one of type $X_i P$ is given by

$$\begin{aligned}\mathcal{M}_{\overline{k}}(X_iP) &= \sum_{\substack{0\leqslant p_j\leqslant k_j\\ 1\leqslant j\leqslant m}}\prod_{j=1}^{n} C_{k_j}^{p_j}\tau^{\overline{p}}\otimes\tau^{\overline{k}-\overline{p}}(\partial_iP)+\sum_{1\leqslant j\leqslant m}k_j\tau^{\overline{k}-\mathbb{1}_j}((D_iq_j)P)\\ &= \tau^{\overline{k}}(X_iP)\end{aligned}$$

for all $i\in\{1,\cdots,m\}$. This shows that the graphical interpretation holds for all L and $|\overline{k}|\leqslant M+1$. We can start the induction since we know that $\tau^{\overline{k}}(1)=\mathbb{1}_{\overline{k}=0}$. This completes the proof.

□

Remark: Note that this graphical approach can be generalized to matrix models with more complex potentials involving tensor products. For example, one can consider a potential V which is a sum of monomials and of tensor products of monomials:

$$V_{\mathbf{t}}=\sum_i t_iq_i^1\otimes\cdots\otimes q_i^d$$

and the associated measure with density $Z_N^{-1}e^{-N^{2-d}(\mathrm{tr})^{\otimes d}V_{\mathbf{t}}}$ with respect to $\mu_N^{\otimes m}$. Then one can write the generalized Swinger Dyson's equation:

$$\tau\otimes\tau(\partial_iP)=\tau(X_iP)+\sum_{k,j}t_k\tau^{\otimes d_k}(q_k^1\otimes\cdots\otimes D_iq_k^jP\otimes\cdots\otimes q_k^d)$$

The previous results remain valid up to a graphical interpretation of the new term. For example $q^1\otimes\cdots\otimes q^k$ will be a bunch of k loops, the first one containing the half-edges of the star of q^1, in the clockwise order, the first of which is the marked one, the second one the half-edges of q^2 ... The additional constraint being that vertices which will be placed in a loop can not be linked to any vertices in an other loop.

2.7. *Existence of an analytic solution to Schwinger-Dyson's equation.* The aim of this section is to prove that for all monomials $(q_j)_{1\leqslant j\leqslant n}$, there exists a convex neighborhood of the origin (actually an open ball) and a finite constant R so that hypothesis **(H)** of section 2.6 is satisfied. Moreover, we show that it depends analytically on $\mathbf{t}$ in a neighborhood of the origin. Let $V_{\mathbf{t}}$ be as before.

Theorem 2.3. *There exists an open neighborhood $U\subset\mathbb{C}^n$ of the origin (a ball of positive radius) such that for $\mathbf{t}\in U$, there exists $\tau_{\mathbf{t}}\in\mathbb{C}\langle X_1,\cdots,X_m\rangle^*$ satisfying* $\mathbf{SD}[V_{\mathbf{t}}]$ *such that:*

- $\mathbf{t}\to\tau_{\mathbf{t}}$ *is analytic on U, i.e. there exists $(\tau^{\overline{k}},\overline{k}\in\mathbb{N}^n)$ in $\mathbb{C}\langle X_1,\cdots,X_m\rangle^*$ such that for all P in $\mathbb{C}\langle X_1,\cdots,X_m\rangle$, t in U,*

$$\tau_{\mathbf{t}}(P)=\sum_{\overline{k}\in\mathbb{N}^n}\prod_{1\leqslant i\leqslant n}\frac{(-t_i)^{k_i}}{k_i!}\tau^{\overline{k}}(P)\tag{2.7}$$

  and the serie converges absolutely on U.
- $\tau^{\overline{k}}(P)=(-1)^{\Sigma k_i}\partial_{t_1}^{k_1}\cdots\partial_{t_n}^{k_n}\tau_{\mathbf{t}}(P)|_{\mathbf{t}=0}=\mathcal{M}_{\overline{k}}(P)$
- *There exists $R<\infty$ so that for all $\mathbf{t}\in U$, all $i_i\cdots i_l\in\{1,\cdots,m\}^l$, all $l\in\mathbb{N}$,*

$$|\tau_{\mathbf{t}}(X_{i_1}\cdots X_{i_l})|\leqslant R^l.$$

Remark: Using (2.3), one can obtain inside the domain of convergence, for all monomial P:

$$\tau_{\mathbf{t}}(P) = \sum_{\overline{k}\in\mathbb{N}^n} \prod_{1\leqslant i\leqslant n} (-s(q_i)t_i)^{k_i}\mathcal{D}_{\overline{k}}(P).$$

Proof.

Note that if we assume that $\tau_{\mathbf{t}}$ can be written as a serie like in (2.7) then according to Theorem 2.2 the $\tau^{\overline{k}}$ are defined uniquely by equation (2.4) so that $\tau^{\overline{k}}(P) = \mathcal{M}_{\overline{k}}(P)$ for all $\overline{k}$, P and reciprocally these equalities imply that $\tau_{\mathbf{t}}$ satisfy $\mathbf{SD}[V_{\mathbf{t}}]$ inside the domain of convergence. The only point is thus to control the growth of the $\tau^{\overline{k}}(P)$'s to show that

$$\sum_{\overline{k}\in\mathbb{N}^n} \prod_{1\leqslant i\leqslant n} \frac{(-t_i)^{k_i}}{k_i!}\tau^{\overline{k}}(P)$$

has a strictly positive radius of convergence. We write, if $\overline{k}! = \prod k_i!$,

$$\frac{\tau^{\overline{k}}(X_iP)}{\overline{k}!} = \sum_{\substack{0\leqslant p_j\leqslant k_j\\ 1\leqslant j\leqslant n}} \sum_{P=P_1X_iP_2} \frac{\tau^{\overline{p}}(P_1)}{\overline{p}!}\frac{\tau^{\overline{k}-\overline{p}}(P_2)}{(\overline{k-p})!} + \sum_{\substack{1\leqslant j\leqslant m\\ k_j\neq 0}} \frac{\tau^{\overline{k}-\mathbb{1}_j}((D_iq_j)P)}{(\overline{k}-\mathbb{1}_j)!}$$

where the second sum runs over all monomials P_1, P_2 so that P decomposes into $P_1X_iP_2$.

Our induction hypothesis will be that for $\overline{k}$ so that $\sum_i k_i \leqslant M-1$ and all monomial P, as well as for $\sum k_i = M$ and monomials P of degree smaller than L,

$$\left|\frac{\tau^{\overline{k}}(P)}{\overline{k}!}\right| \leqslant A^{\sum k_i}B^{degP}\prod_i C_{k_i}C_{degP}$$

where the C_k are the Catalan's numbers which satisfy

$$C_{k+1} = \sum_{p=0}^{k} C_pC_{k-p}, \quad C_0 = 1, \quad \frac{C_{k+l}}{C_l} \leqslant 4^k \quad \forall l,k\in\mathbb{N}. \tag{2.8}$$

Here, $\deg P$ denotes the degree of the monomial P and we can assume $B \geqslant 2$ without loss of generality. Our induction is trivially true for $\overline{k}=0$ and all L since $\mathcal{M}_0 = \tau^{\overline{0}} = \sigma^m$ is the law of m free semi-circular variables which are uniformly bounded by 2 so that

$$|\tau^{\overline{0}}(P)| \leqslant 2^{degP}$$

Moreover, it is satisfied for all $\overline{k}$ and $L=0$ since then $\tau^{\overline{k}}(1) = \mathbb{1}_{\overline{k}=0}$. Let us assume that it is true for all $\overline{k}$ such that $\sum k_i \leqslant M-1$ and all monomials, and for $\overline{k}$ such that $\sum k_i = M$ and monomials P of degree less than L for some $L\geqslant 0$. Then

$$\begin{aligned}\left|\frac{\tau^{\overline{k}}(X_iP)}{k!}\right| &\leqslant \sum_{\substack{0\leqslant p_j\leqslant k_j\\ 1\leqslant j\leqslant n}} \sum_{P=P_1X_iP_2} A^{\sum k_i}B^{degP-1}\prod_{i=1}^{n} C_{p_j}C_{k_j-p_j}C_{degP_1}C_{degP_2}\\ &+ 2\sum_{1\leqslant l\leqslant n} A^{\sum k_j-1}\prod_j C_{k_j}B^{degP+deqq_l-1}C_{degP+deqq_l-1}\\ &\leqslant A^{\sum k_i}B^{degP+1}\prod_i C_{k_i}C_{degP+1}\left(\frac{4^n}{B^2} + 2\frac{\sum_{1\leqslant j\leqslant n} B^{degq_j-2}4^{degq_j-2}}{A}\right)\end{aligned}$$

where we used (2.8) in the last line. It is now sufficient to choose A and B such that

$$\frac{4^n}{B^2}+2\frac{\sum_{1\leqslant j\leqslant n}B^{degq_j-2}4^{degq_j-2}}{A}\leqslant 1$$

(for instance $B=2^{n+1}$ and $A=4nB^{D-2}4^{D-2}$) to verify the induction hypothesis works for polynomials of all degrees (all L's).

Then

$$\tau_{\mathbf{t}}(P)=\sum_{k\in\mathbb{N}^n}\prod\frac{(-t_i)^{k_i}}{k_i!}\tau^{\overline{k}}(P)$$

is well defined for $|t|<(4A)^{-1}$. Moreover, for all monomial P,

$$|\tau_{\mathbf{t}}(P)|\leqslant\sum_{k\in\mathbb{N}^n}\prod_{i=1}^n(4t_iA)^{k_i}(4B)^{degP}\leqslant\prod_{i=1}^n(1-4At_i)^{-1}(4B)^{degP}.$$

so that for small t, $\tau_{\mathbf{t}}$ has a uniformly bounded support.

□

Hence, we see that the enumeration of planar maps could be reduced to the study of Schwinger-Dyson's equations **SD[V]**. For instance, the asymptotics of such enumeration can be obtained by studying the optimal domain in which the solutions are analytic. Matrix models can be useful to study also the solution, e.g. we shall deduce from this approach that the solutions to **SD[V]** are tracial states (the positivity condition being unclear a priori).

3. Existence of tracial states solutions to Schwinger-Dyson's equations from matrix models

Let $V=V_{\mathbf{t}}$ be a polynomial function as before. Consider

$$Z_V^N=\int e^{-N\mathrm{tr}(V(A_1,\cdots,A_m))}\mu_N(dA_1)\cdots\mu_N(dA_m)$$

and μ_V^N the associated Gibbs measure

$$\mu_V^N(dA_1,\cdots,dA_m)=(Z_V^N)^{-1}e^{-N\mathrm{tr}(V(A_1,\cdots,A_m))}\mu_N(dA_1)\cdots\mu_N(dA_m).$$

We will always assume in this section that $V=V_{\mathbf{t}}$ is self-adjoint so that the potential is real. This means that if $V_{\mathbf{t}}=\sum_i t_iq_i$ then for all i, there exists j such that $q_j^*=q_i$ and $t_j=\overline{t_i}$. Note that Z_V^N is not necessarily finite. We will see various assumptions in order to make μ_V^N a proper probabilty measure. The empirical distribution of m matrices $\mathbf{A}=(A_1,\cdots,A_m)\in\mathcal{H}_N^m$ is defined as the element of $\mathcal{M}_{ST}^m$ such that

$$\hat{\mu}_{\mathbf{A}}^N(F):=\hat{\mu}_{A_1,\cdots,A_m}^N(F)=\frac{1}{N}\mathrm{tr}(F(A_1,\cdots,A_m))$$

for all $F\in\mathcal{C}_{st}^m(\mathbb{C})$. Note that the empirical distribution could be defined as well as an element of $\mathcal{M}^m$ but since the random matrices $(A_1,\cdots,A_m)$ under μ_V^N have a priori no uniformly bounded spectral radius, the topology of weak convergence would not be suitable then.

We shall see that if we know that a limit point of $\hat{\mu}_{\mathbf{A}}^N$ under μ_V^N is compactly supported, then it satisfies **SD[V]**. In a second part, we shall give examples of potential V for which this assumption is satisfied. Finally, we discuss localized matrix integrals and show that bounded solutions to **SD[V]** for small potentials can always be constructed by localized matrix integrals.

3.1. *Limit points of empirical distribution of matrices following matrix models satisfy the* $\mathbf{SD[V]}$ *equations.* The integral Z_V^N is well defined provided that the monomials of highest degree in $V_{\mathbf{t}}$ are even and sufficiently large. We shall assume in this paragraph that

$$V_{\mathbf{t}}(\mathbf{X}) = V_{\mathbf{t}}^*(\mathbf{X}) = \sum_{1 \leqslant i \leqslant n} t_i q_i(\mathbf{X}) + \overline{t_i} q_i^*(\mathbf{X}) + \sum_{n+1 \leqslant i \leqslant n+m} t_i X_{i-n}^D \tag{3.1}$$

with D even, monomial functions q_i of degree less or equal than $D-1$ and $t_i > 0$ for $i \in \{n+1, \cdots, n+m\}$. We shall see in the last paragraph of this section that such assumption can be removed provided a cut-off is added.

For such potentials, we show that we can relate the matrix model to the solutions of $\mathbf{SD[V]}$.

Theorem 3.1. *Assume* (3.1). *Then*

(1) *There exists* $M < \infty$ *so that,* μ_V^N *almost surely for all* $i \in \{1, \cdots, m\}$.

$$\limsup_{N \to \infty} \hat{\mu}^N_{A_1, \cdots, A_m}(X_i^D) \leqslant M.$$

(2) *The limit points of* $\hat{\mu}^N_{A_1, \cdots, A_m}$ *for the* $\mathcal{C}^m_{st}(\mathbb{C})$*-topology satisfy the 'weak' Schwinger-Dyson equation*

$$\tau \otimes \tau(\partial_i F) = \tau((D_i V + X_i) F) \quad (\mathbf{WSD})[\mathbf{V}]$$

for all $F \in \mathcal{C}^m_{st}(\mathbb{C})$, *for all* $1 \leqslant i \leqslant m$. *Moreover for all* $i \in \{1, \cdots, n\}$, $\tau(X_i^D) < +\infty$.

Note here that $(D_i V + X_i)F$ does not belong to $\mathcal{C}^m_{st}(\mathbb{C})$ so that it is not clear what $\mathbf{(WSD)[V]}$ means a priori. We define it by the following; since $\tau(X_i^D) < +\infty$ and $D_i V$ has degree less than $D-1$, there exists a sequence $V^\delta \in \mathcal{C}^m_{st}(\mathbb{C})$ so that

$$\lim_{\delta \to 0} \max_{1 \leqslant i \leqslant m} \sup_{\tau(X_i^D) \leqslant M} \tau(|D_i V^\delta - D_i V - X_i|) = 0$$

from which, since any $F \in \mathcal{C}^m_{st}(\mathbb{C})$ is uniformly bounded,

$$\lim_{\delta \to 0} \max_{1 \leqslant i \leqslant m} \sup_{\tau(X_i^D) \leqslant M} |\tau(F D_i V^\delta) - \tau(F(D_i V + X_i))| = 0$$

is well defined.

Proof.

• The first point is trivial since by Jensen's inequality,

$$Z_V^N \geqslant \exp\{-N^2 \int \frac{1}{N} \mathrm{tr}(V(\mathbf{A})) \prod_{1 \leqslant i \leqslant m} d\mu_N(A_i)\} \geqslant \exp\{cN^2\}$$

for some $c > -\infty$, where the last inequality comes from the fact that (see Voiculescu (1991))

$$\lim_{N \to \infty} \int \frac{1}{N} \mathrm{tr}(V(\mathbf{A})) \prod_{1 \leqslant i \leqslant m} d\mu_N(A_i) = \sigma^m(V) < \infty$$

where σ^m is the law of m free semi-circular variables.

Now, observe that by Hölder's inequality,

$$|\hat{\mu}^N_{\mathbf{A}}(q_i)| \leqslant \max_{1 \leqslant i \leqslant m} \hat{\mu}^N_{\mathbf{A}}(|X_i|^{D-1} + 1)$$

so that we deduce

$$\hat{\mu}^N_{\mathbf{A}}(V) \geqslant \sum_{i=1}^{m} \left(t_{i+n}\hat{\mu}^N_{\mathbf{A}}(A_i^D) - c(\mathbf{t})\hat{\mu}^N_{\mathbf{A}}(|A_i|^{D-1}) - c(\mathbf{t})\right)$$

with a finite constant $c(\mathbf{t})$. Since $t_{i+n} > 0$, we conclude that $\hat{\mu}^N_{\mathbf{A}}(V) \geqslant m|t|M/2$ when $\max_{1\leqslant i\leqslant m} \hat{\mu}^N_{\mathbf{A}}(A_i^D) \geqslant M$ for M large enough. Thus

$$\mu^N_V \left(\max_{1\leqslant i\leqslant m} \hat{\mu}^N_{\mathbf{A}}(A_i^D) \geqslant M \right) \leqslant e^{-2^{-1}N^2 Mm|t|} e^{-cN^2} \tag{3.2}$$

goes to zero exponentially fast when $M > \frac{2c}{m|t|}$. The claim follows by Borel-Cantelli's lemma.

• We proceed as in Cabanal-Duvillard and Guionnet (2003), following a common idea in physics, which is to make, in Z^N_V, the change of variables $X_i \to X_i + N^{-1}F(\mathbf{X})$ for a given $i \in \{1, \cdots, m\}$ and $F \in \mathcal{C}^m_{st}(\mathbb{R})$. Noticing that the Jacobian for this change of variable is

$$|J| = e^{N\hat{\mu}^N_{\mathbf{A}} \otimes \hat{\mu}^N_{\mathbf{A}}(\partial_i F) + O(1)}$$

we get that

$$\int e^{(N\hat{\mu}^N_{\mathbf{A}} \otimes \hat{\mu}^N_{\mathbf{A}}(\partial_i F) - N^2 \hat{\mu}^N_{\mathbf{A}}(N^{-1}X_i F(\mathbf{X}) + V(X_i + N^{-1}F(\mathbf{X})) - V(X_i))} \mu^N_V(d\mathbf{A}) = O(1).$$

If we denote

$$E_N = \hat{\mu}^N_{\mathbf{A}} \otimes \hat{\mu}^N_{\mathbf{A}}(\partial_i F) - N\hat{\mu}^N_{\mathbf{A}}(N^{-1}X_i F(\mathbf{X}) + V(X_i + N^{-1}F(\mathbf{X})) - V(X_i))$$

we deduce that

$$\int_{\max_{1\leqslant i\leqslant m} \hat{\mu}^N_{\mathbf{A}}(A_i^D)\leqslant M} e^{NE_N} \mu^N_V(d\mathbf{A}) = O(1).$$

Hence, we conclude by Chebychev inequality and (3.2) that for M big enough, any $\delta > 0$, there exists $\eta > 0$, so that

$$\mu^N_V \left(\{ \max_{1\leqslant i\leqslant m} \hat{\mu}^N_{\mathbf{A}}(X_i^D) \leqslant M\} \cap \{|E_N| \leqslant \delta\} \right) \geqslant 1 - e^{-\eta N}.$$

Moreover,

$$\hat{\mu}^N_{\mathbf{A}}(V(X_i + N^{-1}F(\mathbf{X})) - V(X_i))) = N^{-1}\hat{\mu}^N_{\mathbf{A}}((D_i V)F) + R_N$$

with a rest R_N of order $N^{-2} \max_{1\leqslant i\leqslant m} \hat{\mu}^N_{\mathbf{A}}(X_i^{D-2})$ which we can neglect on $\max_{1\leqslant i\leqslant m} \hat{\mu}^N_{\mathbf{A}}(A_i^D) \leqslant M$. This shows, by Borel Cantelli's Lemma, that for all $F \in \mathcal{C}^m_{st}(\mathbb{R})$,

$$\hat{\mu}^N_{\mathbf{A}} \otimes \hat{\mu}^N_{\mathbf{A}}(\partial_i F) - \hat{\mu}^N_{\mathbf{A}}(X_i F + D_i V F)$$

goes to zero almost surely. This result extends to $F \in \mathcal{C}^m_{st}(\mathbb{C})$ since it can always be decomposed into the sum of two elements of $\mathcal{C}^m_{st}(\mathbb{R})$. Moreover, if we let $A_i^\varepsilon = A_i(1+\varepsilon A_i^2)^{-1} = A_i(\sqrt{-1} + \sqrt{\varepsilon}A_i)^{-1}(-\sqrt{-1} + \sqrt{\varepsilon}A_i)^{-1} \in \mathcal{C}^m_{st}(\mathbb{C})$, then again by Hölder's inequality $\tau(|D_i V(A_i) - D_i V(A_i^\varepsilon)|)$ goes to zero uniformly on $\max_{1\leqslant i\leqslant m} \tau(A_i^D) \leqslant M$. This shows that $\mu \to \mu((D_i V + X_i)F)$ is continuous for the weak $\mathcal{C}^m_{st}(\mathbb{C})$-topology on $\{\mu(A_i^D) \leqslant M\}$ for any $F \in \mathcal{C}^m_{st}(\mathbb{C})$. Therefore, since $\mathcal{M}^m_{ST}$ is compact, we conclude that any limit point of $\hat{\mu}^N_{\mathbf{A}}$ satisfies

$$\tau \otimes \tau(\partial_i F) = \tau((X_i + D_i V)F)$$

□

We therefore have the

Corollary 3.2. *Assume that there exists a limit point τ_V of $\hat{\mu}^N_{\mathbf{A}}$ under μ^N_V which is compactly supported. Then, it satisfies Schwinger-Dyson's equation* **SD[V]**.

Proof.
The proof is straightforward since if τ_V is compactly supported, it is equivalent to say that τ_V satisfies **WSD[V]** or **SD[V]** since $\mathcal{C}^m_{st}(\mathbb{C})$ is dense in the set of polynomial functions (approximate the A_i's by the A^ε_i's defined in the previous proof).

□

Let us also give the final argument to deduce convergence of the free energy from the previous considerations.

Theorem 3.3. *Let $\gamma : [0,1] \to \mathbb{C}^n$ be a continuously differentiable path from 0 to $\mathbf{t}$ such that for all s, $V_{\gamma_s} = \sum_{i=1}^n \gamma_s(i) q_i$ is self-adjoint. Assume that*

• *$\hat{\mu}^N_{\mathbf{A}}$ converges in $\mathcal{M}^m_{ST}$ almost surely or in expectation under $\mu^N_{V_{\gamma_s}}$ for all s in $[0,1]$.*

• *$\max_p \mu^N_{V_{\gamma_s}}(\hat{\mu}^N_{\mathbf{A}}(|X_p|^l))$ is uniformly bounded for s in $[0,1]$ and N large enough for some l strictly greater than the degree of $V_{\mathbf{t}}$.*

Then

(1) *The free energy*
$$F^N_{V_{\mathbf{t}}} = N^{-2} \log\left(Z^N_{V_{\mathbf{t}}}\right)$$
converges as N goes to infinity towards a limit $F_{V_{\mathbf{t}}}$.

(2) *Moreover, there exists $\varepsilon > 0$, such that, if for all s, γ_s is in $B(0,\varepsilon)$, and if the limit points of $\hat{\mu}^N_{\mathbf{A}}$ under $\mu^N_{V_{\gamma_s}}$ are uniformly compactly supported, then*
$$F^N_{V_{\mathbf{t}}} = N^{-2} \log(Z^N_{V_{\mathbf{t}}})$$
converges as N goes to infinity towards
$$F_{V_{\mathbf{t}}} = \sum_{\overline{k} \in \mathbb{N}^n \setminus (0,..,0)} \prod_{1 \leqslant i \leqslant n} \frac{(-t_i)^{k_i}}{k_i!} \mathcal{M}_{\overline{k}}.$$

Note above that the last serie has a positive radius of convergence according to Theorems 2.2 and 2.3. This emphasizes that the possible divergence of $F^N_{V_{\mathbf{t}}}$ does not survive the large N limit.

Proof.

• First, note that as for all s V_{γ_s} is self adjoint, thus, up to a change of coordinates in $\mathbb{C}^n$, V_{γ_s} can be written as $V_{\gamma_s} = \sum_{i=1}^n (\gamma_s(i) q_i + \overline{\gamma_s(i)} q_i^*)$. By differentiating
$$N^{-2} \log Z^N_{V_{\gamma_s}} = \frac{1}{N^2} \log \int e^{-N \mathrm{tr} \sum_{i=1}^n (\gamma_s(i) q_i + \overline{\gamma_s(i)} q_i^*)} d\mu^N$$
with respect to s we obtain that
$$\partial_s N^{-2} \log Z^N_{V_{\gamma_s}} = -\sum_{i=1}^n \mu^N_{V_{\gamma_s}}(\hat{\mu}^N_{\mathbf{A}}(q_i \partial_s \gamma_s(i) + q_i^* \overline{\partial_s \gamma_s(i)})).$$
But, under our assumption, $(\hat{\mu}^N_{\mathbf{A}}(q_i \partial_s \gamma_s(i) + q_i^* \overline{\partial_s \gamma_s(i)}))_{N \in \mathbb{N}}$ converges almost surely and is uniformly integrable so that $\mu^N_{V_{\gamma_s}}(\hat{\mu}^N_{\mathbf{A}}(q_i \partial_s \gamma_s(i) + q_i^* \overline{\partial_s \gamma_s(i)}))$ is a uniformly bounded sequence which converges as N goes to infinity towards $\tau_{\gamma_s}(q_i \partial_s \gamma_s(i) +$

$q_i^*\overline{\partial_s\gamma_s(i)})$ for $s\in[0,1]$. Integrating with respect to s yields the convergence with $F_{V_{\gamma_s}}$ as above by dominated convergence theorem.

• We can choose $\varepsilon>0$ such that on $B(0,\varepsilon)$ there is an unique solution τ_{γ_s} of $\mathbf{SD}[V_{\gamma_s}]$ and it satisfy the combinatorial interpretation of Theorem 2.3. By Corollary 3.2, our hypothesis implies that for s in $[0,1]$ the limit points of $\hat{\mu}^N_{\mathbf{A}}$ are unique and given by τ_{γ_s}. Hence, $\hat{\mu}^N_{\mathbf{A}}$ converges in $\mathcal{M}^m_{ST}$ almost surely towards τ_{γ_s}. Since we assumed our family uniformly integrable, we deduce that $\mu^N_{V_{\gamma_s}}(\hat{\mu}^N_{\mathbf{A}}(q_i\partial_s\gamma_s(i)+q_i^*\overline{\partial_s\gamma_s(i)}))$ converges as N goes to infinity towards $\tau_{\gamma_s}(q_i\partial_s\gamma_s(i)+q_i^*\overline{\partial_s\gamma_s(i)})$ for all $l\in\{1,\cdots,n\}$. We denote $\mathcal{M}_{\overline{k^1},\overline{k^2}}$ (resp. $\mathcal{M}_{\overline{k^1},\overline{k^2}}(P)$) the number of planar maps with k_i^1 vertices of type q_i and k_i^2 of type q_i^* (resp. and with one of type P),

$$\lim_{N\to\infty}\frac{1}{N^2}\log Z^N_{V_{\gamma_s}}=-\sum_{i=1}^n\int_0^1\tau_{\gamma_s}(q_i\partial_s\gamma_s(i)+q_i^*\overline{\partial_s\gamma_s(i)})ds$$

$$=\sum_{i=1}^n\sum_{k_j^1,k_j^2}\int_0^1\prod_j\frac{(-\gamma_s(j))^{k_j^1}}{k_j^1!}\frac{(-\overline{\gamma_s(j)})^{k_j^2}}{k_j^2!}(\mathcal{M}_{\overline{k^1},\overline{k^2}}(q_i)\partial_s\gamma_s(i)+\mathcal{M}_{\overline{k^1},\overline{k^2}}(q_i^*)\partial_s\overline{\gamma_s(i)})$$

$$=\sum_{i=1}^n\sum_{k_j^1k_j^2\neq 0}\prod_j\frac{(-t_j)^{k_j^1}}{k_j^1!}\frac{(-\overline{t_j})^{k_j^2}}{k_j^2!}\mathcal{M}_{\overline{k^1},\overline{k^2}}$$

where we used in the last line the equality $\mathcal{M}_{\overline{k}}(q_i)=\mathcal{M}_{\overline{k}+\mathbb{1}_i}$.

□

We shall in the next section provide a generic example where the assumption of the second point of Theorem 3.3 is satisfied (in fact, a slightly different version since we do not prove that the almost sure limit points of $\hat{\mu}^N_{\mathbf{A}}$ satisfy our compactness assumption, but their average do, which still guarantees the result).

3.2. *Convex interaction models.* Let us assume that we consider a matrix model with potential V such that for all N in $\mathbb{N}$,

$$\varphi^N_{V,a}:(A_k(ij))\in(\mathbb{R}^{N^2})^m\to\mathrm{tr}(V(A_1,\cdots,A_m))+\frac{a}{2}\sum_{k=1}^m\mathrm{tr}(A_k^2)\qquad(3.3)$$

is convex in all dimensions for some $a<1$, i.e the Hessian of $\varphi^N_{V,a}$ is non negative for all $N\in\mathbb{N}$. An example is V of the form

$$V(A_1,\cdots,A_m)=\sum_{i=1}^n t_iP_i(\sum_{k=1}^m\alpha_k^iA_k)+\sum_{i,j}\beta_{i,j}A_1A_j$$

with non-negative t_i's, convex polynomials of one variable P_i, real α's and β with for all i, $|\sum_j\beta_{i,j}|<1-a$. Indeed, by Klein's lemma (c.f. Guionnet and Zeitouni (2002)), since $x\to P_i(\sum\alpha_kx_k)$ is convex,

$$\mathbf{A}\to\mathrm{tr}P_i(\sum\alpha_k^iA_k)$$

is also convex (Here $\mathbf{A}$, by an abuse of notations, denotes the entries of the m-uple of matrices $\mathbf{A}=(A_1,\cdots,A_m)$).

Then, we shall prove that

Theorem 3.4. *Let V be a self-adjoint polynomial function which satisfies (3.3). Then*

- *There exists $R_V < \infty$ so that*
$$\limsup_{N\to\infty} \mu_V^N(\hat{\mu}_{\mathbf{A}}^N(A_i^{2n})) \leqslant (R_V)^n$$
for all $n \in \mathbb{N}$ and $i \in \{1, \cdots, m\}$. Here, R_V is uniformly bounded by some R_M when the quantities $(a, V(0,0,..,0), (D_iV(0,0,..,0))_{1\leqslant i\leqslant m}, \sigma^m(V))$ are bounded by M.
- *$\mu_V^N[\hat{\mu}_{\mathbf{A}}^N]$ is tight and its limit points satisfy* $\mathbf{SD[V]}$.
- *Take $V = V_{\mathbf{t}} = \sum_{i=1}^n t_i q_i$ and let U_a be the set of t_i's for which $V_{\mathbf{t}}$ satisfies (3.3) for a given $a < 1$. For $\varepsilon > 0$ small enough, when $(t_i)_{1\leqslant i\leqslant n} \in U_a \cap B(0,\varepsilon)$, $\hat{\mu}_{\mathbf{A}}^N$ converges in $L^1(\mu_V^N)$ and almost surely to the unique solution to* $\mathbf{SD[V]}$.
- *Let $a < 1$, there exists $\varepsilon > 0$ such that if there exists a continuously differentiable path $\gamma : [0,1] \to U_a \cap B(0,\varepsilon)$ from 0 to $\mathbf{t}$ then*
$$F_{V_{\mathbf{t}}}^N = N^{-2}\log(Z_{V_{\mathbf{t}}}^N)$$
converges as N goes to infinity towards
$$F_{V_{\mathbf{t}}} = \sum_{\overline{k}\in\mathbb{N}^n\setminus(0,..,0)} \prod_{1\leqslant i\leqslant n} \frac{(-t_i)^{k_i}}{k_i!}\mathcal{M}_{\overline{k}}.$$

Remark : Observe that if $V_{\mathbf{t}}$ has only a quadratic interaction term, U_a contains all the range of parameters such that the self potentials of each matrix is convex. For instance, if we look at the Ising model,
$$V_{\mathbf{t}} = \beta AB + \sum_{i=1}^n t_i A^{2i} + \sum_{i=1}^m u_i B^{2i}$$
then $\{|\beta| < a\} \cap \{t_i \in \mathbb{R}^+\} \cap \{u_i \in \mathbb{R}^+\} \subset U_a$.

Proof.
We can assume without loss of generality that $a = 0$ since otherwise we just make a shift on the covariance of the matrices under μ_N. The idea is to use Brascamp-Lieb inequality (c.f. Hargé (2004) for recent improvements) which shows that since
$$f(\mathbf{A}) = e^{-N\mathrm{tr}V(A_1,\cdots,A_m)}$$
is log-concave, for all convex function g on $(\mathbb{R})^{mN^2}$,
$$\mu_V^N(g(\mathbf{A}-\mathbf{M})) = \int g(\mathbf{A}-\mathbf{M})\frac{f(\mathbf{A})\prod d\mu_N(A_i)}{\int f(\mathbf{A})\prod d\mu_N(A_i)} \leqslant \int g(\mathbf{A})\prod d\mu_N(A_i) \quad (3.4)$$
with
$$\mathbf{M} = \int \mathbf{A} d\mu_V^N.$$
Here $\mathbf{A}$ denotes the set of entries of the matrices $(A_1, \cdots, A_m)$. Let us apply (3.4) with $g(\mathbf{A}) = \mathrm{tr}(A_k^{2p})$ which is convex by Klein's lemma. Hence,
$$\mu_V^N\left(\mathrm{tr}((A_k - \mathbb{E}[A_k])^{2p})\right) \leqslant \mu_N(\mathrm{tr}(A^{2p})) \quad (3.5)$$

where $\mathbb{E}[A_k](ij) = \mu_V^N(A_k(ij))$ for $1 \leqslant i,j \leqslant N$. By Theorem 2 p.17 in Soshnikov (1999), there exists a finite constant C so that for all $p \leqslant \sqrt{N}$,

$$\mu_N(\mathrm{tr}(A^{2p})) \leqslant CN4^p.$$

In particular,

$$\limsup_{N\to\infty} \mu_V^N[\frac{1}{N}\mathrm{tr}((A_k - \mathbb{E}[A_k])^{2p})] \leqslant 4^p. \tag{3.6}$$

Also, by Chebychev's inequality we find that if $||A||_\infty$ denotes the spectral radius of A, for all $k \in \{1, \cdots, m\}$

$$\mu_V^N(||A_k - E[A_k]||_\infty \geqslant 3) \leqslant \mu_V^N(\mathrm{tr}((A_k - \mathbb{E}[A_k])^{2p} \geqslant 3^{2p}) \leqslant CN\left(\frac{2}{3}\right)^{2p}$$

for all $p \leqslant \sqrt{N}$. Taking $p = \sqrt{N}$, we deduce by Borel Cantelli's lemma that

$$\limsup_{N\to\infty} ||A_k - E[A_k]||_\infty \leqslant 3 \quad \text{a.s.} \tag{3.7}$$

We now control $\mathbb{E}[A_k]$ uniformly. Since the law of A_k is invariant by the action of the unitary group, we deduce that for all unitary matrix U,

$$\mathbb{E}[A_k] = \mathbb{E}[UA_kU^*] = U\mathbb{E}[A_k]U^* \Rightarrow \mathbb{E}[A_k] = \mu_V^N(\hat{\mu}_{\mathbf{A}}^N(X_k))I. \tag{3.8}$$

We now bound $\mu_V^N(\hat{\mu}_{\mathbf{A}}^N(X_k))$ independently of N. Since V is convex, there are real numbers $(\gamma_i)_{1\leqslant i\leqslant m}$ and $c > -\infty$, $\gamma_i = D_iV(0,\cdots,0)$ and $c = V(0,\cdots,0)$ so that for all $N \in \mathbb{N}$ and all matrices $(A_1,\cdots,A_m) \in \mathcal{H}_N^m$,

$$\mathrm{tr}(V(A_1,\cdots,A_m)) \geqslant \mathrm{tr}(\sum_{i=1}^m \gamma_i A_i + c).$$

By Jensen's inequality, we know that $Z_V^N \geqslant e^{-dN^2}$ for N sufficiently large, $d = 2\sigma^m(V) < +\infty$ and so Chebychev's inequality implies that for all $y > 0$, all $\lambda > 0$,

$$\begin{aligned}
\mu_V^N\left(|\hat{\mu}_{\mathbf{A}}^N(X_k)| \geqslant y\right) &\leqslant e^{(d-c)N^2-\lambda yN^2}[\int e^{-N\sum_{i=1}^m \gamma_i \mathrm{tr}(A_i)+N\lambda\mathrm{tr}(A_k)}\prod_{i=1}^m d\mu_N(A_i) \\
&\quad + \int e^{-N\sum_{i=1}^m \gamma_i \mathrm{tr}(A_i)-N\lambda\mathrm{tr}(A_k)}\prod_{i=1}^m d\mu_N(A_i)] \\
&\leqslant 2e^{(d-c)N^2-\lambda yN^2} e^{\frac{N^2}{2}\sum_{i\neq k}\gamma_i^2+\frac{N^2}{2}(\gamma_k+\lambda)^2}
\end{aligned}$$

Optimizing with respect to λ shows that there exists $A < \infty$ (which depends only on $D_iV(0,\cdots,0)$, $V_i(0,\cdots,0)$ and $\sigma^m(V)$) so that

$$\mu_V^N\left(|\hat{\mu}_{\mathbf{A}}^N(X_k)| \geqslant y\right) \leqslant e^{AN^2-\frac{N^2}{4}y^2}$$

and so

$$\mu_V^N(|\hat{\mu}_{\mathbf{A}}^N(X_k)|) = \int \mu_V^N\left(|\hat{\mu}_{\mathbf{A}}^N(X_k)| \geqslant y\right)dy \leqslant 4\sqrt{A} + \int_{y\geqslant 4\sqrt{A}} e^{-\frac{N^2}{4}(y^2-4A)}dy \leqslant 8\sqrt{A}$$

where we assumed N large enough in the last line. Hence, we have proved that

$$\limsup_N |\mu_V^N(\hat{\mu}_{\mathbf{A}}^N(X_k))| < 8\sqrt{A}. \tag{3.9}$$

Plugging this result in (3.6) and (3.8) we obtain for all $p \geqslant 1$:

$$\begin{aligned}\limsup_{N\to\infty} \mu_V^N[\hat\mu_{\mathbf{A}}^N((A_k)^{2p})] &\leqslant 2^{2p-1}\limsup_{N\to\infty}\mu_V^N[\frac{1}{N}\mathrm{tr}((A_k-\mu_V^N[A_k])^{2p})] \\ &\quad +2^{2p-1}\limsup_{N\to\infty}(\mu_V^N(\frac{1}{N}\mathrm{tr}[A_k])^{2p}) \\ &\leqslant 2^{2p-1}4^p+2^{2p-1}(8\sqrt{A})^{2p}\leqslant R_V^{2p}\end{aligned}$$

with $R_V=4(1+8\sqrt{A})$. To prove the convergence of $\mu_N^V[\hat\mu_{\mathbf{A}}^N]$, remember that $\mu_N^V[\hat\mu_{\mathbf{A}}^N]$ is tight for the $\mathcal{C}_{st}^m(\mathbb{C})$-topology. To study its limit point, recall $\int xe^{-x^2/2}f(x)dx = \int f'(x)e^{-x^2/2}dx$ so that, for $P\in\mathcal{C}_{st}^m(\mathbb{C})$,

$$\begin{aligned}\int\frac{1}{N}\mathrm{tr}(A_kP)d\mu_N^V(\mathbf{A}) &= \frac{1}{2N^2}\sum_{ij}\int\partial_{A_k(ij)}(Pe^{-N\mathrm{tr}(V)})_{ji}\prod d\mu_N(A_i) \\ &= \frac{1}{2N^2}\sum_{ij}\int\Bigg(\sum_{P=QX_kR}2Q_{ii}R_{jj} \\ &\quad -N\sum_{l=1}^{n}\sum_{q_l=QX_kR}t_l\sum_{h=1}^{N}2P_{ji}Q_{hj}R_{ih}\Bigg)d\mu_V^N(\mathbf{A}) \\ &= \int\left(\frac{1}{N^2}(\mathrm{tr}\otimes\mathrm{tr})(\partial_kP)-\frac{1}{N}\mathrm{tr}(D_kVP)\right)d\mu_V^N(\mathbf{A})\end{aligned}$$

which yields

$$\int\left(\hat\mu_{\mathbf{A}}^N((X_k+D_kV)P)-\hat\mu_{\mathbf{A}}^N\otimes\hat\mu_{\mathbf{A}}^N(\partial_kP)\right)d\mu_N^V(\mathbf{A})=0$$

Now, by convexity of V we have concentration of $\hat\mu_{\mathbf{A}}^N$ under μ_N^V (since log-Sobolev inequality is satisfied uniformly, according to Bakry-Emery criterion, and that Herbst's argument therefore applies, see Ané et al. (2000), sections 6 and 7): for all Lipschitz function f on the entries

$$\mu_V^N\left(\mathbf{A}:|f(\mathbf{A})-\mu_V^N(f)|\geqslant\delta\right)\leqslant e^{-\frac{\delta^2}{2||f||_{\mathcal{L}}^2}} \tag{3.10}$$

where $||f||_{\mathcal{L}}$ is the Lipschitz constant of f. Since for $P\in\mathcal{C}_{st}^m(\mathbb{C})$, $\mathbf{A}\to\hat\mu_{\mathbf{A}}^N(P)$ is Lipschitz with constant of order N^{-1} (see Guionnet and Zeitouni (2000)), we conclude that since $\partial_iP\in\mathcal{C}_{st}^m(\mathbb{C})\otimes\mathcal{C}_{st}^m(\mathbb{C})$, for all $P\in\mathcal{C}_{st}^m(\mathbb{C})$,

$$\lim_{N\to\infty}\left|\int\hat\mu_{\mathbf{A}}^N\otimes\hat\mu_{\mathbf{A}}^N(\partial_kP)d\mu_N^V(\mathbf{A})-\mu_N^V[\hat\mu_{\mathbf{A}}^N]\otimes\mu_N^V[\hat\mu_{\mathbf{A}}^N](\partial_kP)\right|=0.$$

Thus

$$\limsup_{N\to\infty}\left(\mu_V^N(\hat\mu_{\mathbf{A}}^N((X_k+D_kV)P))-\mu_V^N[\hat\mu_{\mathbf{A}}^N]\otimes\mu_V^N[\hat\mu_{\mathbf{A}}^N](\partial_iP)\right)=0$$

If τ is a limit point of $\mu_V^N[\hat\mu_{\mathbf{A}}^N]$ for the weak $\mathcal{C}_{st}^m(\mathbb{C})$-topology, we can use the previous moment estimates to show that even though X_k+D_kV is a polynomial function, $\mu_V^N(\hat\mu_{\mathbf{A}}^N((X_k+D_kV)P))$ converges along subsequences towards $\tau((X_k+D_kV)P))$, and of course $\mu_V^N[\hat\mu_{\mathbf{A}}^N]\otimes\mu_V^N[\hat\mu_{\mathbf{A}}^N](\partial_kP)$ converges towards $\tau\otimes\tau(\partial_kP)$. Hence, we get that the limit points of $\mu_V^N[\hat\mu_{\mathbf{A}}^N]$ satisfy the **WSD[V]**. By the previous moment estimate, these limit points are compactly supported, hence they satisfy **SD[V]**.

Similarly, by (3.10), $\hat{\mu}^N_{\mathbf{A}}$ is almost surely tight and its limit points satisfy $\mathbf{SD}[\mathbf{V}]$ according to (3.7) and (3.9).

When $V = V_{\mathbf{t}}$, observe that $R_{\mathbf{t}}$ is uniformly bounded when $|t| \leqslant M$ since $V_{\mathbf{t}}(0,\cdots,0)$ and $(D_i V_{\mathbf{t}}(0,\cdots,0))_{1\leqslant i\leqslant m}$ depends continuously on $\mathbf{t}$. Thus, the first point of the theorem shows that the limit points of $\mu^N_{V_{\mathbf{t}}}[\hat{\mu}^N_{\mathbf{A}}]$ are uniformly compactly supported. Hence, since also we have seen that they satisfy $\mathbf{SD}[V_{\mathbf{t}}]$, for $\mathbf{t}$ small enough, $\hat{\mu}^N_{\mathbf{A}}$ converges in expectation (and therefore almost surely by concentration), to the unique solution to $\mathbf{SD}[V_{\mathbf{t}}]$. The last point is now a direct consequence of Theorem 3.3.

□

Hence, we see here that convex potentials have uniformly compactly supported limit distributions so that we can apply the whole machinery. We strongly believe that this property extends to much more general potentials. However, we shall see in the next section that we can localize the integral to make sure that all limit points are uniformly compactly supported and still keep the enumerative property, hence bypassing the issue of compactness.

3.3. *The uses of diverging integrals.* In the domain of matrix models, diverging integrals are often considered. For instance, if one wants to consider triangulations, one would like to study the integral

$$Z_N(tx^3) = \int e^{tN\mathrm{tr}(M^3)} d\mu_N(M)$$

which is clearly infinite if t is real. The same kind of problem arises in many other models (c.f. the dually weighted graph model Kazakov et al. (1996)). However, we shall see below that at least as far as planar maps are concerned, we can localize the integrals to make sense of it, while keeping its enumerative property. Namely, let $V_{\mathbf{t}} = V^*_{\mathbf{t}} = \sum t_i q_i$ and let us consider the localized matrix integrals given, for $L < \infty$, by

$$Z^{N,L}_{V_{\mathbf{t}}} = \int_{||\mathbf{A}||_\infty \leqslant L} e^{-N\mathrm{tr}(V_{\mathbf{t}}(\mathbf{A}))} \prod d\mu_N(A_i)$$

and the associated Gibbs measure

$$\mu^{N,L}_{V_{\mathbf{t}}}(d\mathbf{A}) = (Z^{N,L}_{V_{\mathbf{t}}})^{-1} \mathbb{1}_{||\mathbf{A}||_\infty \leqslant L} e^{-N\mathrm{tr}(V_{\mathbf{t}}(\mathbf{A}))} \prod d\mu_N(A_i).$$

Here, $||\mathbf{A}||_\infty = \max_{1\leqslant i\leqslant m} ||A_i||_\infty$ and $||A_i||_\infty$ denotes the spectral radius of the matrix A_i.

We shall prove

Theorem 3.5. *There exists $L_0 > 0$ and $\varepsilon_0 > 0$ so that for $\varepsilon < \varepsilon_0$, there exists $L(\varepsilon) > L_0$, $L(\varepsilon)$ going to infinity as ε goes to zero, so that for $\mathbf{t} \in B(0,\varepsilon) \cap \{\mathbf{t} | V_{\mathbf{t}} = V^*_{\mathbf{t}}\}$, for all $L \in [L_0, L(\varepsilon)]$,*

$$\lim_{N\to\infty} \frac{1}{N^2} \log Z^{N,L}_{V_{\mathbf{t}}} = \sum_{\overline{k}\in\mathbb{N}^n\setminus(0,..,0)} \prod_{1\leqslant i\leqslant n} \frac{(-t_i)^{k_i}}{k_i!} \mathcal{M}_{\overline{k}} \tag{3.11}$$

Moreover, under $\mu^{N,L}_{V_{\mathbf{t}}}$, $\hat{\mu}^N_{\mathbf{A}}$ converges almost surely towards $\tau_{\mathbf{t}}$ described in Theorem 2.3.

This shows that, up to localization, the first order asymptotics of matrix models gives the right enumeration for any polynomials. The diverging integrals often considered in physics should be therefore thought to be conveniently localized to keep their combinatorial virtue, and are then as good as others. In view of Lemma 3.6, this localization procedure should not damage the rest of the large N expansion neither.

Proof.

Fix $M > 0$ and choose η sufficiently small, so that $V_{\mathbf{t}}(0,\cdots,0)$, $D_iV_{\mathbf{t}}(0,\cdots,0)$ and $\sigma^m(V_{\mathbf{t}}V_{\mathbf{t}}^*)$ are uniformly bounded by a constant $M < +\infty$ for $\mathbf{t} \in B(0,\eta)$. We will prove that if L is sufficiently large, there exists $0 < \varepsilon < \eta$ such that Theorem 3.11 is valid for all $\mathbf{t}$ in $B(0,\varepsilon) \cap \{\mathbf{t}|V_{\mathbf{t}} = V_{\mathbf{t}}^*\}$.

We now see our potential as a convex potential in order to find an uniform bound on the support. First we bound the Hessian of

$$\varphi^N_{V_{\mathbf{t}}} : (A_k(ij)) \in (\mathbb{R}^{N^2})^m \cap \{\|\mathbf{A}\|_\infty \leqslant L\} \to \mathrm{tr}(V(A_1,\cdots,A_m)) \tag{3.12}$$

uniformly in N:

$$Hess\varphi^N_{V_{\mathbf{t}}}(A,A) = \sum_{i=1}^{n} t_i \sum_{q_i = RXSXT} \mathrm{tr}(RASAT).$$

Now we use Hölder's inequality:

$$\begin{aligned}|\mathrm{tr}(RASAT)| = |\mathrm{tr}(TRASA)| &\leq \sqrt{\mathrm{tr}((TR)A^*A(TR)^*)}\sqrt{\mathrm{tr}(SA^*AS^*)}\\ &\leqslant \|TR\|_\infty \|S\|_\infty \mathrm{tr}(AA^*).\end{aligned}$$

which implies that for $\{\|\mathbf{A}\|_\infty \leqslant L\}$

$$\|Hess\varphi^N_{V_{\mathbf{t}}}\| \leqslant C|\mathbf{t}|$$

and C depends only on L. Therefore, We can find $\varepsilon > 0$ such that if $\mathbf{t} \in B(0,\varepsilon) \cap \{\mathbf{t}|V_{\mathbf{t}} = V_{\mathbf{t}}^*\}$, for all N, $\varphi^N_{V_{\mathbf{t}}} + \frac{1}{4}\sum_{i=1}^n \mathrm{tr}(X_i^2)$ is convex on $\{\|\mathbf{A}\|_\infty \leqslant L\}$.

Thus $\tilde{V}_{\mathbf{t}}(\mathbf{A}) = V_{\mathbf{t}}(\mathbf{A}) + \infty 1\!\!1_{||\mathbf{A}||_\infty > L}$ is a convex potential and

$$1\!\!1_{||\mathbf{A}||_\infty \leqslant L} e^{-N\mathrm{tr}(V_{\mathbf{t}}(\mathbf{A}))} = e^{-N\mathrm{tr}(\tilde{V}(\mathbf{A}))}$$

is log-concave so that we can use the strategy of the proof of the first point in Theorem 3.4. The only point to check is that there exists $d < +\infty$ such that $Z^{N,L}_{V_{\mathbf{t}}} \geqslant e^{-dN^2}$. According to Jensen's inequality,

$$\begin{aligned}Z^{N,L}_{V_{\mathbf{t}}} &= \int_{||\mathbf{A}||_\infty \leqslant L} e^{-N\mathrm{tr}(V_{\mathbf{t}}(\mathbf{A}))} \prod d\mu_N(A_i)\\ &\geqslant \mu^N(||\mathbf{A}||_\infty \leqslant L) \exp\left(-\frac{N}{\mu^N(||\mathbf{A}||_\infty \leqslant L)} \int_{||\mathbf{A}||_\infty \leqslant L} \mathrm{tr}(V_{\mathbf{t}}(\mathbf{A})) \prod d\mu_N(A_i)\right)\end{aligned}$$

The biggest eigenvalue goes almost surely to 2 and $|\int_{||\mathbf{A}||_\infty \leqslant L} \mathrm{tr}(V_{\mathbf{t}}(\mathbf{A})) \prod d\mu_N(A_i)|$ is bounded by $\mu^N(V_{\mathbf{t}}V_{\mathbf{t}}^*)^{\frac{1}{2}}$ which goes to $\sigma^m(V_{\mathbf{t}}V_{\mathbf{t}}^*) < +\infty$ according to Voiculescu (1991). Thus if $L > 2$, $Z^{N,L}_{V_{\mathbf{t}}} \geqslant e^{-dN^2}$ for a finite constant d. Thus, we can use the same technique than in Theorem 3.4 to show that any limit point of $\hat{\mu}^N_{\mathbf{A}}$ has a bounded support R_M independent of L.

We choose $L > R_M$. Now the proof is very close to that of Theorem 3.1 except that we have to be careful to make perturbations which do not change the constraint $||\mathbf{A}||_\infty \leqslant L$. Let $i \in \{1,\cdots,m\}$ and consider the perturbation $A_i \to A_i + N^{-1}h(A_i)$

and $A_j \to A_j$ for $j \neq i$ with a compactly supported function h which vanishes on $[-R, R]^c$. Then for $L > R$, for sufficiently large N, and $||A_i||_\infty \leqslant L$, $||A_i + N^{-1}h(A_i)||_\infty \leqslant L$ so that we see that the limit points of $\hat{\mu}^N_{\mathbf{A}}$ under the localized Gibbs measure $\mu_V^{N,L}$ satisfy for $i \in \{1, \cdots, m\}$, for all h of support strictly less than L

$$\mu \otimes \mu(\partial_i h(X_i)) = \mu((D_i V + X_i)h(X_i)). \tag{3.13}$$

These limit points are also laws of operators bounded by $R_M < L$. Thus the limit points satisfy (3.13) for arbitrary polynomials P i.e. they satisfy **SD[V]**. Now according, to Theorem 2.3 if $\mathbf{t}$ is sufficiently small, **SD[V]**has an unique solution given by the enumeration of maps. Thus we have shown that for $L > R_M$, there exists $\varepsilon > 0$ such that for $\mathbf{t} \in B(0, \varepsilon))$ $\hat{\mu}^N$ goes almost surely to the soltution of **SD[V]** described in Theorem 2.3.

The formula for the free energy is then derived as in Theorem 3.3 since L is fixed independently of $\mathbf{t}$ small enough.

□

Let us remark that if we define, following Voiculescu (1993), for $\mu \in \mathcal{M}^m$, a microstates $\Gamma(\mu, n, N, \eta)$, $n \in \mathbb{N}$, $N \in \mathbb{N}$, $\eta > 0$, as the set of matrices $A_1, .., A_m$ of $\mathcal{H}_N^m$ such that

$$|\mu(\mathbf{X}_{i_1} .. \mathbf{X}_{i_p}) - \frac{1}{N}\mathrm{tr}(\mathbf{A}_{i_1} .. \mathbf{A}_{i_p})| < \eta \tag{3.14}$$

for any $1 \leqslant p \leqslant n$, $i_1, .., i_p \in \{1, .., m\}^p$, then we have

Lemma 3.6. *For all $\delta > 0$ small enough, for $L \in [L_0(\delta), L(\delta)]$, and $|\mathbf{t}| \leqslant \delta$,*

$$\begin{aligned}
&\lim_{N\to\infty} \frac{1}{N^2} \log \int_{||\mathbf{A}||_\infty \leqslant L} e^{-N tr(V_{\mathbf{t}}(\mathbf{A}))} d\mu_N(A_1) \cdots d\mu_N(A_m) \\
&= \lim_{\eta\to 0, n\to\infty} \lim_{N\to\infty} \frac{1}{N^2} \log \int_{\Gamma(\tau_{\mathbf{t}}, n, N, \eta) \cap ||\mathbf{A}||_\infty \leqslant L} e^{-N tr(V_{\mathbf{t}}(\mathbf{A}))} d\mu_N(A_1) \cdots d\mu_N(A_m) \\
&= \lim_{\eta\to 0, n\to\infty} \lim_{N\to\infty} \frac{1}{N^2} \log \int_{\Gamma(\tau_{\mathbf{t}}, n, N, \eta)} e^{-N tr(V_{\mathbf{t}}(\mathbf{A}))} d\mu_N(A_1) \cdots d\mu_N(A_m)
\end{aligned}$$

Proof.

The first equality is a direct consequence of the previous theorem since it is equivalent to the fact that $\mu_{V_{\mathbf{t}}}^{N,L}(\Gamma(\tau_V, n, N, \varepsilon))$ goes to one. The second comes from the fact that for n greater than the degree of V,

$$\begin{aligned}
&\lim_{\eta\to 0, n\to\infty} \lim_{N\to\infty} \frac{1}{N^2} \log \int_{\Gamma(\tau_{\mathbf{t}}, n, N, \eta) \cap ||\mathbf{A}||_\infty \leqslant L} e^{-N\mathrm{tr}(V_{\mathbf{t}}(\mathbf{A}))} d\mu_N(A_1) \cdots d\mu_N(A_m) \\
&= -\tau_{\mathbf{t}}(V_{\mathbf{t}}) + \lim_{\eta\to 0, n\to\infty} \lim_{N\to\infty} \frac{1}{N^2} \log \mu_N^{\otimes m} \left(\Gamma(\tau_{\mathbf{t}}, n, N, \eta) \cap ||\mathbf{A}||_\infty \leqslant L\right) \\
&= -\tau_{\mathbf{t}}(V_{\mathbf{t}}) + \lim_{\eta\to 0, n\to\infty} \lim_{N\to\infty} \frac{1}{N^2} \log \mu_N^{\otimes m} \left(\Gamma(\tau_{\mathbf{t}}, n, N, \eta)\right)
\end{aligned}$$

where we used in the last equality the result of Belinschi and Bercovici (2003), which hold when τ_V is the law of bounded operators with norms strictly smaller than L (see the last remark in Belinschi and Bercovici (2003)).

□

As a corollary, we also deduce that for all $V_{\mathbf{t}}$ with $\mathbf{t}$ small enough, the limits of empirical distributions of matrices given by localized matrix models provide solutions of $\mathbf{SD}[V_{\mathbf{t}}]$. Since these limits have to be tracial states, we deduce that when there is a unique solution to $\mathbf{SD}[V_{\mathbf{t}}]$ it has to be a tracial state. Thus,

Corollary 3.7. *The compactly supported solutions of* $\mathbf{SD}[V_{\mathbf{t}}]$ *are tracial states when* $\mathbf{t}$ *is sufficiently small.*

Note that if $(P_i)_{1\leqslant i\leqslant m}$ in $\mathbb{C}\langle X_1,\cdots,X_m\rangle^m$ is the conjugate variable of a tracial state, Voiculescu (2002) have shown that $P_i = D_i P$ for $1 \leqslant i \leqslant m$ and some polynomial P. This fact should be compared with our graphical interpretation which works only because P_i is a cyclic derivative.

4. Applications to free entropy

Let us recall that Voiculescu's microstates entropy is defined, for $\tau \in \cup_R \mathcal{M}_R^m$, by

$$\chi(\tau) = \lim_{\substack{\eta\to 0, n\to\infty\\ L\to\infty}} \limsup_{N\to\infty} \frac{1}{N^2} \log \mu_N^{\otimes m}\left(\Gamma(\tau,n,N,\eta)\cap ||\mathbf{A}||_\infty \leqslant L\right)$$

with $\Gamma(\tau,n,\eta,N)$ the microstates defined in (3.14). Note that the original definition of Voiculescu is not with respect to the Gaussian measure, but with respect to the Lebesgue measure. However, both definitions only differ by a quadratic term (see Cabanal-Duvillard and Guionnet (2003)). It is an (important) open problem whether in general one can replace the limsup by a liminf in the definition of χ. However, from the previous considerations, we can see the following

Theorem 4.1. *Let $n\in\mathbb{N}$ and $(q_i)_{1\leqslant i\leqslant n}$ be monomials in m non-commutative variables $\mathbf{X}=(X_1,\cdots,X_m)$. Let $V_{\mathbf{t}} = V_{\mathbf{t}}^* \sum_{i=1}^n t_i q_i$. By Theorem 2.3, we know that there exists $\varepsilon>0$ so that for $|t|<\varepsilon$, there exists a unique compactly supported solution $\tau_{\mathbf{t}}$ to $\mathbf{SD}[V_{\mathbf{t}}]$. Then, also for $|t|\leqslant\varepsilon$,*

$$\chi(\tau_{\mathbf{t}}) = \lim_{\substack{\eta\to 0, n\to\infty\\ L\to\infty}} \liminf_{N\to\infty} \frac{1}{N^2} \log \mu_N^{\otimes m}\left(\Gamma(\tau_{\mathbf{t}},n,N,\eta)\cap \{||\mathbf{A}||_\infty \leqslant L\}\right).$$

Moreover,

$$\chi(\tau_{\mathbf{t}}) = -\sum_{k\in\mathbb{N}^n\setminus(0,\cdots,0)} \prod_{i=1}^n \frac{(-t_i)^{k_i}}{k_i!}\left(\sum_{j=1}^n k_j - 1\right)\mathcal{M}_{\overline{k}}.$$

Proof.
In fact, by Lemma 3.6

$$\begin{aligned}
\chi(\tau_{\mathbf{t}}) &= \lim_{\substack{\eta\to 0, n\to\infty\\ L\to\infty}} \limsup_{N\to\infty} \frac{1}{N^2}\log\int_{\substack{\Gamma(\tau_{\mathbf{t}},n,N,\eta)\\ \cap\{||\mathbf{A}||_\infty\leqslant L\}}} e^{N\mathrm{tr}(V_{\mathbf{t}}(\mathbf{A}))-N\mathrm{tr}(V_{\mathbf{t}}(\mathbf{A}))} d\mu_N^{\otimes m}(\mathbf{A})\\
&= \tau_{\mathbf{t}}(V_{\mathbf{t}}) + \lim_{\substack{\eta\to 0, n\to\infty\\ L\to\infty}} \limsup_{N\to\infty} \frac{1}{N^2}\log\int_{\substack{\Gamma(\tau_{\mathbf{t}},n,N,\eta)\\ \cap\{||\mathbf{A}||_\infty\leqslant L\}}} e^{-N\mathrm{tr}(V_{\mathbf{t}}(\mathbf{A}))} d\mu_N^{\otimes m}(\mathbf{A})\\
&\leqslant \tau_{\mathbf{t}}(V_{\mathbf{t}}) + F_{\mathbf{t}}
\end{aligned}$$

where the last inequality holds with

$$F_{\mathbf{t}} = \limsup_{N\to\infty}\frac{1}{N^2}\log\int_{||\mathbf{A}||_\infty\leqslant L'} e^{-N\mathrm{tr}(V_{\mathbf{t}}(\mathbf{A}))} d\mu_N^{\otimes m}(\mathbf{A})$$

for L' chosen as in Lemma 3.6. On the other hand,

$$\begin{aligned}
&\lim_{\substack{\eta\to 0, n\to\infty\\ L\to\infty}} \liminf_{N\to\infty} \frac{1}{N^2}\log \mu_N^{\otimes m}\left(\Gamma(\tau_{\mathbf{t}}, n, N, \eta)\cap ||\mathbf{A}||_\infty \leqslant L\right)\\
&= \tau_{\mathbf{t}}(V_{\mathbf{t}}) + \lim_{\substack{\eta\to 0, n\to\infty\\ L\to\infty}} \liminf_{N\to\infty} \frac{1}{N^2}\log \int_{\Gamma(\tau_{\mathbf{t}},n,N,\eta)\cap||\mathbf{A}||_\infty\leqslant L} e^{-N\mathrm{tr}(V_{\mathbf{t}}(\mathbf{A}))} d\mu_N^{\otimes m}(\mathbf{A})\\
&= \tau_{\mathbf{t}}(V_{\mathbf{t}}) + F_{\mathbf{t}} + \lim_{\eta\to 0, n\to\infty} \liminf_{N\to\infty} \frac{1}{N^2}\log \mu_{V_{\mathbf{t}}}^{N,L'}\left(\Gamma(\tau_{\mathbf{t}}, n, N, \eta)\right)\\
&= \tau_{\mathbf{t}}(V_{\mathbf{t}}) + F_{\mathbf{t}}
\end{aligned}$$

where we used in the last term Theorem 3.5 which implies

$$\lim_{N\to\infty} \mu_{V_{\mathbf{t}}}^{N,L'}\left(\Gamma(\tau_{\mathbf{t}}, n, N, \eta)\right) = 1$$

for all $\varepsilon > 0, n \in \mathbb{N}$. Thus, we see that χ is equal to its liminf definition and moreover

$$\chi(\tau_{\mathbf{t}}) = \tau_{\mathbf{t}}(V_{\mathbf{t}}) + F_{\mathbf{t}}.$$

Now, by Theorems 3.5 and 2.3,

$$F_{\mathbf{t}} = \sum_{\overline{k}\in\mathbb{N}^n\setminus(0,..,0)} \prod_{1\leqslant i\leqslant n} \frac{(-t_i)^{k_i}}{k_i!}\mathcal{M}_{\overline{k}}$$

whereas

$$\tau_{\mathbf{t}}(V_{\mathbf{t}}) = \sum_{i=1}^n t_i \sum_{\substack{k_j\in\mathbb{N},\\ 1\leqslant j\leqslant n}} \prod_{1\leqslant j\leqslant n} \frac{(-t_j)^{k_j}}{k_j!}\mathcal{M}_{k_1,\cdots,k_{i-1},,k_i+1,k_{i+1},\cdots,k_n}$$

from which the formula for $\chi(\tau_{\mathbf{t}})$ is easily derived.

□

5. Applications to the combinatorics of planar maps

For the sake of completeness, we summarize in this last section, the results of a few papers devoted to the enumeration of planar maps, either by a combinatorial approach or by a matrix model approach.

5.1. *The one matrix case.* We now consider the case $m = 1$ where we only have one matrix. Let $V_{\mathbf{t}}(A) = \sum_{i=1}^{2D} t_i A^i$ with $t_{2D} > 0$ a polynomial potential with an even leading power. Then it has been proven in Ben Arous and Guionnet (1997) Theorem 5.2 that the empirical measure satisfies a large deviation principle:

Theorem 5.1. *Let*

$$J(\mu) = \int \left(\frac{x^2}{2} + V_{\mathbf{t}}(x)\right) d\mu(x) - \int\int \log|x-y| d\mu(y)d\mu(x)$$

and

$$I(\mu) = J(\mu) - \inf_{\nu\in P(\mathbb{R})} J(\nu)$$

then the sequence of empirical measure $\hat{\mu}^N$ satisfies a large deviation principle in the scale N^2 with good rate function I. Moreover, the minimum of I is reached at a unique probability measure $\mu_{\mathbf{t}}$ so that

$$\frac{x^2}{2} + V_{\mathbf{t}}(x) - 2\int \log|y-x| d\mu_{\mathbf{t}}(y) = C_{\mathbf{t}}, \;\; \mu_{\mathbf{t}} a.s.$$

with a finite constant $C_{\mathbf{t}}$, and where the left hand side dominates the right hand side on the whole real line.

We can differentiate in x this last equation to recover Schwinger-Dyson's equation. It is not sufficient in general to determine the solution uniquely; one need the inequality on the whole real line to fix the support of the solution.

These analysis of $\mu_{\mathbf{t}}$ has also been investigated with the method of orthogonal polynomials which give a rather sharp description of the limit measure and emphasizes a structure similar to the semi-circular law. More precisely Theorem 3.1 in Ercolani and McLaughlin (2003) gives:

Theorem 5.2. *Let $V_{\mathbf{t}}$ be a real polynomial of degree $2D$. There exists $t > 0$ and $\gamma > 0$ such that if for all i, $|t_i| < t$ and $t_{2D} > \gamma \sum_{i<2D} t_i$ then $\mu_{\mathbf{t}}$ is absolutely continuous with density $\Psi_{\mathbf{t}}$ of the form:*

$$\Psi_{\mathbf{t}}(x) = \frac{1}{2\pi} \mathbb{1}_{[a,b]}(x) \sqrt{(x-a)(x-b)}\, h(x)$$

with

$$h(z) = \int_{C(z,R)} \frac{V_{\mathbf{t}}'(s)}{\sqrt{(s-a)(s-b)}} \frac{ds}{s-z}$$

where R is such that $a, b \in C(z,R)$. Besides, the boundaries a and b can be find by the equations:

$$\int_a^b \frac{V_{\mathbf{t}}'(s)}{\sqrt{(s-a)(b-s)}} ds = 0$$

$$\int_a^b \frac{s V_{\mathbf{t}}'(s)}{\sqrt{(s-a)(b-s)}} ds = 2\pi$$

We now look at combinatorics of the Schwinger-Dyson's equation with one variable, for $V_{\mathbf{t}}(x) = \sum_{i=1}^{2D} t_i x^i$. Remember that from Theorem 2.3, $\mu_{\mathbf{t}}$ can be seen as the generating function of graphs counted by the numbers of stars of valence i:

$$\mu_{\mathbf{t}}(x^p) = \sum_{k_1,\cdots,k_{2D} \in \mathbb{N}} \prod_{i=1}^{2D} \frac{(-t_i)^{k_i}}{k_i!} \mathcal{M}_{\overline{k}}(P).$$

Hence, Theorem 5.2 allows to estimate the numbers of one color planar maps. A more direct combinatorial approach can be developed by considering for instance the dual of those graphs. The dual of a graph is simply obtained by replacing each face by a star and each edge by a transverse edge which link the two stars which come from the face adjacent to the edge. In that operation each star is replaced by a face of the same valence. As we work on the sphere we can decide that the face which comes from the star X^p is the external face.

Thus $\mu_{\mathbf{t}}(X^{p+1})$ is also the generating function of connected planar graphs with an external face of valence $p+1$ and enumerated by the number of faces of a given valence. Those objects are classical ones in combinatorics and we can follow Tutte (1968) to find an equation on these generating functions. The idea is to try to cut the first edge of the external face, then two cases may occur: either the graph is disconnected and we obtain two graphs or it isn't disconnected and the external face has grown. This two cases corresponds in the dual graph to the fact that the first half-edge of the root is a loop or not which is exactly what we use to build our combinatorial interpretation so that we can retrieve the Schwinger-Dyson's equation from this fact. Just by using the equation given by this decomposition and some algebraic tools, combinatorialists have solved some models. For example Bender and Canfield (1994) gives an equation on the generating function $M(u,v)$ of maps whose internal faces have degree living in a fixed set $\mathcal{D} \subset \mathbb{N}$ and enumerated by their number of edges and the degree of the external face. To translate this in our framework, one can consider for a finite $\mathcal{D}$ with an even maximal element,

$$V_{\mathbf{t}}(X) = \sum_{d \in \mathcal{D}} t_d X^d$$

Then under this potential, for small t, the limit measure $\mu_{\mathbf{t}}$ will satisfy our combinatorial interpretation. Define

$$M(u,v) = \sum_{p \in \mathbb{N}} \mu_{(-u^{\frac{d}{2}})_{d \in \mathcal{D}}}(X^p) v^p$$

which counts maps according to the degree of the external face and its number of edges. Theorem 1 of Bender and Canfield (1994) states:

Theorem 5.3. *For a serie $F(z) = \sum_i a_i z^i$ we will note $[z^i]F(z)$ the i^{th} coefficient a_i. Then there exists a unique power serie R satisfying*

$$R = 1 - 4R_1 v - 4R_2 v^2$$

with

$$R_1 = \frac{u}{2} \sum_{i \in D} [v^{i-1}](R^{\frac{1}{2}}) \text{ and } R_2 = \frac{u}{2} \sum_{i \in D} [v^i](R^{\frac{1}{2}}) + u - 3R_1^2.$$

The number m_n of maps with n edges such that every degree of internal face lies in D is then

$$m_n = [u^n] \frac{(R_2(u) + R_1(u)^2)(R_2(u) + 9R_1(u)^2)}{(n+1)u^2}$$

The techniques to prove these results are most often purely algebraic. The main difference in nature that we could meet between the approaches by matrix models or by combinatorics to these enumerations is that the first provides for free additional structure; it shows that these enumerations can be expressed in terms of a probability measure $\mu_{\mathbf{t}}$. This point generalizes to any number of colors where the enumeration can be expressed in terms of tracial states. One may hope that this positivity condition could help in solving these combinatorial problems.

5.2. *Ising model on random graphs.* This model is defined by $m = 2$ and

$$V(A,B) = V_{Ising}(A,B) = -cAB + V_1(A) + V_2(B).$$

In the sequel, we denote in short A for X_1 and B for X_2. It is clear that for $|c| < 1$, V is a convex potential as defined in (3.3) if V_1, V_2 are convex (write $-2AB = (A-B)^2 - A^2 - B^2$ or $2AB = (A+B)^2 - A^2 - B^2$ to see that up to a quadratic term $2^{-1}|c|A^2 + 2^{-1}|c|B^2$, V is convex) Hence we deduce from Theorem 3.4

Corollary 5.4. *For $c \in \mathbb{R}$ and $V_i(x) = \sum_{j=1}^{D} t_j^i x^{2j}$, $i = 1, 2$, set $V_{\mathbf{t},c}(A,B) = -cAB + V_1(A) + V_2(B)$. Let, for $\delta > 0$, $U_\delta = \cap_{i,j}\{0 \leqslant t_j^i \leqslant \delta\} \cap \{|c| < 1-\delta\}$. Then, for $\delta > 0$ small enough and $(\mathbf{t}, c) \in U_\delta$, $\mu_V^N(\hat{\mu}_{\mathbf{A}}^N)$ converges towards the solution $\mu_{\mathbf{t},c}$ of $\mathbf{SD}[V_{\mathbf{t},c}]$ as N goes to infinity. Moreover*

$$\mu_{\mathbf{t},c}(P) = \sum_{\substack{\overline{k} \in \mathbb{N}^{2D} \\ r \in \mathbb{N}}} \prod_{i,j} \frac{(-t_j^i)^{k_j^i}}{k_j^i!} \frac{c^r}{r!} \mathcal{M}_{\overline{k^1}, \overline{k^2}, r}(P)$$

and

$$F(\mathbf{t}, c) - F(\mathbf{t}, 0) = \sum_{\substack{\overline{k} \in \mathbb{N}^{2D} \\ r \geqslant 1}} \prod_{i,j} \frac{(-t_j^i)^{k_j^i}}{k_j^i!} \frac{c^r}{r!} \mathcal{M}_{\overline{k^1}, \overline{k^2}, r}$$

where $\mathcal{M}_{\overline{k^1}, \overline{k^2}, r}$ (resp. $\mathcal{M}_{\overline{k^1}, \overline{k^2}, r}(P)$) is the number of planar maps with k_j^1 vertices of type A^{2j}, k_j^2 of type B^{2j} and r of type AB (resp. and one of type P).

Remark: Note that we took potentials V_1 and V_2 as polynomials with even powers to guarantee our convexity relation but this condition could easily be relaxed by taking more sophisticated domains than U_δ in which the polynomials would remain convex.

Proof.

This result is a consequence of Theorem 2.2, 3.4 and 3.3. Note here that the control on $\mu_V^N(N^{-1}\mathrm{tr}(AB))$ assumed in Theorem 3.3 is satisfied due to Theorem 3.4 which provides a uniform bound when $|c| < \xi$ for $\xi < 1$.

□

According to the graphical interpretation, the limiting measure is linked to planar maps with stars whose type are the monomial of V_1, V_2 and stars of type AB. Those maps are very close from Ising configuration on planar graphs except that two stars of type AB can be linked together. For integers $(k_j^i)_{i\in\{A,B\}, 1\leqslant i \leqslant D}$, define

$$\begin{aligned} \mathcal{I}_{\{k_j^i\},r}(P) \quad = \sharp\{ \quad & \text{planar maps with } k_j^i \text{ stars of color } i \text{ and degree } 2j, \\ & \text{one star of type } P \text{ (if } P \neq 0) \text{ and } r \text{ stars of type } AB \\ & \text{such that there's no link between any of the } r\ AB\text{-stars. } \} \end{aligned}$$

and its rooted counterpart:

$$\begin{aligned} \mathcal{J}_{\{k_j^i\},r}(P) \quad = \sharp\{ \quad & \text{rooted planar maps with } k_j^i \text{ stars of color } i \text{ and degree } 2j, \\ & \text{one star of type } P \text{ wich is the root and } r \text{ stars of type } AB \\ & \text{such that there's no link between any of the } r\ AB\text{-stars. } \} \end{aligned}$$

There's a relation between these quantities similar to (2.3):

$$\mathcal{I}_{\{k_j^i\},r}(P) = \mathcal{J}_{\{k_j^i\},r}(P) r! \Pi_{i,j} k_j^i! (2j)^{k_j^i} \tag{5.1}$$

We can now relate these numbers to our limit measure:

Proposition 5.5. *Let $\mu_{\mathbf{t},c}$ be as in Corollary 5.4, then on its radius of convergence,*

$$\mu_{\mathbf{t},c}(P) = \left(\frac{1}{1-c^2}\right)^{\frac{deg\ P}{2}} \sum_{\substack{k_j^i \in \mathbb{N}^{2D} \\ r\in\mathbb{N}}} \prod_{i,j} \frac{1}{k_j^i!} \left(\frac{-t_j^i}{(1-c^2)^j}\right)^{k_j^i} \frac{c^r}{r!} \mathcal{I}_{\{k_j^i\},r}(P)$$

and

$$F(\mathbf{t},c) - F(\mathbf{t},0) = \frac{1}{1-c^2} \sum_{\substack{k_j^i \in \mathbb{N}, i\in\{1,2\}, \\ j\in\{1,D\}, r\geqslant 1}} \prod_{i,j} \frac{1}{k_j^i!} \left(\frac{-t_j^i}{(1-c^2)^j}\right)^{k_j^i} \frac{c^r}{r!} \mathcal{I}_{\{k_j^i\},r}(0)$$

Proof.

First we define a projection π from rooted maps to rooted Ising graphs such that if M is a map $\pi(M)$ is obtained by deleting pairs of AB stars which are glued. We now apply Corollary 5.4, and translate its result in term of rooted diagrams using (2.3):

$$\mu_{\mathbf{t},c}(P) = \sum_{\substack{\overline{k}\in\mathbb{N}^{2D} \\ r\in\mathbb{N}}} \prod_{i,j} (-2jt_j^i)^{k_j^i} c^r \mathcal{D}_{\{k_j^i\},r}(P)$$

All the maps M appearing in that sum are such that $\pi(M)$ is an Ising graph rooted at a star of type P. For a fixed Ising graph G we must find the contribution in that sum of $\pi^{<-1>}(G)$. But we can construct every graph in that set by adding pairs of stars AB on the edges of G. The numbers of edges of G is $e_G = \frac{degP}{2} + \sum_{i,j} jk_j^i$ so that to get the whole contribution of $\pi^{<-1>}(G)$ we have to multiply the contribution of G by

$$\sum_{a_1,\cdots,a_{e_G}\in\mathbb{N}} c^{2\sum a_i} = \left(\frac{1}{1-c^2}\right)^{\frac{degP}{2}+\sum_{i,j} jk_j^i}.$$

In that sum, a_i stands for the number of pairs of AB stars added on the i^{th} edge. Summing on every graphs, we obtain:

$$\mu_{\mathbf{t},c}(P) = \left(\frac{1}{1-c^2}\right)^{\frac{deg\ P}{2}} \sum_{\substack{\overline{k}\in\mathbb{N}^{2D} \\ r\in\mathbb{N}}} \prod_{i,j} \left(\frac{-2jt_j^i}{(1-c^2)^j}\right)^{k_j^i} c^r \mathcal{J}_{\{k_j^i\},r}(P)$$

and the result follows by using (5.1).

The second point can be proven by proceeding in the same way.

□

In the rest of this section, we compare a few different approaches to solve the enumeration problem of the Ising model. In short, let us emphasize that, for the time being, combinatorial and orthogonal polynomials approaches give the more complete and explicit results. However, these techniques are still limited to very few models. The Schwinger-Dyson's equation or the large deviation approaches can be developed for a much wider range of models (such as q-Potts, induced QCD etc). However, it seems to us that these arguments still need some mathematical efforts to provide as transparent and powerful results (namely for the first a mathematical study of the so-called master-loop equations, and for the second a clear understanding of the relations between complex Burgers equations and the master-loop equations).

5.2.1. *Orthogonal polynomial approach.* Here we take $V_1 = V_2 = (g/4)x^4$. By using orthogonal polynomials techniques, it was proved by Mehta (1981) that the corresponding free energy $F_{g,c}$ satisfies

$$F_{g,c} - F_{0,c} = \int_0^1 (1-x)[\log f(x) - \log \frac{cx}{2(1-c^2)}]dx$$

with $f(x) = f_{g,c}$ solution to the algebraic equation

$$f(x)\{(1 - 6\frac{g}{c}f(x))^{-2} - c^2\} + 12g^2 f^3(x) - \frac{1}{2}cx = 0$$

and the root to be taken equals $2^{-1}cx(1-c^2)^{-1}$ when $g = 0$.

Starting from there, a simpler expression as been derived in Boulatov and Kazakov (1987) (equation (16), (17) with $h = z/g$):

$$\begin{aligned} F_{z,c} &= \frac{1}{2}\ln h(z) + \frac{h^2(z)}{2}\left(\frac{z-1}{2(3z-1)^3} + c^2\frac{z+1}{3z-1} + \frac{c^4}{2}(3z^4 - 3z^2 + 1)\right) \\ &\quad -h(z)(\frac{1}{3z-1} + c^2(1-z^2)) + \frac{1}{2}\ln(1-z^2) + \frac{3}{4} \end{aligned}$$

with

$$h(z) = \frac{(1-3z)^2}{1 - c^2(1-3z)^2(1-3z^2)} \tag{5.2}$$

Hence, by the preceding, Mehta's result gives a formula for the generating function of $\mathcal{J}$ in the quadrangulation case. However, it does not a priori gives the limiting spectral measures of the matrices. Moreover, this strategy could be only developed completely and rigorously for the Ising model and the matrix coupled in chain model (see Chadha et al. (1981)).

5.3. *Direct combinatorial approach.* We can also relate this result to the work of Bousquet-Melou and Schaeffer (2002). Their approach is purely combinatorial; they use bijection with well labeled trees (whose generating functions are well understood) to obtain algebraic equations for the generating functions of the Ising model. Let $I(X, Y, u)$ be the generating function of the Ising model on quasi-tetravalent graphs, (i.e. tetravalent graphs except for the root which is bivalent and black) where X (resp. Y) counts the black (resp. white) tetravalent stars and u the bicolored edges:

$$I(X,Y,u) = \sum_{m,n,r\in\mathbb{N}} X^m Y^n u^r \sharp \left\{ \begin{array}{c} \text{quasi -tetravalent maps with } m \text{ tetravalent} \\ \text{black stars, } n \text{ tetravalent white stars and} \\ r \text{ bi-colored edges} \end{array} \right\}.$$

If $P(x, y, u)$ is the solution to the algebraic equation:

$$P = 1 + 3xyP^3 + \frac{P(1+3xP)(1+3yP)}{u^2(1-9xyP^2)^2} \tag{5.3}$$

Then, by Bousquet-Melou and Schaeffer (2002), Proposition 1 p.4, I can be written in function of $P(x, y, u)$ with $x = X(u - \frac{1}{u})^2$ and $y = Y(u - \frac{1}{u})^2$ as

$$I(X,Y,u) = \frac{u^2-1}{u^2}\left(xP^3 + \frac{P(1-3xP-2xP^2-6xyP^3)}{1-9xyP^2} - \frac{yP^3(1+3xP)^3}{u^2(1-9xyP^2)^3}\right).$$

On the other hand, according to Proposition 5.5, if $V = tA^4 + uB^4 - cAB$ and $\mu_{t,u,c}$ is the associated limit measure then on its domain of convergence,

$$I(X,Y,u) = (1-u^2)\mu_{X(1-u^2)^{-2},Y(1-u^2)^{-2},u}(A^2).$$

If we make the following change of variable in (5.3):

$$x = y = \frac{-z}{3c^2h(z/3)}, P = -c^2h(z/3), u = c$$

then we find (5.2). Hence, a combinatorial approach can be developed to solve the problem of the enumeration of planar maps of the Ising model, a strategy which requires some combinatorial insight. The next approach we present, developed in particular by Staudacher, Kazakov and Eynard, is a direct analysis of the **SD[V]** equations. It is a purely analytical and rather robust strategy.

5.4. *Direct study of the* $\mathbf{SD}[V_{Ising}]$ *equations.* Here, the analysis is based on Theorem 3.4 which asserts that if V_1, V_2 are convex, for small parameters, $\hat{\mu}^N_{A,B}$ converges almost surely towards the solution $\mu_{\mathbf{t},c}$ of $\mathbf{SD}[V_{\mathbf{t},c}]$ which is a generating function for the enumeration of maps. Hereafter we take $c = 1$ up to a rescaling $\bar{x} = \sqrt{c}x$, $\bar{y} = \sqrt{c}x$, $V_1(x) = \bar{V}_1(\bar{x})$, $\mu_{\mathbf{t}}(P(A,X_2)) = \mu_{\mathbf{t},1}(P(\sqrt{c}^{-1}A, \sqrt{c}^{-1}X_2))$. Following Eynard (2003), we shall analyze the solutions of the Schwinger-Dyson's equation. Observe that the following considerations hold for any range of parameters, not only small parameters. For large parameters, we do not know that the Schwinger-Dyson's equation has a unique solution but we still know that any limit point of the empirical measure of the random matrices still satisfies it. In the next section, we shall see that for the Ising model and any range of parameters, there is a unique such limit point, and it will therefore enjoy the properties described below. We here summarize the main result, as found in Eynard (2003). Let $\mu_{\mathbf{t}}$ be a solution of $\mathbf{SD}[V_{Ising}]$

$$\mu_{\mathbf{t}}((W_1'(A) - B)P) = \mu_{\mathbf{t}} \otimes \mu_{\mathbf{t}}(\partial_A P),$$
$$\mu_{\mathbf{t}}((W_2'(B) - A)P) = \mu_{\mathbf{t}} \otimes \mu_{\mathbf{t}}(\partial_B P),$$

with ∂_A (resp. ∂_B) the non-commutative derivative with respect to A (resp. B) μ_A (resp. μ_B) and $W_i(z) = z^2/2 + V_i(z)$. Now, let μ_A (resp. μ_B) be the spectral measure of the matrix A (resp. B) then we shall obtain an algebraic equation for $H\mu_A(x)$ (resp. $H\mu_B(x)$) the Stieljes transform of the limiting measure μ_A (resp. μ_B) given, for $x \in \mathbb{C}\backslash\mathbb{R}$ by:

$$H\mu_A(x) = \mu_{\mathbf{t}}(\frac{1}{x-A}) = \int \frac{1}{x-y} d\mu_A(y)$$

Proposition 5.6. *Let for* $x, y \in \mathbb{C}\backslash\mathbb{R}$, $Y(x) = W_1'(x) - H\mu_A(x)$ *and* $X(y) = W_2'(y) - H\mu_B(y)$. *Then, there exists a polynomial function* $E(x,y)$ *so that for all* $x, y \in \mathbb{C}\backslash\mathbb{R}$

$$E(X(y), y) = 0 \qquad E(x, Y(x)) = 0.$$

In particular, μ_A *and* μ_B *are absolutely continuous with respect to Lebesgue measure, with Hilbert transform* $H\mu_A$ *and* $H\mu_B$ *so that* $Y(x) = W_1'(x) - H\mu_A(x)$ *satisfies the same algebraic equation with* $x \in \mathbb{R}$.

Proof.

Note that since we know that $\mu_{\mathbf{t}}$ is compactly supported, we can take Stieljes functions in $\mathbf{SD}[V_{Ising}]$ instead of polynomials P since the latest are dense by

Weirstrass theorem. We choose $P = P(A) = (x-A)^{-1}$ in the second equation in $\mathbf{SD}[V_{Ising}]$ to obtain:

$$\mu_{\mathbf{t}}\left(\frac{W_2'(B)}{x-A}\right) = -1 + xH\mu_A(x)$$

Then we use this in the first equation written with

$$P(A,B) = \frac{1}{(x-A)}\frac{(W_2'(y) - W_2'(B))}{(y-B)}$$

to get after some calculation

$$U(x,y)(y - Y(x)) = (Y(x) - W_1'(x))(x - W_2'(y)) + 1 - Q(x,y) \tag{5.4}$$

where

$$U(x,y) = \mu_{\mathbf{t}}\left(\frac{1}{(x-A)}\frac{W_2'(y) - W_2'(B)}{(y-B)}\right),$$

and

$$Q(x,y) = \mu_{\mathbf{t}}\left(\frac{W_1'(x) - W_1'(A)}{(x-A)}\frac{W_2'(y) - W_2'(B)}{(y-B)}\right).$$

To obtain our algebraic equation, we simply define

$$E(x,y) = (y - W_1'(x))(x - W_2'(y)) + 1 - Q(x,y)$$

and we obtain the famous "Master-loop equation"

$$E(x, Y(x)) = 0$$

by taking $y = Y(x)$ in (5.4). In a symmetric way, we can show that if $X(y) = W_2'(y) - H\mu_B(x)$ then we also have $E(X(y), y) = 0$. Note that E is a polynomial function. Hence, this shows that $Y(x)$, $X(y)$ and so the generating functions $H\mu_A(x)$ and $H\mu_B(y)$ are solution to an algebraic equation. However, this equation still contains a certain numbers of unknown; $\{\mu_{\mathbf{t}}(A^p B^q), p \leqslant \deg(V_1) - 2, q \leqslant \deg(V_2) - 2\}$. It is argued in physics that when $\mathbf{t}$ is small, the supports of μ_A and μ_B should be connected and therefore $(x, Y(x))$ and $(X(y), y)$ should then be genus zero curves. Then, these unknowns should be determined by the asymptotic behavior of $X(y)$ and $Y(x)$ at infinity

$$X(y) \simeq W_2'(y) - \frac{1}{y}(1 + o(1)), \quad Y(x) \simeq W_1'(x) - \frac{1}{x}(1 + o(1)).$$

Note in passing that, as solutions of an algebraic equation, $H\mu_A$ and $H\mu_B$ extends continuously (but in general not differentially) to the real line (as an extended complex number). As a consequence, μ_A and μ_B have densities with respect to the Lebesgue measure, as the limits of the imaginary part of the Stieljes transform on the real line.

□

5.5. *Large deviations approach.* An approach using large deviation was developed in Guionnet (2003), see also Matytsin (1994). Again, we take $c = 1$ up to rescaling and denote $W_i(x) = x^2/2 + V_i(x)$ for $i = 1, 2$. The main advantage of this strategy is to be valid in the whole range of the parameters. Otherwise, it should provide the same type of information than in the previous paragraph. Namely,

Proposition 5.7. *For any polynomials V_1, V_2 going to infinity faster than x^2, $\hat{\mu}^N_{A,B}$ converges almost surely towards $\mu_{\mathbf{t},1} = \mu_{\mathbf{t}}$ which is uniquely defined by the Schwinger-Dyson's equations*

$$\mu_{\mathbf{t}} \otimes \mu_{\mathbf{t}}(\partial_A P) = \mu_{\mathbf{t}}((W_1'(A) - B)P), \qquad \mu_{\mathbf{t}} \otimes \mu_{\mathbf{t}}(\partial_B P) = \mu_{\mathbf{t}}((W_2'(B) - A)P) \quad (5.5)$$

and by the fact that $\mu_{\mathbf{t}}|_A$ and $\mu_{\mathbf{t}}|_B$ (which are the limits of $\hat{\mu}^N_A$ and $\hat{\mu}^N_B$ respectively) are the unique minimizers of

$$\begin{aligned} S^{V_1,V_2}(\mu) &= \mu_A(W_1) + \mu_B(W_2) - 2^{-1} \int\int \log|x-y| d\mu^A(x) d\mu^A(y) \\ &- 2^{-1} \int\int \log|x-y| d\mu^B(x) d\mu^B(y) \\ &+ \frac{1}{2} \inf_{\rho,m} \{ \int_0^1 \int \frac{m_t(x)^2}{\rho_t(x)} dx dt + \frac{\pi^2}{3} \int_0^1 \int \rho_t(x)^3 dx dt \} \end{aligned}$$

where the inf is taken over m, ρ so that $\mu_t(dx) = \rho_t(x)dx \in \mathcal{P}(\mathbb{R})$, $\mu_0(x \in .) = \mu_A(x \in .)$, $\mu_1(x \in .) = \mu_B(x \in .)$, and

$$\partial_t \rho_t(x) + \partial_x m_t(x) = 0.$$

The infimum in $(\rho_., m_.)$ is taken along the solution to a complex Burgers equation; let $\Omega = \{x \in \mathbb{R}, t \in (0,1) : \rho_t(x) > 0\}$ and define on Ω $u_t(x) = \rho_t(x)^{-1} m_t(x)$ and $f_t(x) = u_t(x) + i\pi\rho_t(x)$. Then on Ω,

$$\partial_t f_t(x) + f_t(x) \partial_x f_t(x) = 0.$$

Moreover, with $\mu_A = \mu_{\mathbf{t}}|_A$ and $\mu_B = \mu_{\mathbf{t}}|_B$, for μ_A-almost all x

$$W_1'(x) - u_0(x) = H\mu_A(x), \quad \mu_A \text{ a.s.}, \quad W_2'(x) + u_1(x) = H\mu_B(x), \quad \mu_B \text{ a.s.} \quad (5.6)$$

In comparison with the previous statements, we note that the above results hold for all c and V_1, V_2, and not only for small parameters.

Proof.

Most of the proof is contained in Guionnet (2003) where the convergence of $\hat{\mu}^N_A$, $\hat{\mu}^N_B$ towards the unique minimizers of S^{V_1,V_2} was proved (see Theorem 3.3 in Guionnet (2003)), as well as the fact that the limit is compactly supported and that $\mu_{\mathbf{t}}$ satisfies (5.5) but for $P \in \mathcal{C}^m_{st}(\mathbb{R})$ (see section 3.2.1, p. 555 and 558, in Guionnet (2003)). It clearly extends to polynomial functions since $\mu_{\mathbf{t}}$ is compactly supported as its marginals are. The only point we stress here is that this imply that $\mu_{\mathbf{t}}$ is also uniquely determined. Indeed, by proceeding by induction over the degree in B of a monomial function P, we see that

$$\tau(BP) = -\tau \otimes \tau(\partial_A P) + \tau(W_1'(A)P)$$

defines uniquely all the moments $\tau(P(A,B))$ from those of $\tau(Q(A))$. Note here that this is specific to the interaction under consideration; in general the solutions of **SD[V]** is not determined by their restriction to one variable.

□

Using for instance the fact that if we let $g_t(x) = tf_t(x) + x$, the Wronskian of (f,g) is null, we find that on each connected component of Ω, there exists an analytic function F so that

$$tf_t(x) + x = F(f_t(x)).$$

In a small parameter region, it should easily be arguable that Ω is connected, as it is when the parameters are null (where the solution at time t can be seen to be a semi-circular variable with variance $1-t+t^2$). One ca argue that f_t extends continuously to $t=0$ and $t=1$ which yields

$$x = F(f_0(x)) \quad f_1(y) + y = F(f_1(y)) \tag{5.7}$$

for all x in the support of μ_A and all y in the support of μ_B. Noting that $f_0(x) = W_1'(x) - \overline{H\mu_A}(x) = -\overline{Y(x)}$, $f_1(x) = -W_2'(x) + H\mu_B(x) = -X(x)$ it is tempting to hope that (5.7) yields the same result that Property 5.6, namely that $(Y(x), x)$ and $(y, X(y))$ satisfy the same algebraic equation. Our knowledge of this field is much too limited to enable us to get this conclusion.

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