High order expansion of matrix models and enumeration of maps.

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Abstract Perturbation of the **GUE** are known in physics to be related to enumeration of graphs on surfaces. Following [14] and [15], we investigate this idea and show that for a small convex perturbation we can perform a genus expansion: the free energy and the moments of the empirical measure can be developed into a series whose g-th term is a generating function of graphs embedded on a surface of genus g.

1 Introduction

Wick's calculus allows to easily compute any moments of Gaussian variables and gives them a combinatorial interpretation since the p-th moment of a Gaussian can be seen as the number of partitions in pairs of $[|1,p|]$. This fact can be used to find moments of the **GUE**, the Gaussian unitary model. Let μ^N be the law on $\mathcal{H}_N(\mathbb{C})^m$ the set of m-tuple $A_1,\cdots,A_m$ of $N\times N$ hermitian matrices such that $\Re e A_i(kl), k<l$, $\Im m A_i(kl), k<l$, $2^{-\frac{1}{2}}A_i(kk)$ is a family of independent real Gaussian variables of variance $(2N)^{-1}$ or more directly

$$\mu^N(d\mathbf{A}) = \frac{1}{Z^N} e^{-\frac{N}{2}\operatorname{tr}(\sum_{i=1}^m A_i^2)} d^N\mathbf{A}$$

with $d^N\mathbf{A}$ the Lebesgue measure on $\mathcal{H}_N(\mathbb{C})^m = (\mathbb{R}^{N^2})^m$ and Z^N a constant of normalization.

For a edge-colored graph on an orientated surface we say that a vertex is of type $q = X_{i_1}\cdots X_{i_p}$ for a monomial q if this vertex is of degree p and when we look at the half-edges going out of it, starting from a distinguished

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one and going in the clockwise order the first half-edge is of color i_1, the second of color $i_2,\ldots$, the p-th of color i_p. A graph on a surface is a map if it is connected and its faces are homeomorphic to discs (see section 3 for a precise definition of these notions). Then, a computation (see [17] for the one matrix case and [20] for the general case) using Wick's calculus shows that for all non commutative monomials, $X_{i_1}\cdots X_{i_p}$,

$$\mu^N(\frac{1}{N}\operatorname{tr}(A_{i_1}\cdots A_{i_p})) = \sum_{g\in\mathbb{N}} \frac{1}{N^{2g}}\mathcal{M}_g(X_{i_1}\cdots X_{i_p}) \tag{1}$$

where $\mathcal{M}_g(X_{i_1}\cdots X_{i_p})$ is the number up to isomorphism of maps with colored edges on a surface of genus g with one vertex of type $X_{i_1}\cdots X_{i_p}$. The contribution for higher asymptotics is given by graphs of higher genus. This is called a genus expansion or a topological expansion. Besides, one can use Euler's formula to show that the sum in the right hand side is always finite. We see that the first asymptotic of the moments of the **GUE** leads to an enumeration of planar object. The link between limit moments of matrices and combinatorial structure already appeared in the first works on random matrices since [26] proved that the moments of hermitian matrices with i.i.d. entries are catalan's numbers. In the multi-matrix case this first asymptotic can be described by the notion of freeness, a crucial property in operator algebra, see [25] for an introduction and [24] which proves this type of asymptotic is satisfied not only for the **GUE** but for a far more larger class of matrices. This freeness has also a combinatorial interpretation as a sum over non-crossing partitions (see [21]).

Can such an interpretation be generalized beyond the Gaussian case? More general genus expansions are of particular interest in physics (see [22] which introduce such a concept). The links between them and matrix integrals were discovered in [6]. We present them here in a general setting. Take a potential $V(X_1,\cdots,X_m) = \sum_i t_i q_i$ with complex parameters $t_1,\cdots,t_n$ and non-commutative monomials q_i. We are interested in the following perturbation of the **GUE**

$$\mu_V^N(dA_1,\cdots,dA_m) = \frac{1}{Z_V^N} e^{-N\operatorname{tr}(V(A_1,\cdots,A_m))} d\mu^N(A_1,\cdots,A_m) \tag{2}$$

where Z_V^N is the normalizing constant making μ_V^N a probability measure. The derivatives of the moments of this model at $\mathbf{t}=0$ are exactly moments of the **GUE** and thus can be computed using Wick's calculus and the limit can be formally expressed as a generating function of graphs. For example, for a quartic potential $V = tX^4$, we can obtain the following formal expansion for the free energy (see [4])

$$\log Z^N_{tX^4} = \sum_{g\in\mathbb{N}} N^{2-2g} \sum_{k\in\mathbb{N}} \frac{(-t)^k}{k!} \mathcal{C}^k_g \tag{3}$$

with $\mathcal{C}^k_g$ the number up to isomorphism of connected graphs on a surface of genus g with k vertex of degree 4 and such that faces are homeomorphic to discs (the so-called maps). Note that we have to be careful since the right hand side of (3) is divergent for $t \neq 0$. Thus, this equality is purely formal but we will be able to give it a precise mathematical meaning (at least, Wick's calculus shows that the derivatives of both sides are equal at $t = 0$). Such a formal identity can also be stated for general potential $V_{\mathbf{t}}$.

This paper is the sequel of the two articles [14] and [15]. One aim of this series is to investigate what can be said of the previous equality beyond the identification of the formal series. More precisely we would like to know if for some parameters $\mathbf{t}$ the genus expansion is the large N expansion of the free energy:

$$F^N_{V_{\mathbf{t}}} := \frac{1}{N^2} \ln Z^N_{V_{\mathbf{t}}}.$$

First we need to make some assumptions in order for our probability measure to be well defined. We will always assume that:

1. The perturbation is small: we will restrict ourselves to small coefficients t_i in V. Note that we can not get rid of this condition, as the generating functions of combinatorial objects that we consider have arbitrary small radius of convergence.

2. The potential $V + \frac{1}{2}\sum_i X_i^2$ is "uniformly" convex: there exists $c > 0$ such that for all N in $\mathbb{N}$,

$$\varphi^N_V : \begin{array}{ccc} \mathcal{H}_N(\mathbb{C})^m & \longrightarrow & \mathbb{C} \\ (X_1, \cdots, X_m) & \longrightarrow & \mathrm{tr}(V(X_1, \cdots, X_m) + \frac{1-c}{2}\sum_{i=1}^m X_i^2) \end{array}$$

   is a real and convex function. If V satisfies this condition, we say that V is c-convex.

Thus, for $V_{\mathbf{t}} = \sum_{i=1}^n t_i q_i$ with $\mathbf{t} = (t_1, \cdots, t_n)$ complex numbers and q_i non-commutative monomials we define

$$B_{\eta,c} = \{\mathbf{t} \in \mathbb{C}^n \,||\mathbf{t}| = \max_i |t_i| \leqslant \eta, V_{\mathbf{t}} \text{ is } c\text{-convex}\}.$$

Examples of c-convex potentials can be built using Klein's lemma (see [16]) is

$$V(\mathbf{X}) = \sum_i P_i(\sum_j \alpha_{ij} X_j) + \sum_{k\ell} \beta_{k\ell} X_k X_\ell$$

with real and convex polynomials P_i, real α_{ij}, β_{kl} and for all l, $\sum |\beta_{kl}| < 1-c$.

We proved in the previous articles the first two terms of the expansion converge: Let $V_{\mathbf{t}} = \sum_{i=1}^n t_i q_i$, and $c > 0$, there exists $\eta > 0$ such that for all $\mathbf{t}$ in $B_{\eta,c}$, the free energy has the following expansion

$$F_{V_{\mathbf{t}}}^N = F^0(\mathbf{t}) + \frac{1}{N^2} F^1(\mathbf{t}) + o(\frac{1}{N^2}).$$

We also showed that $F^i(\mathbf{t}), i = 0, 1$ enumerates maps of genus i with vertices associated to the monomials of V. More precisely, [14] tackled the first asymptotic for the free energy and for the moments of the measure. In [15], we looked at the second asymptotic for those quantities and in addition we proved a central limit theorem for the moment of the empirical measure which will be crucial in this paper.

Our objective is to generalize this expansion to any genus. Our main concern is the multi-matrix case since the one matrix case is already well understood. For the latter, the first asymptotic of the empirical measure has been studied from a non-perturbative perspective (that is with assumptions only on the growth of V at infinity and not on the size of its coefficients). A large deviation principle was obtained in [3] and a central limit theorem in [18]. An explicit description of the density of the limit measure is given in [9]. The next orders in the expansion have also been studied. In [2], a recursive procedure based on the loop equation is given to compute recursively the asymptotics of observables such as the free energy. Our proofs also rely on this loop equation, called Schwinger-Dyson's equation. More recently, [1] shows an expansion for the expectation of the Stieltjes transform of the empirical measure. Finally, [9] gave a genus expansion for the free energy using Riemann-Hilbert methods. This is exactly this expansion that we would like to obtain in the multi-matrix case. Our tools are very different from those of [9] but the hypotheses are comparable. (In [9] they assume that $V = t_{2m} x^{2m} + \sum_{i<2m} t_i x^i$ with t_{2m} which dominates the other t_i while we assume the convexity of V).

Many techniques used in these articles, such as the use of orthogonal polynomials, can not be generalized to the multi-matrix case. However, there is a huge litterature which tackles some specific models, such as the so called two matrix model $V = V_1(A) + V_2(B) + cAB$, whose combinatorics is of crucial importance for models of statistical physics on random graphs. The Ising model on random graphs was solved by physicists in [19] and then by combinatoricians in [5] At a non-perturbative level the first order was studied using large deviation technique in [13]. A recent series of papers ([7], [11], [10]) introduces tools of algebraic geometry and gives recursive formula to study the asymptotics of these models.

The interested reader should consult the review [8] and [12] and the book [27]. The last part of [14] also aims to present the many approaches to this problem.

As in [14] and [15], our main tool will be the so-called Schwinger-Dyson's equation and we will try to interpret it as a higher genus Tutte's equation (see [23]). This will lead us to the combinatorial series.

The main result of this paper is that we can go beyond the first two asymptotics, up to any order.

Theorem 1.1 *Let $V_{\mathbf{t}} = \sum_{i=1}^{n} t_i q_i$, and $c > 0$, for all $g \in \mathbb{N}$, there exists $\eta_g > 0$ such that for all $\mathbf{t}$ in $B_{\eta_g,c}$, the free energy has the following expansion*

$$F_{V_{\mathbf{t}}}^N := \frac{1}{N^2} \log Z_{V_{\mathbf{t}}}^N = F^0(\mathbf{t}) + \frac{1}{N^2} F^1(\mathbf{t}) + \cdots + \frac{1}{N^{2g}} F^g(\mathbf{t}) + o(\frac{1}{N^{2g}})$$

with F^g the generating function for maps of genus g associated with V:

$$F^g(\mathbf{t}) = \sum_{\mathbf{k} \in \mathbb{N}^n \setminus \{0\}} \frac{(-\mathbf{t})^{\mathbf{k}}}{\mathbf{k}!} \mathcal{C}_g^{\mathbf{k}}(P)$$

where $\mathbf{k}! = \prod_i k_i!$, $(-\mathbf{t})^{\mathbf{k}} = \prod_i (-t_i)^{k_i}$ and $\mathcal{C}_g^{\mathbf{k}}$ is the number of maps on a surface of genus g with k_i vertices of type q_i.

Note that increasing the order of the expansion is done at the cost of reducing the radius of convergence since the full series in power of g is not convergent.

To tackle this problem, we will look at asymptotics of other observables (like in [14] and [15]). In particular we will be interested by the asymptotic of the non-commutative moments of our measure $E_{\mu_{V_{\mathbf{t}}}^N}[\frac{1}{N}\operatorname{tr}(P)]$ for a non-commutative polynomial P. Such moments appear as derivatives of the free energy since

$$E_{\mu_{V_{\mathbf{t}}}^N}[\frac{1}{N}\operatorname{tr}(P)] = -\left.\frac{\partial}{\partial u}\right|_{u=0} F_{V_{\mathbf{t}}+uP}^N.$$

Theorem 1.2 *With the same hypotheses than in the previous theorem, for all for all $g \in \mathbb{N}$, there exists $\eta > 0$, such that for all $\mathbf{t}$ in $B_{\eta,c}$, for all monomials P*

$$E_{\mu_{V_{\mathbf{t}}}^N}[\frac{1}{N}\operatorname{tr}(P)] = \mathcal{C}_{\mathbf{t}}^0(P) + \cdots + \frac{1}{N^{2g}} \mathcal{C}_{\mathbf{t}}^g(P) + o(\frac{1}{N^{2g}})$$

with $\mathcal{C}_g$ the generating function maps of genus g with some fixed vertices:

$$\mathcal{C}^g_{\mathbf{t}}(P) = \sum_{\mathbf{k}\in\mathbb{N}^n} \frac{(-\mathbf{t})^{\mathbf{k}}}{\mathbf{k}!}\mathcal{C}^{\mathbf{k}}_g(P)$$

where $\mathcal{C}^{\mathbf{k}}_g(P)$ is the number of maps on a surface of genus g with k_i vertices of type q_i and one of type P.

In fact, we will be able to find the asymptotics of many more observables, such as the higher derivatives of the free energy. Indeed, we show that we can differentiate term by term the expansion of Theorem 1.1. Let us introduce for $\mathbf{j} = (j_1, \cdots, j_n) \in \mathbb{N}^n$, the operator of derivation

$$\mathcal{D}_{\mathbf{j}} = \frac{\partial^{\sum_i j_i}}{\partial t_1^{j_1} \cdots \partial t_n^{j_n}}.$$

Theorem 1.3 *With the same hypothesis than in the previous theorem, for all $\mathbf{j} = (j_1, \cdots, j_n) \in \mathbb{N}^n \backslash \{0\}$, for all $g \in \mathbb{N}$, there exists $\eta > 0$ such that for all $\mathbf{t}$ in $B_{\eta,c}$,*

$$\mathcal{D}_{\mathbf{j}} F^N_{V_{\mathbf{t}}} = \mathcal{D}_{\mathbf{j}} F^0(\mathbf{t}) + \cdots + \frac{1}{N^{2g}}\mathcal{D}_{\mathbf{j}} F^g(\mathbf{t}) + o(\frac{1}{N^{2g}}).$$

Besides, $\mathcal{D}_{\mathbf{j}} F^g$ is the generating function maps of genus g with some fixed vertices:

$$\mathcal{D}_{\mathbf{j}} F^g(\mathbf{t}) = \sum_{\mathbf{k}\in\mathbb{N}^n} \frac{(-\mathbf{t})^{\mathbf{k}}}{\mathbf{k}!}\mathcal{C}^{\mathbf{k}+\mathbf{j}}_g.$$

In the next section, we will define some useful notations from non-commutative probability theory and we will recall the main result of [14]. Next, we will look for recursive relations between the asymptotics of the non-commutative moments of our model. This will lead us to study some combinatorial objects in section 4 whose generating functions satisfy these relations. In the sections 5 and 6, we will prove the equality of these moments and these enumerating functions before proving our main results. Finally the last section will be devoted to the proof of Theorem 1.3.

2 Notations and reminder

We denote by $\mathbb{C}\langle X_1, \cdots, X_m\rangle$ the set of complex polynomials on the non-commutative unknown $X_1, \ldots, X_m$ i.e. the complex linear combination of monomials which are simply the set of finite words on $X_1, \ldots, X_m$. Monomials must be thought as non-commutative moments. Let $*$ denotes the linear involution on $\mathbb{C}\langle X_1, \cdots, X_m\rangle$ such that for all complex z and all monomials

$$(zX_{i_1} \ldots X_{i_p})^* = \overline{z} X_{i_p} \ldots X_{i_1}.$$

A polynomial P is self-adjoint if $P = P^*$. We will denote $\mathbb{C}\langle X_1, \cdots, X_m\rangle^*$ the dual of $\mathbb{C}\langle X_1, \cdots, X_m\rangle$. If there exists $R > 0$ such that for all monomial $|\tau(X_{i_1} \ldots X_{i_p})| \leqslant R^p$ we will say that τ has a compact support. By analogy with the one variable case, the infimum of the R's which satisfy this inequality for all monomials will be called the radius of the support of τ.

For a polynomial P and a monomial q, we define $\lambda_q(P)$ as the coefficient of q in the decomposition of P. For $M > 0$, we define the norm $\|.\|_M$ on polynomials:

$$||P||_M = \sum_{l \in \mathbb{N}} \sum_{\substack{q \text{ monomial} \\ deg q = l}} |\lambda_q(P)| M^l.$$

This norm $\|.\|_M$ is an algebra norm, i.e. for all polynomials P, Q,

$$\|PQ\|_M \leqslant \|P\|_M \|Q\|_M.$$

Note that an element τ of $\mathbb{C}\langle X_1, \cdots, X_m\rangle^*$ has a support of radius less than R if and only if for all polynomials P,

$$|\tau(P)| \leqslant \|P\|_R.$$

We extend $\|.\|_M$ on $\mathbb{C}\langle X_1, \cdots, X_m\rangle \otimes \mathbb{C}\langle X_1, \cdots, X_m\rangle$ by defining this norm on the decomposition in monomials:

$$\| \sum \lambda_{q_1,q_2} q_1 \otimes q_2 \|_M = \sum |\lambda_{q_1,q_2}| M^{\deg q_1 + \deg q_2}$$

with this definition for all polynomials P, Q, $\|P \otimes Q\|_M = \|P\|_M \|Q\|_M$.

For $1 \leqslant i \leqslant m$, we define the non-commutative derivatives ∂_i from $\mathbb{C}\langle X_1, \cdots, X_m\rangle$ to $\mathbb{C}\langle X_1, \cdots, X_m\rangle^{\otimes 2}$ by the Leibniz rule

$$\partial_i(PQ) = \partial_i P \times (1 \otimes Q) + (P \otimes 1) \times \partial_i Q$$

and $\partial_i X_j = \mathbb{1}_{i=j} 1 \otimes 1$. For a monomial P, we will often use the convenient expression

$$\partial_i P = \sum_{P = R X_i S} R \otimes S$$

where the sum runs over all possible monomials R, S so that P decomposes into RX_iS. We also define another operator of derivation on polynomials, the cyclic derivative D_i which is linear and such that for all monomials:

$$D_iP = \sum_{P=RX_iS} SR.$$

Alternatively, D can be defined as $m \circ \partial$ where $m(A \otimes B) = BA$. We will see that these two operators appear naturally when we differentiate products of matrices and they both possesses a nice combinatorial interpretation. An important fact we will use later is that for all $M' > M$, both ∂_i and D_i are continuous from $(\mathbb{C}\langle X_1, \cdots, X_m\rangle, \|.\|_{M'})$ to $(\mathbb{C}\langle X_1, \cdots, X_m\rangle, \|.\|_M)$. For example for a monomial q,

$$\frac{\|D_iq\|_M}{\|q\|_{M'}} = \deg q \frac{M^{\deg q - 1}}{M'^{\deg q}} = M^{-1} \deg q \left(\frac{M}{M'}\right)^{\deg q}$$

which is bounded. Note also that due to the particular form of this form, in order to show that an operator θ has a norm bounded by C with respect to this norm, it is sufficient to show that for all monomials q, $\|\theta q\|_M \leqslant C\|q\|_M$

The main object of our study is the law $\mu^N_{V_{\mathbf{t}}}$ on $\mathcal{H}_N(\mathbb{C})^m$

$$\mu^N_{V_{\mathbf{t}}}(dA_1, \cdots, dA_m) = \frac{1}{Z^N_{V_{\mathbf{t}}}} e^{-N \operatorname{tr}(V_{\mathbf{t}}(A_1, \cdots, A_m))} d\mu^N(A_1, \cdots, A_m)$$

and we are particulary interested by the behavior of the random variable

$$\hat{\mu}^N : \begin{array}{ccc} \mathbb{C}\langle X_1, \cdots, X_m\rangle & \longrightarrow & \mathbb{C} \\ P & \longrightarrow & \frac{1}{N} \operatorname{tr}(P(A_1, \cdots, A_m)) \end{array}$$

and its mean:

$$\overline{\mu}^N_{\mathbf{t}} : \begin{array}{ccc} \mathbb{C}\langle X_1, \cdots, X_m\rangle & \longrightarrow & \mathbb{C} \\ P & \longrightarrow & E_{\mu^N_{V_{\mathbf{t}}}}[\frac{1}{N} \operatorname{tr}(P(A_1, \cdots, A_m))] \end{array}$$

We can now state precisely the main result of [14], we will use it very frequently it in the next sections. For any $c > 0$,$R > 0$ there exists $\eta > 0$ such that for all $\mathbf{t} \in B_{\eta,c}$, for all polynomials P, $\hat{\mu}^N(P)$ goes when N goes to $+\infty$, almost surely and in expectation towards $\mu_{\mathbf{t}}(P)$ with $\mu_{\mathbf{t}}$ a solution of the Schwinger-Dyson equation

$$\mu_{\mathbf{t}} \otimes \mu_{\mathbf{t}}(\partial_i P) = \mu_{\mathbf{t}}((X_i + D_iV_{\mathbf{t}})P) \quad \forall P \in \mathbb{C}\langle X_1, \cdots, X_m\rangle, \forall i \in \{1, \cdots, m\}.$$

Besides, this solution is the unique one which has a support bounded by R: for all monomial $X_{i_1} \cdots X_{i_p}$

$$|\mu_{\mathbf{t}}(X_{i_1} \cdots X_{i_p})| \leqslant R^p. \tag{4}$$

Moreover, on $B_{\eta,c}$, $\mu_{\mathbf{t}}$ can be seen as a generating function of planar maps: for all polynomials P,

$$\mu_{\mathbf{t}}(P) = \sum_{k_1,\cdots,k_n \in \mathbb{N}} \prod_i \frac{(-t_i)^{k_i}}{k_i!} \mathcal{M}^{k_1,\cdots,k_n}(P)$$

where $\mathcal{M}^{k_1,\cdots,k_n}(P)$ is the number of planar connected graphs with k_i vertices of type q_i and one of type P. For the rest of the paper we will work in this domain $B_{\eta,c}$ where the convergence holds. In order to shorten a little the notations the subscript $\mathbf{t}$ will be most of the time implicit, for example we will often write μ instead of $\mu_{\mathbf{t}}$, V instead of $V_{\mathbf{t}}$, $\overline{\mu}^N$ instead of $\overline{\mu}_{\mathbf{t}}^N$ ...

3 First order observable

The starting point is a relation already used in [14] for the matrix model when N is fixed: for all polynomial P, for all i,

$$E[\hat{\mu}^N((X_i + D_i V)P)] = E[(\hat{\mu}^N \otimes \hat{\mu}^N)(\partial_i P)].$$

We will give the proof of a generalization of this equality later. Using this equality and some concentration inequalities we were able to prove that for $\mathbf{t}$ in $B_{\eta,c}$ for a well chosen η, for all polynomial P, $E[\hat{\mu}^N(P)]$ was converging towards $\mu(P)$ with μ the unique solution of the Schwinger-Dyson's equation **SD[V]**:

$$\mu((X_i + D_i V)P)) = (\mu \otimes \mu)(\partial_i P). \tag{5}$$

In order to find the next asymptotic, we study the difference between the equation for finite N and the limit equation, if $\nu^N = N^2(\overline{\mu}^N - \mu)$, we obtain by substracting the two equations:

$$\nu^N((X_i + D_i V)P) - (I \otimes \mu + \mu \otimes I)\partial_i P) = N^2 E[((\hat{\mu}^N - \mu) \otimes (\hat{\mu}^N - \mu))(\partial_i P)] \tag{6}$$

Here, an important operator shows up in the left hand side. For $P = X_{i_1} \cdots X_{i_p}$ a monomial, define the following operators:

$$\Xi_1 P = \frac{1}{p} \sum_i D_i V D_i P$$

$$\Xi_2 P = \frac{1}{p}\sum_i (I\otimes\mu + \mu\otimes I)\partial_i D_i P.$$

We extend them by linearity on $\mathbb{C}\langle X_1,\cdots,X_m\rangle$ and we define $\Xi_0 = I - \Xi_2$ and $\Xi = \Xi_0 + \Xi_1$. These operators were introduced in [15] to obtain a central limit theorem for the matrix model. We also define the operator of division by the degree i.e. the linear operator $P \to \overline{P}$ such that for all monomial $P = X_{i_1}\cdots X_{i_p}$, $\overline{P} = \frac{1}{p}P$ and $\overline{1} = 0$. These operators allow us to state the relation for the first correction (6) in a simpler form. If we apply (6) to $D_i\overline{P}$ and then sum on i, we get:

$$\nu^N(\Xi P) = \sum_i N^2 E[((\hat{\mu}^N - \mu)\otimes(\hat{\mu}^N - \mu))(\partial_i D_i \overline{P})]. \tag{7}$$

Then, the strategy is simple, we only have to understand the asymptotic of $N^2E[((\hat{\mu}^N - \mu)\otimes(\hat{\mu}^N - \mu))(R\otimes S)]$ and then "invert" Ξ. The first order asymptotic of $N^2E[((\hat{\mu}^N - \mu)\otimes(\hat{\mu}^N - \mu))(R\otimes S)]$ is easy to compute using [15] as it was shown that $N(\hat{\mu}^N - \mu)(Q)$ converges in law towards a Gaussian law when N goes to infinity. The main issue is that when we try to investigate the next asymptotic, terms of type $N^3E[(\hat{\mu}^N - \mu)^{\otimes 3}(R\otimes S\otimes T)]$ will appear and at their turn they will create terms of greater complexity. That's why we are interested more generally in all the $N^\ell E[(\hat{\mu}^N - \mu)^{\otimes \ell}(P_1\otimes\cdots\otimes P_\ell)]$'s and we will eventually find their full asymptotic. First remark that according to [15], for all P, $N(\hat{\mu}^N(P) - \mu(P))$ converges to a gaussian variable and this convergence occurs in moments (see Corollary 4.8 in [15]). Thus $N^\ell E[(\hat{\mu}^N - \mu)^{\otimes \ell}(P_1\otimes\cdots\otimes P_\ell)]$ has a finite limit when N goes to infinity and this limit is 0 if ℓ is odd. But we need a more precise result which state that this convergence is uniform for all monomials P of reasonable degree.

Lemma 3.1 *For all $\ell \in \mathbb{N}^*$, $\alpha > 0$ there exists $C, \eta, M_0 > 0$, such that for all $\mathbf{t} \in B_{\eta,c}$, $M > M_0$ and all polynomials $P_1,\ldots,P_\ell$ of degree less than $\alpha N^{\frac{1}{2}}$,*

$$\left| E[N^\ell(\hat{\mu}^N - \mu)^{\otimes \ell}(P_1\otimes\cdots\otimes P_\ell)]\right| \le C\|P_1\|_M\cdots\|P_\ell\|_M.$$

Proof.
First using Hölder's inequality, write

$$E[N^\ell(\hat{\mu}^N - \mu)^{\otimes \ell}(P_1\otimes\cdots\otimes P_\ell)] \le \prod_{r=1}^{\ell} E[|N(\hat{\mu}^N - \mu)(P_r)|^\ell]^{\frac{1}{\ell}}.$$

Thus we only have to prove the claim if the P_r are equals. Then we substract the mean

$$E[|N(\hat{\mu}^N-\mu)(P)|^\ell] \leq 2^\ell\{E[|N(\hat{\mu}^N(P)-\overline{\mu}^N(P)|^\ell]+|N(\overline{\mu}^N(P)-\mu(P))|^\ell\} \tag{8}$$

Proposition 3.1 in [15] state a rate of convergence of $\overline{\mu}^N(P)$ to $\mu(P)$: there exists $C, M_0>0$ such that for $M>M_0$, for all polynomials P of degree less than $\varepsilon N^{\frac{2}{3}}$,

$$|N(\overline{\mu}^N(P)-\mu(P))| \leqslant C\frac{\|P\|_M}{N}. \tag{9}$$

Thus in inequality (8) above, we only have to control the first term. We decompose it into the sum

$$\begin{aligned} E[|N(\hat{\mu}^N(P)-\overline{\mu}^N(P)|^\ell] &= E[|N(\hat{\mu}^N(P)-\overline{\mu}^N(P)|^\ell \mathbb{1}_{\|\mathbf{A}\|\leqslant M}] \\ &\quad + E[|N(\hat{\mu}^N(P)-\overline{\mu}^N(P)|^\ell \mathbb{1}_{\|\mathbf{A}\|> M}] \end{aligned} \tag{10}$$

with $\|\mathbf{A}\|$ is the max of the operator norm of the A_i's. The second term can be bounded by

$$(2N)^\ell \overline{\mu}^N((PP*)^\ell)^{\frac{1}{2}}\mathbb{P}(\|\mathbf{A}\|>M) \leqslant (2N)^\ell\|P\|_M^\ell\mathbb{P}(\|\mathbf{A}\|>M)$$

Now according to Lemma 2.2 in [15], we have a controll on the decay of the largest eigenvalue: there exists $a>0$ such that if M is sufficiently large, $\mathbb{P}(\|\mathbf{A}\|>M)\leqslant e^{-aMN}$ and since $e^{-aMN}(2N)^\ell$ is uniformly bounded in N we get:

$$E[|N(\hat{\mu}^N(P)-\overline{\mu}^N(P)|^\ell \mathbb{1}_{\|\mathbf{A}\|> M}] \leqslant C\|P\|_M^\ell. \tag{11}$$

with a constant which may depends on a, ℓ and M.

We are only left with the term

$$E[|N(\hat{\mu}^N(P)-\overline{\mu}^N(P)|^\ell \mathbb{1}_{\|\mathbf{A}\|\leqslant M}].$$

We can use the concentration inequality result of Lemma 2.3 in [15]. Borrowing the notations from this lemma we have:

$$\begin{aligned} &E[|N(\hat{\mu}^N(P)-\overline{\mu}^N(P)|^\ell \mathbb{1}_{\|\mathbf{A}\|\leqslant M}] \\ &= \int_0^{+\infty} \ell x^{\ell-1}\mathbb{P}(|N(\hat{\mu}^N(P)-\overline{\mu}^N(P)|>x, \|\mathbf{A}\|\leqslant M)dx \\ &\leqslant \ell(\varepsilon_{P,M}^N+m_{P,M}^N)^\ell + \int_0^{+\infty}\{\ell(x+\varepsilon_{P,M}^N+m_{P,M}^N)^{\ell-1} \\ &\times \mathbb{P}(|N(\hat{\mu}^N(P)-\overline{\mu}^N(P))-m_{P,M}^N|>x+\varepsilon_{P,M}^N, \|\mathbf{A}\|\leqslant M)\}dx \\ &\leqslant \ell(\varepsilon_{P,M}^N+m_{P,M}^N)^\ell + 2\int_0^{+\infty}\ell(x+\varepsilon_{P,M}^N+m_{P,M}^N)^{\ell-1}e^{\frac{-cx^2}{2(\|P\|_{\mathcal{L}}^M)^2}}dx \end{aligned}$$

Now observe that up to a little change in M, our norm $\|.\|_M$ controll the lipschitz norm of P:

$$\|P\|_{\mathcal{L}}^M \leqslant (\sum_k \|D_k P D_k P^*\|_M)^{\frac{1}{2}} \leqslant C\|P\|_{M'}$$

for a $M < M'$. This is a direct consequence of the continuity of the derivation from $\mathbb{C}\langle X_1, \cdots, X_m\rangle_{M'}$ to $\mathbb{C}\langle X_1, \cdots, X_m\rangle_M$ for $M < M'$. Thus

$$E[|N(\hat{\mu}^N(P) - \overline{\mu}^N(P)|^\ell \mathbb{1}_{\|\mathbf{A}\| \leqslant M}] \leqslant C((\varepsilon_{P,M}^N)^\ell + (m_{P,M}^N)^\ell + \|P\|_{M'}^\ell)$$

Note that since $M < M'$, for all polynomial P, $\|P\|_M \leqslant \|P\|_{M'}$ so that if in the end we have a controll in terms of these two norms we will be able to convert it into a controll with respect to the norm $\|.\|_{M'}$.

The last step is to controll $\varepsilon_{P,M}^N$ and $m_{P,M}^N$. The bound computed in Lemma 2.3 in [15] for monomials is easily extended to polynomials: for all P of degree less than αN,

$$\varepsilon_{P,M}^N \leqslant C\|P\|_M \text{ and } m_{P,M}^N \leqslant C\|P\|_M.$$

Thus we get

$$E[|N(\hat{\mu}^N(P) - \overline{\mu}^N(P)|^\ell \mathbb{1}_{\|\mathbf{A}\| \leqslant M}] \leqslant C\|P\|_M$$

which with (11) give the controll on the left hand side of (10). This with (9) allow to controll (8) which conclude the proof.

□

We now try to find some relation between the $N^\ell E[(\hat{\mu}^N - \mu)^{\otimes \ell}(P_1 \otimes \cdots \otimes P_\ell)]$'s which generalize (7). Remember that, for ℓ odd, those quantities vanish when N goes to infinity. Thus in order to obtain non-trivial limits, we have to distinguish the normalisation according to the parity of ℓ.

For $\ell \geq 1$ we define:

$$\hat{\ell} = \begin{cases} \ell + 1 & \text{if } \ell \text{ is odd} \\ \ell & \text{otherwise.} \end{cases}$$

Note that $\hat{\ell}$ is always an even integer. We now define a function from $\sqcup_{\ell \in \mathbb{N}} \mathbb{C}\langle X_1, \cdots, X_m\rangle^{\otimes \ell}$ to $\mathbb{C}$,

$$\nu^N(P_1 \otimes \cdots \otimes P_\ell) = E_{\mu_V^N}[N^{\hat{\ell}}(\hat{\mu}^N - \mu)^{\otimes \ell}(P_1 \otimes \cdots \otimes P_\ell)].$$

On $\mathbb{C}\langle X_1, \cdots, X_m\rangle^{\otimes \ell}$, this is a ℓ-linear symmetric function which is tracial in each P_r. Our convention will be that for λ in $\mathbb{C} = \mathbb{C}\langle X_1, \cdots, X_m\rangle^{\otimes 0}$, $\nu^N(\lambda) = \lambda$. The relation that will appear as our main tool are the aim of the next property. In a tensor product $P_1 \otimes \cdots \check{P}_r \cdots \otimes P_\ell$ denotes the tensor product of $P_1,\ldots,P_{r-1},P_{r+1}, \ldots,P_\ell$ i.e. the term P_r is omitted.

Property 3.2 *For all ℓ, for all polynomial $P_1,\ldots,P_\ell$, for all N, if ℓ is even*

$$\begin{aligned}\nu^N(\Xi P \otimes P_2 \otimes \cdots \otimes P_\ell) &= \sum_{i,r} \mu_{\mathbf{t}}(D_i\overline{P}D_iP_r)\nu^N(P_2 \otimes \cdots \check{P}_r \cdots \otimes P_\ell) \\ &+ \frac{1}{N^2}\sum_{i,r} \nu^N(D_i\overline{P}D_iP_r \otimes P_2 \otimes \cdots \check{P}_r \cdots \otimes P_\ell) \\ &+ \frac{1}{N^2}\sum_{i} \nu^N(\partial_i D_i\overline{P} \otimes P_2 \otimes \cdots \otimes P_\ell)\end{aligned}$$

and if ℓ is odd

$$\begin{aligned}\nu^N(\Xi P \otimes P_2 \otimes \cdots \otimes P_\ell) &= \sum_{i,r} \mu_{\mathbf{t}}(D_i\overline{P}D_iP_r)\nu^N(P_2 \otimes \cdots \check{P}_r \cdots \otimes P_\ell) \\ &+ \sum_{i,r} \nu^N(D_i\overline{P}D_iP_r \otimes P_2 \otimes \cdots \check{P}_r \cdots \otimes P_\ell) \\ &+ \sum_{i} \nu^N(\partial_i D_i\overline{P} \otimes P_2 \otimes \cdots \otimes P_\ell)\end{aligned}$$

This property is the generalization of the equation (7). One may wonder why we stress so much the difference between the odd and the even case. The point is to keep in mind which terms are of order 1 and which are negligible. In view of this, the ν_1^N are convenient as they should all be of order 1 and thus the previous equation will lead us to find their limit by induction.

Proof.

To sum up the property in a shorter way, we have to prove that for all ℓ for all polynomials $P_1,\ldots,P_\ell$ and for all N,

$$\begin{aligned}&N^\ell E[(\hat{\mu}^N - \mu)^{\otimes \ell}(\Xi P_1 \otimes P_2 \otimes \cdots \otimes P_\ell)] \\ &= \sum_{i,r} \mu_{\mathbf{t}}(D_i\overline{P_1}D_iP_r)N^{\ell-2}E[(\hat{\mu}^N - \mu)^{\otimes \ell-2}(P_2 \otimes \cdots \check{P}_r \cdots \otimes P_\ell) \\ &+ \sum_{i,r} N^{\ell-2}E[(\hat{\mu}^N - \mu)^{\otimes \ell-1}(D_i\overline{P_1}.D_iP_r \otimes P_2 \otimes \cdots \check{P}_r \cdots \otimes P_\ell)] \\ &+ \sum_{i} N^\ell E[(\hat{\mu}^N - \mu)^{\otimes \ell+1}(\partial_i D_i\overline{P_1} \otimes P_2 \otimes \cdots \otimes P_\ell)].\end{aligned}$$

We will use the integration by part formula:

$$\int x f(x) e^{-x^2/2} dx = \int f'(x) e^{-x^2/2} dx.$$

We generalize this formula into

$$\int \operatorname{tr}(A_i P) f(\operatorname{tr} Q) d\mu^N = \frac{1}{N} \sum_{\alpha,\beta} \int (\partial_{A_i(\alpha\beta)} P_{\beta\alpha}) f(\operatorname{tr} Q) + P_{\beta\alpha} \partial_{A_i(\alpha\beta)} \operatorname{tr} Q f'(\operatorname{tr} Q) d\mu^N.$$

Two useful computations show the importance of the non-commutative derivatives and their links with the derivation of polynomials of hermitian matrices: if P is a monomial,

$$\sum_{\alpha\beta} \partial_{A_i(\alpha\beta)} P_{\beta\alpha} = \sum_{\alpha\beta} \sum_{P=RX_iS} R_{\beta\beta} S_{\alpha\alpha} = \operatorname{tr} \otimes \operatorname{tr}(\partial_i P)$$

and

$$\partial_{A_i(\alpha\beta)} \operatorname{tr} P = \sum_{P=RX_iS,\gamma} R_{\gamma\beta} S_{\alpha\gamma} = (DP)_{\alpha\beta}.$$

Thus, for $P_1, \ldots, P_\ell$ polynomials:

$$\begin{aligned}
&N^\ell E[\hat\mu^N(X_i P_1)(\hat\mu^N - \mu)^{\otimes \ell-1}(P_2 \otimes \cdots \otimes P_\ell)] \\
&= N^\ell E[(\hat\mu^N \otimes \hat\mu^N)(\partial_i P_1)(\hat\mu^N - \mu)^{\otimes \ell-1}(P_2 \otimes \cdots \otimes P_\ell)] \\
&+ \sum_r N^{\ell-2} E[\hat\mu^N(P_1 D_i P_r)(\hat\mu^N - \mu)^{\otimes \ell-1}(P_2 \otimes \cdots \check{P}_r \cdots \otimes P_\ell)] \\
&- N^\ell E[\hat\mu^N(P_1 D_i V)(\hat\mu^N - \mu)^{\otimes \ell-1}(P_2 \otimes \cdots \otimes P_\ell)].
\end{aligned}$$

Now remember that according to Schwinger-Dyson's equation we have:

$$\mu((X_i + D_i V)P_1) - \mu \otimes \mu(\partial_i P_1) = 0$$

Then, we substract the two equalities and use the identity

$$\hat\mu^N \otimes \hat\mu^N - \mu \otimes \mu = (\hat\mu^N - \mu)(I \otimes \mu + \mu \otimes I) + (\hat\mu^N - \mu) \otimes (\hat\mu^N - \mu)$$

to obtain

$$\begin{aligned}
&N^\ell E[(\hat\mu^N - \mu)((X_i + D_i V)P_1 - (I \otimes \mu + \mu \otimes I)\partial_i P_1) \\
&\qquad (\hat\mu^N - \mu)^{\otimes \ell-1}(P_2 \otimes \cdots \otimes P_\ell)] \\
&= N^\ell E[(\hat\mu^N - \mu)^{\otimes \ell+1}(\partial_i P \otimes P_2 \otimes \cdots \otimes P_\ell)] \\
&+ \sum_r N^{\ell-2} E[(\hat\mu^N - \mu)^{\otimes \ell-1}(P D_i P_r \otimes P_2 \otimes \cdots \check{P}_r \cdots \otimes P_\ell)] \\
&+ \sum_r N^{\ell-2} \mu(P D_i P_r) E[(\hat\mu^N - \mu)^{\otimes \ell-2} \otimes P_2 \otimes \cdots \check{P}_r \cdots \otimes P_\ell)].
\end{aligned}$$

To get the result it is now sufficient to aplly this equality with $P_1 = D_i\bar{P}$ and then to sum on i.

□

This property gives us some precious hints on the limit ν of the ν^N. It should satisfy the "limit equation", if ℓ is even

$$\nu(\Xi P \otimes P_2 \otimes \cdots \otimes P_\ell) = \sum_{i,r} \mu_{\mathbf{t}}(D_i\overline{P}D_iP_r)\nu(P_2 \otimes \cdots \check{P}_r \cdots \otimes P_\ell) \qquad (12)$$

and if ℓ is odd

$$\begin{aligned}\nu(\Xi P \otimes P_2 \otimes \cdots \otimes P_\ell) &= \sum_{i,r} \mu_{\mathbf{t}}(D_i\overline{P}D_iP_r)\nu(P_2 \otimes \cdots \check{P}_r \cdots \otimes P_\ell) \qquad (13)\\ &+ \sum_{i,r} \nu(D_i\overline{P}D_iP_r \otimes P_2 \otimes \cdots \check{P}_r \cdots \otimes P_\ell)\\ &+ \sum_{i} \nu(\partial_i D_i\overline{P} \otimes P_2 \otimes \cdots \otimes P_\ell).\end{aligned}$$

Hopefully, we will be able to study the solutions ν of these equations. In fact, following [14] we would be able to prove that for $R, L > 0$, there exists $\varepsilon > 0$ such that for $|\mathbf{t}| < \varepsilon$ there exists an unique $\nu : \sqcup_{\ell=0}^{2L}\mathbb{C}\langle X_1, \cdots, X_m\rangle^{\otimes \ell} \to \mathbb{C}$ linear on each set $\mathbb{C}\langle X_1, \cdots, X_m\rangle^{\otimes \ell}$ with support bounded by R and which satisfy each of the previous equation for $\ell \le 2L$. But we will proceed in a different way. Looking at these equations, we will try to recognize in them some relations between enumeration of combinatorial objects. This is the aim of the next section.

4 Maps of high genus

In this section, we describe the combinatorials objects that appear in the asymptotic of our measure. Remember that it was shown in [14] that the first asymptotic can be viewed as a generating function for the enumeration of planar maps with vertices of a given type.

First, we choose m colors $\{1, \cdots, m\}$, one for each variable X_i. A star must be thought as the neighbourhood of a vertex in a plane graph. More precisely, it is a vertex with the half-edges coming out of it. One of these half-edge is distinguished and starting from it the other one are clockwisely ordered. Besides, each of these half-edges is colored.

We say that a star is of type q for a monomial $q = X_{i_1} \cdots X_{i_p}$ if it has p half-edges, the first half-edge is distinguished and of color i_1 and then in the clockwise order the second half-edge is of color i_2, the third of color i_3, ..., the p-th of color i_p. This gives a bijection between monomials and stars.

The combinatorial objects that will appear in the asymptotic of our matrix model are maps. A map is a connected graph on a compact orientated connected surface such that edges do not cross each other and faces are homeomorphic to discs. We will consider edge-colored maps such that each vertex as a distinguished edge going out of it so that we can associate a star and a well defined type to any vertex. The genus of the map is the genus of the surface. We will count maps up to homeomorphism of the surface which preserves the graph.

The typical way to construct a map is to put some stars $q_1, \cdots, q_p$ on a surface of genus g. Then we consider all the half-edges that goes outside the stars and glue them two by two while respecting the following constraints:

- Two half-edges can only be glued if they are of the same color
- The edges created by gluing two half-edges mustn't cross any other edge.
- At the end of the process faces must be homeomorphic to discs.

For example, one can ask how many maps of genus 1, we can construct above two stars of type $X_1X_2X_1X_2$. The answer as shown in figure 1 is 4. Note that as faces are homeomorphic to discs, it is sufficient to know which pairs of half-edges are glued together to build the map.

Definition 4.1 *For ℓ in $\mathbb{N}$, P a monomial and integers $\mathbf{k} = (k_1, \cdots, k_n)$, let $\mathcal{M}^{\mathbf{k}}(P)$ be the number of maps of genus 0 (or planar maps) with for all i, k_i stars of type q_i and one of type P where $q_1, \cdots, q_n$ are the monomials which appear in the potential V.*

We could define the same kind of quantities with the condition of being of genus g for $g > 1$ but then we won't be able to find any closed relation of induction between these quantities. In order to get relation induction on enumeration of maps we follow an idea of Tutte (see [23]). We try to decompose a map in smaller ones by contracting one edge (Note that Tutte used to work on the dual of the graph we are considering, thus his operation is a little different).

Imagine a map of genus 1 with a root of type $P = XRXS$ and that the two half-edges corresponding to the X are glued together. Imagine also that the loop resulting from this operation is not retractable on the surface. How does the contraction of this edge decomposes the map? Now R and

PSfrag replacements

?

X_1

X_2

first star

second star

distinguished edges

Figure 1: Maps of genus 1 above two stars of type $X_1X_2X_1X_2$.

S are separated by that loop, we will have to remind these two monomials. That's why we will introduce maps above a root of type $R \otimes S$. Besides R and S must be linked together, otherwise there would be a face (touched by the loop) which is not a disc, something to avoid for a map.

Thus we define some more complex vertices which will appear when we will try to decompose our maps. Let $P_1, \ldots, P_\ell$ be a family of monomials. We associate to this family a bunch of $\ell - 1$ circles such that outside the circles we put the half-edges of P_1 and in the m-th circle we put the half-edges of a star of type P_{m+1} going out of the central point and in the same order.

This object will be called the root and we will name each P_r a vertex of the root (look at figure 2, to see a root of type $X_1X_2X_1X_2 \otimes X_1^2 \otimes X_2X_1$).

PSfrag replacements

X_1

X_2

distinguished edge

Figure 2: Root of type $X_1X_2X_1X_2 \otimes X_1^2 \otimes X_2X_1$.

This corresponds to a star coming from a vertex which have ℓ prescribed loops, the m-th having the germs of edges corresponding to a star of type P_r. Now we construct maps with a root of type $P_1 \otimes \cdots \otimes P_\ell$ and some other vertices of type q_i. We say that such a map is minimal if when we cut the surface along the $\ell - 1$ loops of the root, we do not obtain any component homeomorphic to a disc. This means that for any P_i the component of P_i is not planar i.e. either it is linked to another P_j or it is linked to some other vertices in a way that can't be embedded on a sphere.

Definition 4.2 *For ℓ, g in $\mathbb{N}$, a family of monomials $P_1, \cdots, P_\ell$ and integers $\mathbf{k} = (k_1, \cdots, k_n)$, let $\mathcal{M}_g^{\mathbf{k}}(P_1 \otimes \cdots \otimes P_\ell)$ be the number of minimal maps of genus g with a root of type $P_1 \otimes \cdots \otimes P_\ell$ and for all i, k_i stars of type q_i.*

For example for $V = tX_1X_2X_1X_2$, the figure 1 shows that $\mathcal{M}_1^1(X_1X_2X_1X_2)$ the number of minimal maps of genus 1 with a star of type $X_1X_2X_1X_2$ and a root of type $X_1X_2X_1X_2$ is 4.

We extend by linearity $\mathcal{M}_{\mathbf{k}}$ and $\mathcal{M}_g^{\mathbf{k}}$ so that we can compute them on polynomials P_i instead of monomials and we define the power series for these enumerations:

$$\mathcal{I}(P) = \sum_{\mathbf{k} \in \mathbb{N}^n} \prod_{i=1}^{n} \frac{(-t_i)^{k_i}}{k_i!} \mathcal{M}^{\mathbf{k}}(P)$$

and

$$\mathcal{I}_g(P_1 \otimes \cdots \otimes P_\ell) = \sum_{\mathbf{k}\in\mathbb{N}^n} \prod_{i=1}^n \frac{(-t_i)^{k_i}}{k_i!} \mathcal{M}_g^{\mathbf{k}}(P_1 \otimes \cdots \otimes P_\ell).$$

By convention we define for λ in $\mathbb{C}$, $\mathcal{I}_g(\lambda) = \lambda \mathbb{1}_{g=0}$ and $\mathcal{I}_g \equiv 0$ ig $g < 0$.

Recall that it was proved in [14] that for $\mathbf{t}$ sufficiently small $\mathcal{I}(P) = \mu(P)$ for all P. This was proved using the fact that these two quantities satisfy the same induction relation. The induction relation for the enumeration of maps where given by a decomposition of maps following the strategy of Tutte. We now try to generalize this fact and we begin by looking at the relation given by decomposing maps. First, some values can be directly computed

$$\mathcal{M}_g^{\mathbf{k}}(1 \otimes P_2 \otimes \cdots \otimes P_\ell) = \mathbb{1}_{g=\mathbf{k}=\ell=0}$$

because the component of 1 is automatically planar.

We now want to count maps that contribute to $\mathcal{M}_g^{\mathbf{k}}(X_i P_1 \otimes \cdots \otimes P_\ell)$ with P_i monomials. We look at the first half-edge of the root $X_i P_1 \otimes \cdots \otimes P_\ell$ and see where it is glued. Remember that it must not be planar.

Then three cases may occur (see figure 3):

1. Either (upper right picture in fig 3) the half-edge is glued to a vertex of type $q_j = RX_iS$ for a given j. First we have to choose between the k_j vertices of this type, then we contract the edge coming from this gluing to form a vertex of type SRP_1. This creates

   $$\sum_{1\le j\le n, k_j\neq 0} k_j \mathcal{M}_g^{k_1,\cdots,k_j-1,\cdots,k_n}(D_i q_j P_1 \otimes P_2 \otimes \cdots \otimes P_\ell)$$

   possibilities.

2. The second case (bottom left picture in fig 3) occurs if the half-edge is glued to another half-edge of $P_1 = RX_iS$. It cuts P_1 in two: R and S. It occurs for all decomposition of P_1 into $P_1 = RX_iS$. To write the expression that will arise in a more convenient way, we will use the non-commutative derivative ∂ which satisfy for P a monomial

   $$\partial_i P = \sum_{P=RX_iS} R \otimes S.$$

   We are now left with two separate circles, one for R and one for S. Either both are non-planar which leads to

   $$\sum_{P_1=RX_iS} \mathcal{M}_g^{\mathbf{k}}(R \otimes S \otimes P_2 \otimes \cdots \otimes P_\ell) = \mathcal{M}_g^{\mathbf{k}}(\partial_i P_1 \otimes P_2 \otimes \cdots \otimes P_\ell)$$

PSfrag replacements

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Where can we glue the first half-edge ?

(1): To an other vertex.

(2): To a half-edge of the same vertex.

(3): To another vertex of the root.

X_1

X_2

distinguished edge

Figure 3: The decomposition process for maps.

possibilities or one of the component is planar then the two components can not be linked thus we have to share the vertices of type q_i between them. They are $\binom{\mathbf{k}}{\mathbf{k}'} = \prod_j \binom{k_j}{k'_j}$ ways of choosing for all j, k'_j vertices of type q_j for the component of R.

If the component of R is planar and the one of S is not this leads to

$$\sum_{\mathbf{k}'+\mathbf{k}''=\mathbf{k}} \binom{\mathbf{k}}{\mathbf{k}'} \mathcal{M}_g^{\mathbf{k}''}((\mathcal{M}^{\mathbf{k}'} \otimes I)(\partial_i P_1) \otimes \cdots \otimes P_\ell)$$

possibilities or S is planar and R is not,

$$\sum_{\mathbf{k}'+\mathbf{k}''=\mathbf{k}} \binom{\mathbf{k}}{\mathbf{k}'} \mathcal{M}_g^{\mathbf{k}'}((I \otimes \mathcal{M}^{\mathbf{k}''})(\partial_i P_1) \otimes \cdots \otimes P_\ell)$$

possibilities.

3. The last case occurs if the half-edge is glued with another vertex $P_r = RX_iS$ of the root. This create a vertex of type $D_iP_rP_1$. Note that the edge can not cross the circles of the root so it must go through a handle of the surface thus it changes the genus by one. But this vertex is now free from the condition of non planarity so it can either be planar and thus be separate from the other vertices of the root:

$$\sum_{2\leq m\leq l, \mathbf{k}'+\mathbf{k}''=\mathbf{k}} \binom{\mathbf{k}}{\mathbf{k}'} \mathcal{M}^{\mathbf{k}'}(D_iP_rP_1)\mathcal{M}_{g-1}^{\mathbf{k}''}(P_2 \otimes \cdots \check{P}_r \cdots \otimes P_\ell)$$

possibilities or it may still be non-planar:

$$\sum_{2\leq m\leq \ell} \mathcal{M}_{g-1}^{\mathbf{k}}(D_iP_rP_1 \otimes P_2 \otimes \cdots \check{P}_r \cdots \otimes P_\ell)$$

possibilities.

We can sum these identities and then sum on the k_i's to obtain the following equality, for all g, for all $P_1 \ldots, P_\ell$,

$$\begin{aligned}\mathcal{I}_g(X_iP_1 \otimes \cdots \otimes P_\ell) = &\sum_j (-t_j)\mathcal{I}_g(D_iq_jP_1 \otimes \cdots \otimes P_\ell) \\ &+ \mathcal{I}_g((I \otimes I + \mathcal{I} \otimes I + I \otimes \mathcal{I})\partial_i P_1 \otimes \cdots \otimes P_\ell) \\ &+ \sum_{m\geq 2} \mathcal{I}_{g-1}((I+\mathcal{I})D_iP_rP_1 \otimes \cdots \check{P}_r \cdots \otimes P_\ell)\end{aligned}$$

We can reformulate this by applying it to $P_1 = D_i\bar{P}$ and then summing on

i:

$$\mathcal{I}_g(\Xi P \otimes \cdots \otimes P_\ell) = \sum_{m \geq 2, i} \mu(D_i P_r D_i \bar{P}) \mathcal{I}_{g-1}(P_2 \otimes \cdots \check{P}_r \cdots \otimes P_\ell) \qquad (14)$$
$$+ \sum_{m \geq 2, i} \mathcal{I}_{g-1}(D_i P_r D_i \bar{P} \otimes \cdots \check{P}_r \cdots \otimes P_\ell)$$
$$+ \sum_i \mathcal{I}_g(\partial_i D_i \bar{P} \otimes \cdots \otimes P_\ell).$$

where we used the identity $\mathcal{I} = \mu$. Note that maps that appear in the enumeration must satisfy the condition of non-planarity, this imposes a high genus. We have to break the "planarity" of ℓ components, this can't be done without at least $[\frac{\ell+1}{2}]$ handles on the surface (each handle allow one edge to cross from one vertex of the root to another one, breaking the planarity of two components at most). Thus if $g < Ent(\frac{\ell+1}{2})$, $\mathcal{I}_g(P_1 \otimes \cdots \otimes P_\ell) = 0$. This allow us to write the previous equation in a special case which will appear to be useful, if ℓ is even,

$$\mathcal{I}_{\frac{\ell}{2}}(\Xi P \otimes \cdots \otimes P_\ell) = \sum_{m \geq 2, i} \mu(D_i P_r D_i \bar{P}) \mathcal{I}_{\frac{\ell-2}{2}}(P_2 \otimes \cdots \check{P}_r \cdots \otimes P_\ell). \qquad (15)$$

Thus for ℓ even, $\mathcal{I}_{\frac{\ell}{2}}$ satisfy the limit equation (12) of the matrix model. One can easily check that for ℓ odd $\mathcal{I}_{\frac{\ell}{2}} = \mathcal{I}_{\frac{\ell+1}{2}}$ satisfy also the limit equation (13):

$$\mathcal{I}_{\frac{\ell+1}{2}}(\Xi P \otimes \cdots \otimes P_\ell) = \sum_{m \geq 2, i} \mathcal{I}(D_i P_r D_i \overline{P}) \mathcal{I}_{\frac{(\ell-2)+1}{2}}(P_2 \otimes \cdots \check{P}_r \cdots \otimes P_\ell)$$
$$+ \sum_{m \geq 2, i} \mathcal{I}_{\frac{\ell-1}{2}}(D_i P_r D_i \overline{P} \otimes P_2 \otimes \cdots \check{P}_r \cdots \otimes P_\ell)$$
$$+ \sum_i \mathcal{I}_{\frac{\ell+1}{2}}(\partial_i D_i \bar{P} \otimes \cdots \otimes P_\ell).$$

We can deduce from these identities a control on these enumerations:

Lemma 4.3 *For all $g \geq 0$, there exists $\varepsilon > 0$ such that for $\mathbf{t} \in B(0, \varepsilon)$, $\mathcal{I}^g$ is absolutely convergent and has a bounded support i.e. there exists $M > 0$ such that for all polynomials $P_1, \cdots, P_\ell$,*

$$|\mathcal{I}_g(P_1 \otimes \cdots \otimes P_\ell)| \leqslant \|P_1\|_M \cdots \|P_\ell\|_M$$

.

Proof.

It is sufficient to show that for all $g \geq 0$, there exists $A_g, B_g > 0$ such that for all h, for all monomials P_1,...,P_ℓ, and all integers k_i:

$$\frac{\mathcal{M}_g^{\mathbf{k}}(P_1 \otimes \cdots \otimes P_\ell)}{\prod_i k_i!} \leq A_g^{\Sigma deg P_i} B_g^{\Sigma k_i}$$

This is easy by induction using the decomposition of maps.

□

Finally we need to know the effect of derivation on these generating function. In fact, derivation adds some vertices to the enumeration.

Lemma 4.4 *For all* $\mathbf{j} = (j_1, \cdots, j_n)$,

$$\mathcal{D}_{\mathbf{j}}\mathcal{I}(P) = (-1)^{\mathbf{j}} \sum_{\mathbf{k} \in \mathbb{N}^n} \prod_{i=1}^{n} \frac{(-t_i)^{k_i}}{k_i!} \mathcal{M}^{\mathbf{k}+\mathbf{j}}(P)$$

and

$$\mathcal{D}_{\mathbf{j}}\mathcal{I}_g(P_1 \otimes \cdots \otimes P_\ell) = (-1)^{\mathbf{j}} \sum_{\mathbf{k} \in \mathbb{N}^n} \prod_{i=1}^{n} \frac{(-t_i)^{k_i}}{k_i!} \mathcal{M}_g^{\mathbf{k}+\mathbf{j}}(P_1 \otimes \cdots \otimes P_\ell).$$

Besides, these series are absolutely convergent and has a bounded support.

Proof.

The proof is straightforward, $\mathcal{I}$,$\mathcal{I}_g$ are analytic in a neighbourhood of the origin thus their derivatives are analytic and their series are given by differentiating term by term $\mathcal{I}$ and $\mathcal{I}_g$.

□

Thus derivatives fix some vertices in the enumeration a fact often used in combinatorics to find relation between generating functions of graphs.

5 High order observable

We have already seen that $\nu(P_1 \otimes \cdots \otimes P_\ell) = \mathcal{I}_{\frac{\ell}{2}}(P_1 \otimes \cdots \otimes P_\ell)$ satisfy the limit equation of ν^N. This is our candidate for the limit of the ν^N's. In fact this suggests a statement closely related to 1.1.

Property 5.1 *For all ℓ, for all $g \in \mathbb{N}$, there exists $\eta > 0$ such that for all* $\mathbf{t}$ *in $B_{\eta,c}$, for all polynomials $P_1,\ldots,P_\ell$,*

$$\nu^N(P_1 \otimes \cdots \otimes P_\ell) = \mathcal{I}_{\frac{\ell}{2}}(P_1 \otimes \cdots \otimes P_\ell) + \frac{1}{N^2}\mathcal{I}_{\frac{\ell}{2}+1}(P_1 \otimes \cdots \otimes P_\ell) + \cdots + \frac{1}{N^{2g}}\mathcal{I}_{\frac{\ell}{2}+g}(P_1 \otimes \cdots \otimes P_\ell) + o(\frac{1}{N^{2g}}).$$

To prove this we have to define all the correction to the convergence. We define $\nu_1^N = \nu^N$ and by induction on h, for all N, all polynomials $P_1, \ldots, P_\ell$,

$$\nu_{h+1}^N(P_1 \otimes \cdots \otimes P_\ell) = N^2(\nu_h^N - \mathcal{I}_{\frac{\ell}{2}+h-1})(P_1 \otimes \cdots \otimes P_\ell).$$

Those quantities satisfy also some induction relation similar to those of property 3.2

Property 5.2 *For all $h \geqslant 2$, ℓ in $\mathbb{N}$, for all polynomial $P_1,\ldots,P_\ell$, for all N, the "finite Schwinger-Dyson's equation of order h" $(\mathbf{SD}_{h,\ell}^N)$ is satisfied by ν^N: if ℓ is even*

$$\begin{aligned}\nu_h^N(\Xi P \otimes P_2 \otimes \cdots \otimes P_\ell) &= \sum_{i,r} \mu(D_i\overline{P}D_iP_r)\nu_h^N(P_2 \otimes \cdots \check{P}_r \cdots \otimes P_\ell) \\ &+ \sum_{i,r} \nu_{h-1}^N(D_i\overline{P}D_iP_r \otimes P_2 \otimes \cdots \check{P}_r \cdots \otimes P_\ell) \qquad (\mathbf{SD}_{h,\ell}^N) \\ &+ \sum_i \nu_{h-1}^N(\partial_i D_i\overline{P} \otimes P_2 \otimes \cdots \otimes P_\ell)\end{aligned}$$

and if ℓ is odd

$$\begin{aligned}\nu_h^N(\Xi P \otimes P_2 \otimes \cdots \otimes P_\ell) &= \sum_{i,r} \mu(D_i\overline{P}D_iP_r)\nu_h^N(P_2 \otimes \cdots \check{P}_r \cdots \otimes P_\ell) \\ &+ \sum_{i,r} \nu_h^N(D_i\overline{P}D_iP_r \otimes P_2 \otimes \cdots \check{P}_r \cdots \otimes P_\ell) \qquad (\mathbf{SD}_{h,\ell}^N) \\ &+ \sum_i \nu_h^N(\partial_i D_i\overline{P} \otimes P_2 \otimes \cdots \otimes P_\ell)\end{aligned}$$

Proof.

Remember that we have shown in Property 3.2, for ℓ even

$$\nu_1^N(\Xi P\otimes P_2\otimes\cdots\otimes P_\ell)=\sum_{i,r}\mu_{\mathbf{t}}(D_i\overline{P}D_iP_r)\nu_1^N(P_2\otimes\cdots\check{P}_r\cdots\otimes P_\ell)$$

$$+\frac{1}{N^2}\sum_{i,r}\nu_1^N(D_i\overline{P}D_iP_r\otimes P_2\otimes\cdots\check{P}_r\cdots\otimes P_\ell)$$

$$+\frac{1}{N^2}\sum_i\nu_1^N(\partial_iD_i\overline{P}\otimes P_2\otimes\cdots\otimes P_\ell)$$

and according to (15)

$$\mathcal{I}_{\frac{\ell}{2}}(\Xi P\otimes\cdots\otimes P_\ell)=\sum_{m\geq 2,i}\mathcal{I}(D_iP_rD_i\bar{P})\mathcal{I}_{\frac{\ell-2}{2}}(P_2\otimes\cdots\check{P}_r\cdots\otimes P_\ell).$$

Thus if we substract these two equalities and multiply the result by N^2 we obtain ($\mathbf{SD}_{2,\ell}^N$) (Observe that with our convention $\nu^N(\lambda)=\mathcal{I}_0(\lambda)$).

$$\nu_2^N(\Xi P\otimes P_2\otimes\cdots\otimes P_\ell)=\sum_{i,r}\mu(D_i\overline{P}D_iP_r)\nu_2^N(P_2\otimes\cdots\check{P}_r\cdots\otimes P_\ell)$$

$$+\sum_{i,r}\nu_1^N(D_i\overline{P}D_iP_r\otimes P_2\otimes\cdots\check{P}_r\cdots\otimes P_\ell)$$

$$+\sum_i\nu_1^N(\partial_iD_i\overline{P}\otimes P_2\otimes\cdots\otimes P_\ell).$$

Now suppose that for ℓ even, $h\geqslant 2$, for all polynomial $P_1,\ldots,P_\ell$, for all N, ($\mathbf{SD}_{h,\ell}^N$) is satisfied. Then according to (14):

$$\mathcal{I}_{\frac{\ell}{2}+h-1}(\Xi P\otimes\cdots\otimes P_\ell)=\sum_i\mathcal{I}_{\frac{(\ell+1)+1}{2}+h-2}(\partial_iD_i\bar{P}\otimes\cdots\otimes P_\ell)$$

$$+\sum_{r\geq 2,i}\mathcal{I}_{\frac{(\ell-1)+1}{2}+h-2}(D_iP_rD_i\bar{P}\otimes\cdots\check{P}_r\cdots\otimes P_\ell)$$

$$+\sum_{r\geq 2,i}\mu(D_iP_rD_i\bar{P})\mathcal{I}_{\frac{(\ell-2)}{2}+h-1}(D_iP_rD_i\bar{P}\otimes\cdots\check{P}_r\cdots\otimes P_\ell).$$

and this can be translated into

$$\mathcal{I}_{\frac{\hat{\ell}}{2}+h-1}(\Xi P\otimes\cdots\otimes P_\ell)=\sum_i\mathcal{I}_{\frac{\widehat{\ell+1}}{2}+h-2}(\partial_iD_i\bar{P}\otimes\cdots\otimes P_\ell)$$

$$+\sum_{r\geq 2,i}\mathcal{I}_{\frac{\widehat{\ell-1}}{2}+h-2}(D_iP_rD_i\bar{P}\otimes\cdots\check{P}_r\cdots\otimes P_\ell)$$

$$+\sum_{r\geq 2,i}\mu(D_iP_rD_i\bar{P})\mathcal{I}_{\frac{\widehat{\ell-2}}{2}+h-1}(D_iP_rD_i\bar{P}\otimes\cdots\check{P}_r\cdots\otimes P_\ell).$$

Substracting this equality from ($\mathbf{SD}^N_{h,\ell}$) we get ($\mathbf{SD}^N_{h+1,\ell}$). This proves by induction ($\mathbf{SD}^N_{h,\ell}$) for all h, and for all ℓ even.

We proceed in the same way for ℓ odd. Observe that the equation for ℓ odd and $h=1$ is satisfied according to Property 3.2. Then observe that for ℓ odd, ($\mathbf{SD}^N_{h+1,\ell}$) can be obtained by substracting (14) with $g = \frac{\ell+1}{2} + h - 1$

$$\begin{aligned}\mathcal{I}_{\frac{\ell}{2}+h-1}(\Xi P \otimes \cdots \otimes P_\ell) &= \sum_i \mathcal{I}_{\widehat{\frac{\ell+1}{2}}+h-2}(\partial_i D_i \bar{P} \otimes \cdots \otimes P_\ell) \\ &+ \sum_{r \geq 2, i} \mathcal{I}_{\widehat{\frac{\ell-1}{2}}+h-2}(D_i P_r D_i \bar{P} \otimes \cdots \check{P}_r \cdots \otimes P_\ell) \\ &+ \sum_{r \geq 2, i} \mu(D_i P_r D_i \bar{P}) \mathcal{I}_{\widehat{\frac{\ell-2}{2}}+h-1}(D_i P_r D_i \bar{P} \otimes \cdots \check{P}_r \cdots \otimes P_\ell).\end{aligned}$$

from ($\mathbf{SD}^N_{h,\ell}$).

□

6 Asymptotic of the matrix model

The issue with the previous relations is that they only give us the moments of products of polynomials such that the first polynomial is in the image of Ξ. Thus we need to invert Ξ. We define the operator norm with respect to $\|.\|_M$:

$$|||A|||_M = \sup_{\|P\|_M \leqslant 1} \|AP\|_M.$$

In [14], we give some estimates on the operator norm of Ξ.

Lemma 6.1 *1. The operator Ξ_0 is invertible on $\mathbb{C}\langle X_1, \cdots, X_m\rangle$.*

2. There exists $M_0 > 0$ such that for all $M > M_0$, the operators Ξ_2, Ξ_0 and Ξ_0^{-1} are continuous and their norm are uniformly bounded for $\mathbf{t}$ in B_η.

3. For all polynomials P, $\deg \Xi_0^{-1} P \leqslant \deg P$ and $\deg \Xi_1 P \leqslant \deg P + \deg V - 2$

4. For all $\varepsilon, M > 0$, there exists $\eta_\varepsilon > 0$ such for $|\mathbf{t}| < \eta_\varepsilon$, Ξ_1 is continuous on $\mathbb{C}\langle X_1, \cdots, X_m\rangle$ and $|||\Xi_1|||_M \leqslant \varepsilon$.

The last step to proves Theorem 1.1 is to control the ν^h_N.

Lemma 6.2 *For all $\ell, h \in \mathbb{N}^*$, $\alpha > 0$, there exists $C, \eta, M_0 > 0$, such that for all $\mathbf{t} \in B_{\eta,c}$, $M > M_0$ and all polynomials $P_1,\dots,P_\ell$ of degree less than $\alpha N^{\frac{1}{2}}$,*

$$|\nu_h^N(P_1 \otimes \cdots \otimes P_\ell)| \leq C\|P_1\|_M \cdots \|P_\ell\|_M$$

Proof.

The case $h = 1$, ℓ even is a direct consequence of Lemma 3.1. We treat the other cases by induction using Property 5.2. As the equations are different according to the parity of ℓ, we have to be careful: we prove the result by an induction on h and for a fixed h we deal first with the case ℓ even and then with the case ℓ odd (Note that both time we will do an induction on ℓ). Now we choose $\ell, h \in \mathbb{N}^*$, $\alpha > 0$ and polynomials $P_1,\dots,P_\ell$ of degree less than $\alpha N^{\frac{1}{2}}$.

Then, the idea to nearly invert Ξ on a polynomial P is to approximate P_1 by $\Xi Q_n = (\Xi_0 + \Xi_1)Q_n$ with

$$Q_n = \sum_{k=0}^{n-1}(-\Xi_0^{-1}\Xi_1)^k \Xi_0^{-1} P_1.$$

The remainder is

$$R_n = P_1 - \Xi Q_n = (-\Xi_1\Xi_0^{-1})^n P_1.$$

As Ξ_1 is the multiplication by a derivative of V it should have a small norm and the remainder should be easily controlled.

We can make the decomposition:

$$\nu_h^N(P_1 \otimes \cdots \otimes P_\ell) = \nu_h^N(\Xi Q_n \otimes \cdots \otimes P_\ell) + \nu_h^N(R_n \otimes \cdots \otimes P_\ell) \tag{16}$$

Now we let n goes to infinity with N, for example $n = [\sqrt{N}]$. It is important that n goes to infinity no too slowly but we must have $n = O(\sqrt{N})$ in order to use all the induction hypothesis. An important fact is that the degrees of R_n and Q_n are $O(\sqrt{N})$ since $\Xi_0^{-1}\Xi_1$ change the degree by at most $D-2$.

We first control the term with R_n, by definition of the ν_N^h,

$$\nu_h^N(R_n \otimes \cdots \otimes P_\ell) = (N^{2(h-1)}\nu_1^N + N^{2(h-1)}\mathcal{I}_{\frac{\ell}{2}} + \cdots + \mathcal{I}_{\frac{\ell}{2}+h-1})(R_n \otimes \cdots \otimes P_\ell).$$

Each of the $\mathcal{I}_g$ are compactly supported according to Lemma 4.3 so that if η is sufficiently small, for $\mathbf{t}$ in $B_{\eta,c}$, $\mathcal{I}_{\frac{\ell}{2}}$, $\dots$,$\mathcal{I}_{\frac{\ell}{2}+h-1}$ are convergent and we can take M bigger than the radius of their support. Besides, Property 3.1 shows that for polynomials P of degree of order $N^{\frac{1}{2}}$,

$$|\nu_1^N(P_1 \otimes \cdots \otimes P_\ell)| \leqslant CN\|P_1\|_M \cdots \|P_\ell\|_M.$$

Thus according to Lemma 6.1, for η small, $\|R_n\|_M \leqslant |||\Xi_1\Xi_0^{-1}|||^n\|P_1\|_M$ decrease exponentially fast in n and this uniformly for $\mathbf{t} \in B_{\eta,c}$. Then since $n \sim \sqrt{N}$

$$|\mathcal{I}_g(R_n \otimes \cdots \otimes P_\ell)| \leqslant \|R_n\|_M\|P_2\|_M \ldots \|P_\ell\|_M \leqslant Ce^{-C'\sqrt{N}}\|P_1\|_M \ldots \|P_\ell\|_M.$$

Thus,

$$|\nu_h^N(R_n \otimes \cdots \otimes P_\ell)| \leqslant N^{2h}Ce^{-C'\sqrt{N}}\|P_1\|_M \ldots \|P_\ell\|_M \tag{17}$$

and $N^{2h}Ce^{-C'\sqrt{N}}$ is bounded.

Finally we have to deal with $\nu_h^N(\Xi Q_n \otimes \cdots \otimes P_\ell)$. We can use $(\mathbf{SD}_{h,\ell}^N)$:

$$\begin{aligned}\nu_h^N(\Xi Q_n \otimes P_2 \otimes \cdots \otimes P_\ell) = \sum_{i,m} \mu_{\mathbf{t}}(D_i\bar{Q}_nD_iP_r)\nu_h^N(P_2 \otimes \cdots \check{P}_r \cdots \otimes P_\ell) \\ + \sum_{i,m} \nu_{h-\mathbb{1}_{\ell \text{ even}}}^N(D_i\bar{Q}_nD_iP_r \otimes P_2 \otimes \cdots \check{P}_r \cdots \otimes P_\ell) \\ + \sum_{i} \nu_{h-\mathbb{1}_{\ell \text{ even}}}^N(\partial_iD_i\bar{Q}_n \otimes P_2 \otimes \cdots \otimes P_\ell)\end{aligned} \tag{18}$$

We now use the induction hypothesis. Indeed if h is even, on the right hand side either h decreases or h remains constant and ℓ decreases and remains even. If h is odd, either ℓ becomes even or ℓ decreases. Now, let C be an uniform bound on the norm of Ξ_0^{-1} (which exists according to Lemma 6.1) by definition of Q_n,

$$\|Q_n\|_M \leqslant \sum_{k=0}^{n-1} |||\Xi_0^{-1}\Xi_1|||_M^k|||\Xi_0^{-1}|||_M\|P\|_M \leqslant \frac{C}{1 - C|||\Xi_1|||_M}\|P\|_M.$$

Thus, using Lemma 6.1, if η is sufficiently small, for $\mathbf{t}$ in $B_{\eta,c}$, $\|Q_n\|_M \leqslant 2\|P\|_M$. Note that

$$\deg Q_n \leqslant \deg P_1 + 2\sqrt{N}(D-1) \leqslant (\alpha + 2(D-1))\sqrt{N}.$$

We can now apply the induction hypothesis with $\alpha' = \alpha + 2(D-1)$, there exists M, C, η such that for $\mathbf{t}$ in $B_{\eta,c}$,

$$\begin{aligned}&|\mu_{\mathbf{t}}(D_i\bar{Q}_nD_iP_r)\nu_h^N(P_2 \otimes \cdots \check{P}_r \cdots \otimes P_\ell)| \\ &\leqslant C\|D_i\bar{Q}_nD_iP_r\|_M\|P_2\|_M \cdots \|\check{P}_r\|_M \cdots \|P_\ell\|\end{aligned} \tag{19}$$

where we have assumed that M is bigger than the radius of the support of $\mu_{\mathbf{t}}$ which is bounded according to (4). Besides, by the induction hypothesis we can obtain with the same constant,

$$\begin{aligned}
&|\nu^N_{h-\mathbb{1}_{\ell \text{ even}}}(D_i\bar{Q}_n D_i P_r \otimes P_2 \otimes \cdots \check{P}_r \cdots \otimes P_\ell)| \\
&\leqslant C\|D_i\bar{Q}_n D_i P_r\|_M \|P_2\|_M \cdots \|\check{P}_r\|_M \cdots \|P_\ell\|_M
\end{aligned} \tag{20}$$

and

$$\begin{aligned}
&|\nu^N_{h-\mathbb{1}_{\ell \text{ even}}}(\partial_i D_i\bar{Q}_n \otimes P_2 \otimes \cdots \otimes P_\ell)| \\
&\leqslant \|\partial_i D_i\bar{Q}_n\|_M \|P_2\|_M \cdots \|\check{P}_r\|_M \cdots \|P_\ell\|_M .
\end{aligned} \tag{21}$$

Now remember, that if $M' > M$,

$$D_i : (\mathbb{C}\langle X_1, \cdots, X_m\rangle, \|.\|_{M'}) \to (\mathbb{C}\langle X_1, \cdots, X_m\rangle, \|.\|_M)$$

is continuous, thus

$$\begin{aligned}
\|D_i\bar{Q}_n D_i P_r\|_M &\leqslant \|D_i\bar{Q}_n\|_M \|D_i P_r\|_M \\
&\leqslant C\|Q_n\|_{M'}\|P_r\|_{M'} \leqslant C\|P_1\|_{M'}\|P_r\|_{M'}
\end{aligned}$$

and

$$\|\partial_i D_i \bar{Q}_n\|_M \leqslant C\|P_1\|_{M'}.$$

If we use inequalities (19), (20) and (21) in the decomposition (18), we get

$$|\nu^N_h(\Xi Q_n \otimes P_2 \otimes \cdots \otimes P_\ell)| \leqslant C\|P_1\|_{M'}\|P_2\|_{M'} \cdots \|P_\ell\|_{M'}.$$

Finally, we conclude with (17) and (16)

□

Now, for all ℓ, h there exists $\eta > 0$ such that for $\mathbf{t} \in B_{\eta,c}$, for all polynomials $P_1, \cdots, P_\ell$,

$$\nu^N_g(P_1 \otimes \cdots \otimes P_\ell) = N^2(\nu^N_g - \mathcal{I}_{\frac{\hat{\ell}}{2}+g-1})(P_1 \otimes \cdots \otimes P_\ell)$$

is a bounded sequence for all $g \leqslant h+1$. Thus for all $g \leqslant h$, $\nu^N_g(P_1 \otimes \cdots \otimes P_\ell)$ goes to $\mathcal{I}_{\frac{\hat{\ell}}{2}+g-1}$ and

$$\nu^N(P_1 \otimes \cdots \otimes P_\ell) = \mathcal{I}_{\frac{\hat{\ell}}{2}} + \frac{1}{N^2}\mathcal{I}_{\frac{\hat{\ell}}{2}+1} + \cdots + + \frac{1}{N^{2h}}\mathcal{I}_{\frac{\hat{\ell}}{2}+h} + o(\frac{1}{N^{2h}}).$$

Thus Property 5.1 is proved. The special case $\ell = 1$ is exactly Theorem 1.2:

$$\begin{aligned} E[\hat{\mu}^N(P)] &= \mathcal{I}(P) + \frac{1}{N^2}\nu_1^N(P) \\ &= \mathcal{I}(P) + \frac{1}{N^2}\mathcal{I}_1(P) + \cdots + \frac{1}{N^{2g}}\mathcal{I}_g(P) + o(\frac{1}{N^{2g}}). \end{aligned}$$

Thus we can prove Theorem 1.1.

Theorem 6.3 *For all $g \in \mathbb{N}$, there exists $\eta > 0$ such that for all $\mathbf{t}$ in $B_{\eta,c}$,*

$$F^N_{V_\mathbf{t}} = F^0(\mathbf{t}) + \cdots + \frac{1}{N^{2g}}F^g(\mathbf{t}) + o(\frac{1}{N^{2g}})$$

and F^g is the generating function for maps of genus g associated with V:

$$F^g(\mathbf{t}) = \sum_{k_1,\cdots,k_n \in \mathbb{N}} \prod_i \frac{(-t_i)^{k_i}}{k_i!}\mathcal{C}^{\mathbf{k}}$$

where if $\mathbf{k} = (k_1, \cdots, k_n)$, $\mathcal{C}^{\mathbf{k}}$ is the number of maps on a surface of genus g with k_i vertices of type q_i.

Proof.

Note that the estimate we get in Lemma 7.2 are uniform in $\mathbf{t}$ provided we are in $B_{\eta,c}$. Now observe that if $V_\mathbf{t}$ is c-convex then for α in $[0,1]$, $V_{\alpha\mathbf{t}}$ is c-convex if $c \leqslant 1$ and 1-convex if $c > 1$. Thus if $\mathbf{t}$ is in $B_{\eta,c}$, for all $0 < \alpha < 1$, $\alpha\mathbf{t} \in B_{\eta,\min(c,1)}$. This allow us to use Property 7.3 with an uniformly bounded remainder.

$$\begin{aligned} F^N_{V_\mathbf{t}} &= \int_0^1 -E_{\mu^N_{V_{\alpha\mathbf{t}}}}[\hat{\mu}^N(V_\mathbf{t})]d\alpha \\ &= -\int_0^1 \mu_{\alpha\mathbf{t}}(V_\mathbf{t})d\alpha - \frac{1}{N^2}\int_0^1 \nu^N_{\alpha\mathbf{t}}(\hat{\mu}^N(V_\mathbf{t}))d\alpha \\ &= F^0(\mathbf{t}) + \cdots + \frac{1}{N^{2g}}F^g(\mathbf{t}) + o(\frac{1}{N^{2g}}) \end{aligned}$$

with

$$\begin{aligned} F^0 &= -\int_0^1 \sum_{k_1,\cdots,k_n \in \mathbb{N}} \prod_i \frac{(-\alpha t_i)^{k_i}}{k_i!}\mathcal{M}^{\mathbf{k}}(V_\mathbf{t})d\alpha \\ &= \sum_{k_1,\cdots,k_n \in \mathbb{N}} \prod_i \frac{(-t_i)^{k_i}}{k_i!}\mathcal{C}^{\mathbf{k}} \end{aligned}$$

since it can be easily checked that

$$\sum_{k_1,\cdots,k_n\in\mathbb{N}}\prod_i \frac{(-ut_i)^{k_i}}{k_i!}\mathcal{C}^{\mathbf{k}}$$

has the same derivative in u than $-\int_0^u \sum_{k_1,\cdots,k_n\in\mathbb{N}}\prod_i \frac{(-\alpha t_i)^{k_i}}{k_i!}\mathcal{M}^{\mathbf{k}}(V_{\mathbf{t}})d\alpha$. With the same technique,

$$\begin{aligned} F^g &= -\int_0^1 \sum_{k_1,\cdots,k_n\in\mathbb{N}}\prod_i \frac{(-\alpha t_i)^{k_i}}{k_i!}\mathcal{M}_g^{\mathbf{k}}(V_{\mathbf{t}})d\alpha \\ &= \sum_{k_1,\cdots,k_n\in\mathbb{N}}\prod_i \frac{(-t_i)^{k_i}}{k_i!}\mathcal{C}_g^{\mathbf{k}} \end{aligned}$$

This proves the Theorem:

$$F_{V_{\mathbf{t}}}^N = F^0 + \cdots + \frac{1}{N^2 g}F^g + o(\frac{1}{N^{2g}}).$$

□

7 Higher derivatives.

In this section we will show that one can differentiate these expansions term by terms. Indeed, the family of the ν_h^N's is sufficiently rich to express any of its own derivatives. Thus, we will be able to find a recursive decomposition of this derivatives.

Property 7.1 *For all $1 \leqslant j \leqslant n$, for all polynomials $P_1,\cdots,P_\ell$, if ℓ is even,*

$$\begin{aligned} \frac{\partial}{\partial t_j}\nu_1^N(P_1\otimes\cdots\otimes P_\ell) = &-\nu_1^N(P_1\otimes\cdots\otimes P_\ell\otimes q_j) \\ &+\sum_{r=1}^{\ell}\frac{\partial}{\partial t_j}\mu(P_r)\nu_1^N(P_1\otimes\cdots\check{P}_r\cdots\otimes P_\ell) \\ &+\nu_1^N(P_1\otimes\cdots\otimes P_\ell)\nu_1^N(q_j) \end{aligned}$$

and if ℓ is odd,

$$\frac{\partial}{\partial t_j}\nu_1^N(P_1\otimes\cdots\otimes P_\ell) = -\nu_2^N(P_1\otimes\cdots\otimes P_\ell\otimes q_j) + \sum_{r=1}^{\ell}\frac{\partial}{\partial t_j}\mu(P_r)\nu_2^N(P_1\otimes\cdots\check{P}_r\cdots\otimes P_\ell) + \nu_1^N(P_1\otimes\cdots\otimes P_\ell)\nu_1^N(q_j).$$

Proof.

We simply need to differentiate

$$\frac{1}{Z_V^N}\int\prod_r(\frac{1}{N}\operatorname{tr}-\mu)(P_r)e^{-N\operatorname{tr}V}d\mu^N.$$

In that expression we can either differentiate $\frac{1}{Z_V^N}$, the potential $e^{-N\operatorname{tr}V}$ or one of the $\mu(P_r)$, this leads to

$$\begin{aligned}\frac{\partial}{\partial t_j}E[\prod_r(\frac{1}{N}\operatorname{tr}-\mu)(P_r)] &= N^2E[\prod_r(\frac{1}{N}\operatorname{tr}-\mu)(P_r)(\frac{1}{N}\operatorname{tr}-\mu)(q_j)] \\ &- E[\prod_r(\frac{1}{N}\operatorname{tr}-\mu)(P_r)]N^2E[(\frac{1}{N}\operatorname{tr}-\mu)(q_j)] \\ &+ \sum_r\frac{\partial}{\partial t_j}\mu(P_r)E[\prod_{r'\neq r}(\frac{1}{N}\operatorname{tr}-\mu)(P_{r'})]\end{aligned}$$

Where one can notice that we have added in the two first terms of the right hand side the quantity $\mu(q_j)$ but these two modifications cancel each other.

Now multiply by the normalisation N^ℓ to get the equation in the case ℓ even. In the case ℓ odd, if we multiply by $N^{\ell+1}$ we get

$$\begin{aligned}\frac{\partial}{\partial t_j}\nu_1^N(P_1\otimes\cdots\otimes P_\ell) &= -N^2\nu_1^N(P_1\otimes\cdots\otimes P_\ell\otimes q_j) \\ &+ N^2\sum_{r=1}^{\ell}\frac{\partial}{\partial t_j}\mu(P_r)\nu_1^N(P_1\otimes\cdots\check{P}_r\cdots\otimes P_\ell) \\ &+ \nu_1^N(P_1\otimes\cdots\otimes P_\ell)\nu^N(q_j) \\ &= -\nu_2^N(P_1\otimes\cdots\otimes P_\ell\otimes q_j) \\ &+ \sum_{r=1}^{\ell}\frac{\partial}{\partial t_j}\mu(P_r)\nu_2^N(P_1\otimes\cdots\check{P}_r\cdots\otimes P_\ell) \\ &+ \nu_1^N(P_1\otimes\cdots\otimes P_\ell)\nu^N(q_j) \\ &+ N^2r_N\end{aligned}$$

with by definition of ν_2^N,

$$r_N = -\mathcal{I}_{\frac{l+1}{2}}(P_1 \otimes \cdots \otimes P_\ell \otimes q_j) + \sum_r \frac{\partial}{\partial t_j}\mu(P_r)\mathcal{I}_{\frac{l-1}{2}}(P_1 \otimes \cdots \check{P}_r \cdots \otimes P_\ell).$$

Let's have a closer look at this expression, $\mathcal{I}_{\frac{l+1}{2}}(P_1 \otimes \cdots \otimes P_\ell \otimes q_j)$ counts maps with $\frac{l+1}{2}$ handles and such none of the $l+1$ components P_1,..., P_ℓ, q_j is planar. Since each handle can break the planarity of at most two components, the only way to obtain such a configuration is to form $\frac{l+1}{2}$ couples among these l components and in each of these couples to put a handle between the two components. For example, one can decompose these maps according to the other vertex in the couple of q_j:

$$\mathcal{I}_{\frac{l+1}{2}}(P_1 \otimes \cdots \otimes P_\ell \otimes q_j) = \sum_r \mathcal{I}_1(P_r \otimes q_j)\mathcal{I}_{\frac{l-1}{2}}(P_1 \otimes \cdots \check{P}_r \cdots \otimes P_\ell).$$

Now observe that $\mathcal{I}_1(P_r \otimes q_j)$ counts maps of genus 1 such that the component of P_r is linked to the component of q_j, thus it is equivalent to the counting of planar maps with two precribed vertices, one of type P_r and one of type q_j. According to Lemma 4.4, that exactly what count $\frac{\partial}{\partial t_j}\mu(P_r)$. Thus $r_N = 0$ and the Property is proved.

□

With this decomposition we can now show that the derivatives of the ν_h^N's are of order 1.

Lemma 7.2 *For all* $\mathbf{j} = (j_1, \cdots, j_n)$ *, for all* $\ell, h \in \mathbb{N}^*$*,* $\alpha > 0$*, there exists constants* $C, \eta, M_0 > 0$*, such that for all* $\mathbf{t} \in B_{\eta,c}$*,* $M > M_0$ *and all polynomials* P_1*,...,*P_ℓ *of degree less than* $\alpha N^{\frac{1}{2}}$*,*

$$|\mathcal{D}_{\mathbf{j}}\nu_h^N(P_1 \otimes \cdots \otimes P_\ell)| \leq C\|P_1\|_M \cdots \|P_\ell\|_M$$

Proof.

The proof is essentially the same than the proof of Lemma 6.2. We just use in addition an induction on $\sum_i j_i$. If $\sum_i j_i = 0$ then we are exactly in the case of the Lemma 6.2. Otherwise, Assume that we have already proved this lemma up to the $\sum_i j_i - 1$-th derivative. To prove it for $\sum_i j_i$, we proceed by induction. First we prove it for ν_1^N, and it is straightforward since we can express the derivatives of ν_1^N as a function of derivatives of lesser degree according to Property 7.1. Then for $h > 1$ we need to find an induction relation on the $\mathcal{D}_{\mathbf{j}}\nu_N^h$. We start form Property 5.2:

$$\begin{aligned}\nu_h^N(\Xi P_1 \otimes P_2 \otimes \cdots \otimes P_\ell) = &\sum_{i,m} \mu_{\mathbf{t}}(D_i \bar{P}_1 D_i P_r)\nu_h^N(P_2 \otimes \cdots \check{P}_r \cdots \otimes P_\ell) \\ &+ \sum_{i,m} \nu^N_{h-\mathbb{1}_{\ell \text{ even}}}(D_i \bar{P}_1 D_i P_r \otimes P_2 \otimes \cdots \check{P}_r \cdots \otimes P_\ell) \\ &+ \sum_{i} \nu^N_{h-\mathbb{1}_{\ell \text{ even}}}(\partial_i D_i \bar{P}_1 \otimes P_2 \otimes \cdots \otimes P_\ell)\end{aligned}$$

and we take the derivative:

$$\begin{aligned}&\mathcal{D}_{\mathbf{j}}\nu_h^N(\Xi P_1 \otimes P_2 \otimes \cdots \otimes P_\ell) \\ &= \sum_{\substack{\mathbf{j}'+\mathbf{j}''=\mathbf{j} \\ \mathbf{j}''\neq 0}} \binom{\mathbf{j}'}{\mathbf{j}} \mathcal{D}_{\mathbf{j}'}\nu_h^N((\mathcal{D}_{\mathbf{j}''}\Xi)P_1 \otimes P_2 \otimes \cdots \otimes P_\ell) \\ &+ \sum_{\substack{\mathbf{j}'+\mathbf{j}''=\mathbf{j} \\ \mathbf{j}''\neq 0}} \sum_{i,m} \binom{\mathbf{j}'}{\mathbf{j}} \mathcal{D}_{\mathbf{j}'}\mu_{\mathbf{t}}(D_i \bar{P}_1 D_i P_r)\mathcal{D}_{\mathbf{j}''}\nu_h^N(P_2 \otimes \cdots \check{P}_r \cdots \otimes P_\ell) \\ &+ \sum_{i,m} \mathcal{D}_{\mathbf{j}}\nu^N_{h-\mathbb{1}_{\ell \text{ even}}}(D_i \bar{P}_1 D_i P_r \otimes P_2 \otimes \cdots \check{P}_r \cdots \otimes P_\ell) \\ &+ \sum_{i} \mathcal{D}_{\mathbf{j}}\nu^N_{h-\mathbb{1}_{\ell \text{ even}}}(\partial_i D_i \bar{P}_1 \otimes P_2 \otimes \cdots \otimes P_\ell)\end{aligned}$$

with

$$\mathcal{D}_{\mathbf{j}''}\Xi P = (I \otimes (\mathcal{D}_{\mathbf{j}''}\mu) + (\mathcal{D}_{\mathbf{j}''}\mu) \otimes I)\partial_i D_i \bar{P} - (\mathcal{D}_{\mathbf{j}''} D_i V) D_i \bar{P}.$$

Thus $\mathcal{D}_{\mathbf{j}}\nu_h^N$ has a decomposition similar to the one of Property 5.2 and we can use it in similar way we use it in the proof of Lemma 6.2. The rest of the proof is identical to the one of Lemma 6.2 and we only give the main steps.

We define the same Q_n and R_n as in the previous proof to approximate $\Xi^{-1}P$. Then, $\mathcal{D}_{\mathbf{j}}\nu_h^N(\Xi Q_n \otimes P_2 \otimes \cdots \otimes P_\ell)$ is controlled using the previous decomposition, using the fact that $\mathcal{D}_{\mathbf{j}}\mu$ has a bounded support and that the norm of $(\mathcal{D}_{\mathbf{j}''} D_i V)$ is also bounded. Now write

$$\begin{aligned}\mathcal{D}_{\mathbf{j}}\nu_h^N(R_n \otimes P_2 \otimes \cdots \otimes P_\ell) = (&\mathcal{D}_{\mathbf{j}} N^{2(h-1)}\nu_1^N + N^{2(h-1)}\mathcal{D}_{\mathbf{j}}\mathcal{I}_{\hat{\ell}} + \cdots \\ &+ \mathcal{D}_{\mathbf{j}}\mathcal{I}_{\hat{\ell}+h-1})(R_n \otimes P_2 \otimes \cdots \otimes P_\ell)\end{aligned}$$

Then for $\mathbf{t}$ sufficiently small these series are convergent and have uniformly bounded support according to Lemma 4.4 and we can use Property 7.1 to control one more time $\mathcal{D}_{\mathbf{j}}\nu_1^N$.

□

From there we deduce

Property 7.3 *For all ℓ, for all $g \in \mathbb{N}$, for all $\mathbf{j} = (j_1, \cdots, j_n)$ there exists $\eta > 0$ such that for all $\mathbf{t}$ in $B_{\eta,c}$, for all polynomials $P_1, \ldots, P_\ell$,*

$$\mathcal{D}_{\mathbf{j}}\nu^N(P_1 \otimes \cdots \otimes P_\ell) = \mathcal{D}_{\mathbf{j}}\mathcal{I}_{\frac{\ell}{2}}(P_1 \otimes \cdots \otimes P_\ell) + \frac{1}{N^2}\mathcal{D}_{\mathbf{j}}\mathcal{I}_{\frac{\ell}{2}+1}(P_1 \otimes \cdots \otimes P_\ell) + \cdots + \frac{1}{N^{2g}}\mathcal{D}_{\mathbf{j}}\mathcal{I}_{\frac{\ell}{2}+g}(P_1 \otimes \cdots \otimes P_\ell) + o(\frac{1}{N^{2g}}).$$

Finally we prove Theorem 1.3,

Theorem 7.4 *For all $\mathbf{j} = (j_1, \cdots, j_n)$, for all $g \in \mathbb{N}$, there exists $\eta > 0$ such that for all $\mathbf{t}$ in $B_{\eta,c}$,*

$$\mathcal{D}_{\mathbf{j}}F^N_{V_{\mathbf{t}}} = \mathcal{D}_{\mathbf{j}}F^0(\mathbf{t}) + \cdots + \mathcal{D}_{\mathbf{j}}F^g(\mathbf{t}) + o(\frac{1}{N^{2g}}).$$

Besides, $\mathcal{D}_{\mathbf{j}}F^g$ is the generating function for rooted maps of genus g associated with V:

$$\mathcal{D}_{\mathbf{j}}F^g(\mathbf{t}) = \sum_{k_1,\cdots,k_n \in \mathbb{N}} \prod_i \frac{(-t_i)^{k_i}}{k_i!}\mathcal{C}_g^{\mathbf{k}+\mathbf{j}}(q_{i_1}, \cdots, q_{i_p})$$

where $\mathcal{C}_g^{k_1,\cdots,k_n}$ is the number of maps on a surface of genus g with k_i vertices of type q_i.

Proof.

The case $\mathbf{j} = 0$ is just Theorem 1.1. Thus we can assume $\mathbf{j} \neq 0$, for example $j_1 \neq 0$. Observe that for all i,

$$\frac{\partial}{\partial t_i}F^N_{V_{\mathbf{t}}} = -E[\hat{\mu}^N(q_i)] = -\mathcal{I}(q_i) - \frac{1}{N^2}\nu^N(q_i).$$

we can use the Property 7.3: there exists $\eta > 0$ such that for $\mathbf{t} \in B_{\eta,c}$,

$$\begin{aligned}\mathcal{D}_{\mathbf{j}}F^N_{V_{\mathbf{t}}} &= -\mathcal{D}_{\mathbf{j}-\mathbb{1}_{i=1}}(\mathcal{I}(q_1) + \frac{1}{N^2}\nu^N(q_1)) \\ &= -\mathcal{D}_{\mathbf{j}-\mathbb{1}_{i=1}}\mathcal{I}(q_1) - \frac{1}{N^2}\mathcal{D}_{\mathbf{j}-\mathbb{1}_{i=1}}\nu^N(q_1) \\ &= -\mathcal{D}_{\mathbf{j}-\mathbb{1}_{i=1}}\mathcal{I}(q_1) - \frac{1}{N^2}\mathcal{D}_{\mathbf{j}-\mathbb{1}_{i=1}}\mathcal{I}_1(q_1) - \cdots - \frac{1}{N^{2g}}\mathcal{D}_{\mathbf{j}-\mathbb{1}_{i=1}}\mathcal{I}_g(q_1) \\ &\quad + o(\frac{1}{N^{2g}})\end{aligned}$$

Observe now that according to Lemma 4.4

$$\mathcal{D}_{\mathbf{j}-\mathbb{1}_{i=1}}\mathcal{I}_g(q_1) = \sum_{k_1,\cdots,k_n\in\mathbb{N}}\prod_i \frac{(-t_i)^{k_i}}{k_i!}\mathcal{M}_g^{\mathbf{k}+\mathbf{j}-\mathbb{1}_{i=1}}(q_1)$$

$$= -\sum_{k_1,\cdots,k_n\in\mathbb{N}}\prod_i \frac{(-t_i)^{k_i}}{k_i!}\mathcal{C}_g^{\mathbf{k}+\mathbf{j}} = -\mathcal{D}_{\mathbf{j}}F_g$$

and by the same method, $\mathcal{D}_{\mathbf{j}-\mathbb{1}_{i=1}}\mathcal{I}(q_1) = -\mathcal{D}_{\mathbf{j}}F_0$. Thus we get,

$$\mathcal{D}_{\mathbf{j}}F_{V_{\mathbf{t}}}^N = \mathcal{D}_{\mathbf{j}}F_0 + \cdots + \frac{1}{N^{2g}}\mathcal{D}_{\mathbf{j}}F_g + o(\frac{1}{N^{2g}}).$$

□

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