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UNITARY MATRIX INTEGRALS

BENOÎT COLLINS, ALICE GUIONNET, AND EDOUARD MAUREL-SEGALA

Abstract. We prove that the limit of various unitary matrix integrals, including the Itzykson-Zuber integral, exists in a small parameters region and is analytic in these parameters.

Introduction

Unitary matrix integrals are of fundamental importance for mathematical physics. Their large N limit is both supposed to describe physical systems (2D quantum gravitation, gauge theory, renormalization, etc...), and on a more pedestrian point of view, to be generating series enumerating combinatorial objects (see e.g [14, 18]).

The Gaussian matrices are the most studied by physicists, and their limits have been proved to exist and match with the formal power limit on a mathematical level of rigor by two authors [10, 11, 16] and previously in the one matrix case (corresponding to non-colored graphs) in [1, 2] and [7].

In this paper we will rather concentrate on unitary matrix integrals

$$I_N(V, A_i^N) := \int_{\mathcal{U}(N)} e^{N\mathrm{Tr}(V(U_i,U_i^*,A_i^N,1\leq i\leq m))} dU_1 \cdots dU_m$$

where A_i^N are $N \times N$ deterministic uniformly bounded matrices, dU denotes the Haar measure on the unitary group $\mathcal{U}(N)$ and V is a non-commutative polynomial. To simplify the exposition, we shall assume hereafter that the A_i^N are Hermitian matrices. We shall only consider non-oscillatory integrals and thus assume that the polynomial V is such that $\mathrm{Tr}(V(U_i, U_i^*, A_i^N))$ is real for all $U \in \mathcal{U}(N)$, all $N \in \mathbb{N}$.

We shall assume that the joint distribution of the $A_i^N, 1 \leq i \leq m$ converges; namely for all polynomial function P in m non-commutative indeterminates

$$\lim_{N\to\infty} \frac{1}{N}\mathrm{Tr}(P(A_i^N, 1 \leq i \leq m)) = \tau(P) \tag{1}$$

for some linear functionnal τ on the set of polynomials. We shall then consider the problem of the asymptotic behaviour of $I_N(V, A_i^N)$ as N goes to infinity.

In the most general framework, nothing was known about the convergence of these matrix integrals, except the formal convergence. Namely, it was proved [5] by one

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author that for each k, the limit

$$\frac{\partial^k}{\partial z^k} N^{-2} \log \int_{\mathcal{U}(N)} e^{zNTr(V(U_i,U_i^*,A_i^N))} dU_1 \cdots dU_m|_{z=0}$$

converges towards an integer $f_k(V,\tau)$ depending only on the limiting distribution of the A_i^N's and V.

The goal of this article is to show the following theorem

Theorem 0.1. *Under the above hypotheses and if we further assume that the spectral radius of the matrices $(A_i^N, 1 \le i \le m, N \in \mathbb{N})$ is uniformly bounded (by say M), there exists $\varepsilon = \varepsilon(M,V) > 0$ so that for $z \in [-\varepsilon, \varepsilon]$,*

$$\lim_{N\to\infty} \frac{1}{N^2} \log \int_{\mathcal{U}(N)} e^{zN\,Tr(V(U_i,U_i^*,A_i^N,1\le i\le m))} dU_1 \cdots dU_m := F_{V,\tau}(z).$$

Moreover, $F_{V,\tau}(z)$ is an analytic function of $z \in \mathbb{C} \cap B(0,\varepsilon)$.

We will hopefully prove rapidly that the derivatives of $F_{V,\tau}(z)$ at $z = 0$ match with the coefficients $f_k(V,\tau)$ of the formal expansion. Our approach is based on non-commutative differential calculus and perturbation analysis as developped in the context of Gaussian matrices in [10, 11, 16].

The most important example of unitary matrix integral is the so-called spherical integral, studied by Harisch-Chandra and re-computed by Itzykson and Zuber

$$HCIZ(A,B) = \int_{U\in\mathbb{U}_n} e^{NTr(AUBU^*)} dU.$$

This integral is of fundamental importance in analytic Lie theory and was computed for the first time by Harish-Chandra in [13]. In the last two decades it has also become an issue to compute its large dimension asymptotics.

Theorem 0.1 holds true for the HCIZ integral as well. It thus relate the results of [5] who computed the formal limit of the HCIZ integral and those of [12] where the limit of $HCIZ(A,B)$ was computed (regardless of any small parameters assumptions) by using large deviations techniques. In fact, it implies that the free energy found in [12] is analytic in the vicinity of the origin. Let

$$I(\mu) = \frac{1}{2}\mu(x^2) + \frac{1}{2}\int\int \log|x-y| d\mu(x)d\mu(y).$$

If μ_A (resp. μ_B) denote the limiting spectral measure of A (resp. B), assume that $I(\mu_A)$ and $I(\mu_B)$ are finite. Then, the limit of $N^{-2}\log HCIZ(A,B)$ is given, according to [12], by

$$(2)\quad I(\mu_A,\mu_B) = -I(\mu_A) - I(\mu_B) - \frac{1}{2}\inf_{\rho,m}\{\int_0^1 \int \frac{m_t(x)^2}{\rho_t(x)} dxdt + \frac{\pi^2}{3}\int_0^1 \int \rho_t(x)^3 dxdt\}$$

where the inf is taken over m, ρ so that $\mu_t(dx) = \rho_t(x)dx \in \mathcal{P}(\mathbb{R})$ is a continuous process, $\mu_0 = \mu_A$, $\mu_1 = \mu_B$ and

$$\partial_t \rho_t(x) + \partial_x m_t(x) = 0.$$

The inf over (ρ_t, m_t) is taken (see [8]) at the solution of an Euler equation for isentropic flow with negative pressure $-\frac{\pi^2}{3}\rho^3$.

When μ_A and μ_B have a small compact support of width ρ, our result shows also that $I(\mu_A, \mu_B)$ expends analytically in ρ, a result which is not obvious from formula (2).

The convergence of other integrals was still unknown and it is one of the points of this paper to show their convergence. We use it to study Voiculescu's microstates entropy at laws which are small perturbations of the law of free variables (this generalizes section 4 in [10]).

The paper is organized as follows.

We first study the action of perturbations upon the integral $I_N(V, A_i^N)$ and deduce some properties of the related Gibbs measure; namely that the so-called empirical distribution of the matrices under this Gibbs measure satisfies asymptotically an equation called the Schwinger-Dyson equation. In a second section, we study this equation and in particular the uniqueness of the solution to this equation upon the assumptions that the parameters of V are small enough. This allows us in a third section to obtain the convergence of the integrals $I_N(V, A_i^N)$. Finally, we point out some consequence of our result about the free entropy.

1. Notations

We let m be a fixed integer number throughout this article.

(1) We denote by $(A_i^N)_{1\le i\le m}$ a m-uple of $N\times N$ Hermitian matrices. We shall assume that the sequence $(A_i^N)_{1\le i\le m}$ is uniformly bounded for the operator norm, and without loss of generality that they are bounded by one;
$$\|A_i^N\|_\infty = \lim_{p\to\infty}\left(\mathrm{Tr}((A_i^N)^{2p})\right)^{\frac{1}{2p}} \le 1$$

(2) $\mathcal{U}_N(\mathbb{C})$ denotes the set of unitary matrices, $\mathcal{M}_N$ the set of $N\times N$ matrices with complex entries, $\mathcal{H}_N$ the subset of Hermitian matrices of $\mathcal{M}_N$ and $\mathcal{A}_N$ the subset of antihermitian matrices of $\mathcal{M}_N$;
$$A^* = -A \quad \text{for } A\in\mathcal{A}_N.$$

(3) We denote $\mathbb{C}\langle (U_i, U_i^{-1}, A_i)_{1\le i\le m}\rangle$ the set of polynomial functions in the non-commutative indeterminates $(U_i, U_i^{-1}, A_i)_{1\le i\le m}$. $\mathbb{C}\langle (U_i, U_i^{-1}, A_i)_{1\le i\le m}\rangle$ is equipped with the involution $*$ so that $A_i^* = A_i$ and $U_i^* = U_i^{-1}$ and for any $X_1, \cdots, X_n \in (U_i, U_i^{-1}, A_i)_{1\le i\le m}$, any $z\in\mathbb{C}$,
$$(zX_1X_2\cdots X_{n-1}X_n)^* = \bar{z}X_n^*X_{n-1}^*\cdots X_2^*X_1^*.$$
Note that for any $U_i\in\mathcal{U}(N)$ and $A_i\in\mathcal{H}_N$, any $P\in\mathbb{C}\langle (U_i, U_i^{-1}, A_i)_{1\le i\le m}\rangle$,
$$\left(P(U_i, U_i^{-1}, A_i, 1\le i\le m)\right)^* = P^*(U_i, U_i^{-1}, A_i, 1\le i\le m)$$
where in the left hand side $*$ denotes the standard involution on $\mathcal{M}_N$. We denote $\mathbb{C}\langle (U_i, U_i^{-1}, A_i)_{1\le i\le m}\rangle_h$ the set of Hermitian polynomials; $P = P^*$, and $\mathbb{C}\langle (U_i, U_i^{-1}, A_i)_{1\le i\le m}\rangle_a$ the set of antihermitian polynomials ; $P^* = -P$.

In the sequel, except when something different is explicitly assumed, we shall assume that the potential V belongs to $\mathbb{C}\langle (U_i, U_i^{-1}, A_i)_{1\le i\le m}\rangle_h$, which insures that $\mathrm{Tr}\left(V(U_i, U_i^{-1}, A_i^N, 1\le i\le m)\right) \in \mathbb{R}$ for all $U_i \in \mathcal{U}(N)$ and $A_i^N \in \mathcal{H}_N$.

(4) $\mathbb{C}\langle (U_i, U_i^{-1}, A_i)_{1\le i\le m}\rangle$ is equipped with the non-commutative derivatives ∂_i, $1 \le i \le m$, given by

$$\partial_i A_j = 0, \quad \forall j, \partial_i U_j = 1_{i=j} U_j \otimes 1 \quad \partial_i U_j^{-1} = -1_{i=j} 1 \otimes U_j^{-1}$$

and satisfying the Leibnitz rule for $P, Q \in \mathbb{C}\langle (U_i, U_i^{-1}, A_i)_{1\le i\le m}\rangle$,

$$\partial_i(PQ) = \partial_i P \times 1 \otimes Q + P \otimes 1 \times 1 \partial_i Q. \tag{3}$$

Here, $\times$ denotes the product $P_1 \otimes Q_1 \times P_2 \otimes Q_2 = P_1P_2 \otimes Q_1Q_2$. We also let D_i be the corresponding *cyclic* derivatives such that if $m(A\otimes B) = BA$, $D_i = m \circ \partial_i$.

If q is a monomial in $\mathbb{C}\langle (U_i, U_i^{-1}, A_i)_{1\le i\le m}\rangle$, we more specifically have

$$\partial_i q = \sum_{q=q_1U_iq_2} q_1U_i \otimes q_2 - \sum_{q=q_1U_i^{-1}q_2} q_1 \otimes U_i^{-1} q_2 \tag{4}$$

$$D_i q = \sum_{q=q_1U_iq_2} q_2q_1U_i - \sum_{q=q_1U_i^{-1}q_2} U_i^{-1} q_2 q_1 \tag{5}$$

(5) $\mathcal{T}$ will denote the set of tracial states on the algebra generated by $(U_i, U_i^{-1}, A_i)_{1\le i\le m}$, that is the set of linear forms on $\mathbb{C}\langle (U_i, U_i^{-1}, A_i)_{1\le i\le m}\rangle$ such that

$$\mu(PP^*) \ge 0, \quad \mu(PQ) = \mu(QP), \quad \mu(1) = 1.$$

Throughout this article, we restrict ourselves to tracial states $\tau \in \mathcal{T}$ such that

$$\tau((A_i^N (A_i^N)^*)^n) \le 1 \quad \forall n \in \mathbb{N}, \forall i \in \{1, \cdots, m\}.$$

We denote $\mathcal{M}$ this subset of $\mathcal{T}$. Note that for any monomial $q \in \mathbb{C}\langle (U_i, U_i^{-1}, A_i)_{1\le i\le m}\rangle$, non-commutative Hölder's inequality implies that for $\tau \in \mathcal{M}$,

$$\tau(qq^*) \le 1. \tag{6}$$

We endow $\mathcal{M}$ with its weak topology; τ_n converges to τ iff for all $P \in \mathbb{C}\langle (U_i, U_i^{-1}, A_i)_{1\le i\le m}\rangle$,

$$\lim_{n\to\infty} \tau_n(P) = \tau(P).$$

By Banach-Alaoglu theorem and (6), $\mathcal{M}$ is a compact metric space.

$\mathcal{M}|_{(A_i)_{1\le i\le m}}$ will denote tracial states of $\mathcal{M}$ restricted to the algebra $(A_i)_{1\le i\le m}$.

We denote $\hat{\tau}^N_{A_i^N, U_i, 1\le i\le m}$ the empirical distribution of matrices $A_i^N \in \mathcal{H}_N$ and $U_i \in \mathcal{U}_N$ which is given by

$$\hat{\tau}^N_{A_i^N, U_i, 1\le i\le m}(P) = \frac{1}{N} \mathrm{Tr}\left(P(U_i, U_i^{-1}, A_i^N)_{1\le i\le m})\right)$$

for all $P \in \mathbb{C}\langle (U_i, U_i^{-1}, A_i)_{1\le i\le m}\rangle$. $\hat{\tau}^N_{A_i^N, U_i, 1\le i\le m}$ belongs to $\mathcal{M}$, whereas $\hat{\tau}^N_{A_i^N, 1\le i\le m}$ belongs to $\mathcal{M}|_{(A_i)_{1\le i\le m}}$ for any $A_i^N \in \mathcal{H}_N$ and $U_i \in \mathcal{U}_N$, $1 \le i \le m$. In particular, the limiting distribution τ given by (1) belongs to $\mathcal{M}|_{(A_i)_{1\le i\le m}}$.

2. Matrix models

Let μ_V^N be the distribution on $\mathcal{U}(N)$ given by

$$\mu_V^N(dU_1, \cdots, dU_m) = I_N(V, A_i^N)^{-1} \exp(N\mathrm{Tr}(V)) dU_1 \cdots dU_m.$$

Let $\hat{\tau}^N$ be the random tracial state (or the empirical distribution of $U_1, \cdots, U_m, A_1, \cdots, A_m$) defined for $P \in \mathbb{C}\langle (U_i, U_i^{-1}, A_i)_{1\le i\le m}\rangle$ by

$$\hat{\tau}^N(P) = \hat{\tau}^N_{A_i, U_i, 1\le i\le m} = \frac{1}{N}\mathrm{Tr}(P(A_i, U_i, U_i^{-1}, 1 \le i \le m)).$$

In this section, we investigate the behavior of $\hat{\tau}^N$ under μ_V^N when N goes to infinity. Note that $\hat{\tau}^N$ belongs to $\mathcal{M}$.

The main result of this section is the following

Theorem 2.1. *Assume that V is Hermitian. For all polynomial $P \in \mathbb{C}\langle (U_i, U_i^{-1}, A_i)_{1\le i\le m}\rangle$,*

$$\limsup_{N\to\infty} \left| \hat{\tau}^N \otimes \hat{\tau}^N(\partial_i P) + \hat{\tau}^N(D_i V P)\right| = 0 \quad a.s.$$

In particular, any limit point $\mu \in \mathcal{M}$ of $\hat{\tau}^N$ satisfies the Schwinger-Dyson equation

$$\mu \otimes \mu(\partial_i P) = -\mu(D_i V P) \quad \forall\, P \in \mathbb{C}\langle (U_i, U_i^{-1}, A_i)_{1\le i\le m}\rangle \tag{7}$$

and $\mu|_{(A_i)_{1\le i\le m}} = \tau$.

The idea of the proof, which is rather common in quantum field theory and was succesfully used in [10, 11, 16], is to obtain equations on $\hat{\tau}^N$ by performing an infinitesimal change of variables in $I_N(V, A_i^N)$. More precisely we make the change of variable $\mathbf{U} = (U_1, \cdots, U_m) \to \Psi(\mathbf{U}) = (\Psi_1(\mathbf{U}), \cdots, \Psi_m(\mathbf{U}))$ with

$$\Psi_j : \mathbf{U} \to U_j e^{\frac{\lambda}{N} P_j(\mathbf{U})}$$

where the P_j are antisymmetric polynomials (i.e. $P^* = -P$). This change of variable becomes very close to the identity when N goes to infinity, reason why it is called "infinitesimal".

Lemma 2.1. *Ψ is a local diffeomorphism and its Jacobian has the following developpement when N goes to infinity*

$$J_\Psi = e^{\frac{\lambda}{N}\sum_i Tr\otimes Tr(\partial_i P_i) + \frac{\lambda^2}{2N^2}\{\sum_i Tr\otimes Tr(\partial_i P_i P_i + P_i \partial_i P_i) - \sum_{ij} Tr\otimes Tr(\partial_i P_j \partial_j P_i))\} + O(\frac{1}{N})}$$

Proof. First we will use some elementary tools about the differentials of matrix functionnals:

(1) The map $exp : \mathcal{M}_N(\mathbb{C}) \longrightarrow \mathcal{M}_N(\mathbb{C})$ is differentiable and:

$$D_M exp.H = (\sum_{p=0}^{+\infty} \frac{(Ad_M)^k}{(k+1)!} H) e^M$$

where Ad_M is the operator defined by $Ad_M H = MH - HM$.

(2) If $P \in \mathbb{C}\langle (U_i, U_i^{-1}, A_i)_{1 \le i \le m} \rangle$ is considered as a function of the U_i's, then it is differentiable and:

$$D(U_i \to P(\mathbf{U})).H = \partial_i P \sharp H.$$

This is a justification of the definition of the non-commutative derivative. Fix A in $\mathcal{A}_N(\mathbb{C})$, $1 \leqslant i \leqslant m$ and

$$\tilde{\mathbf{U}} = (U_1, \cdots, U_{i-1}, U_i e^{sA}, U_{i+1}, \cdots, U_m).$$

Then

$$\Psi_j(\tilde{\mathbf{U}}) - \Psi_j(\mathbf{U}) = U_j(e^{sA} e^{\frac{\lambda}{N} P_j(\tilde{\mathbf{U}})} e^{-\frac{\lambda}{N} P_j(\mathbf{U})} - I) e^{\frac{\lambda}{N} P_j(\mathbf{U})}.$$

Thus, we can compute the differential of Ψ,

$$\begin{aligned} D(U_i \to \Psi(\mathbf{U})).A &= A + D_{\frac{\lambda}{N} P_j(U)} exp.(\frac{\lambda}{N} \partial_i P_j \sharp A) e^{-\frac{\lambda}{N} P_j(U)} \\ &= A + \frac{\lambda}{N} \sum_{k=0}^{+\infty} \frac{(Ad_{\frac{\lambda}{N} P_j(U)})^k}{(k+1)!} (\partial_i P_j \sharp A). \end{aligned}$$

We can then deduce that for sufficiently large N, Ψ is at least a local diffeomorphism and we can compute its Jacobian. Let us define $\tilde{\Psi}$ the linear map:

$$\tilde{\Psi}_{ji} A = \sum_{k=0}^{+\infty} \frac{(Ad_{\frac{\lambda}{N} P_j(\mathbf{U})})^k}{(k+1)!} (\partial_i P_j \sharp A)$$

$$\tilde{\Psi}(A_1, \cdots, A_m) = \sum_{ij} \tilde{\Psi}_{ji} A_i$$

The norm of $\frac{\lambda}{N}\tilde{\Psi}$ goes to 0 when N goes to infinity thus for large N,

$$\begin{aligned} J_\Psi = |\det(I + \frac{\lambda}{N}\tilde{\Psi})| &= \exp(\operatorname{Tr}\ln(I + \frac{\lambda}{N}\tilde{\Psi})) \\ &= \exp\left(\sum_{p \geq 1} \frac{(-\lambda)^p}{p N^p} \operatorname{Tr}(\tilde{\Psi}^p) \right) \end{aligned}$$

Note that since $\tilde{\Psi}$ is a bounded operator on $\mathcal{A}_N(\mathbb{C})$ which is a space of dimension N^2, the p-th term in the previous sum is at most of order N^{2-p}. We only look at the terms up to the order $O(1)$. A quick computation show that if

$$\varphi : \begin{array}{ccc} \mathcal{A}_N(\mathbb{C}) & \to & \mathcal{A}_N(\mathbb{C}) \\ X & \to & \sum_l A_l X B_l \end{array}$$

then considered as a real endomorphism, $\mathrm{Tr}\varphi = \sum_l \mathrm{Tr}A_l \mathrm{Tr}B_l$ (this can be checked by decomposing φ on the canonical base of $\mathcal{A}_N(\mathbb{C})$). This is sufficient to obtain the first terms of the Jacobian:

$$\begin{aligned}\frac{\lambda}{N}\mathrm{Tr}(\tilde{\Psi}) &= \frac{\lambda}{N}\sum_i \mathrm{Tr}(\tilde{\Psi}_{ii}) \\ &= \sum_i \frac{\lambda}{N}\mathrm{Tr}\otimes\mathrm{Tr}(\partial_i P_i) + \frac{\lambda^2}{2N^2}\mathrm{Tr}\otimes\mathrm{Tr}(P_i\partial_i P_i - \partial_i P_i P_i) + O(\frac{1}{N})\end{aligned}$$

and

$$\frac{\lambda^2}{N^2}\mathrm{Tr}(\tilde{\Psi}^2) = \frac{\lambda^2}{N^2}\sum_{ij}\mathrm{Tr}(\tilde{\Psi}_{ij}\tilde{\Psi}_{ji}) = \frac{\lambda^2}{N^2}\sum_{ij}\mathrm{Tr}\otimes\mathrm{Tr}(\partial_i P_j \partial_j P_i) + O(\frac{1}{N}).$$

□

Before trying to make the change of variable we need to know whether Ψ is really a bijection.

Lemma 2.2. *For N large enough, Ψ is a diffeomorphism of $\mathcal{U}_N(\mathbb{C})^m$.*

Proof. The only non-trivial property is the injectivity of Ψ. If $\Psi(U) = \Psi(V)$ then

$$U^*V - I = e^{\frac{\lambda}{N}P(U)}e^{-\frac{\lambda}{N}P(V)} - I$$

thus,

$$\|U - V\| = \|e^{\frac{\lambda}{N}P(U)} - e^{\frac{\lambda}{N}P(V)}\|$$

and the results follows since exp is uniformly lipschitz on $\mathcal{U}_N(\mathbb{C})$. □

Proof of Theorem 2.1. We now perform the change of variables $\mathbf{U} \to \Psi(\mathbf{U})$ in $I_N(V, A_i^N)$;

$$\begin{aligned} I_N(V, A_i^N) &= \int e^{N\left(\mathrm{Tr}(V(\Psi(\mathbf{U})) - \mathrm{Tr}(V)\right)} J_\Psi(\mathbf{U}) e^{N\mathrm{Tr}(V)} dU_1 \cdots dU_m \\ &= \int e^{NY^N(P) + C_N(P) + o(1)} e^{N\mathrm{Tr}(V)} dU_1 \cdots dU_m \end{aligned}$$

where in the density we have expanded $\mathrm{Tr}V(\Psi(\mathbf{U}))$ as

$$\mathrm{Tr}(V(\Psi(\mathbf{U})) - \mathrm{Tr}(V) = \frac{\lambda}{N}\sum_i \mathrm{Tr}(D_i V P_i) + \frac{\lambda^2}{2N^2}(\partial^2_{ij} V.(P_i, P_j)) + O(\frac{1}{N^2})$$

and used Lemma 2.1 to get the formulae

$$Y^N(P) = \sum_i \frac{1}{N}\mathrm{Tr}(D_i V P_i) + (\frac{1}{N}\mathrm{Tr}) \otimes (\frac{1}{N}\mathrm{Tr})(\partial_i P_i)$$

and

$$C_N(P) = \sum_{ij} \frac{1}{N} \mathrm{Tr}(\partial^2_{ij} V.(P_i, P_j)) - \frac{1}{N^2} \mathrm{Tr} \otimes \mathrm{Tr}(\partial_i P_j \partial_j P_i)$$
$$+ \frac{1}{N^2} \sum_i \mathrm{Tr} \otimes \mathrm{Tr}(\partial_i P_i P_i + P_i \partial_i P_i).$$

Note that $C_N(P)$ is uniformly bounded and so we have proved that

$$\int e^{NY^N(P)} d\mu_V^N(d\mathbf{U}) = O(1).$$

Borel-Cantelli's lemma thus insures that

$$\limsup_{N\to\infty} Y^N(P) \leq 0 \quad a.s.$$

and the converse inequality holds by changing P into $-P$. This proves the first statment of Theorem 2.1. The last result is simply based on the compactness of $\mathcal{M}$ and the fact that any limit point must then satisfy the same asymptotic equations than $\hat{\tau}^N$.

Another consequence of this convergence is simply the existence of solution to (7) for any Hermitian potential V (since any limit point of $\hat{\tau}^N$ in the compact metric space $\mathcal{M}$ will satisfy it). Moreover, since these solutions are limit points of $\hat{\tau}^N$, they belong to $\mathcal{M}$ and in particular $|\mu(q)| \leq 1$ for any monomial q. □

3. Study of Schwinger-Dyson's equation

Let $V \in \mathbb{C}\langle (U_i, U_i^{-1}, A_i)_{1\leq i\leq m}\rangle_h$. V will be of the form

$$V = \sum_{i=1}^n t_i q_i(U_j, U_j^{-1}, A_j, 1 \leq j \leq m)$$

with monomial functions q_i and complex numbers t_i. We let D be the maximal degree of the monomials q_i.

The goal of this section is to prove that, if the t_i's are small enough, (7) admits a unique solution, see Theorem 3.1. From this and Theorem 2.1 we deduce the following

Corollary 3.1. *Assume that V is Hermitian. Let $D \in \mathbb{N}$ and $\tau \in \mathcal{M}|_{(A_i)_{1\leq i\leq m}}$ be given. There exists $\varepsilon = \varepsilon(D, m) > 0$ such that if $|t_i| \leq \varepsilon$, $\hat{\tau}^N$ converges almost surely to the unique solution of (7). $\tau_V^N = \mu_V^N(\hat{\tau}^N)$ converges as well to this solution as N goes to infinity.*

This result is obvious since Theorems 2.1 and 3.1 shows that $\hat{\tau}^N$ has a unique limit point, and thus converges almost surely. The convergence of τ_V^N is then a direct consequence of bounded convergence theorem since $\hat{\tau}^N \in \mathcal{M}$.

3.1. **SD-equation - definition and properties.**

Definition 3.1. *Let $\tau \in \mathcal{M}|_{(A_i)_{1\le i\le m}}$. A tracial state $\mu \in \mathcal{M}$ is said to satisfy Schwynger-Dyson equation* $\mathbf{SD[V,}\tau\mathbf{]}$ *iff for all $P \in \mathbb{C}\langle (A_i)_{1\le i\le m}\rangle$,*

$$\mu(P) = \tau(P)$$

and for all $P \in \mathbb{C}\langle (U_i, U_i^{-1}, A_i)_{1\le i\le m}\rangle$, all $i \in \{1, \cdots, m\}$,

$$\mu \otimes \mu(\partial_i P) = -\mu(D_i V\, P).$$

Here we prove that τ is uniquely defined provided V is small enough.

Theorem 3.1. *Let $D \in \mathbb{N}$ and $\tau \in \mathcal{M}|_{(A_i)_{1\le i\le m}}$ be given. There exists $\varepsilon = \varepsilon(D, m) > 0$ such that if $|t_i| \le \varepsilon$, there exists at most one solution μ to* $\mathbf{SD[V,}\tau\mathbf{]}$.

Proof. Let μ be a solution to $\mathbf{SD[V,}\tau\mathbf{]}$. Note that if we take q a monomial in $\mathbb{C}\langle (U_i, U_i^{-1}, A_i)_{1\le i\le m}\rangle$, either q does not depend on $U_j, U_j^{-1}, 1 \le j \le m$ and then $\mu(q) = \tau(q)$ is uniquely defined, or q can be written as $q = q_1 U_i^n q_2$ for some i and $n \in \{-1, +1\}$. Then, by the traciality assumption, $\mu(q) = \mu(q_2 q_1 U_i^n) = \mu(U_i^n q')$ with $q' = q_2 q_1$. Remark that we can assume without loss of generality that the last letter of q' is not U_i^{-n}. We next use $\mathbf{SD[V,}\tau\mathbf{]}$ to compute $\mu(U_i^n q)$ for some monomial q. We assume first $n = -1$. Then, by (3),

$$\partial_i \left(U_i^{-1} q\right) = -1 \otimes (U_i^{-1} q) + U_i^{-1} \otimes 1 \times \partial_i q.$$

Taking the expectation, we thus find that by (4) that

$$\begin{aligned} \mu(U_i^{-1} q) &= \mu \otimes \mu(U_i^{-1} \otimes 1 \partial_i q) + \mu(D_i V q) \\ &= \sum_{q = q_1 U_i q_2} \mu(U_i^{-1} q_1 U_i)\mu(q_2) - \sum_{q = q_1 U_i^{-1} q_2} \mu(U_i^{-1} q_1)\mu(U_i q_2) \\ &\quad + \sum_j t_{ij} \mu(q_{ij} q) \end{aligned} \tag{8}$$

where $D_i V = \sum_j t_{ij} q_{ij}$. Note that the sum runs at most on Dn terms and that all the t_{ij} are bounded by $\max |t_i|$. A similar formula is found when $n = +1$ by differentiating $q U_i$.

We next show that (8) characterizes uniquely $\mu \in \mathcal{M}$ when the t_{ij} are small enough. It will be crucial here that $\mu(q)$ is bounded independently of the t_i's (here by the constant 1).

Now, let $\mu, \mu' \in \mathcal{M}$ be two solutions to $\mathbf{SD[V,}\tau\mathbf{]}$ and set

$$\Delta(l) = \sup_{\deg(q) \le l} |\mu(q) - \mu'(q)|$$

where the supremum holds over monomials of $\mathbb{C}\langle (U_i, U_i^{-1}, A_i)_{1\le i\le m}\rangle$ with total degree in the U_j and U_j^{-1} less than l. Namely, if the monomial (or word) q contains

U_j n_j^+ times and U_j^{-1} n_j^- times, we assume $\sum_{j=1}^m (n_j^+ + n_j^-) \leq l$. Note that

$$\Delta(\ell) = \max_{\substack{1\leq i\leq m\\ n\in\{+1,-1\}}} \sup_{\deg q\leq \ell-1} |\mu(U_i^n q) - \mu'(U_i^n q)| \tag{9}$$

and that by (8), we find that, for q with degree less than $\ell - 1$,

$$\begin{aligned}
|\mu(U_i q) - \mu'(U_i q)| &\leq \sum_{q=q_1 U_i q_2} |(\mu-\mu')(q_1)| + \sum_{q=q_1 U_i q_2} |(\mu-\mu')(q_2)| \\
&\quad + \sum_{q=q_1 U_i^{-1} q_2} |(\mu-\mu')(U_i^{-1} q_1)| + \sum_{q=q_1 U_i^{-1} q_2} |(\mu-\mu')(U_i q_2)| \\
&\quad + \sum_j t_{ij} |(\mu-\mu')(q_{ij} q)| \\
&\leq 2\sum_{p=1}^{\ell-2} \Delta(p) + 2\sum_{p=1}^{\ell-1} \Delta(p) + nD\varepsilon\Delta(\ell+D-1)
\end{aligned}$$

where we used that $\deg(q_1) \in \{0, \cdots, \ell-2\}$, $\deg(q_2) \in \{0, \cdots, \ell-2\}$ (but $\Delta(0) = 0$) and $\deg(q_{ij}) \leq D$ and assumed $|t_i| \leq \varepsilon$. Hence, we have proved that

$$\Delta(\ell) \leq 4\sum_{p=1}^{\ell-1} \Delta(p) + nD\varepsilon\Delta(\ell+D).$$

Multiplying these inequalities by γ^ℓ we get, since $\sum_{\ell\geq 1} \gamma^\ell \Delta(\ell) < \infty$ for $\gamma < 1$,

$$H(\gamma) \leq \frac{\gamma}{1-\gamma} H(\gamma) + \frac{nD\varepsilon}{\gamma^D} H(\gamma)$$

resulting with $H(\gamma) = 0$ for γ so that $1 > \frac{\gamma}{1-\gamma} + \frac{nD\varepsilon}{\gamma^D}$. Such a $\gamma > 0$ exists when ε is small enough. This proves the uniqueness.

□

As a corollary, we characterize asymptotic freeness by a Schwinger-Dyson equation, a result which was already obtained in [20], Proposition 5.17.

Corollary 3.2. *A tracial state μ satisfy* $\mathbf{SD[0,}\tau]$ *if and only if , under μ, the algebra generated by $\{A_i, 1 \leq i \leq m\}$ and $\{U_i, U_i^{-1}, 1 \leq i \leq m\}$ are free and the U_i's are two by two free and satisfy*

$$\mu(U_i^n) = 0 \quad \forall n \in \mathbb{Z}\backslash\{0\}.$$

Proof. By the previous theorem, it is enough to verify that the law μ of free variable (A_i, U_i, U_i^{-1}) verifies $\mathbf{SD[0,}\tau]$. So take $P = U_{i_1}^{n_1} B_1 \cdots U_{i_p}^{n_p} B_p$ with some B_k's in the algebra generated by $(A_i, 1 \leq i \leq m)$. We wish to show that for all $i \in \{1, \cdots, m\}$,

$$\mu \otimes \mu(\partial_i P) = 0.$$

Note that by linearity, it is enough to prove this equality when $\mu(B_j) = 0$ for all j. Now, by definition

$$\partial_i P = \sum_{k: i_k = i, n_k > 0} \sum_{l=1}^{n_k} U_{i_1}^{n_1} B_1 \cdots B_{k-1} U_i^l \otimes U_i^{n_k - l} B_k \cdots U_{i_p}^{n_p} B_p$$

$$- \sum_{k: i_k = i, n_k < 0} \sum_{l=0}^{n_k - 1} U_{i_1}^{n_1} B_1 \cdots B_{k-1} U_i^{-l} \otimes U_i^{n_k + l} B_k \cdots U_{i_p}^{n_p} B_p$$

Taking the expectation on both sides, since $\mu(U_j^i) = 0$ and $\mu(B_j) = 0$ for all $i \neq 0$ and j, we see that freeness implies that the right hand side is null (recall here that in the definition of freeness, two consecutive elements have to be in free algebras but the first and the last element can be in the same algebra). Thus, $\mu \otimes \mu(\partial_i P) = 0$ which proves the claim. □

We next show that the solution μ of $\mathbf{SD}[\mathbf{V_t}, \tau]$ depends analytically on the parameters $(t_i)_{1 \leq i \leq n}$.

Theorem 3.2. *There exists $\epsilon > 0$ such that for $\mathbf{t} \in \mathbb{C}^n, \max_{1 \leq i \leq n} |t_i| \leq \varepsilon$, $\mathbf{SD}[\mathbf{V_t}, \tau]$ has a solution $\mu_{\mathbf{t}}$. Moreover, for all polynomials P, $\mathbf{t} \longrightarrow \mu_{\mathbf{t}}(P)$ is analytic. In other words, there exists a family $\mu^{\mathbf{k}}, \mathbf{k} = (k_1, \cdots, k_n) \in \mathbb{N}^n$ in $\mathbb{C}\langle (U_i, U_i^{-1}, A_i)_{1 \leq i \leq m} \rangle^*$ such that*

$$\mu_{\mathbf{t}}(P) = \sum_{\mathbf{k} \in \mathbb{N}^n} \prod_{i=1}^n \frac{t_i^{k_i}}{k_i!} \mu^{\mathbf{k}}(P)$$

converges absolutely for $\max_{1 \leq i \leq n} |t_i| \leq \varepsilon$.

Proof. The idea is to define inductively the $\mu^{\mathbf{k}}$, based on the intuition of which relation they should satisfy if they were the derivatives of the solution μ to $\mathbf{SD}[\mathbf{V_t}, \tau]$. To make sure that the $\mu^{\mathbf{k}}$ exists, we shall only construct them so that they satisfy a subset of the Schwinger Dyson's equation. Moreover, we need to check that the coefficients $\mu^{\mathbf{k}}(P)$ are well bounded to define an absolutely convergent series. Letting then $\nu_{\mathbf{t}}$ be this series, it is uniquely characterized by this set of equations (which is the subset of the Schwinger-Dyson's equations that we used to prove uniqueness). In particular, for t_i's small enough and such $V = \sum t_i q_i$ is Hermitian, ν has to be equal to the solution of the whole set of Schwinger-Dyson's equations coming from matrix models (see Corollary 3.1). We then conclude that $\nu_{\mathbf{t}}$ satisfies the full set of Schwinger-Dyson's equations in the domain where it depends analytically on $\mathbf{t}$.

If $\mathbf{k} = (k_1, \cdots, k_n)$, $\mathbf{k}' = (k'_1, \cdots, k'_n)$, let us denote

$$\mathbf{k}! = \prod_i k_i!, \binom{\mathbf{k}}{\mathbf{k}'} = \prod_i \binom{k_i}{k'_i}, A^{\mathbf{k}} = A^{\sum_i k_i}$$

(1) If P is in $\mathbb{C}\langle (A_i)_{1 \leq i \leq m} \rangle$, set $\mu^{\mathbf{k}}(P) = 1_{\mathbf{k}=0} \tau(P)$.
(2) If $P = RU_i S$ with S in $\mathbb{C}\langle (A_i)_{1 \leq i \leq m} \rangle$, $\mu^{\mathbf{k}}(P) = \mu^{\mathbf{k}}(SRU_i)$.
(3) If $P = RU_i^* S$ with R in $\mathbb{C}\langle (A_i)_{1 \leq i \leq m} \rangle$ and S does not contain any U_j (but may contain the U_j^*), $\mu^{\mathbf{k}}(P) = \mu^{\mathbf{k}}(U_i^* SR)$.

(4) If q does not contain any U_j, we define

$$\mu^{\mathbf{k}}(U_i^* q) = -\sum_{\substack{q=q_1 U_i^* q_2\\ \mathbf{k}'+\mathbf{k}''=\mathbf{k}}} \binom{\mathbf{k}'}{\mathbf{k}''}\mu^{\mathbf{k}'}(U_i^* q_1)\mu^{\mathbf{k}''}(U_i^* q_2) + \sum_j \mu^{\mathbf{k}-1_j}(D_i q_j q)$$

which corresponds to the equation for the solution (when exists) of $\mathbf{SD}[\mathbf{V_t},\tau]$

$$\mu(U_i^* q) = -\sum_{q=q_1 U_i^* q_2} \mu(U_i^* q_1)\mu(U_i^* q_2) + \sum_j t_i \mu(q_i q)$$

(5) Finally, we set

$$\begin{aligned}\mu^{\mathbf{k}}(qU_i) = &-\sum_{\substack{q=q_1 U_i q_2\\ \mathbf{k}'+\mathbf{k}''=\mathbf{k}}} \binom{\mathbf{k}'}{\mathbf{k}''}\mu^{\mathbf{k}'}(q_1 U_i)\mu^{\mathbf{k}''}(q_2 U_i)\\ &+\sum_{\substack{q=q_1 U_i^* q_2\\ \mathbf{k}'+\mathbf{k}''=\mathbf{k}}} \binom{\mathbf{k}'}{\mathbf{k}''}\mu^{\mathbf{k}'}(q_1)\mu^{\mathbf{k}''}(q_2) + \sum_{k_j\neq 0} k_j \mu^{\mathbf{k}-1_j}(D_i q_j q)\end{aligned}$$

which is a differential way to write

$$\mu(qU_i) = -\sum_{q=q_1 U_i q_2} \mu(q_1 U_i)\mu(q_2 U_i) + \sum_{q=q_1 U_i^* q_2} \mu(q_1)\mu(q_2) + \sum_j t_i \mu(q_i q)$$

One can check that this defines uniquely the $\mu^{\mathbf{k}}$. Note also that it does not imply a priori that they are tracial. We must now check that the $\mu^{\mathbf{k}}$ do not grow too fast, otherwise they would not define a convergent series. To find a bound we will use the Catalan's numbers:

$$C_0 = 1, C_{k+1} = \sum_{0\leqslant p\leqslant k} C_p C_{k-p}$$

and we will use the fact that they do not explode too fast: $C_{k+1} \leqslant 4C_k$. We will use the notation $C_{\mathbf{k}} = \prod_i C_{k_i}$. Let us introduce another closely related sequence, for a well chosen A we define $D_0 = 1$ and for $k \geqslant 1$, $D_k = A^{k-1}C_{k-1}$. The two key properties of this sequence is first that it is sub-geometric ($D_k \leqslant (4A)^k$) and secondly it satisfies:

$$D_k = A \sum_{0<p<k} D_p D_{k-p}.$$

Now our induction hypothesis will be that there exists $A, B > 0$ such that for all $\mathbf{k}$, for all monomial P of degree p,

$$\frac{|\mu^{\mathbf{k}}(P)|}{\mathbf{k}!} \leqslant C_{\mathbf{k}} B^{\mathbf{k}} D_p. \tag{10}$$

We will prove this bound by induction using the definition of the $\mu^{\mathbf{k}}$. We will only show how it works for a polynomial of the form qU_i since it is the most complicated case.

$$\begin{aligned}\frac{|\mu^{\mathbf{k}}(qU_i)|}{\mathbf{k}!} \leqslant & \sum_{\substack{q=q_1U_iq_2\\ \mathbf{k}'+\mathbf{k}''=\mathbf{k}}} \frac{|\mu^{\mathbf{k}'}(q_1U_i)|}{\mathbf{k}'!}\frac{|\mu^{\mathbf{k}''}(q_2U_i)|}{\mathbf{k}''!}\\ &+ \sum_{\substack{q=q_1U_i^*q_2\\ \mathbf{k}'+\mathbf{k}''=\mathbf{k}}} \frac{|\mu^{\mathbf{k}'}(q_1)|}{\mathbf{k}'!}\frac{|\mu^{\mathbf{k}''}(q_2)|}{\mathbf{k}''!} + \sum_{k_j\neq 0}\frac{|\mu^{\mathbf{k}-1_j}(D_iq_jq)|}{(\mathbf{k}-1_j)!}\end{aligned}$$

Now we use the induction hypothesis. Let $r = r_1 + \cdots + r_m$ with r_m the number of monomials of D_jV and let D be the degree of V, If q is of degree $p-1$,

$$\begin{aligned}\frac{|\mu^{\mathbf{k}}(qU_i)|}{\mathbf{k}!C_{\mathbf{k}}B^{\mathbf{k}}D_p} &\leqslant 2\sum_{\substack{0<q<p\\ \mathbf{k}'+\mathbf{k}''=\mathbf{k}}}\frac{C_{\mathbf{k}'}B^{\mathbf{k}'}D_qC_{\mathbf{k}''}B^{\mathbf{k}''}D_{p-q}}{C_{\mathbf{k}}B^{\mathbf{k}}D_p}\\ &\quad + r\frac{C_{\mathbf{k}-1_j}B^{\mathbf{k}-1}D_{p+D}}{C_{\mathbf{k}}B^{\mathbf{k}}D_p}\\ &\leqslant 2\prod_i\frac{C_{k_i+1}}{C_{k_i}}\frac{1}{A} + r\frac{(4A)^D}{B}.\end{aligned}$$

The point is that we can choose $A, B > 0$ such that this last quantity is lesser than 1. For example take $A > 4^{n+1}$ and then $B > 2r(4A)^D$.

Thus, for $\|t\| := \max_i |t_i| < 1/4B$, we can define a linear form on $\mathbb{C}\langle (U_i, U_i^{-1}, A_i)_{1\leq i\leq m}\rangle$ by

$$\nu_{\mathbf{t}}(P) = \sum_{\mathbf{k}}\prod_i \frac{t_i^{k_i}}{k_i!}\mu^{\mathbf{k}}(P)$$

as an absolutely convergent serie. Moreover, for a monomial q with degree p, (10) insures that

$$|\nu_{\mathbf{t}}(q)| \leq [1 - 4B\|t\|]^{-n}D_p \leq [1-4B\|t\|]^{-n}(4A)^p.$$

Besides by definition of the $\mu^{\mathbf{k}}$, for all $\mathbf{t}$ inside the radius of convergence of this serie, $\nu_{\mathbf{t}}$ satisfies:

(1) If P is in $\mathbb{C}\langle (A_i)_{1\leq i\leq m}\rangle$, $\nu_{\mathbf{t}}(P) = \tau(P)$,
(2) If $P = RU_iS$ with S in $\mathbb{C}\langle (A_i)_{1\leq i\leq m}\rangle$, $\nu_{\mathbf{t}}(P) = \nu_{\mathbf{t}}(SRU_i)$,
(3) If $P = RU_i^*S$ with R in $\mathbb{C}\langle (A_i)_{1\leq i\leq m}\rangle$ and S does not contain any U_j (but may contain the U_j^*), $\nu_{\mathbf{t}}(P) = \nu_{\mathbf{t}}(U_i^*SR)$,
(4) If q does not contain any U_j,

$$\nu_{\mathbf{t}}(U_i^*q) = -\sum_{q=q_1U_i^*q_2}\nu_{\mathbf{t}}(U_i^*q_1)\nu_{\mathbf{t}}(U_i^*q_2) + \sum_j t_i\nu_{\mathbf{t}}(q_iq).$$

(5) And for all q,

$$\nu_{\mathbf{t}}(qU_i) = -\sum_{q=q_1U_iq_2}\nu_{\mathbf{t}}(q_1U_i)\nu_{\mathbf{t}}(q_2U_i) + \sum_{q=q_1U_i^*q_2}\nu_{\mathbf{t}}(q_1)\nu_{\mathbf{t}}(q_2) + \sum_j t_i\nu_{\mathbf{t}}(q_iq).$$

Coming back to the proof of Theorem 3.1, one sees that these equalities and the above control are sufficient to prescribe uniquely $\nu_{\mathbf{t}}$ provided $|\mathbf{t}| \le \varepsilon$ for some $\varepsilon > 0$ small enough.

Up to adding some null parameters, we can always assume that $\mathbf{t} = (\mathbf{r}, \mathbf{s})$ with V of the form $V = \sum_{i=1}^{n}(t_i q_i + s_i q_i^*)$.

Let us consider the case where $s_i = \bar{t}_i$. Then, V is Hermitian and therefore we know that there exists a solution $\mu_{\mathbf{t}}$ to the Schwinger-Dyson's equation (see Theorem 2.1). In particular, it satisfies the same equations than $\nu_{\mathbf{t}}$ and therefore, for $|\mathbf{t}| \le \varepsilon$, we must have $\nu_{\mathbf{t}} = \mu_{\mathbf{t}}$. As a consequence, $\nu_{\mathbf{t}}$ satisfies the whole set of Schwinger-Dyson's equations.

Recall that an analytic function of two variables $x, y \in \mathbb{C}$ which vanishes on the set $\Lambda = \{x = \bar{y}, x \in \mathbb{C}\}$ is null everywhere (since Λ is totally real). For all P, $f_P(\mathbf{r}, \mathbf{s}) = \nu_{\mathbf{r},\mathbf{s}} \otimes \nu_{\mathbf{r},\mathbf{s}}(\partial_i P) + \nu_{\mathbf{r},\mathbf{s}}(D_i V_{\mathbf{t}} P)$ is an analytic function of $(\mathbf{r}, \mathbf{s})$ in the ball $B(0, \varepsilon)$ which vanishes on $\mathbf{r} = \bar{\mathbf{s}}$ according to the previous paragraph. Hence, f_P is null on $B(0, \varepsilon)$. We thus can conclude that $\nu_{\mathbf{t}}$ satisfies the full set of Schwinger-Dyson's equations. The same arguments applies to show that $\nu_{\mathbf{t}}$ is tracial.

□

4. Models invariant under the action of the unitary group.

It is a well known fact that if we are given a familly of free variables, $X_1, \cdots, X_n$, any non commutative moments of this family can be expressed as function of the moments of the X_i's. In this section we try to generalize this fact to much more general potentials for which the distribution of the matrices is invariant under the action of the unitary group. We shall in fact specify the law $\mu_{\mathbf{t}}$ of theorem 3.2 in the case where $V(U_i, U_i^{-1}, A_i, 1 \le i \le m) = V(U_i^* A_i U_i, 1 \le i \le m)$.

We look at the special case where we have m self-adjoint matrices A_i, m self-adjoint matrices U_i distributed according to the Haar measure and we only look at polynomials which depends only on the $X_i = U_i^* A_i U_i$. We thus study m matrices with fixed spectral measure and in generic positions. Note that this case encompasses the Harich-Chandra-Itzykson-Zuber integral. We define directly our derivatives on the X_i's:

$$\partial_i X_j = 1_{i=j}(X_i \otimes 1 - 1 \otimes X_i).$$

Now, let V be in $\mathbb{C}\langle (X_i)_{1 \le i \le m}\rangle_h$, and let μ be the solution of $SD[V, \tau]$ given in Corollary 3.1. Note that since $\mathbb{C}\langle (X_i)_{1 \le i \le m}\rangle$ is stable under the non-commutative derivatives, we can look at μ as an element of $\mathbb{C}\langle (X_i)_{1 \le i \le m}\rangle^*$ and the Property of regularity and uniqueness still hold. Near the origin, by Theorem 3.2 we have the expansion of the solution:

$$\mu = \sum_{\mathbf{k}} \frac{\mathbf{t}^{\mathbf{k}}}{\mathbf{k}!} \mu^{\mathbf{k}}$$

where $\frac{\mathbf{t}^{\mathbf{k}}}{\mathbf{k}!}$ denotes $\prod_i \frac{t_i^{k_i}}{k_i!}$. Now we define the space $\mathbb{C}\langle (A_i)_{1\leq i\leq m}\rangle_{sym}$ as the $\mathbb{C}$ vectorial space generated by the primitive elements

$$A_1^{\lambda_1^1} \odot \cdots \odot A_1^{\lambda_{i_1}^1} \odot \cdots \odot A_m^{\lambda_1^m} \odot \cdots \odot A_m^{\lambda_{i_m}^m}$$

where $\odot$ is the symmetric tensor product. The degree of an element is the maximal degree of one of its primitive elements and the degree of the primitive element $A_1^{\lambda_1^1} \odot \cdots \odot A_1^{\lambda_{i_1}^1} \odot \cdots \odot A_m^{\lambda_1^m} \odot \cdots \odot A_m^{\lambda_{i_m}^m}$ is $\sum_{k,1\leqslant j\leqslant i_k} \lambda_i^k$. We will denote

$$\tau_{sym}(A_1^{\lambda_1^1} \odot \cdots \odot A_1^{\lambda_{i_1}^1} \odot \cdots \odot A_m^{\lambda_1^m} \odot \cdots \odot A_m^{\lambda_{i_m}^m}) = \prod_{k,1\leqslant j\leqslant i_k} \tau(A_k^{\lambda_j^k}).$$

And we extend τ_{sym} on $\mathbb{C}\langle (A_i)_{1\leq i\leq m}\rangle_{sym}$ by linearity. Thus $\mathbb{C}\langle (A_i)_{1\leq i\leq m}\rangle_{sym}$ can be seen as product of commutative moments.

We prove in this section that for any P in $\mathbb{C}\langle (X_i)_{1\leq i\leq m}\rangle$, $\mu^{\mathbf{k}}(P)$ can be expressed with universal coefficients (depending only on V) as a finite linear combination of the $\tau_{sym}(Q)$ for Q in $\mathbb{C}\langle (A_i)_{1\leq i\leq m}\rangle_{sym}$. Our aim is to control this decomposition. If P can be decomposed into $P = \sum_q \gamma_P(q)q$ in the basis of monomials of $\mathbb{C}\langle (A_i)_{1\leq i\leq m}\rangle_{sym}$, we let

$$\|P\|_M := \sum_q |\gamma_P(q)| M^{\deg(q)}.$$

Proposition 4.1. *For all* $\mathbf{k}$*, all polynomial* P*, there exists a unique* $P_{\mathbf{k}} \in \mathbb{C}\langle (A_i)_{1\leq i\leq m}\rangle_{sym}$ *independent of the distribution of the* A_i*'s such that*

$$\frac{\mu^{\mathbf{k}}(P)}{\mathbf{k}!} = \tau_{sym}(P_{\mathbf{k}}).$$

Besides, $\deg P_{\mathbf{k}} \leqslant \deg P + |\mathbf{k}| \deg V$ *with* $|\mathbf{k}| = \sum_i k_i$ *and there exists* $A, M_0 > 0$ *such that for all* $\mathbf{k}$*,* $M > M_0$*,*

$$\|P_{\mathbf{k}}\|_M \leqslant A^{|\mathbf{k}|} \|P\|_M.$$

In particular, this gives an explicit bound on the coefficient of $P_{\mathbf{k}}$, for example if P is a monomial of degree p then the coefficent of a primitive element of degree $\deg P + |\mathbf{k}| \deg V - d$ in the decomposition of $P_{\mathbf{k}}$ has an absolute value bounded by $A^{|\mathbf{k}|} M^d$. Moreover this statement could certainly become more precise since we can certainly control the constants by $|\mathbf{t}|$.

Proof. We prove the Proposition by induction. For P of degree 0 and $\mathbf{k} = 0$, there is nothing to prove. Take now $(P, \mathbf{k})$ and suppose that the Proposition has already been proved for any $(Q, \mathbf{k}')$ with $\mathbf{k}' < \mathbf{k}$ (i.e. for all i, $k_i' \leqslant k_i$ and the inequality is strict in at least one case) and for any $(Q, \mathbf{k})$ with $\deg Q < \deg P$. We can suppose without loss of generality that P is a monomial. Besides we will also suppose that $P = QU_i$ and Q does not begin with U_i^*. It is not always possible to assume this but the other cases are similar.

$$\frac{\tau^{\mathbf{k}}(QU_i)}{\mathbf{k}!} = -\sum_{\substack{Q=Q_1U_iQ_2\\ \mathbf{k}'+\mathbf{k}''=\mathbf{k}}} \frac{\tau^{\mathbf{k}'}(Q_1U_i)}{\mathbf{k}'!}\frac{\tau^{\mathbf{k}''}(Q_2U_i)}{\mathbf{k}''!}$$

$$+\sum_{\substack{Q=Q_1U_i^*Q_2\\ \mathbf{k}'+\mathbf{k}''=\mathbf{k}}} \frac{\tau^{\mathbf{k}'}(Q_1)}{\mathbf{k}'!}\frac{\tau^{\mathbf{k}''}(Q_2)}{\mathbf{k}''!} - \sum_{k_j\neq 0}\frac{\tau^{\mathbf{k}-1_j}(D_iq_jQ)}{(\mathbf{k}-1_j)!}$$

This suggests to take

$$P_{\mathbf{k}} = \sum_{\substack{Q=Q_1U_iQ_2\\ \mathbf{k}'+\mathbf{k}''=\mathbf{k}}} (Q_1U_i)_{\mathbf{k}'}.(Q_2U_i)_{\mathbf{k}''} + \sum_{\substack{Q=Q_1U_i^*Q_2\\ \mathbf{k}'+\mathbf{k}''=\mathbf{k}}} (Q_1)_{\mathbf{k}'} \odot (Q_2)_{\mathbf{k}''} - \sum_{k_j\neq 0}(D_iq_jQ)_{\mathbf{k}-1_j}.$$

By the induction property this defines a sequence of elements $P_{\mathbf{k}}$ which satisfy the property

$$\frac{\tau^{\mathbf{k}}(P)}{\mathbf{k}!} = \tau(P_{\mathbf{k}}).$$

The bound on the degree is also immediate by induction since on the right hand side either the degree decrease or $\mathbf{k}$ decrease and we multiply by a monomial of V. Thus the only non trivial part is to prove the continuity. The induction hypothesis is that for a monomial of degree p,

$$\|P_{\mathbf{k}}\| \leqslant A^{|\mathbf{k}|}\prod_i C_{k_i}D_p.$$

Then we can conclude by induction exactly as in the proof of Theorem 3.2 when we have proved that the growth of the $\mu^{\mathbf{k}}$ was not too fast (since in fact the decomposition relation are nearly the same). □

5. Application to the asymptotics of $I_N(V, A_i^N)$

Let $(q_1, \cdots, q_n)$ be fixed monomials in $\mathbb{C}\langle (U_i, U_i^{-1}, A_i)_{1\le i\le m}\rangle$. Let $T(q_1,\cdots,q_n)$ be set of $\mathbf{t}\in\mathbb{C}^n$ so that $V_{\mathbf{t}} := \sum_{i=1}^n t_iq_i$ is Hermitian.

Theorem 5.1. *There exists $\eta = \eta(q_1,\cdots,q_n)$ so that for any $\mathbf{t}\in\mathbb{C}^n\cap B(0,\eta)$ we have*

$$\lim_{N\to\infty}\frac{1}{N^2}\log I_N(V_{\mathbf{t}}, A_i^N) = \sum_{\mathbf{k}\in\mathbb{N}^n\setminus(0,..,0)}\prod_{1\le i\le n}\frac{(-t_i)^{k_i}}{k_i!} I_{\mathbf{k}}(q_1,\cdots,q_n,\tau).$$

Moreover, for any j such that $k_j\neq 0$

$$I_{\mathbf{k}}(q_1,\cdots,q_n,\tau) = \frac{1}{\sum 1_{k_j>0}}\sum_{j=1}^n \mu^{\mathbf{k}-1_j}(q_j)$$

where 1_j is the n dimensional vector with null entries except at position j where it has the value one and if $k_j = 0$, $\mu^{\mathbf{k}-1_j}(q_j) = 0$.

Proof. Let

$$F_{\mathbf{t}}^N = \frac{1}{N^2}\log I_N(V_{\mathbf{t}}, A_i^N).$$

Denote $\gamma : [0,1] \to \mathbb{C}^n$ a smooth path in $T(q_1, \cdots, q_n) \cap B_\eta$ such that $\gamma(0) = \mathbf{0}$ and $\gamma(1) = \mathbf{t}$. Note that since $V_{\gamma(\alpha)}$ is Hermitian for all α, we can write at each time

$$V_{\gamma(\alpha)} = \sum_{i=1}^n \Re(\gamma(\alpha)_i)\frac{q_i + q_i^*}{2} + \sum_{i=1}^n \Im(\gamma(\alpha)_i)\frac{q_i - q_i^*}{2i}$$

so that

$$\begin{aligned} \partial_\alpha \mathrm{Tr}(V_{\gamma(\alpha)}(\mathbf{A})) &= \sum_{i=1}^n \partial_\alpha \Re(\gamma(\alpha)_i)\mathrm{Tr}(\frac{q_i + q_i^*}{2}) + \sum_{i=1}^n \partial_\alpha \Im(\gamma(\alpha)_i)\mathrm{Tr}(\frac{q_i - q_i^*}{2i}) \\ &= \Re\left(\sum_{i=1}^n \partial_\alpha \gamma(\alpha)_i \mathrm{Tr}(q_i)\right) \end{aligned}$$

Therefore

$$\partial_\alpha F_{\gamma(\alpha)}^N = \Re\left(\sum_{i=1}^n \partial_\alpha \gamma(\alpha)_i \mu_{V_{\gamma(\alpha)}}^N \left(\hat{\tau}^N(q_i)\right) d\alpha\right). \tag{11}$$

By Corollary 3.1, we know that for all α (since $\gamma(\alpha) \in T(q_1, \cdots, q_n) \cap B(0,\eta)$)

$$\lim_{N\to\infty} \mu_{V_{\gamma(\alpha)}}^N \left(\hat{\tau}^N(q_i)\right) = \tau_{\gamma(\alpha)}(q_i)$$

whereas since $\hat{\tau}^N \in \mathcal{M}$, we know that $\mu_{V_{\gamma(\alpha)}}^N \left(\hat{\tau}^N(q_i)\right)$ stays uniformly bounded. Therefore, a simple use of dominated convergence theorem shows that

$$F_{\mathbf{t}} = \lim_{N\to\infty} F_{\mathbf{t}}^N = \Re\left(\sum_{i=1}^n \int_0^1 \partial_\alpha \gamma(\alpha)_i \mu_{\gamma(\alpha)}(q_i)\right). \tag{12}$$

A simple computation shows, with Theorem 3.2, that

$$F_{\mathbf{t}} = \sum_{i=1}^n \sum_{k_1,\cdots,k_n} \prod_{j=1}^n \frac{(t_j)^{k_j + 1_i}}{(k_j + 1_i)!} \mu^{\mathbf{k}}(q_i).$$

□

6. Application to Voiculescu free entropy

Voiculescu's microstates free entropy is given as the asymptotic the volume of matrices whose empirical distribution approximate sufficiently well a given tracial state. Up to a Gaussian factor, it is more explicitly given by

$$\chi(\mu) = \limsup_{\substack{\varepsilon\downarrow 0 \\ k\uparrow\infty, R\uparrow\infty}} \limsup_{N\to\infty} \frac{1}{N^2}\log \mu_N^{\otimes m}\left(\Gamma_R(\mu,\varepsilon,k)\right)$$

with μ_N the Gaussian measure on $\mathcal{H}_N$ and $\Gamma_R(\mu,\varepsilon,k)$ the microstates

$$\Gamma_R(\mu,\varepsilon,k)=\{X_1,\cdots,X_m\in\mathcal{H}_N:\|X_i\|_\infty\le R|\frac{1}{N}\mathrm{Tr}(X_{i_1}\cdots X_{i_p})-\mu(X_{i_1}\cdots X_{i_p})|<\varepsilon$$

$$p\le k, i_\ell\in\{1,\cdots,m\}\}.$$

When $m=1$, it is well known [19] that $\mu\in\mathcal{P}(\mathbb{R})$ and

$$\chi(\mu)=I(\mu)=\int\int\log|x-y|d\mu(x)d\mu(y)-\frac{1}{2}\int x^2d\mu(x)+\text{const}.$$

Moreover, one can replace the lim sup by a lim inf in the definition of χ. Such answers (convergence and formulae for χ) are still open in general when $m\ge 2$ (see [4] for bounds). However, if μ is the law of m free variables with respective laws μ_i, then these questions are settled and

$$\chi(\mu)=\sum_{i=1}^m I(\mu_i).$$

We here want to emphasize that our result provide a small step towards dependent variables by showing convergence and giving a formula for the type of laws μ solutions of Schwinger-Dyson's equations. Indeed, let us consider $V=V(U_iA_iU_i^*,1\le i\le m)$ with V a Hermitian polynomial and μ the unique solution of $\mathbf{SD}[\mathbf{V},\tau]$ with τ the law of the $A_i, 1\le i\le m$ which is now chosen to be the law of m free variables with marginals distribution μ_i, $1\le i\le m$. Under the law $\mu_N^{\otimes N}$, we can diagonalize the matrices $X_i=U_iD_iU_i^*$ with U_i following the Haar measure on $\mathcal{U}(N)$, and find

$$\begin{aligned}
Ł_N &:= \mu_N^{\otimes m}\left(\hat{\tau}^N_{X_1,\cdots,X_m}\in\Gamma_R(\mu,\varepsilon,k)\right)\\
&= \mu_N^{\otimes m}\left(d(\hat{\tau}^N_{D_i},\mu_i)<\varepsilon;\hat{\tau}^N_{U_iD_iU_i^*,1\le i\le m}\in\Gamma_R(\mu,\varepsilon,k)\right)\\
&= Z_N^{-m}\int_{d(\hat{\tau}^N_{D_i},\mu_i)<\varepsilon,\|D_i\|_\infty\le R}\left(\int 1_{\hat{\tau}^N_{U_iD_iU_i^*,1\le i\le m}\in\Gamma_R(\mu,\varepsilon,k)}dU_1\cdots dU_m\right)\prod\Delta(\lambda^i_j)e^{-\frac{N}{2}\sum(\lambda^j_i)^2}d\lambda^j_i
\end{aligned}$$

where we denoted $\Delta(\lambda_j)=\prod_{k\ne j}|\lambda_k-\lambda_j|$ and

$$Z_N=\int\prod\Delta(\lambda_j)e^{-\frac{N}{2}\sum_{j=1}^N(\lambda_j)^2}d\lambda^j_i.$$

In these notations, $D_i=\mathrm{diag}(\lambda^i_1,\cdots,\lambda^i_N)$.

As a consequence, applying the large deviations result of [3] to the diagonal matrices D_i

$$\begin{aligned}
Ł_N &\le e^{N^2\sum_{i=1}^m I(\mu_i)+N^2o(\varepsilon)}\sup_{d(\hat{\tau}^N_{D_i},\mu_i)<\varepsilon,\|D_i\|_\infty\le R}\int 1_{\hat{\tau}^N_{U_iD_iU_i^*,1\le i\le m}\in\Gamma_R(\mu,\varepsilon,k)}dU_1\cdots dU_m\\
&:= e^{N^2\sum_{i=1}^m I(\mu_i)+N^2o(\varepsilon)}Ł^1_N
\end{aligned}$$

with

$$
\begin{aligned}
Ł_N^1 &= \sup_{d(\hat{\tau}^N_{D_i},\mu_i)<\varepsilon,\|D_i\|_\infty\le R} \int 1_{\hat{\tau}^N_{U_iD_iU_i^*,1\le i\le m}\in\Gamma_R(\mu,\varepsilon,k)} e^{N\mathrm{Tr}(V)-N\mathrm{Tr}(V)} dU_1\cdots dU_m \\
&= e^{-N^2\mu(V)+N^2o(\varepsilon)} \sup_{d(\hat{\tau}^N_{D_i},\mu_i)<\varepsilon,\|D_i\|_\infty\le R} \int 1_{\hat{\tau}^N_{U_iD_iU_i^*,1\le i\le m}\in\Gamma_R(\mu,\varepsilon,k)} e^{N\mathrm{Tr}(V)} dU_1\cdots dU_m \\
&\le e^{-N^2\mu(V)+N^2o(\varepsilon)} \sup_{d(\hat{\tau}^N_{D_i},\mu_i)<\varepsilon,\|D_i\|_\infty\le R} \int e^{N\mathrm{Tr}(V)} dU_1\cdots dU_m \\
&= e^{-N^2\mu(V)+N^2o(\varepsilon)} \sup_{d(\hat{\tau}^N_{D_i},\mu_i)<\varepsilon,\|D_i\|_\infty\le R} I_N(V,D_i)
\end{aligned}
$$

Now, for fixed R, any D_i, D_i' in $d(\hat{\tau}^N_{D_i},\mu_i)<\varepsilon, \|D_i\|_\infty\le R$

$$
\left|\frac{1}{N^2}\log I_N(V,D_i) - \frac{1}{N^2}\log I_N(V,D_i')\right| \le \eta(\varepsilon,R)
$$

with $\eta(\varepsilon,R)$ going to zero as ε goes to zero for any fixed R. Hence,

$$
\limsup_{N\to\infty} \frac{1}{N^2}\log I_N(V,D_i) \le F(V,\mu_i) + \eta(\varepsilon,R)
$$

with $F(V,\mu_i)$ the limit of $N^{-2}\log I_N(V,A_i)$ given in Theorem 5.1 when the distribution of the A_i converges to free variables with marginal distribution μ_i. We thus have proved, letting ε going to zero and then R,k to infinity, that

$$
\chi(\mu) \le \sum_{i=1}^m I(\mu_{A_i}) - \mu(V) + F(V,\mu_i).
$$

Conversely,

$$
Ł_N \quad\ge\quad e^{N^2\sum_{i=1}^m I(\mu_i)+N^2o(\varepsilon)} Ł_N^2
$$

with

$$
\begin{aligned}
Ł_N^2 &:= \inf_{d(\hat{\tau}^N_{D_i},\mu_i)<\varepsilon,\|D_i\|_\infty\le R} \int 1_{\hat{\tau}^N_{U_iD_iU_i^*,1\le i\le m}\in\Gamma_R(\mu,\varepsilon,k)} dU_1\cdots dU_m \\
&= e^{-N^2\mu(V)+N^2o(\varepsilon)} \inf_{d(\hat{\tau}^N_{D_i},\mu_i)<\varepsilon,\|D_i\|_\infty\le R} \int 1_{\hat{\tau}^N_{U_iD_iU_i^*,1\le i\le m}\in\Gamma_R(\mu,\varepsilon,k)} e^{N\mathrm{Tr}(V)} dU_1\cdots dU_m \\
&\ge e^{-N^2\mu(V)+N^2o(\varepsilon)} \inf_{d(\hat{\tau}^N_{D_i},\mu_i)<\delta,\|D_i\|_\infty\le R} \int 1_{\hat{\tau}^N_{U_iD_iU_i^*,1\le i\le m}\in\Gamma_R(\mu,\varepsilon,k)} e^{N\mathrm{Tr}(V)} dU_1\cdots dU_m
\end{aligned}
$$

for any $\delta < \varepsilon$. Now, choosing δ and using the continuity of $\hat{\tau}^N_{U_i D_i U_i^*, 1\le i\le m}$ in the distribution of the uniformly bounded variables D_i, we find by Corollary 3.1 that

$$\liminf_{N\to\infty} \frac{\int 1_{\hat{\tau}^N_{U_i D_i U_i^*,1\le i\le m}\in\Gamma_R(\mu,\varepsilon,k)} e^{N\mathrm{Tr}(V)} dU_1\cdots dU_m}{\int e^{N\mathrm{Tr}(V)} dU_1\cdots dU_m} = 1$$

which insures that

$$\chi(\mu) \geq \sum_{i=1}^m I(\mu_{A_i}) - \mu(V) + F(V,\mu_i).$$

Thus we have proved that

$$\chi(\mu) = \sum_{i=1}^m I(\mu_{A_i}) - \mu(V) + F(V,\mu_i).$$

Note that $\mu(V)$ and $F(V,\mu_i)$ can be written in terms of the $\mu^{\mathbf{k}}$ of Theorem 3.2 by Theorem 5.1.

Hence, we have proved that

Theorem 6.1. *Let $V = \sum_{i=1}^n t_i q_i$ be an Hermitian polynomial and assume that the t_i's are small enough so that Corollary 3.1 holds. Assume also that the hypotheses of Theorem 5.1 hold. Then,*

$$\chi(\tau) = \liminf_{\substack{\varepsilon\downarrow 0\\ k\uparrow\infty}} \liminf_{N\to\infty} \frac{1}{N^2}\log \mu_N^{\otimes m}\left(\Gamma_R(\mu,\varepsilon,k)\right)$$

and a formula of $\chi(\tau)$ can be given in terms of the $\mu^{\mathbf{k}}$ of Theorem 3.2.

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Department of Mathematics, Claude Bernard University, 43 bd du 11 novembre 1918, 69622 Villeurbanne, France
E-mail address: collins@math.univ-lyon1.fr

UMPA, CNRS UMR 5669, ENS Lyon,46 allée d'Italie, 69007 Lyon, France
E-mail address: aguionne@umpa.ens-lyon.fr

UMPA, CNRS UMR 5669, ENS Lyon,46 allée d'Italie, 69007 Lyon, France
E-mail address: emaurel@umpa.ens-lyon.fr