T. Kato 1960s: Non-autonomous parabolic evolution equation

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u(0) &= u_0 \in L^2(\Omega).
\end{aligned}
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- \( A(t) \sim -\nabla_x \cdot \mu(t, x) \nabla_x \) via elliptic form \( a(t) : \mathcal{V}(\Omega) \times \mathcal{V}(\Omega) \to \mathbb{C} \).
- \( u(t)(x) = e^{-tA}u_0(x) \) if \( A(t) = A \) for all \( t > 0 \).
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**Kato Square Root Problem (1961)**

“We do not know whether or not \( \mathcal{D}(A^{1/2}) = \mathcal{D}(A^{*1/2}) \). This is perhaps not true in general. But the question is open even when \( A \) is regularly accretive. In this case it appears reasonable to suppose that both \( \mathcal{D}(A^{1/2}) \) and \( \mathcal{D}(A^{*1/2}) \) coincide with \( \mathcal{D}(\alpha) \), where \( \alpha \) is the regular sesquilinear form which defines \( A \).”
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- Counterexamples by Lions 1962, \( m^c \)Intosh 1982
- Specialize to divergence-form operators.
Let

- $\Omega \subseteq \mathbb{R}^d$ domain, $D \subseteq \partial \Omega$ closed, $\mu \in L^\infty(\Omega)$
- $A \sim -\nabla \cdot \mu \nabla$ accretive operator on $L^2(\Omega)$ associated with

$$W^{1,2}_D(\Omega) \times W^{1,2}_D(\Omega) \to \mathbb{C}, \quad (u, v) \mapsto \int_\Omega \mu \nabla u \cdot \nabla v.$$

- $A^{1/2}$ square root of $A$ defined by e.g.

$$A^{1/2} u = \frac{1}{\pi} \int_0^\infty t^{-1/2} A(t + A)^{-1} \, dt.$$

**Kato conjecture**

It holds $\mathcal{D}(A^{1/2}) = W^{1,2}_D(\Omega)$ with equivalent norms.
Why do we care about the Kato conjecture?

Philosophy

- Elliptic non-regularity results $\mathcal{D}(A) \not\subseteq W^{2,2}(\Omega)$.
- Kato Conjecture $\sim$ optimal Sobolev regularity for $A^{1/2}$. 
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Ex. 1: Elliptic equations on $\mathbb{R}^d$

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\begin{cases}
\frac{\partial^2}{\partial t^2} u(t)(x) + \nabla \cdot \mu(x) \nabla u(t, x) = 0 & (t > 0, \ x \in \mathbb{R}^d), \\
u(0, x) = u_0(x) \in W^{1,2}(\mathbb{R}^d). 
\end{cases}
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- Solution $u(t, x) = e^{-tA^{1/2}} u_0(x)$.
- Kato conjecture $\sim$ Rellich inequality $\| \partial_t u \|_{t=0}^2 \sim \| \nabla u_0 \|_2^2$.
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Ex. 2: Maximal parabolic regularity (e.g. Haller-Dintelmann-Rehberg)

In $L^p$-setting study parabolic equation

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\begin{aligned}
\frac{d}{dt} u(t) + Au(t) &= f \quad (0 < t < T), \\
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Goal: Transport Max. Reg. from $L^p(\Omega)$ to $W_D^{-1:p}(\Omega)$. Many further examples, e.g. Cauchy-Integral along Lipschitz curve, hyperbolic wave equations, ...
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- Adjoint $(-\nabla \cdot \mu \nabla)^{1/2} : L^p(\Omega) \to W^{-1;p}_D(\Omega)$ isomorphism that commutes with parabolic solution operator

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- $L^p'$-Kato conjecture $\sim (−\nabla \cdot \mu^\top \nabla)^{1/2} : W_D^{1,p'}(\Omega) \rightarrow L^{p'}(\Omega)$ isom.

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Many further examples, e.g. Cauchy-Integral along Lipschitz curve, hyperbolic wave equations, . . . .
Positive answers

Self-adjoint operators
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Whole space $\Omega = \mathbb{R}^d$

- $d = 1$: Coifman - McIntosh - Meyer ’82.
- $d \geq 2$: Auscher-Hofmann-Lacey-McIntosh-Tchamitchian ’01, Axelsson-Keith-McIntosh ’06.
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Bounded domains

- $\Omega$ Lipschitz, $D \in \{\emptyset, \partial \Omega\}$: Auscher-Tchamitchian '03, '01 ($p \neq 2$).
- $\Omega$ smooth, smooth $D \leftrightarrow \partial \Omega \setminus D$ interface: Axelsson-Keith-McIntosh '06.
- $\Omega$ Lipschitz around $\overline{\partial \Omega \setminus D}$:
  Auscher-Badr-Haller-Dintelmann-Rehberg ’12 ($p \neq 2$).
Kato for mixed boundary conditions

Theorem (E.-Haller-Dintelmann-Tolksdorf ’14)

Suppose
- $\Omega \subseteq \mathbb{R}^d$ bounded $d$-Ahlfors regular domain,
- $D \subseteq \partial \Omega$ closed and $(d - 1)$-Ahlfors regular,
- $\Omega$ Lipschitz around $\overline{\partial \Omega \setminus D}$.

Then
$$\mathcal{D}(A^{1/2}) = W_{D}^{1,2}(\Omega) \quad \text{with} \quad \|A^{1/2}u\|_2 \sim \|\nabla u\|_2.$$

- First formulated by J.-L. Lions 1962.
- For rough ($= L^\infty$) coefficients new even on Lipschitz domains.
Some ideas of the proof

1. First-order approach via perturbed Dirac operators à la AKM ’06, $H^\infty$-functional calculus.
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Reduction-Theorem (E.-Haller-Dintelmann-Tolksdorf ’14)

In essence, the following holds: If $\mathcal{D} \left( (−\Delta_V)^s \right) \hookrightarrow H^{2s,2}(\Omega)$ for some $s > \frac{1}{2}$, then $\mathcal{D}(A^{1/2}) = W_D^{1,2}(\Omega)$.
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**Reduction-Theorem (E.-Haller-Dintelmann-Tolksdorf ’14)**

In essence, the following holds: If $\mathcal{D}((−Δ)^s) \hookrightarrow H^{2s,2}(Ω)$ for some $s > \frac{1}{2}$, then $\mathcal{D}(A^{1/2}) = W^{1,2}_D(Ω)$.

Extrapolate Kato for $−Δ \implies$ Kato property for general $A$.
Some ideas of the proof

1. First-order approach via perturbed Dirac operators à la AKM ’06, $H^\infty$-functional calculus.

2. Getting rid of the coefficients via Reduction-Theorem (E.-Haller-Dintelmann-Tolksdorf ’14)

In essence, the following holds: If $\mathcal{D} \left( (\Delta_V)^s \right) \hookrightarrow H^{2s,2}(\Omega)$ for some $s > \frac{1}{2}$, then $\mathcal{D}(A^{1/2}) = W^{1,2}_D(\Omega)$.

Extrapolate Kato for $-\Delta_V \Longrightarrow$ Kato property for general $A$ geometry, potential theory $\iff$ harmonic analysis.
\[ \mathcal{D}((−\Delta)_{\nu})^{1/2} = W_{D}^{1,2}(\Omega) \] by self-adjointness. Extrapolate by Snejberg’s stability theorem and the following result.

**Theorem (E.-Haller-Dintelmann-Tolksdorf ’14)**

Let \( \theta \in (0, 1) \) and \( s_0, s_1 \in (\frac{1}{2}, \frac{3}{2}) \). Put \( s_\theta := (1 − \theta)s_0 + \theta s_1 \). Then,

- \( W_{D}^{1,2}(\Omega) = H_{D}^{1,2}(\Omega) \)
- \( [H_{D}^{s_0,2}(\Omega), H_{D}^{s_1,2}(\Omega)]_\theta = H_{D}^{s_\theta,2}(\Omega) \).
- \( [L^{2}(\Omega), H_{D}^{1,2}(\Omega)]_\theta = \begin{cases} H_{D}^{\theta,2}(\Omega), & \text{if } \theta > \frac{1}{2}, \\ H_{D}^{\theta,2}(\Omega), & \text{if } \theta < \frac{1}{2}. \end{cases} \)
\( \mathcal{D}((-\Delta_V)^{1/2}) = W^{1,2}_D(\Omega) \) by self-adjointness. Extrapolate by Sneiberg’s stability theorem and the following result.

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In fact, \( \mathcal{D}((-\Delta_V)^s) = H^{2s,2}_D(\Omega) \) for \( |\frac{1}{2} - s| < \varepsilon \).
Elliptic BVPs on cylindrical domains

Elliptic mixed BVP

\[-\text{div}_{t,x} \mu(x) \nabla_{t,x} U = 0 \quad (\mathbb{R}^+ \times \Omega)\]

\[U = 0 \quad (\mathbb{R}^+ \times D)\]

\[\partial_{\nu\mu} U = 0 \quad (\mathbb{R}^+ \times N)\]

\[\partial_{\nu\mu} U = f \in L^2 \quad (\{0\} \times \Omega)\]
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\[
\uparrow \quad F \sim \begin{bmatrix}
\partial_{\nu\mu} U \\
\nabla_x U
\end{bmatrix}
\]

First order equation

\[
\partial_t F + \begin{bmatrix}
0 & (-\nabla_{\nu})^* \\
-\nabla_{\nu} & 0
\end{bmatrix} \mathbb{B} F = 0 \quad (t > 0)
\]

\[F(0) \perp = f\]
Elliptic BVPs on cylindrical domains

Elliptic mixed BVP

\(- \text{div}_{t,x} \mu(x) \nabla_{t,x} U = 0 \) \quad \left( \mathbb{R}^+ \times \Omega \right)

\( U = 0 \) \quad \left( \mathbb{R}^+ \times D \right)

\( \partial_{\nu} U = 0 \) \quad \left( \mathbb{R}^+ \times N \right)

\( \partial_{\nu} U = f \in L^2 \) \quad \left( \{0\} \times \Omega \right)

\[ F \sim \begin{bmatrix} \partial_{\nu} U \\ \nabla_x U \end{bmatrix} \]

First order equation

\[ \partial_t F + \begin{bmatrix} 0 & (-\nabla \nu)^* \\ -\nabla \nu & 0 \end{bmatrix} \mathbb{D} F = 0 \quad (t > 0) \]

\[ F(0)_{\perp} = f \]

\( \mathbb{L}^2_{\text{loc}}(\mathbb{R}^+; L^2(\Omega)) \) setting
Semigroup solutions via DB-formalism

DB has bounded $H^\infty$-calculus on $\mathcal{H} = \overline{R(DB)}$ (Kato Technology).

**Theorem (Auscher-E. ’14)**

1. For every $F(0) \in \mathcal{H}^+ := 1_{C^+}(DB)\mathcal{H}$ a solution to the first-order system is

   $$F(t) = e^{-tDB}F(0) \quad (t \geq 0).$$

   Via $F \sim \begin{bmatrix} \partial_{\nu\mu} U \\ \nabla_x U \end{bmatrix}$ these functions are in one-to-one correspondence with weak solutions $U$ such that

   $$\tilde{N}_*(|\nabla_{t,x} U|) \in L^2(\mathbb{R}^+ \times \Omega).$$

2. If $\mu$ is either block-diagonal or Hermitean, then for each $f \in L^2(\Omega)$ there exists a unique such solution $u$. 
Thank you for your attention!


