# A computation of invariants of a rational self-map

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**Abstract**

I prove the algebraic stability and compute the dynamical degrees of C. Voisin's rational self-map of the variety of lines on a cubic fourfold.

Je démontre la stabilité algébrique et calcule les degrés dynamiques de l'auto-application rationnelle (construite par C. Voisin) de la variété des droites sur une cubique dans $\mathbb{P}^5$.

Let $V$ be a smooth cubic in $\mathbb{P}^5$ and let $X = \mathcal{F}(V)$ be the variety of lines on $V$. Thus $X$ is a smooth four-dimensional subvariety of the grassmanian $G(1,5)$, more precisely, the zero locus of a section of $S^3U^*$, where $U$ is the tautological rank-two bundle over $G(1,5)$. It is immediate from this description that the canonical class of $X$ is trivial. Let $\mathcal{F} \subset V \times X$ be the universal family of lines on $V$, and let $p : \mathcal{F} \to V$, $q : \mathcal{F} \to X$ be the projections. Beauville and Donagi prove in [BD] that the Abel-Jacobi map $AJ : q_*p^* : H^4(V,\mathbb{Z}) \to H^2(X,\mathbb{Z})$ is an isomorphism, at least after tensoring up with $\mathbb{Q}$; since this is also a morphism of Hodge structures, we obtain, by the Noether-Lefschetz theorem for $V$, that $Pic(X) = \mathbb{Z}$ for a sufficiently general $X$.

Lines on cubics have been studied in [CG]. It is shown there that the normal bundle of a general line on a smooth 4-dimensional cubic is $\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(1)$ ($l$ is then called "a line of the first kind"), and that some special lines ("lines of the second kind") have normal bundle $\mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}(1)$. The lines of the second kind form a two-parameter subfamily $S \subset X$, and all lines on $V$ are either of the first or of the second kind.

An alternative description of lines on a smooth cubic in $\mathbb{P}^n$ from [CG] is as follows: let $F(x_0, \dots x_n) = 0$ define a smooth cubic $V$ in $\mathbb{P}^n$ and consider the Gauss map

$$D_V : \mathbb{P}^n \to (\mathbb{P}^n)^* : \ x \mapsto (\frac{\partial F}{\partial x_0}(x) : \dots : \frac{\partial F}{\partial x_n}(x))$$

(so $D_V(x)$ is the tangent hyperplane to $V$ at $x$). For a line $l \subset V$ there are only two possibilities: either $D_V|_l$ maps $l$ bijectively onto a plane conic, or $D_V|_l$ is two-to-one

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onto a line in $(\mathbb{P}^n)^*$. In the first case, $l$ is of the first kind and the intersection of the hyperplanes $\cap_{x\in l} D_V(x) \subset \mathbb{P}^n$ is a subspace of dimension $n-3$, tangent to $V$ along $l$; in the other case, $l$ is of the second kind and the linear subspace $Q_l = \cap_{x\in l} D_V(x) \subset \mathbb{P}^n$, which is of course still tangent to $V$ along $l$, is of greater dimension $n-2$.

As remarked in [V], $X$ always admits a rational self-map. This map is constructed as follows: from the description of the normal bundle, one sees that for a general line $l \subset V$, there is a unique plane $P_l \subset \mathbb{P}^5$ which is tangent to $V$ along $l$. So $P_l \cap V$ is the union of $l$ and another line $l'$, and one defines $f : X \dashrightarrow X$ by sending $l$ to $l'$. This is only a rational map: indeed, if $l$ is of the second kind, we have a one-parameter family, more precisely, a pencil of planes $\{P :\ l \subset P \subset Q_l\}$ tangent to $V$ along $l$. So the surface $S$ is the indeterminacy locus of the map $f$. Blowing $S$ up, one obtains a resolution of singularities of $f$; it introduces the exceptional divisor $E$ which is a $\mathbb{P}^1$-bundle over $S$, each $\mathbb{P}^1$ corresponding to the pencil of planes tangent to $V$ along $l$. That is, the "image" by $f$ of every exceptional point is a (possibly singular) rational curve on $X$.

In [V], it is shown that the degree of $f$ is 16. The proof is very short: the space $h^{2,0}(X)$ is generated by a nowhere-degenerate form $\sigma$ ([BD]); C.Voisin then uses a Mumford-style argument (saying that a family of rationally equivalent cycles induces the zero map on certain differential forms) to show that $f^*\sigma = -2\sigma$. As $(\sigma\overline{\sigma})^2$ is a volume form, the result follows.

In holomorphic dynamics, one considers the *dynamical degrees* of a rational self-map (the definition below seems to appear first in [RS]). Those are defined as follows: let $X$ be a Kähler manifold of dimension $k$. Fix a Kähler class $\omega$. Define

$$\delta_l(f,\omega) = [f^*\omega^l] \cdot [\omega^{k-l}]$$

(the product of cohomology classes on $X$), and the $l$th *dynamical degree*

$$\lambda_l(f) = \limsup(\delta_l(f^n,\omega))^{1/n}$$

(this does not depend on $\omega$). It turns out that this is the same as

$$\rho_l(f) = \limsup(r_l(f^n))^{1/n},$$

where $r_l(f^n)$ is the spectral radius of the action of $f^n$ on $H^{l,l}(X)$. If $X$ is projective, one can restrict oneself to the subspace of algebraic classes, replacing $\rho_l$ and $r_l$ by the correspondent spectral radii $r_l^{alg}$ and $\rho_l^{alg}$. Moreover, Dinh and Sibony [DS] recently proved that the lim sup in the definition of $\lambda_l(f)$ is actually a limit, and also the invariance of the dynamical degrees by birational conjugation.

The map $f$ is said to be *cohomologically hyperbolic* if there is a dynamical degree which is strictly greater then all the others. In fact the dynamical degrees satisfy a convexity property which says that if $\lambda_l(f)$ is such a maximal degree, then $\lambda_i$ grows together with $i$ until $i$ reaches $l$, and decreases thereafter. It is conjectured that the cohomological hyperbolicity implies certain important equidistribution properties for $f$. The case when the maximal dynamical degree is the topological degree $\lambda_k$ is studied by Guedj in [G] (building on the work of Briend-Duval [BrDv]); see [DS2] for a proof in a more general situation.

In general it is very difficult to compute the dynamical degrees of a rational self-map, because for rational maps it is not always true that $(f^n)^* = (f^*)^n$ (where $f^*$ denotes the transformation induced by $f$ in the cohomology.). In fact few examples of such computations are known.

If $(f^n)^* = (f^*)^n$, the map $f$ is called *algebraically stable*. The computation of its dynamical degrees is then much easier: one only has to study the linear map $f^*$, and it is not necessary to look at the iterations of $f$.

The purpose of this note is to show that the example $f$ considered in the beginning is algebraically stable and to compute its dynamical degrees. The map $f$ turns out to be cohomologically hyperbolic, and the dominating dynamical degree is in this case $\lambda_2$.

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## 1. Algebraic stability of the self-map

From now on let $X = \mathcal{F}(V)$ and $f$ be as described in the beginning. Let $S$ be the indeterminacy locus $I(f)$, that is, the surface of lines of the second kind.

We assume that $V$ is *sufficiently general*. The first goal is to prove that $f$ is algebraically stable.

**Lemma 1:** *For a general $V$, the surface $S$ is smooth.*

*Proof:* Let $T$ be the parameter space of cubics in $\mathbb{P}^5$ Consider the incidence variety $\mathcal{I} \subset G(1,5) \times T$: $\mathcal{I} = \{(l, V) : l \textit{ is of the } 2nd \textit{ kind on } V\}$. It is enough to show that $\mathcal{I}$ is smooth in codimension two. By homogeneity, it is enough to verify that the fiber $I_l = pr_1^{-1} l$ is smooth in codimension two for some $l \in G(1,5)$. Choose coordinates $(x_0 : \ldots : x_5)$ on $\mathbb{P}^5$ so that $l$ is given by $x_2 = x_3 = x_4 = x_5 = 0$; then $l \subset V$ means that four coefficients (those of $x_0^3$, $x_0^2 x_1$, $x_0 x_1^2$ and $x_1^3$) of the equation $F = 0$ of $V$ vanish, and $l$ being of the second type means that the 4 quadratic forms $\partial F/\partial x_i|_l$, $i = 2, 3, 4, 5$ in two variables $x_0, x_1$, span a linear space of dimension 2, rather than 3. This in turn means that a $4 \times 3$ matrix $M$, whose coefficients are (different) coordinates on $T' \subset T$, the projective space of cubics containing $l$, has rang 2. It follows from general facts about determinantal varieties that $Sing(I_l)$ is the locus where $rg(M) \leq 1$, and it has codimension four in $I_l$ (in fact, $Sing(I_l)$ is also contained in the locus of singular cubics, [CG]).

Let us now look at the map $f : X \dashrightarrow X$. The first thing to observe is as follows: $K_X = 0$ means that $f$ cannot contract a divisor. Indeed, let $\pi : Y \to X$, $g : Y \to X$ be the resolution of the indeterminacies of $f$. Then the ramification divisor (i.e. the vanishing locus of the Jacobian determinant) of $\pi$ is equal to that of $g$, so $g$ can only contract a divisor which is already $\pi$-exceptional. By the same token, any subset contracted by $g$ must lie in the exceptional divisor of $\pi$, in other words, anything contracted by $f$ is in $S$.

**Lemma 2:** *The map $g$ does not contract a surface (neither to a point nor to a curve), unless possibly some surface already contracted by $\pi$.*

*Proof (C. Voisin):* If it does, this surface $Z$ projects onto $S$. Recall that $X$ is holomorphic symplectic, with symplectic form $\sigma = AJ(\eta)$, where $\eta$ generates $H^{3,1}$ of the cubic. Notice that $S$ is not lagrangian: $\sigma|_S \neq 0$. This is because, as we shall see in the next section, the cohomology class of $S$ is $5(H^2 - \Delta)$, where $H$ is the hyperplane section class for the Plücker embedding, and $\Delta$ is the Schubert cycle of lines contained in a hyperplane. We have $[\sigma] \cdot \Delta = 0$ in the cohomologies, by projection formula (because $[\eta]$ is a primitive cohomology class). But $[\sigma] \cdot H^2$ is non-zero, because $[\sigma\overline{\sigma}] \cdot H^2$ is the Bogomolov square of an ample $H$. Thus so is $\sigma|_S$.

Now recall ([V]) that $f^*\sigma = -2\sigma$, that is, $g^*\sigma = -2\pi^*\sigma$. This is a contradiction, since $g^*\sigma|_Z = 0$, whereas $\pi^*\sigma|_Z \neq 0$.

**Remark:** The same argument with $\sigma$ shows that $g$ cannot contract the $\pi$-exceptional divisor (even not to a surface). A somewhat more elaborate version of it shows that $g$ does not contract any surface at all; but we won't need this to prove the algebraic stability. See also the remark after Theorem 3.

**Theorem 3:** *The self-map $f$ of $X = \mathcal{F}(V)$ described above is algebraically stable, that is, we have $(f^n)^* = (f^*)^n$, where the upper star denotes the action on the cohomologies, and $n$ is a positive integer.*

*Proof:* The transformation induced by $f$ on the cohomologies is given by the class of the graph $\Gamma(f) \in H^8(X \times X)$. In general, $(hf)^* = h^*f^*$ is implied by $\Gamma(f) \circ \Gamma(h)$ having only one 4-dimensional component (which is then, of course, $\Gamma(hf)$). Recall that the correspondence $\Gamma(f) \circ \Gamma(h)$ is

$$\{(x, z) | \exists y : \ (x, y) \in \Gamma(f), \ (y, z) \in \Gamma(h)\}.$$

Let $h = f^n$. Suppose that $\Gamma(f) \circ \Gamma(h)$ has an extra 4-dimensional component $M$. There are three possibilities for its image under the first projection:

1) It is a divisor $D$. Then $f$ is defined at a general point $d \in D$; but fibers of $M$ over $D$ must be at least one-dimensional. That is, $D$ is contracted by $f$ (to the indeterminacy locus of $h$). This is impossible.

2) It is a surface $Z$; fibers of $M$ over $Z$ must be two-dimensional. But if $Z \not\subset I(f)$, there is only one $y$ corresponding to generic $x \in Z$, and furthermore, since $Z$ is not contracted by $f$, $h(y)$ is at most a curve. Indeed, the locus $I_2(h) = \{y : dim(h(y)) \geq 2\}$ is of codimension at least three in $X$. Thus a fiber of $M$ over a generic $x \in Z$ cannot be two-dimensional. If $Z \subset I(f)$, one concludes in a similar way: for any $x \in Z$, the corresponding $y$ are parametrized by a rational curve $D_x$; for a general $x$, the curve $D_x$ is not the indeterminacy locus of $h$, and, moreover, since no surface is contracted, $D_x$ intersects $I(h)$ in a finite number of points, all outside $I_2(h)$. Thus again the general fiber of $\Gamma(f) \circ \Gamma(h)$ over $T$ is at most one-dimensional.

3) It is a curve $C$. By a simple dimension count as in the previous case, we see that then $C$ must be contracted by $f$ to a point $y$ such that $dim(h(y)) = 3$. Recall that $h = f^n$. We may assume that $n$ is the smallest number with the property $dim(h(y)) = 3$. As for any $x \in X$, $f(x)$ is either a point or a rational curve, $S$ must be contained in $f^{n-1}(y)$ as a component. This implies that $S$ is covered by rational curves. But, as we shall see in the next section, $K_S$ is ample; so this case is also impossible.

**Remark:** In fact one can show that $g$ is finite (and this obviously implies the algebraic stability). Indeed, by the argument of Lemma 2, if $g(C)$ is a point, then $\sigma|_S$ vanishes on $\pi(C)$ and so $\pi(C)$ is a component of a canonical divisor on $S$. On the other hand, all lines on $V$ corresponding to the points of $\pi(C)$ intersect the line corresponding to the point $g(C)$. This means that $\pi(C)$ is contained in a hyperplane section of $S$. But it turns out (see the remark in the end of next section) that $K_S$ is even ampler than the hyperplane section class $H_S$. To get a contradiction, it remains to show that the vanishing locus of $\sigma|_S$ is smooth and irreducible for generic $V$. C. Voisin indicates that this is not very difficult to calculate explicitely using the Griffiths' description of the primitive cohomology of hypersurfaces (here cubics) in terms of residues.

## 2. Some cohomology classes on $X$

Recall that $X \subset G(1,5)$ is the zero-set of a section of the bundle $S^3U^*$ (of rang 4). We shall describe the action of $f^*$ on those cohomology classes which are restrictions of classes on $G(1,5)$.

So let $H = c_1(U^*)|_X = [\mathcal{O}_{G(1,5)}(1)]|_X$; $\Delta = c_2(U^*)|_X$ (so $\Delta$ is the class of the subset of points $x \in X$ such that the corresponding lines $l_x \subset \mathbb{P}^5$ lie in a given hyperplane).

**Lemma 4:** $H^4 = 108$; $H^2\Delta = 45$; $\Delta^2 = 27$.

*Proof:* $[X] = c_4(S^3U^*)$ and

$$c_4(S^3U^*) = 9c_2(U^*)(2c_1^2(U^*) + c_2(U^*)).$$

Let $\sigma_i = c_i(U^*)$; those are Schubert classes on $G(1,5)$: $\sigma_1 = \{$lines intersecting a given $\mathbb{P}^3\}$, $\sigma_2 = \{$lines lying in a given hyperplane$\}$.

Notice that $c_1(U^*)$ is the hyperplane section class in the Plücker imbedding and restricts as such onto a "sub-grassmanian" of lines lying in a linear subspace. So

$$H^4 = 9\sigma_2\sigma_1^4(2\sigma_1^2 + \sigma_2) = 18deg(G(1,4)) + 9deg(G(1,3)) = 108.$$

The other equalities are proved similarly..

For generic $X$, the class $H$ generates $H^{1,1}_{alg}$ over $\mathbb{Z}$, and $H^3$ generates $H^{3,3}_{alg}$ over $\mathbb{Q}$.

One can also show that $H^2$ and $\Delta$ generate $H^{1,1}_{alg}$ over $\mathbb{Q}$; we do not need this for our computation, but see the remark in the end of the last section.

The next step is to compute the indeterminacy surface $S$.

Let $A$ denote the underlying vector space for $\mathbb{P}^5 \supset V$, and let $L \subset A$ be the underlying vector space for a line $l \subset V$. Write the Gauss map associated to $F$ defining $V$ as $D_F : A \to S^2A^*$. The condition $l \subset V$ implies that this map descends to $\phi : A/L \to S^2L^*$:

$$\begin{array}{ccc} A & \to & S^2A^* \\ \downarrow & & \downarrow \\ A/L & \to & S^2L^* \end{array}$$

For a general $L$, the map $\phi$ is surjective with one-dimensional kernel; and it has two-dimensional kernel exactly when $L$ is the underlying vector space of a line of the second kind. Now $L$ is the fiber over a point $x_l \in X$ of the restriction $U_X$ of $U$ to $X$; so, globalizing and dualizing, we get the following resolution for $S$:

$$0 \to S^2U_X \to Q_X^* \to M \to M \otimes \mathcal{O}_S \to 0,$$

where $Q_X^*$ is the restriction to $X$ of the universal quotient bundle over $G(1,5)$, and $M$ is a line bundle. In fact

$$[M] = det(Q_X^*) - det(S^2U_X) = -H + 3H = 2H.$$

So one has a resolution for the ideal sheaf $\mathcal{I}_S$:

$$0 \to S^2U_X(-2H) \to Q_X^*(-2H) \to \mathcal{I}_S \to 0.$$

The cohomology class $[S]$ can be computed by the Thom-Porteous formula: a partial case of this formula identifies the cohomology class of the degeneracy locus $D_{e-1}(\phi)$, where a vector bundle morphism $\phi : E \to F$ ($rg(E) = e \leq rg(F) = f$) is not of maximal rang, as $c_{f-e+1}(F - E)$, where we put formally

$$c(F - E) = 1 + c_1(F - E) + c_2(F - E) + \ldots = c(F)/c(E).$$

So

$$\begin{aligned}[S] = c_2(Q_X^* - S^2U_X) = c_2(Q_X^*) - c_2(S^2U_X) + 2Hc_1(S^2U_X) = \\ = H^2 - \Delta - 4\Delta - 2H^2 + 6H^2 = 5(H^2 - \Delta).\end{aligned}$$

One can also make a direct calculation using the equality

$$[S] = -c_2(\mathcal{O}_S) = c_2(\mathcal{I}_S)$$

(see e.g. [F], chapter 15.3). Later, we shall need its extension

$$i_*c_1(N_{S,X}) = i_*K_S = c_3(\mathcal{I}_S)$$

where $i : S \to X$ is the usual embedding. From the resolution of the ideal sheaf, one computes

$$c_3(\mathcal{I}_S) = c_3(Q_X^*(-2H)) - c_3(S^2U_X(-2H)) - c_2(\mathcal{I}_S)c_1(S^2U_X(-2H)) = 20H^3 - 27H\Delta.$$

Thus we have obtained the following

**Proposition 5:** *The Chern classes of* $\mathcal{I}_S$ *are:* $c_1(\mathcal{I}_S) = 0$; $c_2(\mathcal{I}_S) = 5(H^2 - \Delta)$; $c_3(\mathcal{I}_S) = 20H^3 - 27H\Delta = \frac{35}{4}H^3$.

The last equality is obtained by taking the intersection with $H$ and using lemma 4.

**Remark:** One can also remark that $S$ being the degeneration locus of a map $\phi : S^2U \to Q^*$, its normal bundle is isomorphic to $(Ker(\phi|_S))^* \otimes (Coker(\phi|_S))$, so its determinant is $c_1(Coker(\phi|_S)) - 2c_1(Ker(\phi|_S)) = c_1(Q^*|_S) - c_1(S^2U|_S) - c_1(Ker(\phi|_S)) = 2H_S - c_1(Ker(\phi|_S))$, in particular it is at least as positive than $2H_S$. It is easy to see that the cohomology class of the intersection $HS$ is $\frac{35}{12}H^3$; this suggests that the canonical class of $S$ is equal to $3H_S$, but I have not checked this (one probably can write an analogue of the Thom-Porteous formula, but I could not find a reference, and in our particular case the direct calculation is easy).

### 3. Computation of the inverse images

Let $\pi : \tilde{X} \to X$ be the blow-up of $S$; this resolves the singularities of $f$, that is, $g = f \circ \pi : \tilde{X} \to X$ is holomorphic. Let $E$ denote the exceptional divisor of $\pi$. We have $g^*H = a\pi^*H - bE$; the number $a$ is, by algebraic stability, the first dynamical degree we are looking for. In fact the numbers $a$ and $b$ can be computed by a simple geometric argument; we begin by sketching this computation.

**Proposition 6** $a = 7$ and $b = 3$.

*Proof:* Consider the two-dimensional linear sections $V \cup \mathbb{P}^3$ of $V$; they form a family of dimension $dim(G(3,5)) = 8$. A standard dimension count shows that in this family, there is a one-parameter subfamily of cones over a plane cubic (indeed, one computes that out of $\infty^{19}$ cubics in $\mathbb{P}^3$, $\infty^{12}$ are such cones, so being a cone imposes only 7 conditions on a cubic surface in $\mathbb{P}^3$). Furthermore, some (a finite number) of those plane cubics are degenerate, so on $V$ we have a finite number of cones over a plane cubic with a double point (as it was already mentioned in the first section). Thus on $X$, we have a one-parameter family of plane cubic curves $C_t$.

Another way to see this is to remark that the lines passing through a point of $V$ sweep out a surface which is a cone over the intersection of a cubic and a quadric in $\mathbb{P}^3$ ([CG]). The quadric sometimes degenerates in the union of two planes, so the intersection becomes a union of two plane cubics. Moreover, the two plane cubics have three points in common (e.g. from the arithmetic genus count).

The resulting curves $C_t$ on $X$ are clearly $f$-invariant, and when smooth, map 4:1 to themselves (indeed, through a given point $p$ of an elliptic curve $C \in \mathbb{P}^2$, there are 4 tangents to $C$, because the projection from $p$ is a 2:1 map from $C$ to $\mathbb{P}^1$ and so has 4 ramification points). A smooth $C_t$ has 3 points in common with $S$. Let $C_0$ be singular; then it has at least one point $x$ in common with $S$, and is the image by $g$ of the exceptional $E_x \cong \mathbb{P}^1 \subset \tilde{X}$ over this point; $g$ is 1:1 from $E_x$ to $C_0$.

It follows that $E_x g^*H = 3$, so $b = 3$ (recall that $E$ restricts to itself as $\mathcal{O}_E(-1)$, hence $E_x E = -1$). Also, $C_t H = C_t E = 3$ and $C_t g^* H = 12$, from where $a = 7$.

However, I could not find such a simple way to compute the higher dynamical degrees. The following approach is suggested by A. Kuznetsov.

On $\tilde{X}$, one has a vector bundle $P$ of rank three: its fiber over $x_l$ is the underlying vector space of the plane tangent to $V$ along $l$ when $l$ is of the first kind, and we take the obvious extension to the exceptional divisor. One has $\pi^*U_X \subset P$.

First of all, let us describe the quotient: let $E$ denote the exceptional divisor of $\pi$. Then one can see that we obtain $P$ as the following extension:

$$\begin{array}{ccccccccc} 0 & \to & \pi^*U_X & \to & P & \to & M^*(E) & \to & 0 \\ & & || & & \downarrow & & \downarrow & & \\ 0 & \to & \pi^*U_X & \to & A \otimes \mathcal{O} & \to & \pi^*Q_X & \to & 0, \end{array}$$

where the map $M^*(E) \to \pi^*Q_X$ is the dual of the pull-back of the map $Q_X^* \to M \otimes \mathcal{I}_S$ from the resolution of $S$ in the last section.

(Alternatively, one can compute the cokernel of the natural inclusion $\pi^*U_X \to P$ using the exact sequences from the proof of the next proposition and the knowledge of $c_1(g^*U_X^*) = 7\pi^*H - 3E$).

**Proposition 7:** *The bundle $g^*U_X^*$ fits into an exact sequence*

$$0 \to (M^*)^{\otimes 2}(2E) \to P^* \to g^*U_X^* \to 0.$$

*Proof:* Consider the projective bundle $p : \mathbb{P}_{\tilde{X}}(P) \to \tilde{X}$. It is equipped with a natural map $\psi : \mathbb{P}_{\tilde{X}}(P) \to \mathbb{P}(A)$, and the inverse image of $\mathcal{O}_{\mathbb{P}(A)}(1)$ is the tautological line bundle $\mathcal{O}_{\mathbb{P}_{\tilde{X}}(P)}(1)$, meaning that its direct image under the projection to $\tilde{X}$ is $P^*$. Let $h = c_1(\mathcal{O}_{\mathbb{P}_{\tilde{X}}(P)}(1))$. We have a natural inclusion $\mathbb{P}_{\tilde{X}}(\pi^*U_X) \subset \mathbb{P}_{\tilde{X}}(P)$ . Denote by $d$ the class of $\mathbb{P}_{\tilde{X}}(\pi^*U_X)$ considered as a divisor on $\mathbb{P}_{\tilde{X}}(P)$. The subvariety of $\mathbb{P}_{\tilde{X}}(P)$ swept out by the lines $f(l)$ (where $l$ is a fiber of $\mathbb{P}_{\tilde{X}}(\pi^*U_X)$), is a divisor; let us denote it by $\Sigma$. We clearly have $2d + [\Sigma] = 3h$. Furthermore, $\Sigma$ itself is a projective bundle over $\tilde{X}$, namely, it is the projectivization of $g^*U_X$, in the sense that the vector bundle $g^*U_X^*$ is the direct image of $\mathcal{O}_\Sigma(1) = \mathcal{O}_{\mathbb{P}_{\tilde{X}}(P)}(1)|_\Sigma$ under the projection to $\tilde{X}$. On $\mathbb{P}_{\tilde{X}}(P)$, we have the exact sequence

$$0 \to \mathcal{O}_{\mathbb{P}_{\tilde{X}}(P)}(h - [\Sigma]) \to \mathcal{O}_{\mathbb{P}_{\tilde{X}}(P)}(1) \to \mathcal{O}_\Sigma(1) \to 0.$$

Also, $h - [\Sigma] = 2(d - h)$. So our proposition follows by taking the direct image of the above exact sequence, once we show that

$$\mathcal{O}_{\mathbb{P}_{\tilde{X}}(P)}(h - d) = p^*M(-E).$$

As for this last statement, it is clear that $\mathcal{O}_{\mathbb{P}_{\tilde{X}}(P)}(h-d) = p^*F$ for some line bundle $F$ on $\tilde{X}$, because it is trivial on the fibers. We deduce that $F = M(-E)$ from the exact sequence

$$0 \to \mathcal{O}_{\mathbb{P}_{\tilde{X}}(P)}(h - d) \to \mathcal{O}_{\mathbb{P}_{\tilde{X}}(P)}(1) \to \mathcal{O}_{\mathbb{P}_{\tilde{X}}(P)}(1)|_{\mathbb{P}_{\tilde{X}}(\pi^*U_X)} \to 0 :$$

indeed, it pushes down to

$$0 \to F \to P^* \to \pi^* U_X^* \to 0.$$

We are ready to prove the main result:

**Theorem 8** *We have $f^*H = 7H$, $f^*H^2 = 4H^2 + 45\Delta$, $f^*\Delta = 31\Delta$ and $f^*H^3 = 28H^3$. The dynamical degrees of $f$ are therefore 7, 31, 28 and 16. In particular, $f$ is cohomologically hyperbolic.*

*Proof:* We have already seen (proposition 6) that $f^*H = \pi_* g^* H = 7H$. Notice that $\pi_*(\pi^* HE) = 0$ and $\pi_* E^2 = -S$ (this is because $\mathcal{O}(E)$ restricts to $E$ as $\mathcal{O}_E(-1)$, where we view $E$ as a projective bundle over $S$). This gives

$$f^*H^2 = 49H^2 - 9S = 49H^2 - 45(H^2 - \Delta) = 4H^2 + 45\Delta.$$

Further, $g^*\Delta = c_2(g^*U_X)$, and we can find the latter from the exact sequences describing $g^*U_X$ and $P$. From

$$0 \to \pi^* U_X \to P \to M^*(E) \to 0,$$

we get $c_2(P) = \pi^*\Delta - \pi^*H(-2\pi^*H + E) = \pi^*\Delta - \pi^*HE + 2\pi^*H^2$, and from

$$0 \to g^* U_X \to P \to M^{\otimes 2}(-2E) \to 0,$$

$$c_2(g^*U_X) = c_2(P) + (4\pi^*H - 2E)(7\pi^*H - 3E) = \pi^*\Delta + 30\pi^*H^2 + 6E^2 - 27\pi^*HE,$$

that is,

$$f^*\Delta = \Delta + 30H^2 - 30(H^2 - \Delta) = 31\Delta.$$

Finally,

$$g^*H^3 = (7\pi^*H - 3E)^3 = 343\pi^*H^3 - 441\pi^*H^2E + 189\pi^*HE^2 - 27E^3.$$

We have $\pi_*(\pi^*H^2E) = 0$ and

$$\pi_*(\pi^*HE^2) = -5H(H^2 - \Delta) = -\frac{35}{12}H^3.$$

Let us compute $\pi_*E^3$: this is $\pi_*\xi^2$, where $\xi$ is the tautological class on $E$ viewed as a projective bundle $\mathbb{P}_S(N)$ over $S$. Let $r$ be the projection of $E$ to $S$, i.e. the restriction of $\pi$ to $E$. We have $\xi^2 + c_1(r^*N)\xi + c_2(r^*N) = 0$, which yields $\pi_*E^3 = -i_*c_1(N)$, $i$ being the embedding of $S$ into $X$. But the latter is just $c_3(i_*\mathcal{O}_S)$ ([F]). We have computed this class in the last section:

$$\pi_*E^3 = 27H\Delta - 20H^3 = -\frac{35}{4}H^3.$$

Putting all this together, we get

$$f^*H^3 = 343H^3 - \frac{35}{4}(63 - 27)H^3 = 28H^3.$$

This finishes the proof of Theorem 8.

**Remark 9:** Notice that the eigenvectors of $f^*$ on the invariant subspace generated by $H^2$ and $\Delta$ are orthogonal with respect to the intersection form. There are reasons for this; moreover, at least for $X$ generic the map $f^*$ on the whole cohomology group $H^4(X)$ is self-adjoint with respect to the intersection form. One can see it as follows: $X$ is a deformation of the punctual Hilbert scheme $Hilb^2(S)$ of a $K3$ surface [BD]; this implies a decomposition of $H^4(X)$ into an orthogonal direct sum of Hodge substructures

$$H^4(X) =< H^2 > \oplus (H \cdot H^2(X)^0) \oplus S^2(H^2(X)^0),$$

where $H^2(X)^0$ denotes the orthogonal to $H$ in $H^2(X)$. For $X$ generic, the second summand is an irreducible Hodge structure and the third one is a sum of an irreducible one and a one-dimensional subspace: this follows from the fact that $H^2(X)^0 = H^4(V)^{prim}$ (where $V$ is the corresponding cubic fourfold and "prim" is primitive cohomologies) and a theorem of Deligne which says that the closure of the monodromy group (of a general Lefschetz pencil) acting on $H^4(V)^{prim}$ is the full orthogonal group. Therefore the decomposition rewrites as

$$H^4(X) =< H^2, \Delta > \oplus H \cdot H^2(X)^0 \oplus V,$$

where the last two summands are simple, and so $f^*$ must be a homothety on each of them.

It also follows from this decomposition that on $X$ generic, the space of algebraic cycles of codimension two is only two-dimensional, and that all lagrangian surfaces must have the cohomology class proportional to $\Delta$. In particular, since $f^*$ and $f_*$ must preserve the property of being lagrangian, this explains why $\Delta$ is an eigenvector of $f^*$ and $f_*$.

# References


[BD] A. Beauville, R. Donagi, La variété des droites d'une hypersurface cubique de dimension 4, C. R. Acad. Sci. Paris Sér. I Math., 301 (1985), no. 14, 703–706.

[BrDv] J.-Y. Briend, J. Duval: Deux caractrisations de la mesure d'équilibre d'un endomorphisme de $\mathbb{P}^k(\mathbb{C})$, Publ. Math. Inst. Hautes études Sci. No. 93 (2001), 145-159.

[CG] H. Clemens, Ph. Griffiths: The intermediate jacobian of the cubic threefold, Ann. Math. 95 (1972), 281-356.

[DS] T. C. Dinh, N. Sibony: Une borne supérieure pour l'entropie topologique d'une application rationnelle, Annals of Maths. 161 (2005), 1637-1644.

[DS2] T. C. Dinh, N. Sibony: Distribution des valeurs de transformations méromorphes et applications, Comment. Math. Helvetici 81 (2006), 221-258.

[F] W. Fulton, Intersection theory, Springer-Verlag, Berlin 1984.

[G] V. Guedj, Ergodic properties of rational mappings with large topological degree, Ann. of Math. (2) 161 (2005), no. 3, 1589-1607.

[RS] A. Russakovskii, B. Schiffman, Value distribution for sequences of rational mappings and complex dynamics, Indiana Univ. Math. J. 46 (1997), no. 3, 897-932.

[V] C. Voisin, Intrinsic pseudovolume forms and $K$-correspondences. The Fano Conference, 761–792, Univ. Torino, Turin, 2004