

Potential density of rational points on the variety of lines of a cubic fourfold

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0 Introduction

Recall that for a variety X defined over a field k , rational points are said to be potentially dense in X if for some finite extension k' of k , $X(k')$ is Zariski dense in X . For example, this is the case if X is a rational or a unirational variety.

If k is a number field, a conjecture of Lang and Vojta, proved in dimension 1 by Faltings, but still open even in dimension two, predicts that varieties of general type never satisfy potential density over k . On the contrary, it is generally expected that when the canonical bundle K_X is negative (i.e. X is a Fano variety) or trivial, rational points are always potentially dense in X (see [9] for a conjectural geometric characterization of potentially dense varieties).

In the Fano case, there are many examples confirming this. First of all, many Fano varieties are unirational (in dimension ≤ 2 , all of them are even rational over a finite extension of the definition field). Furthermore, Fano threefolds such as a three-dimensional quartic or a “double Veronese cone” (a hypersurface of degree 6 in $\mathbb{P}(1, 1, 1, 2, 3)$) are birational to elliptic fibrations over \mathbb{P}^2 , which makes it possible to prove potential density (see [13]).

If the canonical bundle is trivial, the known examples are much less convincing. It is well-known that rational points on abelian varieties are potentially dense. But the simply-connected case remains largely unsolved.

A number of results concerning potential density of rational points on $K3$ surfaces defined either over a number field or a function field appeared in the last years. Bogomolov and Tschinkel [6] proved potential density of rational points for $K3$ surfaces admitting an elliptic pencil, or an infinite automorphisms group. Remark that these $K3$ surfaces are rather special; in particular, their geometric Picard number is never equal to 1, whereas it is equal to 1 for a general projective $K3$.

The case of $K3$ surfaces defined over a (complex) function field has been solved for certain types of families by Hassett and Tschinkel [14]. They prove in particular the existence of $K3$ surfaces defined over $\mathbb{C}(t)$, whose geometric Picard group is equal to \mathbb{Z} , and which satisfy potential density. Their method can likely be adapted to produce examples over $\overline{\mathbb{Q}}(t)$.

However, no example of a $K3$ surface defined over a number field, and with geometric Picard group equal to \mathbb{Z} is known to satisfy potential density.

In this paper, we consider certain four-dimensional analogues of $K3$ surfaces, namely, Fano varieties of lines F of cubic 4-folds X . These are 4-dimensional varieties with trivial canonical bundles, and which even possess a non-degenerate holomorphic 2-form ω which will play an important role in our study. Their similarity with $K3$ surfaces is shown by the work of Beauville and Donagi [3], where they prove that

these varieties are deformations of the second punctual Hilbert scheme of a K3 surface. However, for very general X , the geometric Picard group of F is equal to \mathbb{Z} , by an easy deformation argument. By a result due to Terasoma [24], this property holds true even for many X 's defined over \mathbb{Q} . We prove the following result:

Theorem 0.1 *For many cubic 4-folds X defined over a number field, such that the corresponding variety of lines F (which is defined over the same number field) has cyclic geometric Picard group, F satisfies the potential density property.*

Here, the term “many” can be made precise and exactly depends on the structure of the set of points in a certain moduli space satisfying Terasoma’s density condition (see [24]). In lemma 1.3, we give precise conditions under which the conclusions of Theorem 0.1 hold.

Remark 0.2 An interesting problem would be to write down the explicit equation of some cubic fourfold as in Theorem 0.1, (cf. [19] for some equations of K3 surfaces over the rationals, with cyclic geometric Picard group and infinitely many rational points). Our existence result does not lead to such an explicit construction.

Let us describe the strategy of our proof. We will use two geometric ingredients. The first one is the rational self-map $\phi : F \dashrightarrow F$ constructed in [25] (cf section 1); in [2], this map was already used to prove the potential density of rational points for such a variety F over an uncountable field (say, a function field of a complex curve, as in [14]), assuming its geometric Picard group is \mathbb{Z} .

The proof in [2] shows in fact that for a general complex point $t \in F(\mathbb{C})$, its orbit $\{\phi^l(t), l \in \mathbb{N}\}$ is Zariski dense in F . In the case where F is defined over a number field k , this proof does not say anything about points of F over finite extensions of k , because it is based on discarding certain “bad” countable unions of proper subvarieties, and it seems difficult to get any better control over those subsets. As they could contain all points defined over a number field, one cannot conclude from [2] that there is a point $t \in F(\mathbb{Q})$ with Zariski dense orbit. Neither do we prove such a statement, and our strategy uses in fact the following supplementary ingredient: the variety F is covered by a 2-dimensional family of surfaces $(\Sigma_b)_{b \in B}$, which are lagrangian with respect to the non-degenerate holomorphic 2-form ω and birationally equivalent to abelian surfaces (cf [26] and section 1).

Of course, for any surface as above defined over a number field, rational points are potentially dense in it. Furthermore ϕ is defined over the same number field as F . It thus suffices to prove that for one such surface Σ_b , the countably many surfaces $\phi^l(\Sigma_b)$ are Zariski dense in F .

This is done in two steps, and our proof shows that this density result holds once a non-vanishing condition (see Proposition 2.7) for a certain l -adic Abel-Jacobi cycle class holds for the corresponding Σ_b . This non-vanishing is satisfied for many $\overline{\mathbb{Q}}$ -points in the moduli space B of these surfaces, by an argument which is taken from [12] (and attributed there to Bloch and Nori).

In section 2, we prove that for many Σ_b 's defined over a number field, the Zariski closure of the union of the surfaces $\phi^l(\Sigma_b)$ contains at least a divisor (that is, Σ_b is not preperiodic), and this already necessitates this non-vanishing condition.

In the last section, we want to exclude the possibility that the union of the surfaces $\phi^l(\Sigma_b)$ has a Zariski closure which is of dimension 3. We thus have to study

divisors in F invariant under a power of ϕ and containing a Zariski dense union of lagrangian surfaces birationally equivalent to abelian surfaces. We use the restriction of the holomorphic symplectic form of F to such divisors to classify them (cf section 3.1) into two types (see corollary 3.2). The conclusion then comes in two steps, the first one being geometric, the second one being arithmetic : we exclude one type by geometric considerations in section 3.2 and we show in section 3.3 that the remaining case can be excluded using the non-vanishing of the Abel-Jacobi invariant as above.

1 Preliminaries

Let $X \subset \mathbb{P}^5$ be a smooth cubic fourfold, and F the variety of lines of X ; let us denote by $P \subset F \times X$ the universal incidence correspondence, that is, $p : P \rightarrow F$ is the universal family of lines parameterized by F and $q : P \rightarrow X$ is the natural inclusion in $X \subset \mathbb{P}^5$ on each of these lines. Recall from [3] that the Abel-Jacobi map

$$p_*q^* : H^4(X_{\mathbb{C}}, \mathbb{Q}) \rightarrow H^2(F_{\mathbb{C}}, \mathbb{Q})$$

is an isomorphism of rational Hodge structures (of bidegree $(-1, -1)$), which sends h^2 , $h := c_1(\mathcal{O}_X(1))$, to $c_1(\mathcal{L})$, where \mathcal{L} is the restriction to F of the Plücker line bundle. (In the whole paper, we will denote by $X_{\mathbb{C}}$ the complex manifold associated to a variety defined over a subfield of \mathbb{C} , and $H^*(X_{\mathbb{C}}, R)$ will then denote the Betti cohomology of this complex manifold with coefficients in R .) The number $h^{3,1}(X)$ is equal to 1, as shown by Griffiths theory, and we thus have $h^{2,0}(F) = 1$. Let $\omega = q_*p^*\alpha$ be a generator of $H^{2,0}(F) = H^0(F, \Omega_F^2)$, where α generates $H^{3,1}(X)$. Then ω is a non-degenerate 2-form by [3].

On the other hand, it was proved by Terasoma [24] that many cubic fourfolds X defined over number fields satisfy the property that the Hodge classes in $H^4(X_{\mathbb{C}}, \mathbb{Q})$ are rational multiples of h^2 . (Terasoma formulates his result in terms of the non-existence of non-zero primitive algebraic cycle classes, but this is equivalent to the non-existence of non-zero primitive Hodge classes, as the Hodge conjecture is known to be true for cubic 4-folds, see [28], [8].) We deduce from this together with the fact that $H^2(F_{\mathbb{C}}, \mathbb{Z})$ has no torsion and the class $c_1(\mathcal{L})$ is not divisible in $H^2(F_{\mathbb{C}}, \mathbb{Z})$, that the corresponding variety F , which is also defined over a number field, satisfies $\text{Pic } F_{\mathbb{C}} = \mathbb{Z}$, with generator \mathcal{L} . Equivalently, the geometric Picard group of F is equal to $\mathbb{Z}\mathcal{L}$.

For each smooth hyperplane section $Y \subset X$, the variety of lines contained in Y is a smooth surface Σ_Y of class $c := c_2(\mathcal{E})$, where \mathcal{E} is the restriction to F of the universal rank 2 bundle on the Grassmannian. Furthermore, these surfaces are isotropic with respect to ω . Indeed, denoting by

$$p_Y : P_Y \rightarrow \Sigma_Y, \quad q_Y : P_Y \rightarrow Y$$

the incidence variety for Y , we have the following equality:

$$\omega|_{\Sigma_Y} = p_{Y*} \circ q_Y^*(\alpha|_Y) \text{ in } H^{2,0}(\Sigma_Y).$$

But by Lefschetz theorem on hyperplane sections, we have $H^{3,1}(Y) = 0$, and hence $\alpha|_Y = 0$. Thus $\omega|_{\Sigma_Y} = 0$.

Recall next the observation made in [26] : consider first the case where Y has one ordinary double point $y \in Y$. Then from [7], we know that the normalization of

Σ_Y is isomorphic to the symmetric product $C_x^{(2)}$, where C_x is the curve of lines in Y passing through y . This curve is the complete intersection of a cubic and a quadric in \mathbb{P}^3 . The map $C_x^{(2)} \rightarrow \Sigma_Y$ sends a pair of lines l_1, l_2 through y to the residual line l of the intersection $Q \cap Y$ of the plane $Q := \langle l_1, l_2 \rangle$ generated by l_1 and l_2 ($Q \cap Y$ is a plane cubic curve which already contains the two lines l_1, l_2). The inverse of this map associates to a generic line l in Y not passing through y the pair of lines which forms the residual intersection of the plane $\langle l, y \rangle$ with Y (this is indeed a plane conic singular at y).

The curve C_x is smooth and has genus 4.

Assume now that Y has exactly two more double points y' and y'' , (so y, y', y'' are not on the same line, as otherwise this line would be contained in $Sing Y$). Then the curve C_x also acquires exactly two double points, corresponding to the lines l', l'' joining y respectively to y' and y'' , which are obviously contained in Y (we refer to [7], or to the proof of lemma 1.2 below for the geometry of nodal cubic threefolds).

Thus the normalization of C_x has genus 2, and its second symmetric product is birational to an abelian surface.

Remark 1.1 This abelian surface is isomorphic to the intermediate Jacobian of the desingularization \tilde{Y} of Y obtained by blowing-up the nodes. It is also isomorphic via the Abel-Jacobi map to the group $CH_1(\tilde{Y})_{hom}$ (cf [4], [7]).

Imposing three double points to Y imposes three conditions to the corresponding hyperplane of \mathbb{P}^5 . Thus we get a 2-dimensional family $(\Sigma_b)_{b \in B}$ of such (highly singular) surfaces in F , and we will denote by $(A_b)_{b \in B}$ the corresponding family of abelian surfaces, which are birationally equivalent to them. Thus A_b can be defined as $Alb \Sigma'_b$ for some desingularization Σ'_b of Σ_b .

Note that if X is defined over a number field, so are B and the family of abelian surfaces $(A_b)_{b \in B}$.

We will need the following lemma:

Lemma 1.2 *A generic genus 2 curve appears as the normalization of the family of lines through a singular point of some cubic threefold with three non-collinear double points.*

Proof. A generic genus 2 curve C is the normalization of a complete intersection C' of a smooth quadric and a cubic surface in \mathbb{P}^3 , where C' admits exactly 2 double points. Indeed, it suffices to choose two generic line bundles of degree 3 on C , which will provide two maps from C to \mathbb{P}^1 . Then for a generic choice, the induced map from C to $\mathbb{P}^1 \times \mathbb{P}^1$ has for image a nodal curve C' . This curve is in the linear system $|\mathcal{O}(3, 3)|$ on $\mathbb{P}^1 \times \mathbb{P}^1$, which equivalently means that C' is a complete intersection $Q \cap S$, where S has degree 3 in \mathbb{P}^3 and $Q \cong \mathbb{P}^1 \times \mathbb{P}^1$ is a smooth quadric. As the arithmetic genus of C' is 4, C' has 2 double points.

Now, following [7], let Z be the blow-up of \mathbb{P}^3 along C' , and consider the morphism from Z to \mathbb{P}^4 given by the cubic equations vanishing on C' : if q, t are given choices of quadratic, resp. cubic generating equations for the ideal of C' , this morphism is given by the linear system of cubic polynomial on \mathbb{P}^3 vanishing along C' :

$$X_0q, \dots, X_3q, t.$$

Denoting by Y_0, \dots, Y_4 the homogeneous coordinates on \mathbb{P}^4 used above, the image of Z under this morphism is the cubic threefold Y which has for equation

$$t(Y_0, \dots, Y_3) = Y_4 q(Y_0, \dots, Y_3).$$

It is immediate to verify that Y has exactly 3 ordinary double points, one of them being $o := (0, \dots, 0, 1)$, the others satisfying $Y_4 = 0$ and corresponding to the double points of C' . The curve of lines in Y through o is sent isomorphically to C' by the projection from o . ■

Our arguments in the next sections will apply to those X satisfying the conclusions of the following lemma. These X 's will thus satisfy the conclusion of theorem 0.1.

Lemma 1.3 *For many X 's defined over a number field, the following two properties are satisfied:*

- *The corresponding variety of lines F defined over the same number field has geometric Picard group equal to \mathbb{Z} .*
- *For some (and in fact many) closed points $b \in B$ defined over a number field, the abelian surfaces A_b have geometric Néron-Severi group equal to \mathbb{Z} .*

Proof. The proof goes exactly as in Terasoma [24]. Consider the family \mathcal{M} parameterizing pairs (A, X) where A is a principally polarized abelian variety which is the intermediate Jacobian of the desingularization of a hyperplane section Y of X with exactly three independent nodes (with its canonical polarization). \mathcal{M} maps via the first projection to the moduli space of principally polarized abelian varieties \mathcal{A}_2 , and via the second projection, to the open set $U \subset \mathbb{P}(H^0(\mathbb{P}^5, \mathcal{O}_{\mathbb{P}^5}(3)))$ parameterizing smooth cubic 4-folds. Lemma 1.2 shows that both maps are dominating. Replacing \mathcal{M} and \mathcal{A}_2 by finite covers \mathcal{M}' , \mathcal{A}'_2 if necessary, we may assume there are universal families

$$\mathcal{T} \rightarrow \mathcal{A}'_2, \mathcal{X} \rightarrow U, \mathcal{T} \times_{\mathcal{M}'} \mathcal{X}_{\mathcal{M}'} \rightarrow \mathcal{M}'$$

(where $\mathcal{X}_{\mathcal{M}'}$ is the pull-back of \mathcal{X} to \mathcal{M}'). Note furthermore that all families are defined over \mathbb{Q} . We fix compatible base-points o, o_A, o_U for $\mathcal{M}', \mathcal{A}'_2, U$.

For the general polarized abelian complex surface T , one has $NS(T) = \mathbb{Z}\lambda$, which is equivalent to the condition that the only Hodge classes in $H^2(T, \mathbb{Q})$ are multiples of the polarization $\lambda \in H^2(A, \mathbb{Q})$.

This is a consequence of the stronger result saying that the monodromy group

$$Im \rho_A : \pi_1(\mathcal{A}'_{2, \mathbb{C}}, o_A) \rightarrow Aut H^2(T_{o_A, \mathbb{C}}, \mathbb{Q})$$

does not act via a finite group on any non-zero class in λ^\perp .

As the image of $\pi_1(\mathcal{M}'_{\mathbb{C}}, o)$ in $\pi_1(\mathcal{A}'_{2, \mathbb{C}}, o_A)$ has finite index, the same holds for

$$Im \rho'_A : \pi_1(\mathcal{M}'_{\mathbb{C}}, o) \rightarrow Aut H^2(T_{o, \mathbb{C}}, \mathbb{Q}).$$

Similarly, there is no non-zero class in $H^4(X_{o_U, \mathbb{C}}, \mathbb{Q})_{prim}$ on which the monodromy group

$$Im \rho_U : \pi_1(U_{\mathbb{C}}, o_U) \rightarrow Aut H^4(X_{o_U, \mathbb{C}}, \mathbb{Q})_{prim}$$

acts via a finite group, which implies as above the Noether-Lefschetz theorem, saying that for general X , there is no non-trivial Hodge class in $H^4(X, \mathbb{Q})_{prim}$. It follows that there is no non-zero class in $H^4(X_{o_U, \mathbb{C}}, \mathbb{Q})_{prim}$ on which the monodromy group

$$Im \rho'_U : \pi_1(\mathcal{M}'_{\mathbb{C}}, o_U) \rightarrow Aut H^4(X_{o_U, \mathbb{C}}, \mathbb{Q})_{prim}$$

acts via a finite group. Of course, we have the same conclusions for the monodromy acting on cohomology with \mathbb{Q}_l coefficients:

$$\rho'_A : \pi_1(\mathcal{M}'_{\mathbb{C}}, o) \rightarrow Aut H^2(T_{o, \mathbb{C}}, \mathbb{Q}_l), \rho'_U : \pi_1(\mathcal{M}'_{\mathbb{C}}, o_U) \rightarrow Aut H^4(X_{o_U, \mathbb{C}}, \mathbb{Q}_l)_{prim}.$$

Finally, we use the density theorem of Terasoma [24], which says that for infinitely many points o of \mathcal{M} defined over a number field k_o , the image of the Galois group $Gal(\overline{k_o}/k_o)$ acting on

$$\begin{aligned} & H_{et}^2(T_{o, \overline{k_o}}, \mathbb{Q}_l) \oplus H_{et}^4(X_{o, \overline{k_o}}, \mathbb{Q}_l)_{prim} \\ & \cong H^2(T_{o, \mathbb{C}}, \mathbb{Q}_l) \oplus H^4(X_{o, \mathbb{C}}, \mathbb{Q}_l)_{prim} \end{aligned}$$

is equal to the algebraic monodromy group, that is, to the image of $\pi_{1, alg}(\mathcal{M}')$ acting on

$$H_{et}^2(\overline{\mathcal{T}_{k(\mathcal{M}')}}, \mathbb{Q}_l) \oplus H_{et}^4(\overline{\mathcal{X}_{k(\mathcal{M}')}}, \mathbb{Q}_l).$$

As the algebraic monodromy group contains the classical monodromy group $Im(\rho'_A, \rho'_U)$, it thus follows that for such a point o , there is no non-zero cycle class in $H_{et}^2(T_{o, \overline{k_o}}, \mathbb{Q}_l)$ which is not proportional to λ , or in $H_{et}^4(X_{o, \overline{k_o}}, \mathbb{Q}_l)_{prim}$, because on cycle classes, the Galois group acts via a finite group. As cycle classes in $H_{et}^4(X_{o, \overline{k_o}}, \mathbb{Q}_l)$ correspond bijectively via p_*q^* to $(Pic F_{o, \overline{k_o}}) \otimes \mathbb{Q}_l$, the proof is complete. \blacksquare

Fix X satisfying the conclusion of the above lemma, and let us come back to the 2-dimensional variety B parameterizing very singular surfaces Σ_b which are birationally equivalent to abelian surfaces A_b (thus B is the fiber of the map $\mathcal{M} \rightarrow U$ introduced above). We shall take a desingularisation of the total space of the family $(\Sigma_b)_{b \in B}$. Eventually after replacing B by a Zariski open subset, this gives us a new family parameterized by B , with smooth generic fiber. The fibers will be denoted by Σ'_b . We have then $A_b = Alb \Sigma'_b$.

Let $k(B)$ be the function field of B . Consider the following supplementary assumption on the point $b \in B$ defined over a number field $k(b) \subset \mathbb{C}$. We will assume first $b \in B_{reg}$, the Zariski open subset of B over which the family $(\Sigma'_b)_{b \in B}$ is a smooth family of surfaces. Then there is a family of Albanese varieties A_b over B_{reg} , so that we have on one hand the algebraic monodromy acting on the étale cohomology of the generic geometric fiber

$$\begin{aligned} \pi_{1, alg}(B_{reg}) & \rightarrow Aut(H_{et}^1(A_{\overline{k(B)}}, \mathbb{Q}_l) \oplus H_{et}^2(A_{\overline{k(B)}}, \mathbb{Q}_l)_{prim}) \cong \\ & \cong Aut(H_{et}^1(A_{b, \overline{k(b)}}, \mathbb{Q}_l) \oplus H_{et}^2(A_{b, \overline{k(b)}}, \mathbb{Q}_l)_{prim}) \end{aligned}$$

and on the other hand the Galois group action acting on the étale cohomology of the closed geometric fiber

$$Gal(\overline{k(b)}/k(b)) \rightarrow Aut(H_{et}^1(A_{b, \overline{k(b)}}, \mathbb{Q}_l) \oplus H_{et}^2(A_{b, \overline{k(b)}}, \mathbb{Q}_l)_{prim}).$$

By Terasoma [24], the image of the Galois group representation is equal to the algebraic monodromy group for many closed points b of B defined over a number field. When this is satisfied, the abelian surface $A_b = \text{Alb } \Sigma'_b$ is geometrically simple, and in fact has geometric Néron-Severi group equal to \mathbb{Z} . Indeed, by our choice of X , we know by lemma 1.3 that this is true for the abelian surface corresponding to a sufficiently general point of $B(\mathbb{C})$, and arguing as in the proof of lemma 1.3, this implies the same results for those fibers A_b satisfying this condition.

The action on the first cohomology is considered because later it shall be used in the proof of Proposition 2.1, which is the key ingredient for our main theorem. Note that $H_{et}^1(A_{b,\bar{k}(b)}, \mathbb{Q}_l) = H_{et}^1(\Sigma'_{b,\bar{k}(b)}, \mathbb{Q}_l)$, since $A_b = \text{Alb } \Sigma'_b$.

To conclude this preliminary section, recall from [25] that for any cubic 4-fold X , the Fano variety of lines F carries a rational self-map

$$\phi : F \dashrightarrow F$$

which can be described as follows:

If $t \in F$ parameterizes a line $\Delta_t \subset X$, and t is generic, then there exists a unique plane $Q_t \subset \mathbb{P}^5$ containing Δ_t which is everywhere tangent to X along Δ_t . As X is not swept out by planes, Δ_t is not contained in any plane contained in X , and thus $Q_t \cap X = 2\Delta_t + \Delta_{t'}$ for some point $t' =: \phi(t) \in F$.

In the generic case, X does not contain any plane, and thus the indeterminacy locus of ϕ consists of those points $t \in F$ for which the plane Q_t is not unique, which means equivalently that there is a $\mathbb{P}^3 \subset \mathbb{P}^5$ everywhere tangent to X along Δ_t .

One can show that the indeterminacy locus of ϕ is then a smooth surface of general type, and compute its invariants (see for example [1]). The indeterminacy of ϕ is resolved by a single blow-up along this surface.

An obvious but crucial property of ϕ that we will use constantly in the paper is the following (cf [25]) :

Lemma 1.4 *For $t \in F$ away from the indeterminacy locus of ϕ , one has the equality*

$$2\Delta_t + \Delta_{\phi(t)} = h^3 \text{ in } CH^3(X);$$

in particular, the class of $2\Delta_t + \Delta_{\phi(t)}$ in $CH^3(X)$ does not depend on t .

One first consequence is the following (see [25]):

Corollary 1.5 *One has $\phi^*\omega = -2\omega$.*

Indeed, recall the incidence correspondence $P \subset F \times X$. Then the line Δ_t has by definition its class in $CH^3(X)$ equal to $P_*(t)$. Thus lemma 1.4 can be restated by saying that

$$P_*(\phi(t)) = -2P_*t + h^3 \text{ in } CH^3(X) \tag{1.1}$$

for generic $t \in F$. Mumford's theorem conveniently generalized (cf [27], Proposition 10.24) then tells us that we have

$$\phi^*(P^*\alpha) = -2P^*\alpha,$$

for $\alpha \in H^{3,1}(X)$ as above. As we have $P^*\alpha = \omega$, this gives the result. ■

Remark 1.6 We use here a very easy special instance of Mumford's theorem. Indeed, we actually have (away from a codimension 2 subset) an equality of effective 1-cycles

$$P_*(\phi(t)) + 2P_*t = Q_t \cap X,$$

from which one deduces that the $(2,0)$ -form $\phi^*(P^*\alpha) + 2P^*\alpha$ on F is the restriction to F of a $(2,0)$ -form on the Grassmannian of 2-planes in \mathbb{P}^5 , hence is 0.

Another corollary which will be used is the following:

Corollary 1.7 [25] *We have*

$$\deg \phi = 16. \tag{1.2}$$

Proof. Indeed, denoting by $\tau : \tilde{F} \rightarrow F$ the blow-up along the indeterminacy locus, and by $\tilde{\phi} : \tilde{F} \rightarrow F$ the resulting desingularization of the rational map ϕ , it follows from the previous corollary that $\tilde{\phi}^*\omega^2 = 4\tau^*\omega^2$ so that

$$\tilde{\phi}^*(\omega^2 \wedge \bar{\omega}^2) = 16\tau^*(\omega^2 \wedge \bar{\omega}^2).$$

Integrating over \tilde{F} gives the result, as $\omega^2 \wedge \bar{\omega}^2$ is a volume form on F . ■

2 Non-preperiodicity

As mentioned in the introduction, Theorem 0.1 will be obtained as a consequence of the following fact : there exist (many) X 's defined over a number field k , and (many) surfaces Σ_b as in the previous section, also defined over a number field, such that the surfaces $\phi^l(\Sigma_b)$, $l \in \mathbb{N}$, are Zariski dense in X .

Our goal in this section is to show that for many surfaces Σ_b , the set of distinct iterations $\phi^l(\Sigma_b)$ is infinite. This is proved in proposition 2.3. Note that ϕ^l is always generically defined along Σ_b . Indeed, we may assume by induction on l that ϕ^{l-1} is generically defined along Σ_b . Then $\phi^{l-1}(\Sigma_b)$ must be a surface, as proved in [1]. If ϕ^l were not generically defined along Σ_b , then $\phi^{l-1}(\Sigma_b)$ would be contained in the indeterminacy locus of ϕ . But this indeterminacy locus is a surface of general type ([1]), hence cannot be dominated by a surface which is birationally equivalent to an abelian surface. So it does make sense to speak of $\phi^l(\Sigma_b)$, and each $\phi^l(\Sigma_b)$ is indeed a surface.

The strategy is as follows. Recall from lemma 1.7 that we have $\deg \phi = 16$ and thus $\deg \phi^l = 16^l$. What we will show, using proposition 2.1, is the fact that if a surface Σ_b satisfies

$$\phi^l(\phi^k(\Sigma_b)) = \phi^k(\Sigma_b)$$

for some $l > 0$, and if furthermore a certain Terasoma-type condition holds for the point b , then

$$\phi^l : \phi^k(\Sigma_b) \rightarrow \phi^k(\Sigma_b)$$

has to be of degree 16^l . This shall imply the following equality of degree 4 cohomology classes

$$\phi^{l*}([\phi^k(\Sigma_b)]) = [\phi^k(\Sigma_b)],$$

contradicting the results of [1], which show that 1 is not an eigenvalue of ϕ^{l*} acting on $H^4(F, \mathbb{Q})$.

An important step in the proof is the following result concerning the restriction to the surfaces Σ'_b of the incidence correspondence P between F and X :

Proposition 2.1 1) *The morphism*

$$P_{b*} : CH_0(\Sigma'_{b,\mathbb{C}}) \rightarrow CH_1(X_{\mathbb{C}})$$

induced by the pull-back to $\Sigma'_b \rightarrow \Sigma_b \subset F$ of the incidence correspondence $P \subset F \times X$ satisfies: $P_{b*}|_{CH_0(\Sigma'_{b,\mathbb{C}})_{hom}}$ factors through the Albanese map

$$CH_0(\Sigma'_{b,\mathbb{C}})_{hom} \xrightarrow{alb} Alb \Sigma'_{b,\mathbb{C}} \cong A_{b,\mathbb{C}}.$$

2) *For many closed points $b \in B$ defined over a number field, the group morphism induced by $P_{b,*}$*

$$P_{b,alb} : A_{b,\mathbb{C}} = Alb \Sigma'_{b,\mathbb{C}} \rightarrow CH_1(X_{\mathbb{C}})$$

is non-zero.

The closed points of the second part of the proposition are exactly the points satisfying a certain Terasoma-type condition, which shall be stated in the end of this section.

Remark 2.2 It is essential here to consider the Chow groups of the corresponding complex algebraic variety, because conjecturally $CH_1(X)_{hom}$ is trivial for X defined over a number field. Also, $CH_1(X_{\mathbb{C}})$ has no torsion, and thus the second part of the proposition is equivalent to saying that the group morphism $P_{b,alb}$ is not of torsion.

We postpone the proof of proposition 2.1 and prove now the following proposition, which is the main result of this section.

Proposition 2.3 *If b satisfies the conclusion of Proposition 2.1, 2), the Zariski closure of the union of surfaces $\phi^l(\Sigma_b)$, $l \in \mathbb{N}$, contains at least a divisor.*

As mentioned above, because we already know that the $\phi^l(\Sigma_b)$ are surfaces, the statement is equivalent to the fact that there are actually infinitely many distinct surfaces $\phi^l(\Sigma_b)$, $l \in \mathbb{N}$. Equivalently, we have to prove that we do not have $\phi^{l'}(\Sigma_b) = \phi^k(\Sigma_b)$ for some $l', k, l' > k$. This is equivalent to saying that for any $l > 0, k \geq 0$

$$\phi^l(\phi^k(\Sigma_b)) \neq \phi^k(\Sigma_b), \tag{2.1}$$

or that no surface $\phi^k(\Sigma_b)$, $k \in \mathbb{Z}_{\geq 0}$, is periodic for ϕ . In current terminology, one says that Σ_b is not preperiodic for ϕ .

The proof of proposition 2.3 will proceed by contradiction. So let us assume that $\Sigma_{b,k} = \phi^l(\Sigma_{b,k})$ for some $l > 1$, where $\Sigma_{b,k} := \phi^k(\Sigma_b)$. Let us choose a desingularization $\Sigma'_{b,k}$ of $\Sigma_{b,k}$. We first prove:

Lemma 2.4 *The Albanese map*

$$alb : \Sigma'_{b,k} \rightarrow Alb \Sigma'_{b,k}$$

is a birational map.

Proof. First observe that $\Sigma'_{b,k}$ is rationally dominated via the rational map ϕ^k by the abelian surface $A_b = \text{Alb } \Sigma'_b$. On the other hand, let

$$\phi_*^k : A_{b,\mathbb{C}} = \text{Alb } \Sigma'_{b,\mathbb{C}} \rightarrow \text{Alb } \Sigma'_{b,k,\mathbb{C}}$$

be induced by the rational map ϕ^k between Σ'_b and $\Sigma'_{b,k}$. We claim that ϕ_*^k is an isogeny between $\text{Alb } \Sigma'_{b,k,\mathbb{C}}$ and $\text{Alb } \Sigma'_{b,\mathbb{C}}$.

Indeed, this follows from the following three facts: We know from proposition 2.1, 1) that for Σ'_b (hence also for $\Sigma'_{b,k}$ by property (1.1)), the restriction $P_{b*} : CH_0(\Sigma'_{b,\mathbb{C}})_{\text{hom}} \rightarrow CH_1(X_{\mathbb{C}})$ (resp. $P_{b,k,*} : CH_0(\Sigma'_{b,k,\mathbb{C}})_{\text{hom}} \rightarrow CH_1(X_{\mathbb{C}})$) factors through $\text{Alb } \Sigma'_{b,\mathbb{C}}$ (resp. $\text{Alb } \Sigma'_{b,k,\mathbb{C}}$). Let

$$P_{b,k,alb} : \text{Alb } \Sigma'_{b,k,\mathbb{C}} \rightarrow CH_1(X_{\mathbb{C}}), \quad P_{b,alb} : A_{b,\mathbb{C}} \rightarrow CH_1(X_{\mathbb{C}})$$

be induced respectively by the maps $P_{b,k,*}$ and P_{b*} .

By formula (1.1), factoring everything through the Albanese maps, we get

$$P_{b,k,alb} \circ \phi_*^k = (-2)^k P_{b,alb} : A_{b,\mathbb{C}} \rightarrow CH_1(X_{\mathbb{C}}). \quad (2.2)$$

On the other hand, we know by Proposition 2.1, 2) that $P_{b,alb} : A_{b,\mathbb{C}} \rightarrow CH_1(X_{\mathbb{C}})$ is non-zero. As $P_{b,*}$ is a group morphism which is induced by a correspondence, its kernel is a countable union of translates of an abelian subvariety of $A_{b,\mathbb{C}}$. Thus it must be countable because we know that A_b is a geometrically simple abelian variety and that $P_{b,*}$ is non-zero. Formula (2.2) then shows that the surjective morphism ϕ_*^k has countable kernel, hence that it is an isogeny.

In conclusion, the surface $\Sigma'_{b,k}$ sits between two abelian surfaces:

$$A_b \dashrightarrow \Sigma'_{b,k} \xrightarrow{\text{alb}} \text{Alb } \Sigma'_{b,k},$$

where the composed map is an isogeny. It follows immediately that $\text{alb} : \Sigma'_{b,k} \rightarrow \text{Alb } \Sigma'_{b,k}$ is birational. ■

As a consequence of this lemma and formula (2.2), we get now:

Corollary 2.5 *The rational map*

$$\phi^l : \Sigma'_{b,k} \dashrightarrow \Sigma'_{b,k}$$

has degree $(-2)^{4l} = (16)^l$.

Proof. Indeed, formula (2.2) applied to the maps ϕ^k and ϕ^{k+l} tells us that

$$P_{b,k,alb} \circ \phi_*^l = (-2)^l P_{b,k,alb} : \text{Alb } \Sigma'_{b,k,\mathbb{C}} \rightarrow CH_1(X_{\mathbb{C}}). \quad (2.3)$$

Here $\phi_*^l : \text{Alb } \Sigma'_{b,k,\mathbb{C}} \rightarrow \text{Alb } \Sigma'_{b,k,\mathbb{C}}$ is induced by the rational self-map $\phi^l : \Sigma'_{b,k} \rightarrow \Sigma'_{b,k}$. Furthermore, we know that $\text{Ker } P_{b,k,alb}$ is countable. From formula (2.3), we thus deduce that ϕ_*^l must be multiplication by $(-2)^l$, hence must have degree $(-2)^{4l}$. As the surface $\Sigma'_{b,k}$ is birationally equivalent to its Albanese variety by lemma 2.4, the same result holds for $\phi^l : \Sigma'_{b,k} \rightarrow \Sigma'_{b,k}$. ■

Proof of Proposition 2.3. We know that ϕ^l is generically well-defined along $\Sigma'_{b,k}$. Take the standard desingularization

$$\tilde{\phi} : \tilde{F} \rightarrow F, \pi : \tilde{F} \rightarrow F$$

obtained by blowing-up the indeterminacy locus of ϕ . It is proved in [1] that $\tilde{\phi}$ does not contract divisors and can only contract a surface which is already contracted by π (in fact even this is impossible, but needs some extra argument). It follows that for the natural desingularization $\tilde{\phi}^l : \tilde{F} \rightarrow F, \pi' : \tilde{F} \rightarrow F$ of ϕ^l obtained by taking the graph, the following holds: For any subvariety $\Sigma \subset \tilde{F}$ which is contracted under $\tilde{\phi}^l$, the image of Σ under π' is at most 1-dimensional. We apply this to any component Σ of $\tilde{\phi}^{l-1}(\Sigma_{b,k})$. We conclude that if either $\dim \Sigma \geq 3$ or Σ does not dominate $\Sigma_{b,k}$ via $\tilde{\phi}^l$, $\pi'(\Sigma)$ has dimension at most 1. On the other hand, $\pi'(\tilde{\phi}^{l-1}(\Sigma_{b,k}))$ contains $\Sigma_{b,k}$ by periodicity, and we have just seen that the restriction of ϕ^l to $\Sigma_{b,k}$ is already of degree $(16)^l = \deg \tilde{\phi}^l$. It thus follows that the only 2-dimensional component of $\tilde{\phi}^{l-1}(\Sigma_{b,k})$ dominating $\Sigma_{b,k}$ is sent birationally by π' to $\Sigma_{b,k}$, while the non-dominating components or the three dimensional component are contracted to curves or points in F by π' .

This implies that we have the equality of 2-dimensional cycles:

$$(\phi^l)^*(\Sigma_{b,k}) = \Sigma_{b,k}$$

and thus

$$(\phi^l)^*([\Sigma_{b,k}]) = [\Sigma_{b,k}],$$

where $[\cdot]$ denotes the cohomology class of a cycle. This contradicts the computation of the eigenvalues of $(\phi^l)^*$ on $H^4(F_{\mathbb{C}}, \mathbb{Q})$ performed in [1].

The proof is now complete, assuming Proposition 2.1. ■

Proof of Proposition 2.1. Statement 1) follows from the fact that the surface Σ_b is the surface of lines of a nodal cubic threefold Y_b , for which we know that the Abel-Jacobi map on a desingularization \tilde{Y}_b induces an isomorphism:

$$CH_1(\tilde{Y}_{b,\mathbb{C}})_{hom} \xrightarrow{\Phi_{\tilde{Y}_b}} J^3(\tilde{Y}_{b,\mathbb{C}}).$$

The correspondence from Σ'_b to X thus factors to a correspondence from Σ'_b to \tilde{Y}_b , which, when restricted to $CH_0(\Sigma'_{b,\mathbb{C}})_{hom}$ has to factor through $Alb \Sigma'_{b,\mathbb{C}} = J^3(\tilde{Y}_{b,\mathbb{C}})$.

For the proof of statement 2), we will follow the argument of [12]. In fact this will be as in [12] a consequence of an even stronger statement (Proposition 2.7), the proof of which is attributed there to Bloch and Nori, and which we will also need in section 3.3.

Let us first introduce a corrected codimension 3 cycle

$$P' \in CH^3(F \times X)_{\mathbb{Q}}$$

which has the property that its cohomology class satisfies

$$[P'] \in H^2(F_{\mathbb{C}}, \mathbb{Q}) \otimes H^4(X_{\mathbb{C}}, \mathbb{Q})_{prim}, \tag{2.4}$$

and which differs from the incidence cycle P by a combination of cycles of the form $\Gamma_i \times h^i$, $\Gamma_i \in CH^{3-i}(F)$. This cycle is obtained by considering the cycle j_*P on $F \times \mathbb{P}^5$, where j is the inclusion of X in \mathbb{P}^5 (or of $F \times X$ in $X \times \mathbb{P}^5$). Denoting by $H \in CH^1(\mathbb{P}^5)$ the class of $\mathcal{O}_{\mathbb{P}^5}(1)$, this cycle $j_*P \in CH^4(F \times \mathbb{P}^5)$ can be written as $\sum_{i=1}^{i=4} pr_1^* \Gamma_i \cdot pr_2^* H^i$, with $\Gamma_i \in CH^{4-i}(F)$ (cf [27], Theorem 9.25). The fact that the sum is only over $i \geq 1$ is due to the fact that $H^5 j_*P = 0$. Let

$$P' := P - \frac{1}{3} \left(\sum_{i=0}^{i=3} pr_1^* \Gamma_{i+1} \cdot pr_2^* h^i \right) \in CH^3(F \times X)_{\mathbb{Q}},$$

where now the projectors are those of $F \times X$ to its factors. Then $j_*P' = 0$ because $j_*h^i = 3H^{i+1}$, and property (2.4) follows by Lefschetz theorem on hyperplane sections, which tells that $\text{Ker}(j_* : H^*(X_{\mathbb{C}}, \mathbb{Q}) \rightarrow H^{*+2}(\mathbb{P}_{\mathbb{C}}^5, \mathbb{Q}))$ is zero for $* \neq 4$.

We can see, using Poincaré duality on X , the vector space $H^2(F_{\mathbb{C}}, \mathbb{Q}) \otimes H^4(X_{\mathbb{C}}, \mathbb{Q})_{\text{prim}}$ as

$$\text{Hom}(H^4(X_{\mathbb{C}}, \mathbb{Q})_{\text{prim}}, H^2(F_{\mathbb{C}}, \mathbb{Q})).$$

Thus, for any morphism $f : T \rightarrow F$, the cycle f^*P' is cohomologous to 0 if and only if the map

$$f^* \circ P^* : H^4(X_{\mathbb{C}}, \mathbb{Q})_{\text{prim}} \rightarrow H^2(T_{\mathbb{C}}, \mathbb{Q})$$

is 0. (Note that, by definition of P' , $P^* = P'^*$ on $H^4(X_{\mathbb{C}}, \mathbb{Q})_{\text{prim}}$.)

Assume now that this last condition is satisfied. (This is true in particular if T is one of the surfaces Σ'_b , by the same argument as in section 1, where we prove that the surfaces Σ_b are Lagrangian with respect to ω .)

Then the cycle f^*P' is cohomologous to 0. If furthermore T and f are defined over a number field k , then by the comparison theorems between étale l -adic and Betti cohomology for algebraically closed fields of characteristic 0, the cycle f^*P' has trivial l -adic étale cohomology class in $H_{\text{ét}}^6(T_{\bar{k}} \times_{\bar{k}} X_{\bar{k}}, \mathbb{Q}_l(3))$. On the other hand, the cycle f^*P' defined over k has a *continuous étale* (cf [15], [21]) cohomology class

$$[f^*P'] \in H^6(T_k \times_k X_k, \mathbb{Q}_l(3)).$$

(Continuous étale cohomology is first defined for étale coefficients sheaves like \mathbb{Z}_l , which are inverse limits of finite coefficients sheaves. Continuous étale cohomology with coefficients in a \mathbb{Q}_l -module is then defined by tensoring with \mathbb{Q}_l .)

Following [21], we omit in the sequel both “continuous” and “étale” in the notation for these cycle classes.

The Hochschild-Serre spectral sequence for continuous étale cohomology then gives us the l -adic Abel-Jacobi invariant

$$\gamma_{f^*P'} \in H^1(k, H^5(T_{\bar{k}} \times_{\bar{k}} X_{\bar{k}}, \mathbb{Q}_l(3))),$$

which lies in fact (because the cycle P' is deduced from P by a Chow-Künneth decomposition) in the subspace $H^1(k, H^1(T_{\bar{k}}, \mathbb{Q}_l) \otimes_{\mathbb{Q}_l} H^4(X_{\bar{k}}, \mathbb{Q}_l(3)))$.

(Here $H^1(k, \cdot)$ denotes the continuous étale cohomology of $\text{Spec } k$.)

Coming back to the case where T is one of our surfaces Σ'_b , with $b \in B$ a point defined over a number field $k(b)$, let us denote by $P'_b \in CH^3(\Sigma'_b \times X)_{\mathbb{Q}}$ the cycle τ_b^*P' , where $\tau_b : \Sigma'_b \times X \rightarrow F \times X$ is the natural map. Using the Bloch-Srinivas lemma (cf [5] or [27], Corollary 10.20) and the fact that there is no non-zero algebraic cycle class in $H_{\text{ét}}^4(X_{\bar{k}(b)}, \mathbb{Q}_l(2))_{\text{prim}}$, we get as in [12] the following lemma:

Lemma 2.6 *If the l -adic Abel-Jacobi invariant*

$$\gamma_{P'_b} \in H^1(k(b), H^1(\Sigma'_{k(b)}, \mathbb{Q}_l) \otimes_{\mathbb{Q}_l} H^4(X_{k(b)}, \mathbb{Q}_l)(3))$$

is non zero, then the morphism

$$P_{b,alb} : Alb \Sigma'_{b,\mathbb{C}} = A_{b,\mathbb{C}} \rightarrow CH_1(X_{\mathbb{C}})$$

is non zero.

Using this lemma, the statement 2) of proposition 2.1 is a consequence of the following proposition:

Proposition 2.7 *For many closed points $b \in B$ defined over a number field, the l -adic Abel-Jacobi invariant*

$$\gamma_{P'_b} \in H^1(k(b), H^1(\Sigma'_{k(b)}, \mathbb{Q}_l) \otimes_{\mathbb{Q}_l} H^4(X_{k(b)}, \mathbb{Q}_l)(3))$$

is non-zero.

Proof. The proof of this statement goes as in [12], once we notice the following facts: We have a 2-dimensional family of surfaces which sweep out F :

$$\mathcal{S}_B := \cup_{b \in B} \Sigma'_b \xrightarrow{s} F$$

where s is dominating. For a generic curve $B' \subset B$ defined over a number field, the family of surfaces

$$\mathcal{S}_{B'} := \cup_{b \in B'} \Sigma'_b \xrightarrow{s} F$$

covers an hypersurface in F , and it thus follows that the pull-back map by s :

$$H^{2,0}(F) \rightarrow H^{2,0}(\mathcal{S}_{B'})$$

is non-zero because the $(2,0)$ -form ω is non-degenerate, hence cannot vanish on a divisor of F .

More precisely, for any Zariski open set $B'_0 \subset B'$, the cohomology class $[s^*\omega]$ does not vanish in the Betti cohomology space $H^2(\mathcal{S}_{B'_0,\mathbb{C}}, \mathbb{C})$, because it is of type $(2,0)$. We will take for B'_0 the locus where the fiber of our family of surfaces $\pi : \mathcal{S}_{B'} \rightarrow B'$ is smooth.

Hence we conclude similarly that the Betti cohomology class of the cycle

$$s^*P'_{|\mathcal{S}_{B'_0}} \in CH^3(\mathcal{S}_{B'_0} \times X)$$

does not vanish in

$$H^6(\mathcal{S}_{B'_0,\mathbb{C}} \times X_{\mathbb{C}}, \mathbb{Q}).$$

However, we know already that the cycle s^*P' restricted to the fibers $\Sigma'_b \times X$ of π has vanishing cohomology class. Hence $[s^*P'_{|\mathcal{S}_{B'_0}}]$ lies at least in the L^1 level of the Leray filtration associated to

$$\pi' = \pi \circ pr_1 : \mathcal{S}_{B'_0,\mathbb{C}} \times X_{\mathbb{C}} \rightarrow B'_{0,\mathbb{C}},$$

and gives an element of

$$H^1(B'_{0,\mathbb{C}}, R^5\pi'_{\mathbb{C}*}\mathbb{Q})$$

and in fact more precisely (using the fact that P' was obtained from P by applying a Chow-Künneth projector on X) in

$$H^1(B'_{0,\mathbb{C}}, R^1\pi_{\mathbb{C}*}\mathbb{Q} \otimes H^4(X_{\mathbb{C}}, \mathbb{Q})_{prim}).$$

This last class is non-zero because the Leray spectral sequence has only two terms, as the affine curve B'_0 has the homotopy type of a CW-complex of dimension 1.

We can now make the same construction in l -adic étale cohomology for the corresponding algebraic varieties defined over $\overline{\mathbb{Q}}$. Using the comparison theorems between Betti cohomology of the complex manifold and étale cohomology of the algebraic variety, we conclude that the étale cohomology class

$$[p^*P'_{|_{\mathcal{S}_{B'_0}}}]_{et} \in H_{et}^6(\mathcal{S}_{B'_0, \overline{\mathbb{Q}}} \times_{\overline{\mathbb{Q}}} X_{\overline{\mathbb{Q}}}, \mathbb{Q}_l(3))$$

is non-zero, lies in the L^1 level of the Leray filtration for étale cohomology, and has a non-zero image in

$$H_{et}^1(B'_{0, \overline{\mathbb{Q}}}, R^1\pi_{et*}\mathbb{Q}_l \otimes_{\mathbb{Q}_l} H^4(X_{\overline{\mathbb{Q}}}, \mathbb{Q}_l)_{prim}(3)).$$

This provides a non-trivial extension E of the étale local system $R^1\pi_{et*}\mathbb{Q}_l \otimes_{\mathbb{Q}_l} H^4(X_{\overline{\mathbb{Q}}}, \mathbb{Q}_l(3))$ on B'_0 by the trivial local system. As in [12], let us choose $b \in B'_0(\overline{\mathbb{Q}})$ satisfying the Terasoma condition (that is, the coincidence of the images of the monodromy and Galois representations) for E . Then we conclude as in [12], Lemma 7, that the corresponding extension of $Gal(\overline{k(b)}/k(b))$ -modules is also non-trivial, and it follows that the class

$$\gamma_{P'_b} \in H^1(k(b), H^1(\Sigma'_{k(b)}, \mathbb{Q}_l) \otimes_{\mathbb{Q}_l} H^4(X_{k(b)}, \mathbb{Q}_l)(3))$$

is non-zero. ■

3 Study of invariant divisors

We have now established that the Zariski closure of the union of surfaces $\phi^l(\Sigma_b)$ contains at least a divisor. Our goal in this section is to exclude the case where the Zariski closure of the union of the surfaces $\Sigma_{b,i}$ is of dimension 3. Observe now that if this closure is not equal to F , the union of its divisorial components is invariant under ϕ , hence each of its irreducible components must be invariant under some power ϕ^l .

Reasoning by contradiction, we will denote in the sequel by $\tau : D \rightarrow F$ a desingularization of a 3-dimensional irreducible component of the Zariski closure of the union of the surfaces $\Sigma_{b,i}$, where b is fixed and is supposed to satisfy the conclusion of Proposition 2.3. As this Zariski closure has only finitely many 3-dimensional components and is stable under ϕ , it follows that each of its irreducible components is stable under some power ϕ^l of ϕ .

We will denote by Σ_{b,n_i} the surfaces $\Sigma_{b,k}$ contained in $\tau(D)$ and Σ'_{b,n_i} their proper transforms under τ .

We shall also denote by ω_D the holomorphic 2-form $\tau^*\omega$. Notice that ω_D is a non-zero form, because ω is non-degenerate. Thus at a point where τ is of maximal rank 3, ω_D cannot vanish.

3.1 A Jouanolou type argument

The kernel of ω_D at the generic point is one-dimensional. This provides a line bundle L and an injective map $L \rightarrow T_D$ with torsion-free cokernel, generically defined as $\text{Ker } \omega_D : T_D \rightarrow \Omega_D$. As the surfaces Σ'_{b,n_i} are isotropic for ω_D , the image of L must be tangent to any $\Sigma'_{b,i}$, or at least to those not contained in the singular set of $\tau(D)$. Recall that the surfaces Σ'_{b,n_i} are Zariski dense in D . In particular there are infinitely many such surfaces which are distinct and not contained in the singular set of $\tau(D)$.

We start by proving an analogue of Jouanolou's theorem (cf [16]) for this situation:

Proposition 3.1 *There exists a rational map $f : D \dashrightarrow \mathbb{P}^1$ such that L is tangent to the fibers of f .*

Proof: For convenience of the reader, we recall at the same time the construction of Jouanolou. For large N , the classes of the surfaces Σ'_{b,n_i} , $i \leq N$ are not linearly independent in $NS(D)$. Therefore one can find integers m_i such that the line bundle associated to the divisor $M = \sum_{i=i_0}^N m_i \Sigma'_{b,n_i}$ has zero Chern class in $H^1(D, \Omega_D)$. Here, i_0 is chosen in such way that ω_D does not vanish on Σ'_{b,n_i} for $i \geq i_0$. Let U_α be an open covering of D , and f_α be a meromorphic function defining M on U_α . The vanishing of the first Chern class of the cocycle $g_{\alpha\beta} \in H^1(D, \mathcal{O}_D^*)$ means that on the intersections $U_{\alpha\beta} = U_\alpha \cap U_\beta$

$$\frac{df_\alpha}{f_\alpha} - \frac{df_\beta}{f_\beta} = \sigma_\alpha - \sigma_\beta$$

for some $\sigma_\alpha \in \Omega_{U_\alpha}^1$, $\sigma_\beta \in \Omega_{U_\beta}^1$. Therefore, the $\frac{df_\alpha}{f_\alpha} - \sigma_\alpha$ patch together into a meromorphic 1-form ψ_M with logarithmic singularities along the components of the support of M , defined up to a holomorphic 1-form, that is, we have a map

$$\psi : \text{Div}_\Sigma^0(D) \otimes \mathbb{C} \rightarrow H^0(D, \Omega_D^1 \otimes \mathcal{M}_D^*) / H^0(D, \Omega_D^1),$$

where \mathcal{M}_D^* denotes the sheaf of meromorphic functions and where $\text{Div}_\Sigma^0(D)$ is the kernel of the cycle class map, restricted to the subspace generated by the Σ'_{b,n_i} , $i \geq i_0$. Moreover, on $U_{\alpha\beta}$, $d\sigma_\alpha = d\sigma_\beta$, so there is a well-defined holomorphic 2-form $d\sigma$. It follows from the degeneration of ‘‘Hodge to de Rham’’ spectral sequence that $d\sigma$ is actually zero (cf. [16]). Therefore ψ_M is closed.

Moreover, the map ψ is injective (this is [16], Lemma 2.8).

The meromorphic 1-form ψ_M induces a meromorphic section of L^* . Since outside of the singularities of the foliation, that is, outside of an analytic subset of codimension at least two in D , L is tangent to Σ'_{b,n_i} along Σ'_{b,n_i} for $i \geq i_0$, this section is actually holomorphic, because ψ_M has logarithmic singularities along the smooth part of the Σ'_{b,n_i} 's appearing in the support of M . So the map ψ induces a map

$$\chi : \text{Div}_\Sigma^0(D) \otimes \mathbb{C} \rightarrow H^0(D, L^*) / V,$$

where V is the image in $H^0(D, L^*)$ of the space of global holomorphic 1-forms on D . The kernel of χ is clearly infinite-dimensional.

Let M and M' be two non-proportional elements of $\text{Ker } \chi \subset \text{Div}_\Sigma^0(D) \otimes \mathbb{C}$. Then we can choose the meromorphic forms ψ_M and $\psi_{M'}$ in such a way that

$$\psi_M|_L = 0, \psi_{M'}|_L = 0.$$

It then follows that L is contained in the kernel of the meromorphic 2-form $\psi_M \wedge \psi_{M'}$. But this property is also satisfied by our original 2-form ω_D . Therefore $\psi_M \wedge \psi_{M'} = f\omega_D$ for some meromorphic function f on D . We meet now two cases:

1) Assume that for all M and M' , f is a constant function on D . In this case, the meromorphic form $\psi_M \wedge \psi_{M'}$ is nonsingular. On the other hand, the meromorphic 1-form $\psi_{M'}$ has logarithmic singularities along the surfaces Σ'_{b,n_j} appearing in M' . The absence of singularities in $\psi_M \wedge \psi_{M'}$ then implies that the restriction of ψ_M to the surfaces Σ'_{b,n_j} appearing in M' and not in M is zero. But as M' is arbitrary, it then follows that there are infinitely many surfaces Σ'_{b,n_i} which are integral surfaces for the 1-form ψ_M . Jouanolou's theorem [16] then tells that there exists a non-constant meromorphic function from D to \mathbb{P}^1 , whose fibers are leaves of ψ_M . As ψ_M vanishes on L , we are done in this case.

2) In the second case, we have for some M' a non-constant rational map $f : D \dashrightarrow \mathbb{P}^1$. Finally, differentiating the equality $\psi_M \wedge \psi_{M'} = f\omega_D$, and using the fact that all our forms are closed, we get $df \wedge \omega_D = 0$, so df vanishes on $\text{Ker } \omega_D = L$. ■

Corollary 3.2 *We have two possibilities: either*

(1) *D is rationally fibered in (birationally) abelian varieties, and Σ'_{b,n_i} are fibers for all but finitely many i 's; or*

(2) *D is rationally fibered in curves over a surface T , the fibers are integral curves of the foliation L , and Σ'_{b,n_i} project onto curves in T . Furthermore the 2-form ω_D is pulled-back from a 2-form on T .*

Here, “rationally fibered” means that the fibration map is only rational, not necessarily regular. We can, of course, arrange for it to be regular by blowing-up D .

Proof: Consider the Stein factorisation $g : D \dashrightarrow C$ of the map f . If all but finitely many Σ'_{b,n_i} 's are fibers, we are done; if not, consider the general fiber D_c . On D_c , we still have the foliation L from the proposition 3.1. This foliation has infinitely many integral curves, which are components of $\Sigma'_{b,n_i} \cap D_c$, where i is such that Σ'_{b,n_i} is not a fiber. So, by the original Jouanolou's theorem, there must be a fibration $h_c : D_c \dashrightarrow Z_c$ over a curve, whose fibers are generically connected and tangent to L . A countability argument for the Chow varieties of curves in D shows that we can assume that the fibers of h_c , $c \in C$, form a family of algebraic curves covering D and consisting of integral leaves of L . It is then immediate that these curves provide a fibration of D to a surface T , which we may assume to have connected fibers.

As L is tangent to these fibers, and to the surfaces Σ'_{b,n_i} , it follows that the Σ'_{b,n_i} 's are contracted to curves in T . Finally, as the fibration has connected fibers, whose tangent space is generically the kernel of ω_D , it follows that ω_D comes from a form on T . ■

3.2 Case where D is fibered into abelian surfaces

In this subsection, we will exclude case 1) of Corollary 3.2. We argue again by contradiction, and assume that we have a divisor D , invariant under some power ϕ^l of ϕ , and admitting a fibration over a curve, whose fibers are surfaces birationally equivalent to abelian surfaces. We know furthermore that infinitely many fibers are surfaces Σ'_{b,n_i} , which are not periodic under ϕ^l .

The starting point is the following

Proposition 3.3 *The Kodaira dimension of D is zero.*

Proof: D is (rationally) fibered over a curve in varieties of Kodaira dimension zero. By the universal property of the Iitaka fibration, $\kappa(D)$ is thus at most one.

To exclude the case where $\kappa(D) = 1$, we use the result of [20], saying that a rational self-map ψ of any variety X induces an automorphism of finite order on the base B of the Iitaka fibration (note that the fact that ψ induces a polarization-preserving automorphism of B is clear because the pull-back ψ^* is an automorphism of the vector space of pluricanonical forms; the non-trivial statement is the finiteness of the order of the induced automorphism on B . Also, we do not need the full strength of Theorem A of [20]; the crucial Proposition 2.3 of *loc. cit.* for an abelian fibration suffices, and this is shown by a monodromy argument).

We now apply this to $X = D$ and $\psi = \phi^l$. If $\kappa(D) = 1$, then the map $f : D \dashrightarrow \mathbb{P}^1$ constructed in 3.1 is the Iitaka fibration, and the finiteness of the order of the induced map on \mathbb{P}^1 implies that the Σ_{b,n_i} (which are fibers of f) are periodic, whereas we know that this is not the case.

Suppose now that $\kappa(D) = -\infty$. Then D is uniruled. Consider the maximal rationally connected fibration $r : D \dashrightarrow Q$. As the fibers are rationally connected, every holomorphic form on D must be pulled-back from Q . Since D carries a non-zero holomorphic 2-form ω_D , Q is a surface, and $\omega_D = r^*\eta$ for some $0 \neq \eta \in H^{2,0}(Q)$. The surfaces Σ'_{b,n_i} cannot project onto curves on Q , since they are not uniruled. Therefore they dominate Q via r . But as they are isotropic for ω_D , we get

$$\omega_D|_{\Sigma'_{b,n_i}} = 0 = r^*\eta|_{\Sigma'_{b,n_i}},$$

hence $\eta = 0$, which is a contradiction. ■

If D had numerically trivial canonical class, the natural idea to continue would be to use the Bogomolov-Beauville decomposition; it would imply, because of the existence of a holomorphic 2-form, that up to a finite covering, D is either an abelian threefold or a product of an elliptic curve and a $K3$ -surface. In our situation, we can take the minimal model of D , but it can have terminal singularities, and at this moment, no analogue of Bogomolov-Beauville decomposition is known for terminal varieties. Nevertheless, as observed already by Campana and Peternell [10], it exists for threefolds with a holomorphic 2-form.

The following proposition is proved in [10]; we give a somewhat more detailed argument for reader's convenience.

Proposition 3.4 *(Theorem 3.2 of [10]) For any smooth projective threefold D such that $\kappa(D) = 0$ and $h^{2,0}(D) \neq 0$, there exists a dominant rational map $\mu : Y \dashrightarrow D$,*

where Y is either an abelian threefold, or a product $E \times S$ of a projective K3 surface S and an elliptic curve E .

Proof. By minimal model theory, there is a birational map $D \dashrightarrow D^0$, such that D^0 has at most canonical singularities, the Weil divisor K_{D^0} is \mathbb{Q} -Cartier and $mK_{D^0} = 0$ for some $m > 0$. Take the canonical cover $D' = \text{Spec}(\oplus_{i=0}^{m-1} \mathcal{O}(iK_{D^0}))$ (where $\mathcal{O}(iK_{D^0})$ are sheaves associated to Weil divisors; see [22], appendix to section 1, for a discussion of the definition of the pluricanonical sheaves etc. in this case). It is well-known (cf [22]) that it is étale in codimension 2 (wherever K_{D^0} is Cartier), the singularities of D' are canonical and that $K_{D'} = 0$. Let Y be a resolution of D' .

Since canonical singularities are rational by [11], that is, there are no higher direct images of the structure sheaf, we have $h^2(\mathcal{O}_{D^0}) = h^2(\mathcal{O}_D) > 0$. Therefore $h^2(\mathcal{O}_{D'}) \geq h^2(\mathcal{O}_{D^0}) > 0$. Since canonical class of D' is trivial, we obtain by Serre duality that $h^1(\mathcal{O}_{D'}) > 0$, and so, by rationality of the singularities, $h^1(\mathcal{O}_Y) > 0$.

Thus the Albanese map of Y is non-trivial. It factors through D' and gives a fibration $\alpha : D' \rightarrow \text{Alb } D' := \text{Alb } Y$ ([17]). Now the results of [17], section 8, say that this becomes a product after a finite étale covering. Thus if the dimension of fibers of α is zero or one, D is dominated by an abelian threefold; if this dimension is two, then either the fibers are birational to a K3 surface S and D is dominated by a product of S and an elliptic curve, or, repeating the process (i.e. considering the Albanese map) for the fiber of α , we get again that D is dominated by an abelian threefold. ■

Let us denote by $H_0^2(F, \mathbb{Q})$ the orthogonal of $c_1(\mathcal{L})$ with respect to the Beauville-Bogomolov form. This is just the image of the map $p_*q^* : H^4(X, \mathbb{Q})_{\text{prim}} \rightarrow H^2(F, \mathbb{Q})$ (cf section 1).

Lemma 3.5 *For F such that $\text{Pic}(F) = \mathbb{Z}$, $H_0^2(F, \mathbb{Q}) = H_{\text{tr}}^2(F, \mathbb{Q})$ is a simple Hodge structure of rank 22.*

Proof: The simplicity follows from $h^{2,0}(F) = 1$ by a standard argument: let V be a non-trivial Hodge substructure of $H_0^2(F, \mathbb{Q})$. If V is of level zero, that is, $V \otimes \mathbb{C} \subset H^{1,1}(F)$, then there are integral $(1, 1)$ classes in $H_0^2(F, \mathbb{C})$, contradicting $\text{Pic}(F) = \mathbb{Z}$. If V is not of level zero, $V \otimes \mathbb{C}$ contains $H^{2,0}(F)$, so the orthogonal of V is of level zero, hence must be trivial, and we conclude that $V = H_0^2(F, \mathbb{Q})$.

The rank is computed in [3]: indeed, it is proved there that F is deformation equivalent to the Hilb^2 of a K3 surface, so that $b_2(F) = 23$; and since $\text{Pic}(F) = \mathbb{Z}$, the $b_2^{\text{tr}}(F)$ is only one less. ■

In the above lemma and in what follows, we denote by $b_2^{\text{tr}}(Y)$ the difference between $b_2(Y)$ and the Picard number $\rho(Y)$, where Y is a projective manifold.

Lemma 3.6 *Let Y, Z be smooth projective of the same dimension. If there exists a dominant rational map $f : Y \dashrightarrow Z$, then $b_2^{\text{tr}}(Z) \leq b_2^{\text{tr}}(Y)$.*

Proof: Let $p : Y' \rightarrow Y$ be a resolution of f , so that p is a composition of blow-ups with smooth centers and $q : Y' \rightarrow Z$ is a generically finite morphism, $q = f \circ p$. Since $q_*q^* = \text{deg}(q) \cdot \text{Id}$ on cohomology groups, $q^* : H^2(Z, \mathbb{C}) \rightarrow H^2(Y', \mathbb{C})$ is

injective and takes transcendental cohomology to transcendental cohomology. So $b_2^{tr}(Z) \leq b_2^{tr}(Y')$. Since p is a composition of blow-ups, $b_2^{tr}(Y') = b_2^{tr}(Y)$. ■

We now conclude as follows. From propositions 3.3 and 3.4, our divisor D is rationally dominated either by an abelian threefold, or by a product of an elliptic curve and a K3-surface. Consider the morphism of Hodge structures

$$\tau^* : H_{tr}^2(F, \mathbb{C}) \rightarrow H^2(D, \mathbb{C})$$

As ω restricts non-trivially to D , this morphism is non-zero. By irreducibility of the Hodge structure on $H_{tr}^2(F, \mathbb{Q})$, it is injective and its image is contained in the transcendental part of $H^2(D, \mathbb{Q})$, so $b_2^{tr}(D)$ is at least 22 (lemma 3.5). This is clearly greater than b_2^{tr} of a three-dimensional torus. Finally, let E be an elliptic curve and S a projective K3-surface; we have

$$H^2(E \times S, \mathbb{Q}) = H^2(E, \mathbb{Q}) \otimes H^0(S, \mathbb{Q}) \oplus H^0(E, \mathbb{Q}) \otimes H^2(S, \mathbb{Q});$$

this is of dimension 23 and has at least a 2-dimensional subspace of algebraic classes, so that $b_2^{tr}(E \times S)$ is again strictly less than 22, contradicting lemma 3.6.

We thus proved that the case of a fibration in abelian surfaces is impossible.

3.3 Case where D is fibered over a surface

In this section we consider the remaining situation, where D admits a rational fibration over a surface T , in such a way that the 2-form ω_D is pulled-back from a 2-form on T and the countably many surfaces $\Sigma'_{b, n_i} \subset D$ contained in D are inverse images of curves $T_i \subset T$:

$$\Sigma'_{b, n_i} = \pi^{-1}(T_i).$$

Note that the curves T_i are then Zariski dense in T . Our goal is to show that this last case cannot occur, if the point $b \in B$ satisfies the Terasoma-type condition from the previous section. This will be done in several steps.

We claim first that this fibration $\pi : D \dashrightarrow T$ is preserved by ϕ^l . Indeed, the fibration is determined by the data of the 2-form ω_D up to a scalar : the fibers of π are the integral leaves of the vector field defined generically on D as the kernel of ω_D . As we have $(\phi^l)^*\omega_D = (-2)^l\omega_D$, the claim follows.

As D is defined only up to bimeromorphic transformations, we may assume π is actually a morphism. We will denote by $\psi : T \dashrightarrow T$ the rational map induced by ϕ^l on the basis of the fibration.

Lemma 3.7 *The generic fiber of π is a curve of general type.*

Proof. As the T_i are Zariski dense in T , it suffices to prove the same result for the fibration $\pi : \Sigma'_{b, n_i} \rightarrow T_i$.

Recall now that the surfaces $\Sigma'_{b, j}$ are birationally equivalent to an abelian surface A_b , and that we proved in section 1 that we could assume A_b to be geometrically simple. We then simply use the fact that curves in a simple abelian surface are of general type. ■

Let $g \geq 2$ be the genus of the generic fiber of π . Let $m : T \dashrightarrow \mathcal{M}_g$ the rational map associated to the fibration π and let Z be its image. By definition, the fibration becomes isotrivial when restricted to the fibers of m .

As ϕ^l acts on D preserving the fibration π , and the fibers of π are of general type, ϕ^l induces for generic $t \in T$ an isomorphism

$$C_t \cong C_{\psi(t)}, t \in T,$$

where $C_t := \pi^{-1}(t)$. This isomorphism and the definition of m shows that $m \circ \psi = m$.

Lemma 3.8 *Z has dimension 1, and the restriction of m to $T_i \subset T$ maps each T_i onto Z .*

Proof. As the rational self-map ψ is not of finite order on T (because the T_i are of the form $\psi^k(T_1)$ and are Zariski dense in T), the equality $m \circ \psi = m$ implies that m is not generically finite-to-one.

Thus either m is constant or its image Z has dimension 1. We will show that the restriction of m to T_i cannot be constant, which implies the full statement of lemma 3.8. Recall that we have $\pi^{-1}(T_i) = \Sigma'_{b,n_i}$, which is birationally equivalent to an abelian surface A_b . If m were constant on T_i , then the rational fibration $A_b \dashrightarrow T_i$ would be isotrivial with fiber C of general type. This does not exist on any abelian surface A , because otherwise, A would be dominated by a product $C \times C'$, which would send the curves $C \times c'$ to the fibers of the fibration. But a rational map

$$\alpha : C \times C' \dashrightarrow A$$

is necessarily a morphism which is a “sum morphism”, that is of the form

$$\alpha(c, c') = \alpha_1(c) + \alpha_2(c').$$

Thus the fibers of this isotrivial fibration would be translates in A (in the directions given by $\alpha_2(C')$) of a given curve of general type. But two fibers must intersect because they are of general type, and they can intersect only along the 0-dimensional indeterminacy locus of the considered fibration on A . Their intersection is thus stable under translation by $\alpha_2(C')$, which is a contradiction. ■

Corollary 3.9 *The fibers $T_z := m^{-1}(z)$, $z \in Z$ intersect the curves T_i .*

Indeed, this is equivalent to the surjectivity of the restricted map $m : T_i \rightarrow Z$. ■

Let now $z \in Z$ be a point defined over a number field and consider the curve $T_z = m^{-1}(z) \subset T$. The surface $\pi^{-1}(T_z)$ admits an isotrivial fibration with fiber C_z defined over a number field k , and thus is dominated by a product $C_z \times_k T'_z$, where T'_z is a finite cover of T_z .

For each point $y \in T'_z$ defined over a number field $k(y)$, the curve $D_y := \pi^{-1}(y)$ is isomorphic to C_z and the restricted cycle

$$P'_y := P'_{|D_y \times X} \subset D_y \times X_{k(y)} \cong C_{z,k(y)} \times X_{k(y)}$$

is homologically trivial in the sense that its étale cohomology class in $H_{\text{ét}}^6(D_{y,\bar{k}} \times X_{\bar{k}}, \mathbb{Q}_l(3))$ is zero. Thus it admits an Abel-Jacobi l -adic invariant

$$\begin{aligned} \gamma_y &\in H^1(k(y), H^5(D_{y,\bar{k}(y)} \times X_{\bar{k}(y)}, \mathbb{Q}_l(3))) \\ &= H^1(k(y), H^5(C_{z,\bar{k}(y)} \times X_{\bar{k}(y)}, \mathbb{Q}_l(3))) \end{aligned}$$

which is obtained as in the previous section by considering the continuous étale cycle class of P'_y in $H^6(D_y \times X_{k(y)}, \mathbb{Q}_l(3))$ and applying the Hochschild-Serre spectral sequence to it.

We choose a point $z \in Z$ defined over a number field and which is sufficiently general (so as to avoid singular fibers). We will prove the following two propositions:

Proposition 3.10 *The class γ_y does not depend on $y \in T'_z(\bar{\mathbb{Q}})$.*

This makes sense, as for $y, y' \in T'_z(\bar{\mathbb{Q}})$, one may choose a common definition field k for y and y' , on which the isomorphism $D_y \cong D_{y'} \cong C_z$ is also defined. Then we compare both classes in $H^1(k, H^5(C_{z,\bar{k}} \times X_{\bar{k}}, \mathbb{Q}_l(3)))$.

Proof of Proposition 3.10. Indeed, consider the surface $\pi^{-1}(T_z)$ which admits as a rational finite cover $C_z \times_k T'_z$. Let us denote by $r : C_z \times_k T'_z \dashrightarrow F$ the natural rational map. We claim that $(r, Id)^* P' \in CH^3(C_z \times_k T'_z \times X)_{\mathbb{Q}}$ is a cycle homologous to 0, which means that its étale cohomology class on $C_{z,\bar{k}} \times_{\bar{k}} T'_{z,\bar{k}} \times_{\bar{k}} X_{\bar{k}}$ or equivalently its Betti cohomology class on $C_{z,\mathbb{C}} \times T'_{z,\mathbb{C}} \times X_{\mathbb{C}}$ vanishes. Indeed, we know that the 2-form $\omega = P'^* \alpha$ on F has the property that its restriction ω_D to D is of the form $\pi^* \omega_T$ for some holomorphic 2-form on T . Of course ω_T vanishes on the curve T_z . It follows that ω_D vanishes on the surface $\pi^{-1}(T_z)$. Let us work in the complex setting: by lemma 3.5, the Hodge structure on $H^4(X, \mathbb{Q})_{\text{prim}}$ is simple. As the morphism

$$r^* \circ P'^* : H^4(X_{\mathbb{C}}, \mathbb{Q})_{\text{prim}} \rightarrow H^2(C_{z,\mathbb{C}} \times T'_{z,\mathbb{C}}, \mathbb{Q})$$

is not injective, because its complexification vanishes on the $(3, 1)$ -part, it must be 0, which proves the claim.

Having this, we conclude that the cycle $(r, Id)^* P' \in CH^3(C_z \times_k T'_z \times X)$ has an l -adic Abel-Jacobi invariant

$$\gamma \in H^1(k, H^5(C_{z,\bar{k}} \times_{\bar{k}} T'_{z,\bar{k}} \times_{\bar{k}} X_{\bar{k}}, \mathbb{Q}_l(3)))$$

which in fact lies in

$$H^1(k, H^1(C_{z,\bar{k}} \times_{\bar{k}} T'_{z,\bar{k}}, \mathbb{Q}_l) \otimes_{\mathbb{Q}_l} H^4(X_{\bar{k}}, \mathbb{Q}_l)(3))$$

because P' is obtained from P by applying a Chow-Künneth projection on X .

We have

$$H_{\text{ét}}^1(C_{z,\bar{k}} \times_{\bar{k}} T'_{z,\bar{k}}, \mathbb{Q}_l) \cong H_{\text{ét}}^1(C_{z,\bar{k}}, \mathbb{Q}_l) \oplus H_{\text{ét}}^1(T'_{z,\bar{k}}, \mathbb{Q}_l),$$

and thus the first projection of γ gives us a class

$$\gamma_z \in H^1(k, H^1(C_{z,\bar{k}}, \mathbb{Q}_l) \otimes_{\mathbb{Q}_l} H^4(X_{\bar{k}}, \mathbb{Q}_l)(3))$$

which by definition restricts to γ_y , for any $y \in T'_z$. ■

Proposition 3.11 *The class γ_y , $y \in T'_z$ is non-zero, for many closed points $b \in B$ defined over a number field.*

Proof. The fiber T_z meets the countably many curves T_i , by corollary 3.9. Let $y \in T_1 \cap T_z$ be defined over a number field k . The class γ_y is equal to the class γ_z by Proposition 3.10. Now, note that, as $y \in T_1$, the fiber D_y is a curve in the surface $\Sigma'_{b,n_1} = \pi^{-1}(T_1)$. This surface is birationally equivalent to a simple abelian surface, and thus we have an injective morphism of étale cohomology groups:

$$H_{\text{ét}}^1(\Sigma'_{b,n_1,\bar{k}}, \mathbb{Q}_l) \rightarrow H_{\text{ét}}^1(D_{y,\bar{k}}, \mathbb{Q}_l),$$

where k is a common field of definition of T_1 and y , and \bar{k} is its Galois closure. This restriction map is $\text{Gal}(\bar{k}/k)$ -equivariant and it makes the left hand side into a direct summand of right hand side, using the existence of a non-degenerate $\text{Gal}(\bar{k}/k)$ -invariant intersection pairing on the right hand side. It follows that there is an induced injection

$$\begin{aligned} & H^1(k, H^1(\Sigma'_{b,n_1,\bar{k}}, \mathbb{Q}_l) \otimes_{\mathbb{Q}_l} H^4(X_{\bar{k}}, \mathbb{Q}_l)(3)) \\ & \hookrightarrow H^1(k, H^1(D_{y,\bar{k}}, \mathbb{Q}_l) \otimes_{\mathbb{Q}_l} H^4(X_{\bar{k}}, \mathbb{Q}_l)(3)), \end{aligned}$$

which obviously sends the Abel-Jacobi l -adic invariant of the cycle $P'_{\Sigma'_{b,n_1}}$ to the l -adic invariant of the cycle P'_y , that is γ_y . We already proved in proposition 2.7 that the first one is non-zero, because it is equal to $(-2)^{n_1} \gamma_{P'_y}$ by property (1.1), hence $\gamma_y \neq 0$. \blacksquare

Let us now explain how to use these propositions to get a contradiction, which will exclude the case considered in this section.

The fiber T_z meets the countably many curves T_i , as we proved in the proof of Lemma 3.8. This fiber is acted on by ψ , as proved in lemma 3.8. Let $y \in T_1 \cap T_z$ be defined over a number field k .

By proposition 3.10, its l -adic Abel-Jacobi invariant

$$\gamma_y \in H^1(k, H^5(D_{y,\bar{k}} \times X_{\bar{k}}, \mathbb{Q}_l(3)))$$

must be equal to the l -adic Abel-Jacobi invariant

$$\gamma_{y'} \in H^1(k, H^5(D_{y',\bar{k}} \times_{\bar{k}} X_{\bar{k}}, \mathbb{Q}_l(3)))$$

of the cycle $P'_{y'} \subset D_{y'} \times X$, where $y' = \psi(y) \in \psi(T_1) \cap T_z$, and D_y and $D_{y'}$ are identified via the trivialisation of the fibration of π over T_z . Up to replacing the power l by an higher power, we may assume that this identification is given by $\phi^l : D_y \cong C_{y'}$ because the automorphisms group of these curves is finite. Thus we get

$$\phi^{l*} \gamma_{y'} = \gamma_y, \tag{3.1}$$

where we see ϕ^l as a map from D_y to $D_{y'}$.

On the other hand, we get from (1.1) and Bloch-Srinivas decomposition theorem (cf [5] and [27], Corollary 10.20) the equality

$$\phi^{l*}(P'_{y'}) = (-2)^l P'_y + \Gamma \text{ in } CH^3(D_y \times X)_{\mathbb{Q}},$$

where $\Gamma \subset D_y \times X$ is a “vertical” cycle, supported over a divisor of D_y . As vertical cycles have trivial l -adic Abel-Jacobi invariant in the considered group, this implies that

$$\phi^{l*} \gamma_{y'} = (-2)^l \gamma_y. \quad (3.2)$$

As

$$0 \neq \gamma_y \text{ in } H^1(k, H^5(D_{y,\bar{k}} \times_{\bar{k}} X_{\bar{k}}, \mathbb{Q}_l(3)))$$

by proposition 3.11, (3.1) and (3.2) provide a contradiction. ■

References

- [1] E. Amerik. A computation of invariants of a rational self-map, preprint 2007, to appear in *Annales de la Faculté des Sciences de Toulouse*.
- [2] E. Amerik, F. Campana. Fibrations méromorphes sur certaines variétés de classe canonique triviale, *Pure and Applied Mathematics Quarterly*, Volume 4, Number 2, 2008 (Fedor Bogomolov special issue, part I).
- [3] A. Beauville, R. Donagi. La variété des droites d’une hypersurface cubique de dimension 4, *C. R. Acad. Sci. Paris Sér. I Math.* 301 (1985), no. 14, 703–706.
- [4] S. Bloch. *Lectures on algebraic cycles*, Duke University Mathematics Series, IV. Duke University, Mathematics Department, Durham, N.C., 1980.
- [5] S. Bloch, V. Srinivas. Remarks on correspondences and algebraic cycles, *Amer. J. Math.* 105 (1983), no. 5, 1235–1253.
- [6] F. Bogomolov, Yu. Tschinkel. Density of rational points on elliptic $K3$ surfaces, *Asian J. Math.* 4 (2000), no. 2, 351–368.
- [7] H. Clemens, Ph. Griffiths. The intermediate Jacobian of the cubic threefold, *Ann. of Math. (2)* 95 (1972), 281–356.
- [8] A. Conte, J. Murre. The Hodge conjecture for fourfolds admitting a covering by rational curves, *Math. Ann.* 238, no. 1, 79–88 (1978).
- [9] F. Campana. Orbifolds, special varieties and classification theory, *Ann. Inst. Fourier (Grenoble)* 54 (2004), no. 3, 499–630.
- [10] F. Campana, Th. Peternell. Complex threefolds with non-trivial holomorphic 2-forms, *J. Algebraic Geom.* 9 (2000), no. 2, 223–264
- [11] R. Elkik. Rationalité des singularités canoniques, *Invent. Math.* 64 (1981), no. 1, 1–6.
- [12] M. Green, Ph. Griffiths, K. Paranjape. Cycles over fields of transcendence degree 1, *Michigan Math. J.* 52 (2004), no. 1, 181–187.
- [13] J. Harris, Y. Tschinkel. Rational points on quartics, *Duke Math. J.* 104 (2000), no. 3, 477–500.

- [14] B. Hassett, Yu. Tschinkel. Potential density of rational points for K3 surfaces over function fields, arXiv:math/0604222, to appear in the American Journal of Mathematics.
- [15] U. Jannsen. *Mixed motives and algebraic K-theory*, with appendices by S. Bloch and C. Schoen, Lecture Notes in Mathematics, 1400, Springer-Verlag, Berlin, (1990).
- [16] J.P. Jouanolou. Hypersurfaces solutions d'une équation de Pfaff analytique, Math. Ann. 232 (1978), no. 3, 239–245.
- [17] Y. Kawamata. Minimal models and the Kodaira dimension of algebraic fiber spaces, J. Reine Angew. Math. 363 (1985), 1–46.
- [18] Y. Kawamata. On the plurigenera of minimal algebraic 3-folds with $K \equiv 0$, Math. Ann. 275 (1986), no. 4, 539–546.
- [19] R. van Luijk: K3 surfaces with Picard number one and infinitely many rational points, Algebra and Number Theory 1 (2007), 1-15.
- [20] N. Nakayama, D-Q. Zhang. Building blocks of étale endomorphisms, RIMS preprint 1577, 2007.
- [21] W. Raskind. Higher l -adic Abel-Jacobi mappings and filtrations on Chow groups, Duke Math. J. 78 (1995), 33-57.
- [22] M. Reid. Canonical 3-folds, Journées de Géométrie Algébrique d'Angers, Juillet 1979/Algebraic Geometry, Angers, 1979, pp. 273–310, Sijthoff and Noordhoff.
- [23] J. Silverman. Rational points on K3 surfaces: a new canonical height, Invent. Math. 105 (1991), no. 2, 347–373.
- [24] T. Terasoma. Complete intersections with middle Picard number 1 defined over \mathbb{Q} , Math. Z. 189 (1985), no. 2, 289–296.
- [25] C. Voisin. Intrinsic pseudo-volume forms and K -correspondences, in *The Fano Conference*, 761–792, Univ. Torino, Turin, 2004.
- [26] C. Voisin. On some problems of Kobayashi and Lang; algebraic approaches, in *Current developments in mathematics*, 2003, 53–125, Int. Press, Somerville, MA, 2003.
- [27] C. Voisin. *Hodge theory and complex algebraic geometry II*, Cambridge Studies in Advanced Mathematics, 77. Cambridge University Press, Cambridge, 2003.
- [28] S. Zucker. The Hodge conjecture for cubic fourfolds, Compositio Math. 34, 199-209 (1977).