CHARACTERISTIC FOLIATION ON NON-UNIRULED SMOOTH DIVISORS ON PROJECTIVE HYPERKÄHLER MANIFOLDS

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Abstract. We prove that the characteristic foliation $\mathcal{F}$ on a non-singular divisor D in an irreducible projective hyperkähler manifold X cannot be algebraic, unless the leaves of $\mathcal{F}$ are rational curves or X is a surface. More generally, we show that if X is an arbitrary projective manifold carrying a holomorphic symplectic 2-form, and D and $\mathcal{F}$ are as above, then $\mathcal{F}$ can be algebraic with non-rational leaves only when, up to a finite étale cover, X is the product of a symplectic projective manifold Y with a symplectic surface and D is the pull-back of a curve on this surface.

When D is of general type, the fact that $\mathcal{F}$ cannot be algebraic unless X is a surface was proved by Hwang and Viehweg. The main new ingredient for our results is the observation that the canonical bundle of the orbifold base of the family of leaves must be torsion. This implies, in particular, the isotriviality of the family of leaves of $\mathcal{F}$.

RÉSUMÉ: Nous montrons que si le feuilletage caractéristique $\mathcal{F}$ d'un diviseur D lisse d'une variété projective complexe symplectique irréductible X est algébrique, alors ou bien X est une surface, ou bien les feuilles de $\mathcal{F}$ sont des courbes rationnelles. Lorsque D est de type général, ce résultat est dû à Hwang-Viehweg. Nous en déduisons, lorsque X est une variété projective complexe arbitraire munie d'une 2-forme symplectique holomorphe, et D, $\mathcal{F}$ comme ci-dessus, que si les feuiles X sont des courbes algébriques non-rationnelles, que X est le produit d'une surface $K3$ ou abélienne S par une variété symplectique Y, et $D = C \times Y$ pour une courbe $C \subset S$.

L'ingrédient principal nouveau de la démonstration est l'observation que le fibré canonique de la base orbifolde de la famille des feuilles est de torsion. Ceci implique, en particulier, l'isotrivialité de la famille des feuilles de $\mathcal{F}$.

1. Introduction

Let X be a projective manifold equipped with a holomorphic symplectic form σ. Let D be a smooth divisor on X. At each point of D, the restriction of σ to D has one-dimensional kernel. This gives a non-singular foliation $\mathcal{F}$ on D, called *the characteristic foliation.* We say that $\mathcal{F}$ is *algebraic* if all its leaves are compact complex curves.

In [H-V08], J-M Hwang and E. Viehweg proved the following theorem:

Theorem 1.1. *([H-V08], Theorem 1.2) Let (X, σ) be a holomorphic symplectic manifold as above. Assume $dim(X) \geq 4$. Let $D \subset X$ be a smooth hypersurface of general type. Then the characteristic foliation on D induced by σ cannot be algebraic.*

If D is a uniruled divisor on a holomorphic symplectic X, the characteristic foliation $\mathcal{F}$ is always algebraic. Indeed, its leaves are the fibres of the rational quotient fibration on D (see for example [A-V14], Section 4). The aim of this article is to classify the examples where $\mathcal{F}$ is algebraic and D is not uniruled.

Our main results are as follows.

Theorem 1.2. *Let X be a projective manifold with a holomorphic symplectic form σ and let D be a smooth hypersurface in X. If $\mathcal{F}$ is algebraic and the genus of its general leaf is $g > 0$, then the associated fibration is isotrivial and K_D is nef and abundant, with $\nu(K_D) = \kappa(D) = 1$ when $g \geq 2$ and $\nu(K_D) = \kappa(D) = 0$ when $g = 1$.*

Here ν denotes the numerical dimension and κ the Kodaira dimension. In general, $\kappa(D)$ does not exceed $\nu(K_D)$, and K_D is said to be abundant when the two dimensions coincide. By a result of Kawamata, this implies the semiampleness of K_D.

Theorem 1.3. *Let X, D, $\mathcal{F}$ be as above, and suppose moreover that $h^{2,0}(X) = 1$. If $\mathcal{F}$ is algebraic and D is not uniruled, then $dim(X) = 2$.*

By Bogomolov decomposition theorem, the fundamental group of such a manifold is finite. A simply-connected manifold X as above is called *irreducible holomorphic symplectic* or *simple hyperkähler*. In the proof, we shall actually assume that X is simply-connected, since the conditions do not change when X is replaced by a finite covering.

Remark 1.4. *The smoothness assumption is essential, as one sees by considering the Hilbert square X of an elliptic $K3$-surface $g : S \to \mathbb{P}^1$: one has a fibration $h : X \to \mathbb{P}^2 = Sym^2(\mathbb{P}^1)$. If $C \subset \mathbb{P}^2$ is the ramification conic of the natural 2-cyclic cover $(\mathbb{P}^1)^2 \to \mathbb{P}^2$, and $L \subset \mathbb{P}^2$ is a line tangent to C, then the characteristic foliation on the singular divisor $D := h^{-1}(L)$ is algebraic with $g = 1$.*

Theorem 1.5. *Let X, D, $\mathcal{F}$ be as above. Suppose that D is non-uniruled and $\mathcal{F}$ is algebraic. Then, possibly after a finite étale cover, $X = S \times Y$, where $dim(S) = 2$, both S and Y are complex projective manifolds carrying holomorphic symplectic forms σ_S, σ_Y, and $D = C \times Y$, where $C \subset S$ is a curve.*

Remark 1.6. *We shall see from the proof that there is a corresponding decomposition of the form σ. Namely, the surface S from theorem 1.5* *is, up to a finite cover, either* $K3$ *or abelian. In the first case, or if* $g > 1$ *in the second case,* $\sigma = \sigma_S \oplus \sigma_Y$ *on* $TX \cong TS \oplus TY$.

When S *is an abelian surface and* $g = 1$*, write* $X = A \times Z$*, where* A *is an abelian variety and* Z *a product of irreducible hyperkähler manifolds: this is always possible up to a finite cover, by Bogomolov decomposition. Then* $\sigma = \sigma_A \oplus \sigma_Z$ *on* $TX \cong TA \oplus TZ$*. In general,* $A = S \times T$ *for some Abelian variety* T*, but* σ_A *does not have to be the direct sum of symplectic forms on* S *and on the Poincaré-complement* T *of* S *in* A *(see example 4.1).*

The main ideas of the proof of Theorem 1.2 are as follows. Suppose that D is not uniruled and that $\mathcal{F}$ is algebraic. Then $\mathcal{F}$ defines a holomorphic fibration $f : D \to B$ such that its non-singular fibers are curves of genus $g > 0$, and the singular fibers are multiple curves with smooth reduction. We prove that the orbifold canonical bundle of this fibration is trivial. Together with Reeb stability and the generic nefness of the orbifold cotangent bundle from [C-P13], this implies that the Iitaka dimension of the determinant of any subsheaf of the conormal bundle of $\mathcal{F}$ is non-positive. On the other hand, Hwang and Viehweg construct such a subsheaf, coming from the Kodaira-Spencer map, on a certain covering of D, and show that its Iitaka dimension is equal to the number of moduli of the fibres of f. Using the functoriality of the Kodaira-Spencer map, we show that such a sheaf exists already on D, and therefore the family f must be isotrivial. Theorem 1.2 follows easily. The two other theorems are more elementary consequences of the first one.

As an application, we deduce in section 5 a certain case of the Lagrangian conjecture on a projective irreducible holomorphic symplectic manifold of dimension $2n$ from the Abundance conjecture in dimension $2n - 1$ (thus proving this case unconditionally for $n = 2$, as the Abundance conjecture is known for threefolds). This was our initial motivation for this research. When the research has been completed, Chenyang Xu has informed us that this case of the Lagrangian conjecture is a special case of a fundamental result of Demailly, Hacon and Paun ([D-H-P12]). As no algebraic proof of [D-H-P12] is known, our result gives a simple algebro-geometric alternative for hyperkähler manifolds (see section 5 for statements and proofs).

The three next sections are devoted to the proofs of the three theorems. In the last one, we treat our application to the Lagrangian conjecture.

2. Some numerical invariants of the characteristic foliation

2.1. **Smooth rank 1 foliations.** Let D be an m-dimensional ($m \geq 2$) connected projective manifold carrying a non-singular holomorphic foliation $\mathcal{F}$ of rank 1. The foliation $\mathcal{F}$ is called algebraic when all its leaves are compact complex curves. A non-singular algebraic foliation induces a proper holomorphic map $f : D \to B$ onto an $(m-1)$-dimensional normal analytic subspace with quotient singularities B in the Chow scheme of D. Moreover, f is "uni-smooth", that is, the reduction of any of its fibres is a smooth projective curve. Let g denote the genus of a non-singular fiber of f. If $g = 0$, all fibres of f are smooth reduced rational curves, B is smooth, f submersive. If $g > 0$, f may have multiple fibres, of genus one when $g = 1$ and of genus greater than one (but possibly smaller than g) when $g > 1$ (this is easily seen from Reeb stability, for which we refer to [H-V08], section 2).

If $g = 1$, then f is isotrivial (indeed, otherwise one would get other types of singular fibres, besides multiple elliptic curves).

A pair $D, \mathcal{F}$ as above arises when D is a smooth (connected) divisor in a $2n$-dimensional projective manifold X carrying a holomorphic symplectic 2-form σ. The foliation $\mathcal{F}$ is then given, at each $x \in D$, as the σ-orthogonal to TD_x at x. In this case $m = 2n - 1$. In general, $\mathcal{F}$ will not be algebraic. One particular case when $\mathcal{F}$ is algebraic is that of a uniruled D: the leaves of $\mathcal{F}$ are then precisely the fibres of the rational quotient fibration of D (see for instance [A-V14], section 4), so $g = 0$.

We will elucidate below the situation when $\mathcal{F}$ is algebraic and $g > 0$.

Note that in this example, the quotient bundle $T_D/\mathcal{F}$ carries a symplectic form, so has trivial determinant. Therefore the line bundle $\mathcal{F}$ is isomorphic to the anticanonical bundle of D (by adjunction, this is $\mathcal{O}_D(-D)$).

The purpose of this section is to prove the following result, which is stated as theorem 1.2 in the introduction.

Theorem 2.1. *Let $X, \sigma, D, \mathcal{F}$ be as above. If D is non-uniruled, and if $\mathcal{F}$ is algebraic, then the corresponding fibration $f : D \to B$ is isotrivial, K_D is nef and abundant, $\nu(K_D) = \kappa(D) = 1$ if $g \geq 2$, and $\nu(K_D) = \kappa(D) = 0$ if $g = 1$.*

2.2. **Orbifold base.** We define (as in [Ca 04], in a much more general situation there) the *orbifold base* (B, Δ) for f as follows: for each irreducible reduced Weil divisor $E \subset B$, set $E' = f^{-1}(E)$. This is is an irreducible divisor, and $f^*(E) = m_f(E)E'$ for some positive integer $m_f(E)$. This integer is equal to 1 for all but finitely many E. Set $\Delta = \sum_{E \subset B}(1 - \frac{1}{m_f(E)})E$.

Lemma 2.2. *The orbifold pair (B, Δ) has klt singularities, and $K_B + \Delta$ is $\mathbb{Q}$-Cartier. If $D \subset Y$ is a smooth divisor in a manifold Y of dimension $2n$ carrying a holomorphic symplectic 2-form σ, and if the fibres of $f : D \to B$ are tangent to the characteristic foliation $\mathcal{F}$ defined by σ, then $N(K_B + \Delta) \cong \mathcal{O}_B$ for some positive integer N (for example, equal to the l.c.m of the $m_f(E)'s$).*

Proof. The first statement follows from [K-M98], 5.20, applied to a local smooth multisection of f passing through any given of its fibres. To prove the second statement, notice that σ descends (see for example [S09]) to a holomorphic symplectic form σ' on $B - Supp(\Delta)$. Moreover $[(\sigma')^{\wedge(n-1)}]^{\otimes N}$ extends as a non-vanishing section of $N(K_B + \Delta)$ at a sufficiently general point of $Supp(\Delta)$ (for instance, at any smooth point of B which is also a smooth point of $Supp(\Delta)$). Indeed, assume for simplicity of notation that $n = 2$; above such a point, the map f is locally given by $(x, y, z) \mapsto (u, v)$, where $u = x^k, v = y$, and one has

$$(dx \wedge dy)^{\otimes k} = f^*(\frac{du}{u^{1-1/k}} \wedge dv)^{\otimes k}.$$

One therefore obtains that $N(K_B + \Delta) \cong \mathcal{O}_B$ in codimension one, and thus everywhere. □

One regards $K_B + \Delta$ as the *orbifold canonical divisor* $K_{(B,\Delta)}$. For $f : D \to B$ as above, the *relative canonical bundle* $K_{D/(B,\Delta)}$ is well-defined.

The following lemma places this observation in the context of [H-V08], section 2.

Lemma 2.3. *Let $f : D \to B$ be as above. Let $U \subset B$ be a nonempty open subset, and let $U' \subset f^{-1}(U) \subset D$ be a complex submanifold of $f^{-1}(U)$ (which shall be also denoted by D_U) which is proper over U and meets transversally every fibre of f over U. Let D'_U be the normalisation of $U' \times_U D_U$, and $s : D'_U \to D_U$ be the natural (proper) map. Then s is étale, and the natural map $f' : D'_U \to U'$ is smooth. Moreover, for N as in the previous lemma, $NK_{D'_U/U'} = Ns^*K_{D/(B,\Delta)} = Ns^*K_D$.*

Proof. The first assertion is a consequence of Reeb stability theorem (see [H-V08], lemma 2.6). The second claim follows immediately, since the geometric degree of the map s over a fibre of $f' : D' \to U'$ is equal to the ramification order of $f : U' \to U$ at the corresponding point u' of U', for u' outside a codimension-two subset. By our transversality assumption, this order of ramification is also the multiplicity of the component of Δ going through $u = f(u')$, for u outside a codimension-two subset. We thus obtain that $K_{D'/U'} = s^*K_{D/(B,\Delta)}$. The second equality follows from $NK_{D/(B,\Delta)} = NK_D$, since $NK_{(B,\Delta)} = \mathcal{O}_B$. □

Remark 2.4. *It is clear that one can find such an U' over a "large", that is, Zariski open U. But it is also useful to take for U a small neighbourhood of a point $b \in B$, since then the coverings $U' \to U$ and $D'_U \to D_U$ are Galois, with Galois group equal to the holonomy group of the leaf over b.*

2.3. **Sheaves of orbifold differentials.** The foliation $\mathcal{F}$ considered in §2.1 on D gives rise to an exact sequence of vector bundles

$$0 \to N \to \Omega^1_D \to \mathcal{F}^* \to 0,$$

in which N is the conormal bundle to $\mathcal{F}$ (that is, the dual of the quotient $T_D/\mathcal{F}$).

Let $A \subset B$ be the union of the singular points of B and those of $Supp(\Delta)$. It has codimension at least 2 in B, and $A' := f^{-1}(A)$ has codimension at least 2 in D as well, since the fibres of f are all of the same dimension 1.

A local computation shows that, outside of A', N coincides with $f^*(\Omega^1(B,\Delta))$, the pullback of the orbifold cotangent bundle of (B,Δ) as defined in [C-P13]. This orbifold cotangent bundle is in fact a "virtual" coherent sheaf on B, well-defined after a covering; in particular, it is clear from its construction (detailed below outside of a codimension-two locus) that $f^*(\Omega^1(B,\Delta))$ is well-defined. The statement $N = f^*(\Omega^1(B,\Delta))$ is obvious outside of $Sing(B) \cup Supp(\Delta)$, since there both sheaves are equal to $f^*\Omega^1_B$. If now we are near a smooth point of B belonging to the smooth locus of $Supp(\Delta)$, there are local analytic coordinates $x_1, \dots, x_m$ on D, and a positive integer ℓ (the multiplicity of the fibre of f a this point) such that $f(x_1, \dots, x_m) = (u := x_1^\ell, x_2, \dots, x_{m-1})$ on a small open subset $U \subset B$. Now for any proper surjective analytic map $h : U' \to U$ such that $h^*\left(\frac{du}{u^{1-\frac{1}{\ell}}}\right)$ is a well-defined differential on U', the sheaf $h^*(\Omega^1(B,\Delta))$ is generated as an $\mathcal{O}_{U'}$-module by the elements

$$h^*\Big(\frac{du}{u^{1-\frac{1}{\ell}}}\Big), h^*(dx_2), \dots, h^*(dx_{m-1})$$

(see [C-P13] for details, where the particular case where h is finite is considered). The claim is thus obvious, since $f^*\left(\frac{du}{u^{1-\frac{1}{\ell}}}\right) = \ell.dx_1$ in our situation, and since N is generated by the $dx_j, j < m$.

These computations establish the following improvement of lemma 2.3:

Lemma 2.5. *In the situation of lemma 2.3,* $s^*(N) = (f')^*(h^*(\Omega^1(B,\Delta)|_U)$, *and* $(f')_*(s^*(N)) = h^*(\Omega^1(B,\Delta)|_U) = \Omega^1_{U'}$, *where* $f' : D' \to U'$ *and* $h : U' \to U$ *are the natural maps.*

Remark 2.6. *Notice that, by contrast,* $f_*(N) \neq \Omega^1(B,\Delta)$ *as soon as* f *has multiple fibres. For example, if* B *is smooth,* $f_*(N) = \Omega^1_B$.

2.4. **Birational stability of orbifold cotangent sheaves.** The following theorem is proved in [C-P13]:

Theorem 2.7. *(*[C-P13]*, corollary 3.1) Let* (B,Δ) *be a log-canonical pair with* $K_B + \Delta \equiv 0$. *Let* $h : B' \to B$ *be a (global) cyclic cover over which* $h^*(\Omega^1(B,\Delta))$ *is well-defined. Let* L' *be a coherent rank-one sheaf on* B', *and* $k > 0$ *be an integer such that* $L' \subset \otimes^k(h^*(\Omega^1(B,\Delta))$. *Then* $deg_{C'}(L'_{|C'}) \leq 0$ *for any sufficiently general complete intersection curve* C' *cut out on* B' *by the intersection of large enough multiples of*

$H' := h^*(H)$, where H is an ample line bundle on B. In particular, for any integer $m > 0$ one has $h^0(B', (L')^{\otimes m}) \leq 1$.

Recall that $\Omega^1(B, \Delta)$ is a "virtual" sheaf, with well-defined pullback on certain covers $h : B' \to B$, with normal B'. Such global cyclic covers are constructed in [C-P13]; locally at a smooth point of $Supp(\Delta)$, they ramify along $Supp(\Delta)$ with the prescribed multiplicity. It follows from the construction that locally above a small neighbourhood U of a point outside of a codimension-two subset of B, such a covering B' factors through the covering U' of lemmas 2.3 and 2.5, coming from Reeb stability. We may thus assume that the conclusions of these lemmas are satisfied in codimension one over B'.

An immediate consequence is:

Corollary 2.8. *Assume that (B, Δ) is the orbifold base of the fibration $f : D \to B$ associated to the characteristic foliation $\mathcal{F}$ on a smooth divisor D in a manifold carrying a holomorphic symplectic 2-form σ. Let N denote the conormal bundle of $\mathcal{F}$. Then for any coherent subsheaf $\mathcal{G} \subset N$, one has $h^0(D, det(\mathcal{G})) \leq 1$, and so $\kappa(D, det(\mathcal{G})) \leq 0$.*

Proof. From lemma 2.2 above, we deduce that (B, Δ) is klt, with $(K_B + \Delta) \equiv 0$. For any adapted cyclic cover $h : B' \to B$ as in [C-P13] and for any coherent subsheaf $\mathcal{H}' \subset h^*(\Omega^1(B, \Delta))$, we therefore have $\kappa(B', det(\mathcal{H}')) \leq 0$. On such a cover, we also have $N' := s^*(N) = (f')^*(h^*(\Omega^1(B, \Delta))$, outside of a codimension-two subset.

Let $\mathcal{G}' = s^*(\mathcal{G})$. On each fibre of f' (possibly outside of a codimension-two subset), $\mathcal{G}'$ is a subsheaf of the trivial bundle. There are two possible cases:

1. $det(\mathcal{G}')$ has negative degree on the fibres of f' (recall that these are one-dimensional), and then no multiple of $det(\mathcal{G}')$ is effective on D';
2. $det(\mathcal{G}')$ is of degree zero on the fibres of f', and then $\mathcal{G}'$ is trivial on fibres of f', by lemma 2.9 below.

In the first case, $\kappa(D', \det(\mathcal{G}')) = -\infty$. In the second case, possibly after replacing $\mathcal{G}$ by its saturation in N and thus $\mathcal{G}'$ by its saturation in N', we have that $\mathcal{G}' = (f')^*(\mathcal{H}')$, where $\mathcal{H}' = f_*(\mathcal{G}')$, at least over a complement to a codimension-two subspace in B'. But then $\kappa(D', det(\mathcal{G}')) = \kappa(B', det(\mathcal{H}'))$. The latter is non-positive by theorem 2.7.

Since $\kappa(D', det(\mathcal{G}')) = \kappa(D', s^*(det(\mathcal{G}))) = \kappa(D, det(\mathcal{G}))$, the claim is established. □

In the proof, we have used the following simple lemma:

Lemma 2.9. *Let C be a connected complex projective curve, and $\mathcal{M} \subset E$ a coherent subsheaf of a trivial holomorphic vector bundle E on C. Then $deg(det(\mathcal{M})) \leq 0$. If $deg(det(\mathcal{M})) = 0$, then $det(\mathcal{M}) \cong \mathcal{O}_C$, and $\mathcal{M}$ is trivial.*

Proof. If $r = rank(\mathcal{M})$, project E to a generic trivial quotient bundle E' of rank r of E, such that the induced map: $\mathcal{M} \to E'$ is generically isomorphic over C. □

2.5. **The Kodaira-Spencer map.** We have just seen that all coherent subsheaves $\mathcal{G} \subset N$ satisfy $\kappa(det(\mathcal{G})) \leq 0$. In this subsection we shall prove that this implies the isotriviality of the family of curves $f : D \to B$. As we have already remarked, the isotriviality is easy when $g = 1$, where g is the genus of a generic fiber, since only isotrivial elliptic fibrations have no singular fibers besides multiple elliptic curves. So without loss of generality, we may assume that $g \geq 2$.

In [H-V08], Hwang and Viehweg have constructed a generically finite map $s : D' \to D$, such that the sheaf s^*N has a subsheaf with the Iitaka dimension of the determinant equal to the variation of moduli of the curves in the family. We shall observe that this subsheaf is already the inverse image of a subsheaf of N. Together with corollary 2.8, this immediately implies isotriviality.

The construction of [H-V08] starts with the local (Zariski-open) coverings $U' \to U \subset B$ as in lemma 2.3, and a generically finite map $h : B' \to B$ obtained by taking their fibered products and desingularizing. The map $s : D' \to D$ is obtained by base-change. As before, $f' : D' \to B'$ denotes the natural morphism.

The map s is in general not étale and even not necessarily finite, but over suitable small open subsets $U \subset B$, $D_U \subset D$, it factors through the étale coverings $s_U : D'_U \to D_U$ of lemma 2.3.

Since now D' is a smooth family of curves over a smooth base B', one can consider the "Kodaira-Spencer map"

$$(f')_*(\omega_{D'/B'}^{\otimes 2}) \to \Omega^1_{B'}$$

obtained by dualizing the usual Kodaira-Spencer map from $T_{B'}$ to $R^1 f'_* T_{D'/B'}$ associated to the family of curves $f' : D' \to B'$. Let $\mathcal{H}' \subset \Omega^1_{B'}$ be its image: it is a coherent subsheaf of $\Omega^1_{B'}$. Moreover, it is functorial in B', that is, commutes with base change.

Over the non-empty Zariski open subset $B^* \subset B$ over which f is submersive, we similarly get a coherent subsheaf $\mathcal{H}^* \subset \Omega^1_{B^*}$, such that $\mathcal{H}' = h^*(\mathcal{H}^*)$ over $B'^* := h^{-1}(B^*)$.

In the same way, we get the local sheaves $\mathcal{H}'_U$ on local coverings U' of lemma 2.3, and their pullbacks $\mathcal{G}'_U$ on D'_U. Let us call these sheaves *Kodaira-Spencer sheaves.* Our key observation is as follows:

Lemma 2.10. *There exists a unique coherent subsheaf $\mathcal{G} \subset N$ such that, for any $h : B' \to B$ as above, we have: $\mathcal{G}' = (f')^*(\mathcal{H}') = s^*(\mathcal{G})$. Moreover, $\mathcal{G} = f^*(\mathcal{H}^*)$ over B^*.*

Proof. Define indeed $\mathcal{G}' := (f')^*(\mathcal{H}')$. Let $b \in B$ be any point, and let G be the holonomy group of $\mathcal{F}$ near $D_b := f^{-1}(b)$. We have (restricting everything to a suitable neighbourhood U of b) local Galois

G-covers $s : D'_U \to D_U, h_U : U' \to U$, with s étale (by Reeb stability). The local Kodaira-Spencer sheaves $\mathcal{H}'_U$ and $\mathcal{G}'_U$ are G-invariant, and by functoriality, $\mathcal{G}'$ is locally a lift of $\mathcal{G}'_U$.

Define $\mathcal{G}_U := s_*(\mathcal{G}'_U)^G$ to be the G-invariant part of $s_*(\mathcal{G}'_U)$. Because s is étale, we get $\mathcal{G}'_U = s^*(\mathcal{G}_U)$.

Now by functoriality of the Kodaira-Spencer sheaves, $\mathcal{G}_U$ agree on intersections and give the sheaf $\mathcal{G}$ on D.

□

The map $f : D \to B$ determines a holomorphic map $W : B^* \to M_g$, the moduli space of curves of genus g. Let $v := dim(W(B^*)) \subset M_g$. This is also the generic rank of the sheaves $\mathcal{H}^*, \mathcal{G}$ defined above.

Theorem 2.11. *(*[H-V08]*, proposition 4.4). Assume that $g \geq 2$. Then $\kappa(D, det(\mathcal{G})) = v$.*

Remark 2.12. *In* [H-V08]*, this result is stated for $\mathcal{G}'$ above, and not for $\mathcal{G}$ (which is anyway not defined there. But it is crucial for us to descend from $\mathcal{G}'$ to $\mathcal{G}$, and to be able to express $\mathcal{G}$ in the form: $\mathcal{G}' = s^*(\mathcal{G})$). Since $\kappa(D, det(\mathcal{G})) = \kappa(D', s^*(det(\mathcal{G})))$, the equality stated in 2.11 holds as well.*

Corollary 2.13. *The fibration $f : D \to B$ is isotrivial.*

Indeed, by 2.8 we know that the Kodaira dimension of the determinant of any subsheaf of N is non-positive, so $\kappa(D, det(\mathcal{G})) = v = 0$.

2.6. **A more general conjectural isotriviality statement.** The corollary 2.13 is a special case of the following more general conjectural statement, which slightly generalises [T13][1]:

Conjecture 2.14. *Let $f : X \to B$ be a proper, connected, quasi-smooth[2] fibration of quasi-projective varieties, where X is smooth and B is normal. Assume that the (reduced) fibres of f have semi-ample canonical bundle, and that the orbifold base (B, Δ) of f defined as above is special (in the sense of* [Ca 07]*). Then f is isotrivial.*

2.7. **Consequences of isotriviality.** Let us recall the situation we have started with and the results obtained by now.

We consider a fibration $f : D \to B$ between connected projective normal varieties, where D is smooth, such that all of its fibres are curves with smooth reduction and the smooth fibers have genus $g > 0$, and that its orbifold base (B, Δ) satisfies $K_B + \Delta \equiv 0$. This is the case if D is a divisor in a smooth projective manifold carrying a holomorphic symplectic 2-form, and f is the associated characteristic foliation on D, assumed to be algebraic.

We have shown that in this case f is isotrivial.

[1] In [T13], the conjecture is established when B is smooth and $\Delta = 0$.

[2] That is, the reduction of every fibre is smooth.

The construction of [H-V08] gives a finite proper morphism $h : B' \to B$ such that by normalising $D \times_B B'$, we get induced maps $s : D' \to D$ and $f' : D' \to B'$ such that f' is smooth and s satisfies $K_{D'/B'} = s^*(K_{D/(B,\Delta)})$ (see lemma 2.3). More precisely, locally near each point of B, one has $h = h_2 \circ h_1$, $s = s_2 \circ s_1$, where the corresponding map $s_1 : D_1 \to D$ is étale, and $f_1 : D_1 \to B_1$ is smooth (this is the local map from lemma 2.3), whereas the base change $B' \to B_1$, needed to globalize the construction, preserves, together with the smoothness of the family, the only other property used in the sequel: $K_{D'/B'} = s_2^*(K_{D_1/B_1}) = s^*(K_{D/(B,\Delta)})$.

It is well-known that a smooth isotrivial family of curves of genus g, after a suitable finite base change, becomes a product when $g \geq 2$, and a principal fibre bundle when $g = 1$. More precisely, we have the following lemma:

Lemma 2.15. *There exists a finite proper map $h' : B'' \to B'$ such that after base-changing f' by h', we get $f' : D'' \to B''$ and $s' : D'' \to D'$ with the following property: $D'' \cong F \times B''$ over B'' when $g \geq 2$, and $f'' : D'' \to B''$ is a principal fibre bundle if $g = 1$. Moreover, $K_{D''/B''}$ is nef, $\kappa(D'', K_{D''/B''}) = \nu(D'', K_{D''/B''}) = 1$ if $g \geq 2$, and $\kappa(D'', K_{D''/B''}) = \nu(D'', K_{D''/B''}) = 0$ if $g = 1$.*

Here ν denotes the numerical dimension.

Proof. Fix a curve F to which the fibres of f are isomorphic. Take for B'' the closure of a component, in the Chow variety of $F \times D'$, parametrising graphs of the relative isomorphisms of F with the fibres of f'. Thus B'' is projective, by the projectivity of F and D. If $g \geq 2$, this gives the desired isomorphism with a product over B'', since $Aut(F)$ is finite. If $g = 1$, we get the desired principal bundle structure. The second claim is obvious when $g \geq 2$. When $g = 1$, one can remark that $K_{D''/B''}$ is dual to $f''^*(R^1 f''_*(\mathcal{O}_{D''}))$, and the latter is trivial since translations on an elliptic curve operate trivially on cohomology. □

Corollary 2.16. *Let $f : D \to B$ be as above. Then K_D is nef, $\kappa(D) = \nu(D, K_D) = 1$ if $g \geq 2$, and $\kappa(D) = \nu(D, K_D) = 0$ if $g = 1$.*

Proof. Since

$$NK_{D''/B''} = Nh'^*(K_{D'/B'}) = Nh'^*(h^*(K_{D/(B,\Delta)})) = N(h \circ h')^*(K_D),$$

this follows from the preceding lemma, by the preservation of nefness, numerical dimension and Kodaira-Moishezon dimension under inverse images by finite proper morphisms. □

We thus have proved the theorem 1.2. In the next section, we shall give a proof of the theorem 1.3.

3. Divisors on irreducible hyperkähler manifolds.

We suppose now that X is an irreducible holomorphic symplectic manifold of dimension $2n \geq 4$, $D \subset X$ is a smooth non-uniruled divisor on X and the fibres of $f : D \to B$ are tangent to the kernel of the restriction of the holomorphic symplectic form σ to D. Recall that on the second cohomology of X there is a non-degenerate bilinear form q, the *Beauville-Bogomolov form.*

By corollary 2.16 $\nu(K_D) \leq 1 < \frac{dim(X)}{2}$. On the other hand, we have the following well-known lemma.

Lemma 3.1. *Let D be a non-zero nef divisor on a smooth projective irreducible hyperkähler manifold X. Then either $\nu(D) = dim(D)$ (if $q(D, D) > 0$), or $\nu(D) = \frac{dim(X)}{2}$ (if $q(D, D) = 0$).*

Proof. Recall that the numerical dimension of a nef divisor D is the maximal number k such that the cycle D^k is numerically non-trivial. If $q(D, D) \neq 0$, then $\nu(D) = 2n = dim(X)$, since D^{2n} is proportional to $q(D, D)^n$ with non-zero coefficient, by Fujiki formula. If $q(D, D) = 0$, one has $D^n \neq 0$ numerically and $D^{n+1} = 0$ by the results of Fujiki and Verbitsky. Indeed, the natural map from $S^kH^2(X, \mathbb{Z})$ to $H^{2k}(X, \mathbb{Z})$ is an embedding for $k \leq n$, and the Poincare pairing is non-degenerate on the part of the cohomology generated by $H^2(X, \mathbb{Z})$ ([V96]). Together with a version of Fujiki formula, this easily implies the result (see for instance [V09], corollary 2.15).

□

Note that $\nu(X, D) = \nu(D, K_D) + 1$, since $K_D = D|_D$. Therefore $\nu(D) \leq 2$ and the only possibility is $dim(X) = 4$, $\nu(X, D) = 2$, $\nu(D, K_D) = \kappa(D) = 1$, $g \geq 2$. This case can be excluded as follows: since $\kappa(D) = \nu(D, K_D)$, D is a good minimal model and the Iitaka fibration $\phi : D \to C$ is a regular map. Its fibers S are equivalent to D^2 as cycles on X, and therefore are lagrangian. Indeed, it follows from the definition of the Beauville-Bogomolov form σ on X that

$$\int_S \sigma\bar{\sigma} = q(D, D) = 0,$$

and this implies that the restriction of σ to S is zero. So the leaves of the characteristic foliation must be contained in the fibers of ϕ, giving the fibration of S in curves of genus at least 2. But this is impossible on S, since S is a minimal surface of Kodaira dimension zero.

4. Divisors on general projective symplectic manifolds.

The purpose of this section is to prove theorem 1.5.

Recall the setting: (X, σ) is a holomorphic symplectic projective variety, $D \subset X$ is a smooth hypersurface such that its characteristic foliation $\mathcal{F}$ is algebraic and the genus g of the leaves is strictly positive.

We wish to prove that up to a finite étale covering, X is a product with a surface and D is the inverse image of a curve under projection to this surface.

By Bogomolov decomposition theorem, we may assume that X is the product of a torus T and several irreducible hyperkähler manifolds H_j with $q(H_j) = 0$ (here q denotes the irregularity $h^{1,0}$) and $h^{2,0}(H_j) = 1$.

We distinguish two cases:

First case: X is not a torus. We shall proceed by induction on the number of non-torus factors in the Bogomolov decomposition of X.

Since X is not a torus, there is an irreducible hyperkähler factor H in the Bogomolov decomposition. If $X = H$, we are done. Otherwise, write $X = H \times Y$, where Y is the product of the remaining factors. By Künneth formula, we have $\sigma_X = \sigma_H \oplus \sigma_Y$ on $TX \cong TH \oplus TY$, since $q(H) = 0$. For $y \in Y$ general, let $D_y = D \cap (H \times \{y\})$. If this is empty, then $D = H \times D_Y$ for some divisor D_Y of Y, which is smooth with algebraic characteristic foliation. Indeed, at any point of D the σ_X-orthogonal to TD is contained in the σ_X-orthogonal to $TH \subset TD$, whereas $TH^\perp = TY$ since σ_X is a direct sum. We conclude by induction in this case.

Therefore we may suppose that D dominates Y. For $y \in Y$ generic, D_y is a smooth non-uniruled divisor on $H \times y$. At any point $(h, y) \in D$ such that $D_y \neq H \times y$ is smooth at h, we have $TD_y = TD \cap TH$. Moreover, at such a point $TH \not\subset TD$ and thus, taking the σ-orthogonals, $\mathcal{F} \not\subset TY$. We get $(TD_y)^\perp = TD^\perp \oplus TH^\perp = \mathcal{F} \oplus TY$.

Since σ is a direct sum, the σ_H-orthogonal of TD_y in TH is the projection of $\mathcal{F}$ to TH. In other words: the characteristic foliation $\mathcal{F}_{D_y}$ of D_y inside H is the projection on TH of the characteristic foliation $\mathcal{F} \subset TX$ along D_y. The leaves of $\mathcal{F}_{D_y}$ are thus the étale p_H-projections of the leaves of $\mathcal{F}$ along D_y, and so $\mathcal{F}_{D_y}$ is algebraic, with non-uniruled leaves. From theorem 1.3, we deduce that H is a $K3$-surface, and the divisors D_y are curves of genus $g > 0$ for $y \in Y$ generic.

When D_y is singular at h, one has $TH \subset TD$ at (h, y), and therefore at such points $\mathcal{F} \subset TY$.

Fix any $h \in H$ and let C_y denote the leaf of the characteristic foliation of D through (h, y). By isotriviality, all the curves C_y are isomorphic to each other. When y varies in the fibre of D over h, we thus have a positive-dimensional family of nonconstant maps $p_H : C_y \to H$ parameterized by a compact (but possibly not connected) variety D^h, and all images pass through the point $h \in H$. After a base-change $\alpha : Z \to D^h$ (not necessarily finite, but with Z still compact) of the family of the leaves, we have a map $p : C_y \times Z \to H$ mapping a section $c \times Z$ to a point. By the rigidity lemma, all images $p_H(C_y)$ coincide when y varies in a connected component of Z; therefore there is only a finite number of curves C_y through any $h \in H$. By the same reason, such a curve (that is, the projection of a leaf of $\mathcal{F}$ to H) does not intersect its small deformations in the family of the projections of leaves. The family of such curves is thus at most a one-parameter family, and there are only finitely many of them through any given point of H.

We are thus left with two cases: either all leaves of $\mathcal{F}$ project to the same curve on H, so that $p_H(D) = C \subset H$ is a curve and we are finished; or $p_H(D) = H$. In this last case, H is covered by a one-parameter family of curves C_t, which we may suppose irreducible, such that C_t does not intersect its small deformations and there is only a finite number of C_t through a given point.

Notice also that these C_t have to coincide with the connected components of the divisors D_y and therefore the generic C_t is smooth. By adjunction formula, it is an elliptic curve and H is fibered in curves C_t.

We claim that every C_t is non-singular. Indeed, suppose that some C_t is singular at $h \in H$. It has to be a connected component of a D_y for some (h, y) on a leaf of $\mathcal{F}$ projecting to C_t. As we have remarked above, the singularity of D_y at h means that $TH \subset TD$ and therefore $\mathcal{F} \subset TY$ along a connected component of $p_H^{-1}(h)$. But such a component is of strictly positive dimension and therefore would contain a leaf of $\mathcal{F}$. So there are at least two leaves of $\mathcal{F}$ through (h, y), one projecting to C_t and another to a point, which is absurd.

Since H is a $K3$-surface, it does not admit an elliptic fibration without singular fibers by topological reasons (non-vanishing of the Euler number). This is the contradiction excluding $p_H(D) = H$, and thus establishing theorem 1.5 when X is not a torus. $\square$

Second case: $X = T$ is a torus. We shall use Ueno's structure theorem for subvarieties of tori ([U75], Theorem 10.9).

If $g > 1$, then $\kappa(D) = 1$. By Ueno's theorem there is a subtorus K of codimension 2 such that D is the inverse image of a curve on the quotient: $D = p^{-1}(C)$, where $p : T \to S := T/K$ is the projection and $C \subset S$ is a curve of genus $g' > 1$ on the abelian surface S. The σ-orthogonal space to K gives canonically a two-dimensional linear foliation $\mathcal{F}_T$ on T, such that the intersections of its leaves with D are the leaves of $\mathcal{F}$, hence smooth compact curves which project in an étale way by p onto C.

Let us show that the leaves of $\mathcal{F}_T$ are compact. Take a leaf C of $\mathcal{F}$ through a point $x \in T$. It is contained in the leaf L of $\mathcal{F}_T$ through x. Choose a group structure on T in such a way that $x = 0$. The translate of C by any point $a \in C$ is still contained in the leaf L since L is linear; on the other hand, it is not equal to C for a outside of a finite set, since $g(C) > 1$. Since L is two-dimensional and contains a family of compact curves parameterized by a compact base, L must itself be compact.

Therefore the leaves of $\mathcal{F}_T$ are translates of an abelian surface S'. It suffices now to take a finite étale base-change from S to S' to get the desired form $T' = K \times S'$, $D' = K \times C$, σ direct sum of symplectic forms on S' and K.

If $g = 1$, then $\kappa(D) = 0$, and D is a subtorus of codimension 1 with an elliptic fibration. There thus exists an elliptic curve $C \subset T$ and a quotient $\pi : T \to R = T/C$ such that $D = \pi^{-1}(V)$, where V is a codimension 1 subtorus of the torus R. Project $\rho : R \to R/V$, and consider the composition $p : T \to S := R/V$. Then S is an abelian surface, and $C' := p(C)$ is an elliptic curve on it. Moreover, $D = p^{-1}(C)$. Let K be the kernel of p: this is a subtorus of T of codimension 2. By Poincaré reducibility, there exists an abelian surface $S' \subset T$ such that $(S' \cap K)$ is finite. After a finite étale cover, $T' = S' \times K$, and $D' = C \times K$ is of the claimed form. □

Remark 4.1. *In this last case, σ_T is in general not the direct sum of symplectic forms on S' and K. Take for example $T = S \times A$, $D = E \times A$, for $S, A, E \subset S$ Abelian varieties of dimensions $2, (n-2), 1$ respectively, with linear coordinates (x, y) on S, $(z_1, ..., z_{n-2})$ on A, and E given by $x = 0$. Take $\sigma_S := dx \wedge dy$, σ_A arbitrary on A, and $\sigma = \sigma_S + \sigma_A + dx \wedge dz$, for any nonzero linear form z on TA.*

5. Application to the Lagrangian conjecture.

Our aim is corollary 5.2 below. First we prove the following proposition.

Proposition 5.1. *Let $D \subset X$ be a smooth hypersurface in the connected projective*[3] *manifold X of dimension $2n$, carrying a holomorphic symplectic 2-form σ. Denote by $\mathcal{F}$ the characteristic foliation on D defined by σ. Assume that D admits a holomorphic fibration $\psi : D \to S$ onto an $(n-1)$-dimensional connected complex manifold S, such that its general fibre is a lagrangian subvariety of X of zero Kodaira dimension. Then*

1. The foliation $\mathcal{F}$ is ψ-vertical (ie: tangent to the fibres of ψ).

2. Either the smooth fibres of ψ are tori, and then ψ is the restriction to D of a holomorphic Lagrangian fibration ψ' on some open neighborhood of D in X; or their irregularity $q(F)$ is equal to $n-1$. In this case the Albanese map $a_F : F \to Alb(F)$ is surjective and connected, and its fibres are elliptic curves which are the leaves of $\mathcal{F}$. Moreover F has a finite étale covering which is a torus.

Proof. The first claim is obvious, since, at any generic $x \in D$, the σ-orthogonal to TD_x is included into the σ-orthogonal to TF_x (where F

[3]Remark however that the arguments are local near a fibre of ψ, and use only a Kähler assumption there.

denotes the fibre of ψ through x), which is equal to itself since F is Lagrangian.

Since the deformations of our Lagrangian fibres F cover D, we have $q(F) = h^0(Y, \Omega^1_X) = h^0(Y, N_{Y/X}) \geq dim(D) - dim(F) = n - 1$. Note that $q(F) \leq n$, since the Albanese map of a variety with zero Kodaira dimension is surjective with connected fibres by [K81].

If $q(F) = n$, F is bimeromorphic to a torus. Since it admits an everywhere regular foliation, it must be a torus. In this case F deforms in an n-dimensional family and this gives a fibration of a neighbourhood of F in X (indeed, the normal bundle to F in X is trivial since it is isomorphic to the cotangent bundle by the lagrangian condition). Otherwise, $q(F) = n - 1$ and the fibres of the Albanese map a_F are one-dimensional. In fact these are elliptic curves by $C_{n,n-1}$ ([Vi77]), and this also implies that F has a finite étale covering which is a torus.

Finally, the leaves of $\mathcal{F}$ inside F are tangent to the fibres of a_F. Indeed, since $q = n - 1$ and F moves inside an $(n - 1)$-dimensional smooth and unobstructed family of deformations (the fibres of ψ), all deformations of F stay inside D, and the natural evaluation map $ev : H^0(F, N_{F/X}) \otimes \mathcal{O}_F \to TX|_F$ must take its values in $T_{D|F}$.

Assume the leaves of $\mathcal{F}$ are not the fibres of a_F. We can then choose a 1-form u on $Alb(F)$ such that $v = a_F^*(u)$ does not vanish on $\mathcal{F}$ at the generic point z of F. The vanishing hyperplane of v_z in TF_z is however σ-dual to a vector $t_z \in TX_z$, unique and a nonzero modulo TF_z, which corresponds to the 1-form v_z under the isomorphism $(N_F)_z \cong (\Omega^1_F)_z$ induced by σ on the Lagrangian F. Since v does not vanish on $\mathcal{F}_z$ by assumption, $t_z \notin (T_D)_z$, which contradicts the fact that all first-order infinitesimal deformations of F are contained in D. □

Corollary 5.2. *Assume that X is a projective irreducible hyperkähler manifold of dimension $2n$, and $D \subset X$ a smooth and nef reduced and irreducible divisor. Assume that K_D is semi-ample. Then $\mathcal{O}_X(D)$ is semi-ample.*

Proof. If the Beauville-Bogomolov square $q(D, D)$ is positive, then D is big and the statement follows from Kawamata base point freeness theorem. So the interesting case is when D is Beauville-Bogomolov isotropic. We have $K_D = \mathcal{O}_X(D)|_D$. If K_D is semi-ample, its Kodaira dimension is equal to $\nu(D) = n - 1$ (lemma 3.1) and the Iitaka fibration ψ is regular. The relative dimension of ψ is equal to n. In fact $q(D, D) = 0$ implies that ψ is lagrangian in the same way as in [M01] (using that $K_D = \mathcal{O}_D(D)$ and that a suitable positive multiple $m.F$ of the fibre F is $\psi^*(H^{n-1})$ for some very ample line bundle H on S). By proposition 5.1, we have two possibilities: either the characteristic foliation on D is algebraic, which is impossible by theorem 1.3; or F is a torus, and then the fibration ψ extends near D, since F must deform in an n-dimensional family. In this case we conclude by [G-L-R 11], [H-W12] and [M08]. □

Recall that the Lagrangian conjecture affirms that a non-zero nef Beauville-Bogomolov isotropic divisor is semiample (and thus there is a lagrangian fibration associated to some multiple of such a divisor). Corollary 5.2 shows that the Lagrangian conjecture is true for an effective smooth divisor on a holomorphic symplectic manifold of dimension $2n$, if the Abundance conjecture holds in dimension $2n-1$. Since the Abundance conjecture is known in dimension 3 ([K92]), we have the following:

Corollary 5.3. *Let X be a projective smooth irreducible hyperkähler manifold of dimension 4, and D a nef divisor on X. Assume that D is effective and smooth. Then $\mathcal{O}_X(D)$ is semi-ample.*

Notice that if $dim(X) = 4$, we can use [A11] instead of [G-L-R 11] and [H-W12], and [AC05] instead of [M08], so that the proof is, in this case, more elementary.

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