Collections of parabolic orbits in homogeneous spaces, homogeneous dynamics and hyperkähler geometry

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Abstract
Consider the space $M = O(p,q)/O(p) \times O(q)$ of positive p-dimensional subspaces in a pseudo-Euclidean space V of signature (p,q), where $p > 0$, $q > 1$ and $(p,q) \neq (1,2)$, with integral structure: $V = V_{\mathbb{Z}} \otimes \mathbb{R}$. Let Γ be an arithmetic subgroup in $G = O(V_{\mathbb{Z}})$, and $R \subset V_{\mathbb{Z}}$ a Γ-invariant set of vectors with negative square. Denote by $R^\perp$ the set of all positive p-planes $W \subset V$ such that the orthogonal complement $W^\perp$ contains $r \in R$. We prove that either $R^\perp$ is dense in M or Γ acts on R with finitely many orbits. This is used to prove that the squares of primitive classes giving the rational boundary of the Kähler cone (i.e. the classes of "negative" minimal rational curves) on a hyperkahler manifold X are bounded by a number which depends only on the deformation class of X. We also state and prove the density of orbits in a more general situation when M is the space of maximal compact subgroups in a simple real Lie group.

Contents

1 Introduction

In [AV2], the following result of hyperbolic geometry was established.

Theorem 1.1: Let $\mathbb{H} = SO(1,n)/SO(n)$ be the n-dimensional real hyperbolic space, $n > 2$, and $\Gamma \subset SO(1,n)$ an arithmetic subgroup naturally acting on $\mathbb{H}$.

[1]Partially supported by RSCF, grant number 14-21-00053.
[2]Partially supported by RSCF, grant number 14-21-00053.
Keywords: hyperkähler manifold, moduli space, period map, Torelli theorem
2010 Mathematics Subject Classification: 53C26, 32G13

Consider a Γ-invariant collection $\{S_i\}$ of rational hyperplanes $S_i \subset \mathbb{H}$, and let $Z := \bigcup_i S_i$ be their union in $\mathbb{H}$. Assume that Γ acts on $\{S_i\}$ with infintely many orbits. Then $Z = \bigcup_i S_i$ is dense in $\mathbb{H}$.

Here we view $\mathbb{H}$ as the projectivization of the positive cone V^+ in the vector space V with integral structure, $V = \mathbb{Z}^{n+1} \otimes \mathbb{R}$, equipped with an integral quadratic form of signature $(+, -, \dots, -)$).

This result can be understood as a statement about homogeneous geometry of the space $M = G/K$, where $G = SO(1,n)$ and K a maximal compact subgroup; the hyperplanes S_i are orbits, under maximal parabolic subgroups $P_i = \mathrm{Stab}(v_i)$, $v_i^2 < 0$, of certain points $x_i \in v_i^\perp$. In the present paper we generalize this result for $SO^+(p,q)$ (connected component of $SO(p,q)$) for all $p, q \geqslant 2$.

Theorem 1.2: Let $G = SO^+(V)$ be a connected component of the orthogonal group, where $V = V_\mathbb{Z} \otimes_\mathbb{Z} \mathbb{R}$ is a pseudo-Euclidean space of signature (p,q), where $p > 0$, $q > 1$ and $(p,q) \neq (1,2)$, obtained from an integral lattice $V_\mathbb{Z}$, $\Gamma \subset SO^+(V_\mathbb{Z})$ an arithmetic subgroup of G, and $R \subset V_\mathbb{Z}$ a Γ-invariant set of vectors with negative squares. Denote by $\mathrm{Gr}_+ = O(p,q)/O(p) \times O(q)$ the space of all positive p-dimensional planes in V, and let $R^\perp$ be the set of all planes $W \in \mathrm{Gr}_+$ such that the orthogonal complement $W^\perp$ contains some $r \in R$. Then either Γ acts on R with finitely many orbits, or $R^\perp$ is dense on $\mathrm{Gr}_+(V)$.

Proof: It is a special case of Theorem 1.7. ■

Below we state and prove a generalization of this result when G is any simple non-compact algebraic Lie group. For our applications, only the case $G = SO^+(3,q), q > 1$ is interesting, but it seems that this generatization simplifies the problem and makes it conceptually easier.

Our motivation comes from the algebraic geometry of hyperkähler manifolds; see Section 3 for details on those. The connection is as follows: on the second cohomology of a hyperkähler manifold X, there is an integral non-degenerate quadratic form q, the **Beauville-Bogomolov-Fujiki (BBF) form**. The signature of q is $(3, b_2 - 3)$. If M is projective, q is of signature $(+, -, \dots, -)$ on the real Neron-Severi group $\mathrm{NS}(X) \otimes \mathbb{R}$; we view the projectivization of the positive cone in $\mathrm{NS}(X) \otimes \mathbb{R}$ as the hyperbolic space $\mathbb{H}$. An important question of algebraic geometry is to describe the ample cone inside the positive cone. It is well-known that it is cut out by a (possibly infinite) number of rational hyperplanes. One may ask whether there are only finitely many of them up to the action of automorphism group (this is a version of the "cone conjecture" by Kawamata and Morrison). Our theorem from [AV2] implies this as soon as the Picard rank is greater than three. The group Γ of the theorem is the Hodge monodromy group (see subsection Definition 3.9) rather than the automorphism group, but this is handled using the global Torelli theorem ([Ma], [V1]).

Our motivation for the present paper is a refinement of the cone conjecture in the topological context. In fact we have shown in [AV1] that the Kähler cone inside the positive cone is a connected component of the complement to the union of hyperplanes orthogonal to the so-called **MBM classes** of type $(1,1)$; those MBM classes are simply the classes whose orthogonal hyperplane supports a face of the Kähler cone of a birational model of X, as well as their monodromy transforms (so that the set R of MBM classes is invariant by Γ). In this case, Theorem 1.1 says that the set of primitive MBM classes of type $(1,1)$ is finite up to the action of the monodromy group; since the latter acts by isometries, a consequence of Theorem 1.1 is that the BBF squares of primitive MBM classes of type $(1,1)$ are bounded in absolute value by a constant N, which apriori depends on the complex manifold M.

But we also have shown that the MBM property is deformation invariant, that is, an MBM class remains MBM on all deformations where it stays of Hodge type $(1,1)$. It therefore makes sense to introduce the notion of an MBM class in $H^2(X,\mathbb{Z})$ without specifying its Hodge type, by requiring it to be MBM on those deformations where it is of type $(1,1)$. One then can ask whether there exists an upper bound N for the absolute value of the BBF square of a primitive MBM class which depends not on the complex structure, but only on its deformation type. By a result of Huybrechts, this is the same as to ask that the bound only depends on topology (indeed the result affirms that there are only finitely many deformation types in the topological class). We have conjectured the affirmative answer in [AV1] (Conjecture 6.4).

The purpose of this paper is to prove this conjecture. The main ingredient of the proof is the generalization of Theorem 1.1 to the space $M = G/K$, where $G = SO(3,n)$ and $K = SO(3)\times SO(n)$. This is exactly Theorem 1.2 with $p = 3$ which we state below separately in order to introduce some notations and terminology.

Recall that **a lattice** in a Lie group is a discrete subgroup of finite covolume (that is, the quotient has finite volume). Arithmetic subgroups of reductive groups without non-trivial rational characters are lattices by Borel and Harish-Chandra theorem.

Theorem 1.3: Let Γ be an arithmetic lattice in G, where $G = SO^+(3,n)$, $n \geqslant 2$, is the connected component of the unity of the group of linear isometries of a vector space (V,q) of signature $(3,n)$. Consider a Γ-invariant set of rational vectors $R\subset V$ with negative square. Let $\mathrm{Gr}_{+++} = \frac{SO(3,n)}{SO(3)\times SO(n)}$ be the Grassmannian of 3-dimensional positive oriented planes in V. For each $r\in R$, denote by S_r the set of all 3-planes $W\in \mathrm{Gr}_{+++}$ orthogonal to r. Assume that Γ acts on R with infinitely many orbits. Then the union $\bigcup_{r\in R} S_r$ is dense in Gr_{+++}.

A more general theorem is proved in Section 2, and the application to hyperkähler geometry is explained in detail in Section 3. Here we just state the main corollary and give the idea of its proof.

Corollary 1.4: Let X be a hyperkähler manifold with $b_2(X)\geqslant 5$. The monodromy group acts with finitely many orbits on the set of primitive MBM classes in $H^2(X,\mathbb{Z})$. The BBF square of a primitive MBM class on M is therefore bounded in absolute value by a constant N which depends only on the topology of X.

Idea of proof: Let R be the set of primitive MBM classes; then the union $\bigcup_{r\in R} S_r$ cannot be dense in Gr_{+++} since its complement is identified to a connected component of the **Teichmüller space of hyperkähler structures**, open in Gr_{+++} ([AV3]). ∎

Remark 1.5: This corollary also removes a technical assumption $b_2 \neq 5$ needed in [AV2] to prove that the set of faces of the Kähler cone is finite up to automorphism group action; see section 3.

Notice that each S_r is an orbit of some point x_r in Gr_{+++} under the maximal parabolic subgroup $P_r = \mathrm{Stab}(r)$; such an orbit is special in the sense that the subspace corresponding to x_r is orthogonal to r. By analogy with the hyperbolic space setting, we call them **parabolic orbits of hyperplane type**.

Theorem 1.3 and Theorem 1.1 are special cases of a more general statement. Denote by G a connected simple real algebraic Lie group, $\mathfrak{g}$ its Lie algebra, and $\mathfrak{g}_{\mathbb{C}}$ its complexification.

Recall that a connected subgroup $P \subset G$ is called **parabolic** if the complexification $\mathfrak{p}_{\mathbb{C}}$ of its Lie algebra contains a Borel subalgebra of $\mathfrak{g}_{\mathbb{C}}$, and **maximal parabolic** when it is maximal with these properties. In particular there are no other connected closed subgroups between P and G.

Let K be a maximal compact subgroup of G. By Cartan's theorem, G retracts on K, so K is itself connected ([Ho, Theorem 3.1, Chapter XV]). Since the normalizer of a maximal compact subgroup of a simple algebraic Lie group is again compact, K is equal to its normalizer, as follows from the Iwasawa decomposition. Therefore, we have a natural bijection between G/K and the set of conjugates of K, that is, one may view G/K as a set of all maximal compact subgroups of G. The action of G on G/K by left translation becomes, under this identification, the action of G on the set of maximal compact subgroups by conjugation.

Let P be a maximal parabolic subgroup and K' a maximal compact. Call P and K' **compatible** if $P \cap K'$ is maximal compact in P. If P and K' are compatible, then clearly P is compatible with all maximal compacts in the P-orbit of K'.

Definition 1.6: An **orbit of hyperplane type** under the action of P on G/K is a P-orbit consisting of P-compatible maximal subgroups.

Now we can formulate the general statement.

Theorem 1.7: Let Γ be an arithmetic lattice in G, where G is a connected simple real algebraic Lie group, K its maximal compact subgroup, and P a maximal parabolic subgroup which is assumed to be generated by unipotents. We assume that the groups G, K, P are defined over $\mathbb{Q}$. Consider a Γ-invariant set $\{S_i\}$ of orbits of hyperplane type of subgroups $x_i P x_i^{-1}$ acting on G/K. Assume that Γ acts on $\{S_i\}$ with infinitely many orbits. Then the union $\bigcup S_r$ is dense in G/K.

The next section is devoted to the proof of Theorem 1.7.

2 Homogeneous dynamics, parabolic subgroups and Mozes-Shah theorem

We deduce Theorem 1.7 from the general formalism of Ratner, Mozes-Shah and Eskin-Mozes-Shah used in [AV2] to prove Theorem 1.1.

In order to be able to apply this machinery we first prove a simple statement on Lie groups which replaces a set of orbits of hyperplane type of conjugated parabolic subgroups on G/K by a set of orbits of a single one on a suitable fibration over G/K, which is a G-homogeneous space "in between" G/K and G itself.

As above, we identify G/K with the set of all maximal compact subgroups of G.

Proposition 2.1: Let G and K be as above. Consider a maximal parabolic subgroup $P \subset G$, and let $S_i = y_i P y_i^{-1} x_i$, $i \in I$ by a set of orbits of hyperplane type under conjugates of P. Then there exists a single P_0 conjugate to P such that the S_i are projections of P_0-orbits R_i in G.

Remark 2.2: Those projections themselves are of course not P_0-orbits, as the projection map we consider is not equivariant. In the case when G is $SO^+(1,2)$, this construction is known as the **geodesic flow**: the hyperplanes in the hyperbolic plane lift tautologically to the unit tangent bundle as orbits of a single $SO^+(1,1)$, corresponding to the subgroup of diagonal matrices under the identification with $PSL(2,\mathbb{R})$.

Proof of Proposition 2.1: Under the identifications we have made, the action of G on G/K corresponds to the adjoint action on M, that is, $x \in G$ sends a maximal compact subgroup K to xKx^{-1}. Consider the space M_1 of all pairs (K_1, P_1), where $K_1 \subset G$ is a maximal compact subgroup, $P_1 \subset G$ a maximal parabolic subgroup which is conjugate to P, and $K_1 \cap P_1$ a maximal compact subgroup of P_1; that is, K_1 and P_1 are compatible.

(For instance, in the situation of Theorem 1.3, $G = SO(3,n)$, $K = SO(3) \times SO(n)$, the maximal parabolic group $P_1 = \mathrm{St}_G(v)$ is a stabilizer of a vector v with negative square, and $K_1 \cap P_1$ is isomorphic to $SO(3) \times SO(n-1)$.)

Now let M_0 denote the space of maximal parabolic subgroups conjugate to P, that is, subgroups of the form $P_1 = gPg^{-1} \subset G$. There is a natural diagram

$$\begin{tikzcd} & M_1 \arrow[dl, "\pi"'] \arrow[dr, "\sigma"] & \\ M & & M_0 \end{tikzcd}$$

with forgetful maps π and σ. Since the normalizer $N(P)$ is an algebraic group which has the same connected component of the unity as P, and M_0 is naturally identified with $G/N(P)$, the standard map $G/P \longrightarrow M_0 = G/N(P)$ is a finite covering. Therefore the orbits of hyperplane type under $P_1 = gPg^{-1}$ on M are connected components of $\pi(\sigma^{-1}(P_1))$. As P_1 varies in M_0, the connected components of $\sigma^{-1}(P_1)$ give a G-invariant foliation on M_1 with leaves which are mapped to gPg^{-1}-orbits of hyperplane type in M. Taking the preimages of those leaves in G, we obtain a translation-invariant foliation on G with leaves which are mapped to gPg^{-1}-orbits in M. But such a foliation is necessarily by orbits of the action of a subgroup; this subgroup is the P_0 that we are looking for. ■

Proof of Theorem 1.7:
We have seen that the collection of S_i lifts to a Γ-invariant collection of P_0-orbits R_i in G. It suffices to prove that this collection is dense or finite up to the action of Γ, or, in other words, that the corresponding orbits in $\Gamma\backslash G$ are finitely many or dense. To this end, we apply the same argument as in [AV2]. The P_0-orbits R_i give rise to probability measures μ_i on $\Gamma\backslash G$, supported on those orbits; those are simply the translates of μ_{P_0}, the "pushforward" of the Haar measure on P_0. If there are infinitely many of them, then by Mozes-Shah and Dani-Margulis theorems ([MS], Corollaries 1.1, 1.3, 1.4) one can extract from $\{\mu_i\}$ a weakly converging subsequence of measures on the one-point compactification of $\Gamma\backslash G$, possibly converging to the measure μ_∞, concentrated at the infinite point, but otherwise to a probability measure on an orbit of another closed subgroup P_0' containing P or its conjugate ([MS], Theorem 1.1).

We remark that the results of Mozes-Shah from [MS] are valid for any collection of measures ergodic with respect to subgroups generated by unipotents (as can be found in the same paper, see e.g. Lemma 2.3). The crucial point for our theorem is that our converging subsequence consists of measures supported on orbits of the same maximal parabolic subgroup P_0, which is of finite index in its normalizer. Then it cannot converge to infinity by a theorem of Eskin-Mozes-Shah ([EMS], Theorem 1.1). Convergence to another translate of μ_{P_0} is impossible because of the finiteness of the index of P_0 in its normalizer, as in Lemma 4.12 of [AV2]. Therefore it converges to a translate of the ergodic measure μ_Q associated to a closed subgroup Q strictly containing (a conjugate of) P; but such Q can only be G itself, from where the density.

This finishes the proof of Theorem 1.7.

3 Teichmüller space for hyperkähler structures and MBM classes

3.1 Hyperkähler manifolds and monodromy

In this subsection, we recall some basic results on hyperkähler manifolds.

Definition 3.1: A **hyperkähler manifold** is a compact Kähler holomorphically symplectic manifold.

Definition 3.2: A hyperkähler manifold M is called **simple**, or IHS, if $\pi_1(M) = 0$, $H^{2,0}(M) = \mathbb{C}$.

This definition is motivated by Bogomolov's decomposition theorem:

Theorem 3.3: ([Bo1]) Any hyperkähler manifold admits a finite covering which is a product of a torus and several simple hyperkähler manifolds. ■

Remark 3.4: Further on, we shall assume that all hyperkähler manifolds we consider are of maximal holonomy, that is, simple.

An important property of hyperkähler manifolds is the presence of an integral quadratic form of signature $(3, b_2 - 3)$ on their second cohomology, the **Bogomolov-Beauville-Fujiki (BBF) form**. It was defined in [Bo2] and [Bea] using integration of differential forms, but it is easiest to describe it by the Fujiki theorem, proved in [F1]; it stresses the topological origin of the BBF form.

Theorem 3.5: (Fujiki) Let M be a simple hyperkähler manifold, $\eta \in H^2(M)$, and $n = \frac{1}{2}\dim M$. Then $\int_M \eta^{2n} = cq(\eta,\eta)^n$, where q is a primitive integral quadratic form on $H^2(M,\mathbb{Z})$, and $c > 0$ a constant (depending on M). ■

The signature of the BBF form on the space of $(1,1)$-classes is $(+, -, \dots, -)$.

Definition 3.6: A cohomology class $\eta \in H^2_{\mathbb{R}}(M)$ is called **negative** if $q(\eta,\eta) < 0$, and **positive** if $q(\eta,\eta) > 0$. The **positive cone** $\operatorname{Pos}(M) \in H^{1,1}(M)$ is one of the two connected components of the set of positive $(1,1)$-classes which contains the Kähler classes.

Definition 3.7: Let M be a compact complex manifold, and $\mathsf{Diff}_0(M)$ a connected component of its diffeomorphism group (**the group of isotopies**). Denote by Comp the space of complex structures of Kähler type on M (remark here that the set of complex structures of Kähler type is open in the space of all complex structures by Kodaira-Spencer stability theorem), and let $\mathsf{Teich} := \mathsf{Comp} / \mathsf{Diff}_0(M)$. We call it **the Teichmüller space.**

For hyperkähler manifolds, this is a finite-dimensional complex non-Hausdorff manifold ([Cat], [V1]).

Definition 3.8: The **mapping class group** is $\mathsf{Diff}(M)/\mathsf{Diff}_0(M)$, naturally acting on Teich.

It follows from a result of Huybrechts (see [Hu]) that in the hyperkähler case Teich has only finitely many connected components. Therefore, the subgroup of the mapping class group which fixes the connected component of our chosen complex structure is of finite index in the mapping class group.

Definition 3.9: The **monodromy group** Γ is the image of this subgroup in $\operatorname{Aut} H^2(M,\mathbb{Z})$. The **Hodge monodromy group** is the subgroup $\Gamma_{\mathsf{Hdg}} \subset \Gamma$ preserving the Hodge decomposition.

Theorem 3.10: ([V1], Theorem 3.5) The monodromy group is a finite index subgroup in $O(H^2(M,\mathbb{Z}),q)$ (and the Hodge monodromy therefore acts as an arithmetic subgroup of the orthogonal group on the Picard lattice).

In [Ma],, E. Markman obtained the following implication of the global Torelli theorem relating the Hodge monodromy to automorphisms.

Theorem 3.11: If $\gamma \in \Gamma_{\mathsf{Hdg}}$ takes a Kähler class to a Kähler class, then $\gamma = f^*$ for some $f \in \operatorname{Aut}(M)$.

3.2 MBM classes

The notion of an MBM class was introduced in [AV1] in order to understand better how the Kähler cone sits in the positive cone.

Definition 3.12: An integral $(1,1)$-class z on M is MBM if for some $\gamma \in \Gamma_{\mathsf{Hdg}}$, the hyperplane $\gamma(z)^\perp$ supports a (maximal-dimensional) face of the Kähler cone of a birational model of M.

This somewhat mysterious definition has a simple geometric interpretation thanks to the deformation invariance of the MBM property. Note that the BBF form identifies homology with rational coefficients and cohomology. The following two theorems are proven in [AV1].

Theorem 3.13: Let M be a non-algebraic hyperkähler manifold with $\operatorname{Pic}(M) = \langle z\rangle$, where $z \in H_2(M,\mathbb{Z})$ a negative homology class. Then z is an MBM class if and only if for some $\lambda \neq 0$, λz can be represented by a curve.

Theorem 3.14: Let M, M' be hyperkähler manifolds in the same deformation class, such that a negative cohomology class $z \in H^2(M,\mathbb{Z})$ is of type (1,1) on M and M'. Then z is an MBM class on M if and only if it is MBM on M'.

This result allows one to extend the notion of MBM classes to the whole of $H^2(M,\mathbb{Z})$.

Definition 3.15: A negative class $\eta \in H^2(M,\mathbb{Z})$ is called **MBM** if it is MBM for some deformation of M where it is of type $(1,1)$.

By the very definition, the MBM property is deformation-invariant.

Remark 3.16: It follows immediately from the definition that the set of MBM classes in this generalized sense in $H^2(M,\mathbb{Z})$ is also Γ-invariant, where Γ is the monodromy group of M. Indeed given an MBM class z and a monodromy transform $\gamma(z)$ one can find a deformation of M such that both of them are of type $(1,1)$; then $\gamma(z)$ is MBM in the sense of our first definition.

The main result of this paper is the following theorem.

Theorem 3.17: Let $R \subset H^2(M,\mathbb{Z})$ be the set of primitive MBM classes in the cohomology of a hyperkähler manifold whose second Betti number $b_2(M)$ is at least 5. Then the monodromy group Γ acts on R with finitely many orbits. In particular, there is a number N depending only on the deformation type of M, such that for any $z \in R$, $|q(z)| \leqslant N$.

Proof: Consider $G = SO^+(3,n)$ where $n = b_2(M) - 3 > 1$, and the homogeneous space G/K as in Theorem 1.3, identified with the grassmannian of positive 3-planes in the vector space $H^2(M,\mathbb{R})$ of signature $(3,n)$. The subsets formed by 3-planes orthogonal to a given MBM class in $H^2(M,\mathbb{Z})$ form an orbit of hyperplane type. As $n > 1$, by Theorem 1.3 either the set of MBM classes is finite up to Γ-action, or the corresponding orbits of hyperplane type are dense in G/K. The latter is impossible. Indeed, in the same way as the orthogonals to the MBM classes of type $(1,1)$ serve as boundaries of the Kähler cone inside the positive cone, the orthogonals to the all MBM classes are a complement of a meaningful object of modular nature, the Teichmüller space of hyperkähler structures ([AV3]), open in the Grassmannian. In the next and final subsection, we briefly recall this for reader's convenience.

3.3 Hyperkähler Teichmüller space

Definition 3.18: Let (M,g) be a Riemannian manifold, and I,J,K endomorphisms of the tangent bundle TM satisfying the quaternionic relations

$$I^2 = J^2 = K^2 = IJK = -\operatorname{Id}_{TM}.$$

The triple (I,J,K) together with the metric g is called **a hyperkähler structure** if I,J and K are integrable and Kähler with respect to g.

Consider the Kähler forms $\omega_I, \omega_J, \omega_K$ on M:

$$\omega_I(\cdot,\cdot) := g(\cdot, I\cdot), \quad \omega_J(\cdot,\cdot) := g(\cdot, J\cdot), \quad \omega_K(\cdot,\cdot) := g(\cdot, K\cdot).$$

An elementary linear-algebraic calculation implies that the 2-form $\Omega := \omega_J + \sqrt{-1}\,\omega_K$ is of Hodge type $(2,0)$ on (M, I). This form is clearly closed and non-degenerate, hence it is a holomorphic symplectic form.

In algebraic geometry, the word "hyperkähler" is essentially synonymous with "holomorphically symplectic", due to the following theorem, which is implied by Yau's solution of Calabi conjecture ([Bes], [Bea]).

Theorem 3.19: Let M be a compact, Kähler, holomorphically symplectic manifold, ω its Kähler form, $\dim_{\mathbb{C}} M = 2n$. Denote by Ω the holomorphic symplectic form on M. Suppose that $\int_M \omega^{2n} = \int_M (\operatorname{Re}\Omega)^{2n}$. Then there exists a unique hyperkähler metric g with the same Kähler class as ω, and a unique hyperkähler structure (I, J, K, g), with $\omega_J = \operatorname{Re}\Omega$, $\omega_K = \operatorname{Im}\Omega$. ∎

Every hyperkähler structure induces a whole 2-dimensional sphere of complex structures on M, as follows. Consider a triple $a, b, c \in R$, $a^2 + b^2 + c^2 = 1$, and let $L := aI + bJ + cK$ be the corresponging quaternion. Quaternionic relations imply immediately that $L^2 = -1$, hence L is an almost complex structure. Since I, J, K are Kähler, they are parallel with respect to the Levi-Civita connection. Therefore, L is also parallel. Any parallel complex structure is integrable, and Kähler. We call such a complex structure $L = aI + bJ + cK$ **a complex structure induced by the hyperkähler structure**. There is a 2-dimensional holomorphic family of induced complex structures, and the total space of this family is called **the twistor space** of a hyperkähler manifold, its base being **the twistor line** in the Teichmüller space Teich which we are going to define next.

Definition 3.20: Let (M, I, J, K, g) and (M, I', J', K', g') be two hyperkähler structures. We say that these structures are **equivalent** if the corresponding quaternionic algebras in $\operatorname{End}(TM)$ coincide.

Consider the infinite-dimensional space Hyp of all quaternionic triples I, J, K on M which are induced by some hyperkähler structure, with the same C^∞-topology of convergence with all derivatives. The quotient $\mathsf{Hyp}/SU(2)$ (which is probably better to write as $\mathsf{Hyp}/SO(3)$, since -1 acts trivially on the triples) is naturally identified with the set of equivalence classes of hyperkähler structures, up to changing the metric g by a constant multiplier.

Remark 3.21: As shown in [AV3], for hyperkähler manifolds with maximal holonomy the quotient $\mathsf{Hyp}_m := \mathsf{Hyp}/SU(2)$ is also identified with the space of all hyperkähler metrics of fixed volume, say, volume 1.

Definition 3.22: Define **the Teichmüller space of hyperkähler structures** as the quotient $\mathsf{Hyp}_m / \mathsf{Diff}_0$, where Diff_0 is the connected component of the group of diffeomorphisms Diff, and **the moduli of hyperkähler structures** as $\mathsf{Hyp}_m / \mathsf{Diff}$.

Definition 3.23: Let M be a hyperkähler manifold of maximal holonomy, and $\mathsf{Teich}_h := \mathsf{Hyp}_m / \mathsf{Diff}_0$ the Teichmüller space of hyperkähler structures. Consider the space $\mathbb{P}\mathsf{er}_h = Gr_{+++}(H^2(M,\mathbb{R}))$ of all positive oriented 3-dimensional subspaces in $H^2(M,\mathbb{R})$, naturally diffeomorphic to $\mathbb{P}\mathsf{er}_h \cong SO(b_2-3,3)/SO(3)\times SO(b_2-3)$. Let $\mathsf{Per}_h : \mathsf{Teich}_h \longrightarrow \mathbb{P}\mathsf{er}_h$ be the map associating the 3-dimensional space generated by the three Kähler forms $\omega_I, \omega_J, \omega_K$ to a hyperkähler structure (M,I,J,K,g). This map called **the period map for the Teichmüller space of hyperkähler structures**, and $\mathbb{P}\mathsf{er}_h$ **the period space of hyperkähler structures**.

Theorem 3.24: Let M be a hyperkähler manifold of maximal holonomy, and $\mathsf{Per}_h : \mathsf{Teich}_h \longrightarrow \mathbb{P}\mathsf{er}_h$ the period map for the Teichmüller space of hyperkähler structures. Then the period map $\mathsf{Per}_h : \mathsf{Teich}_h \longrightarrow \mathbb{P}\mathsf{er}_h$ is an open embedding for each connected component. Moreover, its image is the set of all spaces $W \in \mathbb{P}\mathsf{er}_h$ such that the orthogonal complement $W^\perp$ contains no MBM classes.

Proof: See [AV3]. ■

Let $V = H^2(M,\mathbb{R})$ be the second cohomology of a hyperkähler manifold, equipped with the BBF form. Denote by Gr_{+++} the space of all positive oriented 3-planes in V. For each primitive MBM class x, the set of $W \in \mathrm{Gr}_{+++}$ orthogonal to x is an orbit of the maximal parabolic group $P_x = \mathrm{St}_{SO(V)}(x)$. By Theorem 3.24, the space Teich_h is identified with $\mathrm{Gr}_{+++} \setminus \bigcup_{x\in R} P_x(W_x)$, where $W_x \in \mathrm{Gr}_{+++}$ is any 3-space orthogonal to x, and R the set of primitive MBM classes. Denote by $\Gamma \subset SO(V)$ the monodromy group of M. As shown in [V1], it is a lattice in $SO(V)$. Since $P_x(W_x)$ is determined by x and determines it uniquely, up to a sign, the number of Γ-orbits on R is infinite if and only if the number of Γ-orbits on $P_x(W_x)$ is infinite. However, the union $\bigcup_{x\in R} P_x(W_x)$ is closed by Theorem 3.24, hence Theorem 1.3 implies that R/Γ is finite. We have proved Theorem 3.17.

Acknowledgements: We are grateful to Alex Eskin, Misha Kapovich and Maxim Kontsevich for illuminating discussions of hyperbolic geometry.

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