ON THE CHARACTERISTIC FOLIATION ON A SMOOTH HYPERSURFACE IN A HOLOMORPHIC SYMPLECTIC FOURFOLD

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1. Introduction

Definition 1. A holomorphic symplectic manifold is a simply-connected compact Kähler manifold X with a nowhere degenerate global holomorphic two-form ω. A holomorphic symplectic manifold is irreducible if in addition $H^0(X, \Omega^2)$ is spanned by such an ω.

This terminology is explained by Bogomolov decomposition theorem which states that, up to a finite étale covering, each holomorphic symplectic manifold is a product of several irreducible ones and a torus.

Let X be a holomorphic symplectic manifold with a holomorphic symplectic form ω. Let D be a smooth divisor on X. At each point of D, the restriction of ω to D has one-dimensional kernel. This gives a non-singular foliation $\mathcal{F}$ on D, called the characteristic foliation. Hwang and Viehweg in [HV] have shown that if X is projective and D is of general type, then $\mathcal{F}$ cannot be algebraic (unless in the trivial case when X is a surface, D is a curve, so $\mathcal{F}$ has a single leaf equal to D; recall that a foliation in curves on a compact Kähler manifold is called algebraic when all its leaves are compact complex curves). This result has been extended by Amerik and Campana in [AC2] to the case when D is not necessarily of general type; in particular, when X is irreducible, they proved that the characteristic foliation $\mathcal{F}$ on a nonsingular irreducible divisor D is algebraic if and only if the leaves of $\mathcal{F}$ are rational curves or X is a surface.

Suppose now that X is an irreducible holomorphic symplectic fourfold and D, $\mathcal{F}$ are as above. It is easy to give an example when the Zariski closure of a general leaf of $\mathcal{F}$ is a surface. Indeed such is the case when X has a Lagrangian fibration $f : X \to Z$ and D is the preimage of a general curve $C \subset Z$. The leaves of $\mathcal{F}$ are contained in the fibers of f; note that the general fiber is a torus by the Arnold-Liouville theorem. The aim of this paper is to prove that all examples where the Zariski closure of a general leaf is two-dimensional are obtained in this way.

Theorem 2. *Let X be an irreducible holomorphically symplectic 4-dimensional manifold and let D be an irreducible smooth divisor on X. Suppose that a general leaf of the characteristic foliation $\mathcal{F}$ on D is non-compact but there exists a rational fibration $p : D \dashrightarrow C$ by surfaces such that every leaf of $\mathcal{F}$ is contained in the closure of some fiber of p. Then there exists an almost holomorphic lagrangian fibration $f : X \dashrightarrow B$ extending p. In particular the general fiber of p is a torus.*

Greb, Lehn and Rollenske in [GLR] in the non-projective case, Amerik in [A] in the 4-dimensional case and Hwang and Weiss in [HW] in the general case proved that any Lagrangian torus in an irreducible holomorphically symplectic manifold is a fibre of an almost holomorphic Lagrangian fibration. The general fiber of p is clearly lagrangian, since the tangent space to S at a general point is contained in the tangent space to D

and contains the kernel of the restriction of the symplectic form to D. Therefore to prove theorem 2 it suffices to show that the general fiber S of p is a torus.

Our argument proceeds as follows. In the next section, using the classification of surfaces and Brunella's results on foliations on surfaces, we reduce the problem to the case when S is an elliptic fibration with some special properties. Then we remark that such an S deforms away from D and its general deformation has similar properties. Finally, in the last section we produce a deformation-theoretic argument leading to the conclusion.

2. Smooth foliations on surfaces and the geometry of S

We assume that the closure of a sufficiently general leaf of the characteristic foliation $\mathcal{F}$ is a surface. There is then a family of such surfaces which rationally fibers D; we denote a general surface of the family by S and the rational fibration by $p : D \dashrightarrow B$. Apriori p can have an indeterminacy locus and S can be singular along this locus. This indeterminacy locus is also the intersection of general surfaces in the family (notice that if p has indeterminacy then B is a rational curve, since certain rational curves coming from the resolution of indeterminacy then dominate B; so S is a general member of a linear system defining p, therefore the base locus of the linear system is the intersection of two such S).

From the fact that $\mathcal{F}$ is a smooth foliation (i.e. in the neighbourhood of each point D looks like a product with a curve) given by the ω-orthogonal to D we deduce an obvious

Observation 3. *S is a lagrangian surface, the indeterminacy locus of p is a union of leaves of $\mathcal{F}$, and $\mathcal{F}$ induces a smooth foliation on the normalization of S.*

Note that lagrangian implies projective in this context (see e.g. [C]). Our next aim is to show that p is regular and hence S is smooth.

Let $\tau : S' \to S$ be the normalization. The image of $T_{S'}$ in $\tau^* T_D$ contains $\tau^*\mathcal{F}$. We denote by $\mathcal{F}'$ the induced foliation on S'.

Lemma 4. *The conormal bundle of $\mathcal{F}'$ is effective.*

Proof. Consider the exact sequence

$$0 \to T_{S'}/\tau^*\mathcal{F} \to \tau^*(T_D/\mathcal{F}) \to \tau^*(T_D)/T_{S'} \to 0.$$

The sheaf in the middle carries a symplectic form and thus, being isomorphic to its dual, has zero determinant. We have the normal bundle of $\mathcal{F}'$ on the left, and the first Chern class of the sheaf on the right is therefore equal to that of the conormal bundle in the sense of algebraic cycles, but the sheafs itself are not necessarily isomorphic since $\tau^*(T_D)/T_{S'}$ can have torsion at the critical points of τ. The set of those critical points is a divisor on S' (it is a union of leaves of the regular foliation $\mathcal{F}'$), so the conormal bundle of $\mathcal{F}'$ is the torsion-free part of $\tau^*(T_D)/T_{S'}$ possibly tensored up with something effective. But $\tau^*(T_D)/T_{S'}$ modulo torsion is already effective by deformation theory, since τ is a general member of a one-dimensional family of maps to D (see e.g. [HM]). □

This lemma and the theory of foliation of surfaces imply the needed

Proposition 5. *The map p is regular, therefore its general fiber S is smooth.*

Proof. Brunella in [B2][Proposition 6.2]) shows that if $h^0(X, N^*_{\mathcal{F}}) \geq 1$ for a foliation $\mathcal{F}$ on a surface X and C a compact curve invariant by the foliation, then either C is contractible or $\mathcal{F}$ is a fibration over a non-rational curve. If p is not regular, apply this to $X = S'$ and C the compact leaf of the foliation coming from the indeterminacy locus of p. By Camacho-Sad formula $C^2 = 0$, so C is not contractible, but $\mathcal{F}'$ is not a fibration either, a contradiction. □

From now on we consider the restriction of the foliation $\mathcal{F}$ on D to the smooth surface S, which we denote by the same letter $\mathcal{F}$. By assumption, $\mathcal{F}$ has at least one non-algebraic leaf on S. As it has been already remarked in the proof of lemma 4, the symplectic form induces an isomorphism between $N_{S/D}$ and the conormal bundle of $\mathcal{F}$. Since $N_{S/D}$ is trivial, so is $N_{\mathcal{F}}$ and we have

Lemma 6. *$N_{\mathcal{F}}$ is the trivial line bundle, $K_S \simeq K_{\mathcal{F}} = (T_{\mathcal{F}})^*$ and $c_2(S) = 0$.*

Proof. Indeed the last two statements follow from the first by the exact sequence of a smooth foliation

$$0 \to T_{\mathcal{F}} \to T_S \to N_{\mathcal{F}} \to 0.$$

□

Corollary 7. *S is not of general type and the minimal model of S can not be a $K3$ surface or an Enriques surface. If the minimal model of S is a torus, a bielliptic or a properly elliptic surface, then S is itself minimal.*

Proof. The first statement is [BPV] [Proposition VII.2.4]; the rest follows from the classification and from the fact that c_2 goes up under blow-ups. Indeed c_2 is strictly positive for K3 and Enriques surfaces and non-negative for the other minimal models mentioned. □

Lemma 8. *$h^0(K_S) > 0$, so the Kodaira dimension of S is not equal to $-\infty$ and S can not be a bielliptic surface or its blow-up.*

Proof. This is an immediate consequence of a lemma by Brunella [B1, Lemma 7] which affirms that a regular foliation $\mathcal{F}$ on a smooth projective surface S with $h^0(T^*_{\mathcal{F}}) = 0$ and $h^0(N_{\mathcal{F}}) > 0$ is a fibration. Since this is not our case we must have $h^0(K_S) = h^0(K_{\mathcal{F}}) > 0$. □

By the Kodaira-Enriques classification, we see that if S is not a torus (in which case we are done), then S is properly elliptic, that is, S is an elliptic surface of Kodaira dimension 1. We denote the elliptic fibration by $\pi : S \to C$. Note that the only singularities of π are multiple fibers, by the fact that $c_2(S) = 0$ (see e.g. [BPV]). Therefore π induces another smooth rank-one foliation on S which we denote by $\mathcal{G}$ and the corresponding map by $\pi : S \to C$. Let us denote by g the genus of C. Note that the arithmetic genus $\chi(\mathcal{O}_S)$ of S is equal to zero by Noether formula.

No fiber of π can be $\mathcal{F}$-invariant. This can be seen using the same proposition by Brunella as in the proof of proposition 5. Indeed the fibers of π are not contractible, and $\mathcal{F}$ is a non-algebraic foliation with effective conormal bundle. Let us now consider the divisor of tangency D_{tan} between $\mathcal{F}$ and $\mathcal{G}$. By definition, the support of D_{tan} consists of points where $\mathcal{F}$ and $\mathcal{G}$ are not transversal (see [B1, page 573]).

Lemma 9. *The divisor of tangency D_{tan} is trivial.*

Proof. Suppose that there exists a point of tangency p. Let E be the leaf of $\mathcal{G}$ (i.e. the reduction of the fiber of π) containing p. By the claim above E is not $\mathcal{F}$-invariant, so we can consider the tangency index $tang(E, \mathcal{F})$ (see e.g. [B2], chapter III) and by supposition it is strictly positive. But by a formula from same reference

$$tang(E, \mathcal{F}) = c_1(N_{\mathcal{F}}) \cdot E - \chi(E) = 0,$$

hence we have a contradiction. □

Corollary 10. *The tangent bundle to S is the direct sum of its subbundles $T_{\mathcal{F}}$ and $T_{\mathcal{G}}$, and $T_{\mathcal{G}}$ is trivial.*

Proof. The first statement is another formulation of the transversality of $\mathcal{F}$ and $\mathcal{G}$ at every point. The second follows from the formula

$$\mathcal{O}(D_{tan}) = T^*_{\mathcal{G}} \otimes N_{\mathcal{F}}$$

(see for example [B1, Lemme 4]). We know that $N_{\mathcal{F}}$ is trivial so we get that $T^*_{\mathcal{G}}$ is also trivial. □

Lemma 11. *The canonical bundle K_S of S is isomorphic to $\pi^*(K_C) \otimes \mathcal{O}(\sum(l_E - 1)E)$, where l_E denotes the multiplicity of E as a fiber of π.*

Proof. We have just seen that $K_{\mathcal{G}}$ is trivial, on the other hand, one can compute this as the canonical bundle of a foliation defined by a fibration:

$$K_{\mathcal{G}} \simeq K_{S/C} \otimes \mathcal{O}(\sum(1 - l_E)E)$$

where $K_{S/C} = K_S \otimes \pi^*(K^*_C)$ is the relative canonical bundle and the sum is over all fibres of π. By Kodaira's canonical bundle formula K_S is equal to $\pi^*(K_C \otimes L) \otimes \mathcal{O}(\sum(l_E - 1)E)$ where $L = R^1\pi_*(\mathcal{O}_S)^* \cong \pi_* K_{S/C}$ is the so-called fundamental line bundle on C (see for example [BPV, Corollary V.12.3]). So $T^*_{\mathcal{G}} \simeq K_{S/C} \otimes \mathcal{O}(\sum(1 - l_E)E) \simeq \pi^*(L)$. Since $T^*_{\mathcal{G}} \simeq \mathcal{O}$ by the Corollary 10, we get that L is trivial. Hence $K_S \simeq \pi^*(K_C) \otimes \mathcal{O}(\sum(l_E - 1)E)$. □

Note that the triviality of the fundamental line bundle $R^1(\pi_*(\mathcal{O}_S))^*$ also has the following geometric consequence (see e.g. [FM][Proposition 3.22]):

Proposition 12. *Let S be an elliptic surface and let $d = \deg(R^1\pi_*\mathcal{O}_S)^*$ and g be the genus of the base curve C. Then $p_g(S) = d + g - 1$, if $(R^1\pi_*\mathcal{O}_S)^*$ is not trivial, and $p_g(S) = g$ if $(R^1\pi_*\mathcal{O}_S)^*$ is trivial.*

Corollary 13. *The geometric genus of S is equal to g.*

In particular we get by the Lemma 8 that $g = g(C) > 0$.

3. Geometry of deformations of S

Let us denote by M an irreducible component of Barlet space of X containing $[S]$. The following lemma follows from the fact that the deformations of lagrangian subvarieties are unobstructed and remain lagrangian (e.g. [V]).

Lemma 14. *(i)The Barlet space $\mathcal{B}(X)$ is smooth of dimension $g+1$ near S. In particular, S is contained in a unique irreducible component M of $\mathcal{B}(X)$.*

(ii)If $[S']\in M$ represents a smooth subvariety S', then S' is an elliptic Lagrangian surface in X and $h^0(\Omega^1_{S'}) = g(C) + 1$.

Proof. (i) See the proof of [GLR][Lemma 3.1]. The tangent space to M at $[S]$ is isomorphic to $H^0(N_{S/X})$. Since S is Lagrangian, the symplectic form induces an isomorphism $\Omega^1_S \simeq N_{S/X}$, so $T_{[S]}(M) \simeq H^0(\Omega^1_S)$. The dimension of this last vector space is equal to $p_g(S) + 1 - \chi(\mathcal{O}_S) = g + 1$.

(ii) By [V], deformations of S in X are unobstructed and are also Lagrangian. So S' is Lagrangian and the symplectic form induces an isomorphism $\Omega^1_{S'} \simeq N_{S'/X}$. The dimension of the tangent space to M at S is equal to the dimension of the tangent space to M at S', i.e. $h^0(\Omega^1_{S'}) = h^0(\Omega^1_S) = g(C) + 1$. The Kodaira dimension is invariant by smooth deformations (even the plurigenera are, by a theorem of Siu [S]), so the Kodaira dimension of S' is equal to one, i.e. $S^{'}$ is a properly elliptic surface over a curve C'. □

We also have $c_2(S^{'}) = c_2(S) = 0$. Hence $S' \to C'$ has only multiple fibers as singularities, since all singular fibers give nontrivial contribution to c_2. The number of multiple fibers and their multiplicities are also invariant under deformation, see for example [FM, Proposition 7.1]. The genus of C' is equal to $g = g(C)$, since a deformation of S induces a deformation of C (see e.g. [FM, Theorem 7.11]).

Corollary 15. *The geometric genus $p_g(S')$ of S' is equal to $g(C)$, where C is the base curve of S. Moreover the fundamental line bundle of the elliptic fibration on $S^{'}$ is trivial.*

Proof. The first statement is the invariance of (pluri)genera; the second follows by $g(C) = g(C')$ and Proposition 12. □

Now we are ready for our main conclusion on deformations of S.

Lemma 16. *The sheaf $\Omega^1_{S'}$ decomposes into the direct sum*

$$\Omega^1_{S'} \simeq \mathcal{O} \oplus (\pi^*(\Omega^1_{C'}) \otimes \mathcal{O}(\sum (l_E - 1)E)),$$

where $\pi : S^{'} \to C'$ is an elliptic fibration of S'. Since $\Omega^1_{S'} \simeq N_{S'/X}$ the same statement is true for the normal bundle to S' in X.

Proof. The elliptic fibration on S' induces a smooth foliation $\mathcal{G}'$. Consider the corresponding exact sequence on S':

$$0 \to N^*_{\mathcal{G}'} \to \Omega^1_{S'} \to K_{\mathcal{G}'} \to 0. \tag{1}$$

First of all $K_{\mathcal{G}'}$ is trivial since it is isomorphic to the pull-back of the fundamental line bundle that is trivial by the Corollary 15, so $K_{S'} \simeq K_{\mathcal{G}'} \otimes N^*_{\mathcal{G}'} \simeq N^*_{\mathcal{G}'}$. Note that $N^*_{\mathcal{G}'}$ is isomorphic to $\pi^*(\Omega^1_{C'}) \otimes \mathcal{O}(\sum (l_E - 1)E)$ and the exact sequence (1) has the following form:

$$0 \to \pi^*(\Omega^1_{C'}) \otimes \mathcal{O}(\sum (l_E - 1)E) \to \Omega^1_{S'} \to \mathcal{O} \to 0. \tag{2}$$

The long exact sequence of cohomologies starts as

$$0 \to H^0(\pi^*(\Omega^1_{C'}) \otimes \sum (l_E - 1)E) \to H^0(\Omega^1_{S'}) \to H^0(\mathcal{O}) \to \mathrm{Ext}^1(\mathcal{O}, \pi^*(\Omega^1_{C'})).$$

But $h^0(\Omega^1_{S'}) = g(C) + 1$ by Lemma 14(ii), and $h^0(\pi^*(\Omega^1_{C'}) \otimes \sum (l_E - 1)E) = g(C)$ since $h^0(\pi^*(\Omega^1_{C'}) \otimes \sum (l_E - 1)E) = h^0(N^*_{\mathcal{G}}) = h^0(K_{S'})$ and the last number is equal to $g(C') = g(C)$ by Corollary 15. The map $H^0(\Omega^1_{S'}) \to H^0(\mathcal{O})$ must then be surjective, hence the exact sequence (2) splits. □

4. Conclusion by deformation theory

Let us denote by $\mathcal{U}$ the universal family over the component M of the Barlet space of X containing the parameter point of S, and by ε and γ the natural maps of $\mathcal{U}$ to X and M. It is well-known that M is compact and ε and γ are proper. The tangent space to $\mathcal{U}$ at a point $(u, S^{'})$, where $u \in S^{'}$ and $S^{'}$ is smooth, can be described as the first term of the following exact sequence:

$$0 \to T_{\mathcal{U},(u,S')} \to T_{X,u} \oplus H^0(N_{S'/X}) \to (N_{S'/X})_u,$$

where the last map sends couple (v, s), where $v \in T_{X_u}$ and $s \in H^0(N_{S'/X})$, to $(v \bmod T_{S'}) - s(u)$. The differentials of ε and γ are compositions of the first morphism with the projections to T_{X_u} and $H^0(N_{S'/X})$ respectively. Note that we can compute the dimension of the kernel of $d\varepsilon$ at every smooth point $(u, S^{'})$: it is equal to the dimension of the space of sections of $N_{S'/X}$ vanishing at u, i.e. $g-1$ if u does not lie on a multiple fiber on $S^{'}$, and g if u does. Consider the case of a point $(u^{'}, S^{'})$ of $\mathcal{U}$ such that $u^{'}$ does not lie on a multiple fiber on $S^{'}$. Let us prove that fibers of ε over $u^{'}$ are smooth reduced varieties of dimension $g-1$: indeed a fiber has dimension at least $g-1$ since the dimension of $\mathcal{U}$ is equal to $g+3$, so the dimension of the fiber and that of the tangent space to the fiber are both equal to $g-1$ by the remark above, so that the fiber is equidimensional, reduced and smooth. Moreover we get that $d\varepsilon$ is surjective at $(u^{'}, S^{'})$ so ε is a submersion on the preimage U of an analytic neighborhood of $u^{'}$ that does not contain points of multiple fibers on $S^{'}$. Hence by Ehresmann's lemma ε is a locally trivial fibration on U.

Lemma 17. *(cf.* [A]*) Whenever $L \cap K \neq \emptyset$ where $L, K \in M$ and L is smooth, the intersection $L \cap K$ is equidimensional. That is, since $c_2(N_{L/X}) = 0$, all irreducible components of $L \cap K$ are curves.*

Proof. Consider $\varepsilon^{-1}(L)$ – the preimage of L in $\mathcal{U}$. Consider one of the irreducible components $\widetilde{\varepsilon^{-1}(L)}$ of $\varepsilon^{-1}(L)$. By the remarks above the dimension of $\widetilde{\varepsilon^{-1}(L)}$ is equal to $g+1$. Note that γ can not be dominant on $\widetilde{\varepsilon^{-1}(L)}$. Indeed suppose that γ is dominant on $\widetilde{\varepsilon^{-1}(L)}$, then, since the dimension of M is also equal to $g+1$, the dimension of a general fibre would be zero. If some components dominate M and other don't, we pick a point in M outside of images of those non-dominating components and obtain that the corresponding surface has purely zero-dimensional non-empty intersection with L, which is impossible since $c_2(N_{L/X}) = 0$ and this is the same as the self-intersection number of L in X. So all components are non-dominating and the dimension of a general fiber is always equal to 1. Since this fiber is exactly the intersection of L and the corresponding surface from the family, we are done. □

The following observation is well-known from deformation theory; it is a consequence of the tubular neighbourhood lemma and the unobstructedness of deformations of lagrangian subvarieties.

Observation 18. *The locus where a smooth lagrangian surface L from the family M intersects its small deformation is the zero locus of the corresponding section $s \in H^0(N_{L/X}) \simeq \Omega^1_L$. The intersection is transversal wherever the section s has no multiple zero.*

We thus have two cases to consider: when the the genus g of the base curves of our elliptic surfaces from the family M is greater than 1, any elliptic fiber on L is in the zero locus of some section of Ω^1_L and therefore is a component of the intersection of L with a small deformation; in the case when $g = 1$, such intersections happen only along multiple fibers since the sections of Ω^1_L then have no other zeros.

By the same type of reasons, we have the following

Lemma 19. *The map ε is dominant.*

Indeed the surface S deforms away from the divisor D, so there is a surface from the family M through a general point of X.

If we assume that S is not a torus, the family of intersections of surfaces from M is also dominant:

Lemma 20. *If S is not a torus, then there is more than one S through a general point of X.*

Proof. Assume the contrary, then the family M gives a rational fibration of X in subvarieties of non-maximal Kodaira dimension. By [AC1] it must be a fibration on lagrangian tori, a contradiction. □

At a general point of $x \in X$, the general intersections are transversal as follows from Sard's lemma. More precisely, consider in $\mathcal{U} \times \mathcal{U}$ the complement to the preimage of the diagonal Δ_M in $M \times M$, let us denote it by $\mathcal{U} \times \mathcal{U} \setminus ((\gamma \times \gamma)^*(\Delta_M))$. Consider the preimage of Δ_X, the diagonal in $X \times X$, in the following diagram:

$$\begin{array}{ccc} \mathcal{U} \times \mathcal{U} \setminus ((\gamma \times \gamma)^*(\Delta_M)) & \xrightarrow{\varepsilon \times \varepsilon} & X \times X \\ {\scriptstyle \gamma\times\gamma}\big\downarrow & & \\ M \times M \setminus (\Delta_M) & & \end{array}$$

Denote this preimage by I. Over $M \times M \setminus (\Delta_M)$, I is the family of intersections of surfaces from M. At a general point $x \in X$, consider the intersection of smooth surfaces corresponding to m_1 and $m_2 \in M$; the point $((x, m_1), (x, m_2))$ is a general point of U. As the intersections dominate X, one deduces from Sard's lemma that ε is submersive at both (x, m_1) and (x, m_2); then $\varepsilon \times \varepsilon$ is submersive at $((x, m_1), (x, m_2))$, so that I is smooth and the intersection is transversal.

With this in mind, let us first treat the case $g > 1$.

Proposition 21. *Assume $g > 1$. Let x be a general point of X and S a general (smooth) surface from the family M through this point. Let E be the fiber of the elliptic fibration on S through x. Then any other surface from the family M passing through x intersects S along E (and nothing else) in the neighbourhood of x.*

Proof. There are certainly surfaces satisfying this condition, namely suitable small deformations of S mentioned in observation 18. Take one of them, say S', intersecting S transversally at x. Suppose there are other intersection curves through x; consider a general L from M such that the intersection component with S is not E. We apply the same argument as in [A]: three tangent planes T_xS, T_xL, T_xS' intersect along three distinct lines and generate a three-space H_x in T_xX; but these are lagrangian planes so we get a contradiction (each of the intersection lines must then be orthogonal to H with respect to the symplectic form). □

Corollary 22. *The case $g > 1$ is impossible.*

Proof. Indeed we see that all surfaces passing through the general point $x \in X$ intersect along the elliptic fiber of one of them; X must then be rationally fibered in elliptic curves, as there is thus only one such a fiber through x. This is impossible by [AC1]. $\square$

It remains to rule out the case when $g = 1$ but S is not a torus, that is, $\pi : S \to C$ has multiple fibers. The difficulty is now that, as we shall see, the multiple fibers of all our surfaces are contained in the divisor D; on the other hand, S now intersects its small deformation S' necessarily along a multiple fiber, so such an intersection is not generic. To deal with this, we go into more details on deformations of elliptic fibers.

So consider the case where S is an elliptic surface with an elliptic base curve and at least one multiple fiber. By lemma 16 for a smooth surface S' from the family M we have $\Omega^1_{S'} \simeq \mathcal{O}(\sum(l_E - 1)E) \oplus \mathcal{O}$, so M is 2-dimensional and the surfaces from M still cover X. Denote by $\mathcal{D}(E)$ the irreducible component of the Douady space that contains a fiber E of the elliptic fibration of S and by $\mathcal{D}(E)_{mul}$ an irreducible component of the Douady space that contains the reduction of a multiple fiber of S (all fibers are in the same component, but there might be several components parameterizing the reductions of fibers, e.g. of different multiplicities).

Lemma 23. *$\mathcal{D}(E)$ is 3-dimensional and the normal bundle $N_{E/X}$ for each (possibly multiple) fiber E of the elliptic fibration of a surface S' from M is isomorphic to $\mathcal{O} \oplus \mathcal{O} \oplus \mathcal{O}$. In particular locally in a neighbourhood U of E one has a fibration of U by small deformations of E.*

Proof. Consider the exact sequence

$$0 \to N_{E/S'} \to N_{E/X} \to N_{S'/X}|_E \to 0,$$

we have $N_{E/S'} \simeq \mathcal{O}$ and $N_{S'/X}|_E \simeq \mathcal{O} \oplus \mathcal{O}$, so

$$0 \to \mathcal{O} \to N_{E/X} \to \mathcal{O} \oplus \mathcal{O} \to 0 \tag{3}$$

Furthermore $h^0(N_{E/X})$ is the tangent to $\mathcal{D}(E)$ and must therefore be at least three-dimensional; we conclude that $h^0(N_{E/X}) = 3$ and the exact sequence 3 splits. The last statement follows from the standard calculation of the differential of the evaluation map from the universal family of elliptic curves to X. $\square$

Lemma 24. *$\mathcal{D}(E)_{mul}$ is 1-dimensional and all elliptic curves from the family $\mathcal{D}(E)_{mul}$ are contained in D, more precisely they are fibers of the elliptic surfaces from our family M contained in D.*

Proof. All multiple fibers are in D since by an easy calculation, $\mathcal{O}_{S'}(D) = \mathcal{O}(\sum_E(m(E)-1)E)$, where m stands for multiplicity. They must coincide with fibers of the elliptic surfaces in D since otherwise we would have surfaces from M intersecting at points and not at curves, contradicting $c_2(S') = 0$. Let E_m be the reduction of a fiber of multiplicity m on S'. Consider the exact sequence

$$0 \to N_{E_m/S'} \to N_{E_m/X} \to N_{S'/X}|_{E_m} \to 0.$$

Since $N_{E_m/S'}$ is a torsion bundle of order m (see e.g. [FM][Lemma III.8.3]) and $N_{S'/X}|_{E_m}$ is isomorphic to the direct sum of the trivial bundle and the inverse of that torsion bundle of order m (see [FM][Lemma III.8.3]), we get that $\dim(\mathcal{D}(E)_{mul}) \leq 1$.

Since multiple fibers of smooth surfaces from D already form a 1-dimensional family, we get that $\dim(\mathcal{D}(E)_{mul}) = 1$. □

Corollary 25. *A smooth curve from $\mathcal{D}(E)$ either does not intersect D or lies in D. In particular any smooth surface from M intersects any surface from D by fibers of the corresponding elliptic fibrations.*

Proof. A general small deformation of an elliptic curve in D does not intersect D, so the intersection number DE is zero. An elliptic curve from $\mathcal{D}(E)$ contained in D must be a fiber of some S by the same reason as in the preceding lemma.

□

Now we are ready to finish the proof of our main result.

End of proof of the main theorem in the case $g = 1$: As we have already seen studying the case $g > 1$, to get a contradiction it is sufficient to show that all surfaces from M passing through a general $x \in X$ intersect at the same elliptic curve E which is the fiber of the elliptic fibration on each of them. Suppose that this is not true: the intersection of two smooth surfaces L and S from M contains a curve C that is not a fiber of the corresponding elliptic fibrations. We claim that L and S share a multiple fiber (which automatically intersects the multisection C of L). Indeed let F be a multiple fiber of L and let c be a point of intersection of F and C. We have seen that F is contained in D. Consider the elliptic curve E on S that contains c. Since $F \subset D$, E intersects D and by the Corollary 25 $E \subset D$, i.e. E is equal to F.

But this is impossible since no neighbourhood of E is then fibered in elliptic curves from $\mathcal{D}(E)$ (as some curves around E, those lying on L and those lying on S, must intersect at the points of C).

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