A remark on a question of Beauville about lagrangian fibrations

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Abstract

This note is a proof of the fact that a lagrangian torus on an irreducible hyperkähler fourfold is always a fiber of an almost holomorphic lagrangian fibration.

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Let X be an irreducible holomorphic symplectic variety of dimension $2n$, with a lagrangian torus $A \subset X$ (such a torus is always an abelian variety, even if X is not projective: this follows from the fact that the space $H^{2,0}(X)$ restricts to zero on A, see [C]). Beauville [B] asked whether there always exists a lagrangian fibration f (holomorphic or almost holomorphic), such that A is a fiber of f. Recently, D. Greb, C. Lehn and S. Rollenske [GLR] proved that it is always the case when X is non-projective, or even admits a non-projective deformation preserving the lagrangian subtorus. It is not clear a priori whether all irreducible holomorphic symplectic varieties admit such deformations. The authors also have announced another paper in preparation, where Beauville's question is answered in the affirmative in dimension four[1]. The purpose of this short note is to provide an answer to the "almost holomorphic version" of Beauville's question in dimension four in a very elementary way.

Since the torus $A \subset X$ is lagrangian, its normal bundle $N_{A,X}$, being isomorphic to its cotangent bundle, is trivial. The deformation theory of lagrangian tori in holomorphic symplectic manifolds is well understood: as it is explained in [GLR],

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[1]Their paper appeared on arxiv.org:1110.2680 a few days after a conversation between myself and one of the authors, and contains a simplified (with respect to the original approach of the authors) argument for the existence of a meromorphic fibration, similar to the one given below. My primary reason for making this note public was to provide a source which they could cite.

by the results of Ran and Voisin the space T of deformations of A in X is of dimension n, it is smooth at A, and any deformation A_t of A which is smooth is itself a lagrangian torus. Moreover, if Y denotes the universal family of the A_t's, the projection $q : Y \to X$ is unramified along each smooth A_t. In particular, one has a finite number d of A_t's through a general point of X, and if through a certain point $x \in X$ there is an infinite number of A_t then almost all of them are singular (in fact, if Y denotes the universal family, the projection $q : Y \to X$ is unramified along each smooth A_t). To show that there is an almost holomorphic fibration with fibers A_t is the same as to show that $d = 1$, or that a general A_t does not intersect any other member A_s.

To say the same things using slightly different words, from the above-mentioned facts on the universal family map q the following is immediate.

Fact: *Locally in a neighbourhood U of A, one has a lagrangian fibration by its small deformations A_t.*

One has to show that this local fibration gives rise to a global, possibly meromorphic, fibration.

Let us list a few immediate consequences of this fact in a lemma. We assume for simplicity that X is projective.

Lemma 1: *a) A lagrangian torus does not intersect its small deformations;*

b) Whenever $A_t \cap A_s \neq \emptyset$ and A_s is smooth, all irreducible components of $A_t \cap A_s$ are positive-dimensional;

c) For a general $A_{s'}$ intersecting A_t, the intersection $A_t \cap A_{s'}$ is equidimensional.

Proof: Whereas a) is immediate from the existence of a local fibration, b) deserves a few words of explanation. Consider the local fibration $f_s : U_s \to T_s$ in a neighbourhood of A_s (here T_s is a small disc around $s \in T$). The intersection of A_t with U_s may consist of several local components. The statement that $A_t \cap A_s$ cannot have a zero-dimensional component is clearly implied by the claim that no component of $A_t \cap U_s$ dominates the base T_s (indeed, such a component would have zero-dimensional general fiber under the map f_s, whereas for a non-dominating component all fibers are of strictly positive dimension). If this claim does not hold, we can choose a sufficiently general point $s' \in T_s$ outside of the image of all non-dominating components of $A_t \cap U_s$. The resulting lagrangian torus $A_{s'}$ has non-empty zero-dimensional intersection with A_t. This is impossible since the self-intersection number $[A] \cdot [A] = 0$.

The proof of c) is similar to b): it suffices to choose $s' \in T_s$ sufficiently general in a "maximal" image of a component of $A_t \cap U_s$. If this image is of dimension c, the intersection $A_t \cap A_{s'}$ is purely $(n - c)$-dimensional.

Consider now the case $dim(X) = 4$, then the intersection $A_t \cap A_s$ is a curve if nonempty. Fix a general lagrangian torus A_t. Since the family of its deformations $A_s, s \in T$ induces a local fibration in the neighbourhood of each smooth member, the intersections $A_t \cap A_s$ induce a fibration or several fibrations on A_t. Indeed, since most of the A_s intersecting A_t are smooth, this is a family of cycles on A_t not intersecting their neighbours, as are the A_s on X itself; since these cycles are divisors on a surface, those are (possibly reducible) curves whose square is equal to zero, that is, our A_t is fibered in elliptic curves and our intersections are unions of fibers.

Of course, the A_s intersecting A_t do not necessarily form an irreducible family, so apriori there can be several (say, $k > 1$) such fibrations on A_t. Nevertheless an easy linear algebra argument shows that there is in fact only one:

Proposition 2: *One has $k = 1$.*

Proof: Otherwise, we can find two more tori A_u and A_s through a general point p of A_t, in such a way that the pairwise intersections C_{ts}, C_{tu} and C_{us} are curves with distinct tangents at p and the intersection of all three tori has p as an isolated point. One deduces easily from Sard's lemma that for general t, s, the intersection $A_s \cap A_t$ is reduced, so we may assume that the tangent planes to A_u, A_s and A_t are distinct. But since the pairwise intersections of those planes in T_pX are distinct lines, the planes only span a hyperplane $V \subset T_pX$. Now the restriction σ_V of the symplectic form σ_p, that is, the value of σ at p, to V has one-dimensional kernel. Recall that T_pA_u, T_pA_s and T_pA_t are σ_p-isotropic. Since $V = T_pA_t + T_pA_s$, one must have $T_pC_{ts} = Ker(\sigma_V)$. But by the same reason the same holds for T_pC_{tu}, a contradiction.

Now we are ready to prove the announced result, which we formulate as a theorem.

Theorem 3: *The answer to the almost holomorphic version of Beauville's question is positive in dimension 4, that is, a general A_t does not intersect any other A_s, so that $d = 1$ and the family A_t induces an almost holomorphic fibration on X.*

Proof: Suppose $d > 1$, then we claim that the irreducible components of the intersections $A_t \cap A_s$ rationally fiber X, that is, there is only one such curve through a general point $p \in X$. Indeed, suppose there are two of them, C_1 and C_2. Then C_1 is a component of the intersection $A_{1,1} \cap A_{1,2}$ and C_2 of $A_{2,1} \cap A_{2,2}$. Since p is general in X, we may suppose that all four of the above abelian surfaces are smooth. Each pair of those four tori intersect at p, and we know that they should then intersect along a curve through p. Looking at the possible configurations and keeping in mind that the intersection of smooth surfaces is a disjoint union of elliptic curves, one is back to what is just ruled out in Proposition 2.

On the other hand, it is proved in [AC], Proposition 3.4, that any rational fibration $f : X \dashrightarrow B$ of an irreducible holomorphic symplectic variety X has fibers of dimension at least $dim(X)/2$, provided that the general fiber is not of general type. Since the irreducible components of $A_t \cap A_s$ are elliptic curves, they cannot rationally fiber X, a contradiction.

Remark 4: This argument generalizes to higher-dimensional X in the case when the intersection of two general (intersecting) lagrangian tori $A_t \cap A_s$ is of codimension one in each; so if $2n = dim(X)$, one wants $A_t \cap A_s$ to be equal to $n-1$; when $n = 2$ this is of course the only option unless X is fibered by the A_t's. More precisely, $A_t \cap A_s$ must be of dimension $n - 1$ for *each* general pair of intersecting lagrangian tori: that is to say, the subset $W = \{(t, s)|A_t \cap A_s \neq \emptyset\} \subset T \times T$ can have several irreducible components $W_i,\ i = 1, \ldots, l$, such that a generic pair of tori in W_i intersects in dimension e_i, and for our argument we need this dimension to be equal to $n - 1$ for any $i = 1, \ldots, l$. As soon as this condition is satisfied, the same reasoning as above applies to show the existence of an almost holomorphic lagrangian fibration with fibers A_t.

We hope to return to Beauville's question in a forthcoming joint work with Frédéric Campana.

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