Morrison-Kawamata cone conjecture for hyperkähler manifolds

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Abstract
Let M be a simple hyperkähler manifold, that is, a simply connected compact holomorphically symplectic manifold of Kähler type with $h^{2,0} = 1$. Assuming $b_2(M) \neq 5$, we prove that the group of holomorphic automorphisms of M acts on the set of faces of its Kähler cone with finitely many orbits. This statement is known as Morrison-Kawamata cone conjecture for hyperkähler manifolds. As an implication, we show that a hyperkähler manifold has only finitely many non-equivalent birational models. The proof is based on the following observation, proven with ergodic theory. Let M be a complete Riemannian orbifold of dimension at least three, constant negative curvature and finite volume, and $\{S_i\}$ an infinite set of complete, locally geodesic hypersurfaces. Then the union of S_i is dense in M.

Contents

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1 Introduction

1.1 Kähler cone and MBM classes

Let M be a hyperkähler manifold, that is, a compact, holomorphically symplectic Kähler manifold. We assume that $\pi_1(M) = 0$ and $H^{2,0}(M) = \mathbb{C}$: the general case reduces to this by Bogomolov decomposition (Theorem 2.3). Such hyperkähler manifolds are known as simple hyperkähler manifolds, or IHS (irreducible holomorphic symplectic) manifolds. The known examples of such manifolds are deformations of punctual Hilbert scheme of K3 surfaces, deformations of generalized Kummer varieties and two sporadic ones discovered by O'Grady. In [AV] we gave a description of the Kähler cone of M in terms of a set of cohomology classes $S \subset H^2(M, \mathbb{Z})$ called **MBM classes** (Definition 2.14). This set depends only on the deformation type of M.

Recall that on the second cohomology of a hyperkähler manifold, there is an integral quadratic form q, called the Beauville-Bogomolov-Fujiki form (see section 2 for details). This form is of signature $(+, -, \dots, -)$ on $H^{1,1}(M)$. Let $\operatorname{Pos} \subset H^{1,1}(M)$ be the positive cone, and $S(I)$ the set of all MBM classes which are of type (1,1) on M with its given complex structure I. Then the Kähler cone is a connected component of $\operatorname{Pos} \backslash S(I)^\perp$, where $S(I)^\perp$ is the union of the orthogonal complements to all $z \in S(I)$.

1.2 Morrison-Kawamata cone conjecture for hyperkähler manifolds

The Morrison-Kawamata cone conjecture for Calabi-Yau manifolds was stated in [Mo]. For K3 surfaces it was already known since mid-eighties by the work of Sterk [St]. Kawamata in [Ka1] proved the relative version of the conjecture for Calabi-Yau threefolds admitting a holomorphic fibration over a positive-dimensional base.

In this paper, we concentrate on the following version of the cone conjecture (see Subsection 5.2 for its relation to the classical one, formulated for the ample cone of a projective variety).

Definition 1.1: Let M be a compact, Kähler manifold, $\operatorname{Kah} \subset H^{1,1}(M, \mathbb{R})$ the Kähler cone, and $\overline{\operatorname{Kah}}$ its closure in $H^{1,1}(M, \mathbb{R})$, called **the nef cone**. A **face** of the Kähler cone is the intersection of the boundary of $\overline{\operatorname{Kah}}$ and a hyperplane $V \subset H^{1,1}(M, \mathbb{R})$ which has non-empty interior.

Conjecture 1.2: (Morrison-Kawamata cone conjecture, Kähler version)
Let M be a Calabi-Yau manifold. Then the group $\operatorname{Aut}(M)$ of biholomorphic

automorphisms of M acts on the set of faces of Kah with finite number of orbits.

The **original Morrison-Kawamata cone conjecture** is formulated for projective Calabi-Yau manifolds and has two versions: **the weak one** states that $\mathrm{Aut}(M)$ acts with finitely many orbits on the set of faces of the ample cone and **the strong one** states that $\mathrm{Aut}(M)$ has a finite polyhedral fundamental domain on the ample cone, or, more precisely, on the cone $\mathrm{Nef}^+(M)$ obtained from the ample cone by adding the "rational part" of its boundary (see [Mo], [T], [MY] for details).

We shall be interested in the case when the manifold M is simple hyperkähler (that is, IHS). Our main purpose is to prove Conjecture 1.2. Notice that the stronger version involving fundamental domains cannot be true in this Kähler setting, as for a very general IHS M the Kähler cone is equal to the positive cone whereas $\mathrm{Aut}(M)$ is trivial. However when M is projective IHS, the Kähler version of the conjecture implies almost immediately not only the weak, but also the strong original version (see section 5).

In [AV], we have shown that the Kähler version of the Morrison-Kawamata cone conjecture holds whenever the Beauville-Bogomolov square of primitive MBM classes is bounded. This is known to be the case for deformations of punctual Hilbert schemes of K3 surfaces and for deformations of generalized Kummer varieties.

The strong version of the cone conjecture for projective IHS under the boundedness assumption for primitive MBM classes has been proved by Markman and Yoshioka in [MY]. In Section 5 we suggest a rapid alternative way to deduce this strong version from ours: the tools are Borel and Harish Chandra theorem on arithmetic subgroups and geometric finiteness results from hyperbolic geometry. To apply the first one, we have to suppose that the Picard number is at least three. The case of Picard number two has to be treated separately, but the argument is fairly easy. Thus it is the boundedness (in absolute value) of squares of primitive MBM classes which is at the heart of all versions of Morrison-Kawamata cone conjecture for IHS.

Let us also briefly mention that this conjecture has a birational version, proved for projective hyperkähler manifolds by E. Markman in [M3] and generalized in [AV] to the non-projective case. In this birational version, the nef cone is replaced by the birational nef cone (that is, the closure of the union of pullbacks of Kähler cones on birational models of M) and the group $\mathrm{Aut}(M)$ is replaced by the group of birational automorphisms $\mathrm{Bir}(M)$.

The key point of the proof of [AV] is the observation that the orthogonal group $O(H^{1,1}_{\mathbb{Z}}(M), q)$ of the lattice $H^{1,1}_{\mathbb{Z}}(M) = H^{1,1}(M) \cap H^2(M, \mathbb{Z})$, and therefore the **Hodge monodromy group** Γ_{Hdg} (see Definition 2.12) which is a subgroup of finite index in $O(H^{1,1}_{\mathbb{Z}}(M), q)$, acts with finitely many orbits on the set of classes of fixed square $r \neq 0$. When the primitive MBM classes have bounded square, we conclude that the monodromy acts with finitely many orbits on the set of MBM classes. As those are precisely the classes whose orthogonal hyperplanes support the faces of the Kähler cone, it is not difficult to deduce that there are only finitely many, up to the action of the monodromy group, faces of the Kähler cone, and also finitely many oriented faces of the Kähler cone (an oriented face is a face together with the choice of normal direction). An element of the monodromy which sends a face F to a face F', with both orientations pointing towards the interior of the Kähler cone, must preserve the Kähler cone. On the other hand, Markman proved ([M3], Theorem 1.3) that an element of the Hodge monodromy which preserves the Kähler cone must be induced by an automorphism, so that the cone conjecture follows.

1.3 Main results

The main point of the present paper is that the finiteness of the set of primitive MBM classes of type $(1,1)$, up to the monodromy action, can be obtained without the boundedness assumption on their Beauville-Bogomolov square.

Our main technical result is the following

Theorem 1.3: Let L be a lattice of signature $(1,n)$ where $n \geqslant 3$, $V = L \otimes \mathbb{R}$. Let Γ be an arithmetic subgroup in $SO(1,n)$. Let $Y := \bigcup S_i$ be a Γ-invariant union of rational hyperplanes S_i orthogonal to negative vectors $z_i \in L$ in V. Then either Γ acts on $\{S_i\}$ with finitely many orbits, or Y is dense in the positive cone in V.

Proof: See Theorem 4.11. ■

Remark 1.4: The assumption $n \geqslant 3$ is important for our argument which is based on Ratner theory. We shall see that Ratner theory applies to our problem as soon as the connected component of the unity of $SO(1,n-1)$ is generated by unipotents, that is, for $n \geqslant 3$.

Taking $H^{1,1}_{\mathbb{Z}}(M) = H^{1,1}(M) \cap H^2(M,\mathbb{Z})$ for L and the Hodge monodromy group for Γ, we easily deduce:

Theorem 1.5: Assume that M is projective, of Picard rank at least 4. The monodromy group acts with finitely many orbits on the set of MBM classes which are of type $(1,1)$.

Proof: See Theorem 5.1. ■

Note that, by a result of Huybrechts, the projectivity assumption for M is equivalent to the signature $(1,n)$ assumption for its Picard lattice L.

The boundedness follows as an obvious corollary.

Corollary 1.6: On a projective M with Picard number at least 4, primitive MBM classes of type $(1,1)$ have bounded Beauville-Bogomolov square.

Proof: Indeed, the monodromy acts by isometries. ■

Using the deformation invariance of MBM property, we can actually drop the assumption that M is projective and has Picard rank at least four. Indeed, if M is a simple hyperkähler manifold with $b_2(M) \geqslant 6$, we can always deform it to a projective manifold M' on which all classes from $H^{1,1}_{\mathbb{Z}}(M)$ stay of type $(1,1)$ (see Proposition 2.31). Since the square of a primitive MBM class is bounded on M', the same is true for M.

The Morrison-Kawamata cone conjecture is then deduced as we have sketched it above, exactly in the same way as in [AV].

Theorem 1.7: Let M be a simple hyperkähler manifold with $b_2(M) \geqslant 6$. The group of automorphisms $\operatorname{Aut}(M)$ acts with finitely many orbits on the set of faces of the Kähler cone $\operatorname{Kah}(M)$.

Proof: See Theorem 5.4. ■

Remark 1.8: The theorem holds trivially for M with $b_2(M) < 5$, so that our result is valid as soon as $b_2(M) \neq 5$. This remaining case can probably be handled using methods of hyperbolic geometry (completely different from those of the present paper; we hope to return to this question in a forthcoming note). One would, though, believe that simple hyperkähler manifolds with $b_2 = 5$ do not exist.

Finally, as observed by Markman and Yoshioka, the boundedness of squares of primitive MBM classes implies the following theorem (we thank Y. Kawamata for indicating us the statement).

Theorem 1.9: Let M be a simple hyperkähler manifold with $b_2(M) \geqslant 6$. Then there are only finitely many simple hyperkähler manifolds birational to M.

Proof: This is just [MY], Corollary 1.5. Indeed, the classes e menitioned in Conjecture 1.1 from [MY] (that is, the classes generating the extremal rays of the Mori cone on the simple hyperkähler birational models of M) are MBM classes in the sense of our Definition 2.14. ■

The crucial tool for the proof of Theorem 1.3 is Ratner theory. We recall this and some other relevant information from ergodic theory in section 3, after some preliminaries on hyperkähler manifolds in section 2. In section 4 we deduce Theorem 1.3 from Mozes-Shah and Dani-Margulis theorems. Finally, in the last section we apply this to hyperkähler manifolds and prove Theorem 1.7.

2 Preliminaries

2.1 Hyperkähler manifolds, monodromy and MBM classes

Definition 2.1: A **hyperkähler manifold** is a compact, Kähler, holomorphically symplectic manifold.

Definition 2.2: A hyperkähler manifold M is called **simple**, or IHS, if $\pi_1(M) = 0$, $H^{2,0}(M) = \mathbb{C}$.

This definition is motivated by Bogomolov's decomposition theorem:

Theorem 2.3: ([Bo1]) Any hyperkähler manifold admits a finite covering which is a product of a torus and several simple hyperkähler manifolds. ■

Remark 2.4: The Bogomolov decomposition theorem can be obtained by applying the de Rham holonomy decomposition theorem and Berger's classification of manifolds with special holonomy to the Ricci-flat hyperkähler metric on a compact holomorphically symplectic Kähler manifold. Then, a hyperkähler manifold is simple if and only if its hyperkähler metric has maximal holonomy group $\operatorname{Hol}(M)$ allowed by the hyperkähler structure, that is $\operatorname{Hol}(M) = Sp(n)$, where $n = \frac{1}{2}\dim_{\mathbb{C}} M$.

Remark 2.5: Further on, we shall assume that all hyperkähler manifolds we consider are simple.

The Bogomolov-Beauville-Fujiki form was defined in [Bo2] and [Bea], but it is easiest to describe it using the Fujiki theorem, proved in [F1].

Theorem 2.6: (Fujiki) Let M be a simple hyperkähler manifold, $\eta \in H^2(M)$, and $n = \frac{1}{2}\dim M$. Then $\int_M \eta^{2n} = cq(\eta,\eta)^n$, where q is a primitive integral quadratic form on $H^2(M,\mathbb{Z})$, and $c > 0$ a constant (depending on M). ■

Remark 2.7: Fujiki formula (Theorem 2.6) determines the form q uniquely up to a sign. For odd n, the sign is unambiguously determined as well. For even n, one needs the following explicit formula, which is due to Bogomolov and Beauville.

$$\begin{aligned}\lambda q(\eta,\eta) = & \int_X \eta\wedge\eta\wedge\Omega^{n-1}\wedge\overline{\Omega}^{n-1} - \\ & - \frac{n-1}{n}\left(\int_X \eta\wedge\Omega^{n-1}\wedge\overline{\Omega}^n\right)\left(\int_X \eta\wedge\Omega^n\wedge\overline{\Omega}^{n-1}\right)\end{aligned} \tag{2.1}$$

where Ω is the holomorphic symplectic form, and $\lambda > 0$.

Definition 2.8: A cohomology class $\eta \in H^{1,1}_{\mathbb{R}}(M)$ is called **negative** if $q(\eta,\eta) < 0$, and **positive** if $q(\eta,\eta) > 0$. Since the signature of q on $H^{1,1}(M)$ is $(1, b_2 - 3)$, the set of positive vectors is disconnected. **The positive cone** $\operatorname{Pos}(M)$ is the connected component of the set $\{\eta \in H^{1,1}_{\mathbb{R}}(M) \ \mid\ q(\eta,\eta) > 0\}$ which contains the classes of the Kähler forms. Using the Cauchy-Schwarz inequality, it is easy to check that the positive cone is convex.

Definition 2.9: Let M be a hyperkähler manifold. The **monodromy group** of M is a subgroup of $GL(H^2(M,\mathbb{Z}))$ generated by the monodromy transforms for all Gauss-Manin local systems.

It is often enlightening to consider this group in terms of the mapping class group action. In the following paragraphs, we recall this description.

Definition 2.10: Let M be a compact complex manifold, and $\mathsf{Diff}_0(M)$ a connected component of its diffeomorphism group (**the group of isotopies**). Denote by Comp the space of complex structures of Kähler type on M (remark here that the set of complex structures of Kähler type is open in the space of all complex structures by Kodaira-Spencer stability theorem), and let $\mathsf{Teich} := \mathsf{Comp} / \mathsf{Diff}_0(M)$. We call it **the Teichmüller space.**

For hyperkähler manifolds, this is a finite-dimensional complex non-Hausdorff manifold ([Cat], [V2]).

Definition 2.11: The **mapping class group** is $\mathsf{Diff}(M)/\mathsf{Diff}_0(M)$. It naturally acts on Teich. The quotient of Teich by this action may be viewed as the "moduli space" for M. However, this space is too non-Hausdorff to be useful: any two open subsets of a connected component of $\mathsf{Teich} / \mathsf{Diff}$ intersect ([V3], [V4]).

It follows from a result of Huybrechts (see [H3]) that in the hyperkähler case Teich has only finitely many connected components. Therefore, the subgroup of the mapping class group which fixes the connected component of our chosen complex structure is of finite index in the mapping class group.

Definition 2.12: The **monodromy group** Γ is the image of this subgroup in $\operatorname{Aut} H^2(M,\mathbb{Z})$. The **Hodge monodromy group** is the subgroup $\Gamma_{\mathsf{Hdg}} \subset \Gamma$ preserving the Hodge decomposition.

The following theorem is crucial for the Morrison-Kawamata cone conjecture.

Theorem 2.13: ([V2], Theorem 3.5) The monodromy group is a finite index subgroup in $O(H^2(M,\mathbb{Z}),q)$ (and the Hodge monodromy is therefore an arithmetic subgroup of the orthogonal group of the Picard lattice).

Next, we recall from [AV] the definition of MBM classes. Remark that any birational map between hyperkähler manifolds $\varphi : M \dashrightarrow M'$ is an isomorphism in codimension one (in general this easily follows from the nefness of the canonical class, which yields that the sets of exceptional divisors for the projections from the resolution of singularities M'' of φ to M and M' coincide; see for instance [Ka2] for details of this argument in a more general situation) and therefore induces an isomorphism on the second cohomology. We say that M and M' are **birational models** of each other.

Definition 2.14: A non-zero negative rational homology class $z \in H^{1,1}(M)$ is called **monodromy birationally minimal** (MBM) if for some isometry $\gamma \in O(H^2(M,\mathbb{Z}))$ belonging to the monodromy group, $\gamma(z)^\perp \subset H^{1,1}(M)$ contains a face of the pull-back of the Kähler cone of one of birational models M' of M.

Remark 2.15: Here the orthogonal is taken with respect to the Beauville-Bogomolov form. A face of $\operatorname{Kah}(M)$ is, by definition, of maximal dimension $h^{1,1}(M)-1$. So the definition of z being MBM means that $\gamma(z)^\perp \cap \partial \operatorname{Kah}(M')$ contains an open subset of $\gamma(z)^\perp$. The MBM classes, or more precisely the rays they generate, are natural analogues of "extremal rays" from projective geometry, up to monodromy and birational equivalence; hence the name. In fact when M is projective, those are exactly the monodromy transforms of extremal rays on birational models of M; this is already implicit in [AV], but see [KLM], Proposition 2.3 and Remark 2.4, for an explicit formulation.

The following theorem has been proved in [AV].

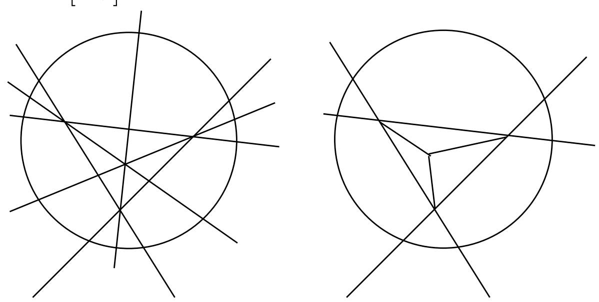
Theorem 2.16: Let M be a hyperkähler manifold, $z \in H^{1,1}(M)$ an integral cohomology class, $q(z,z)<0$, and M' a deformation of M such that z remains of type (1,1) on M'. Assume that z is monodromy birationally minimal on M. Then z is monodromy birationally minimal on M'. ■

Remark 2.17: Keeping this theorem in mind, we say that a class $z \in H^2(M,\mathbb{Z})$ is MBM if such is the case on a deformation M' of M where z becomes of type $(1,1)$.

The MBM classes can be used to determine the Kähler cone of M explicitly.

Theorem 2.18: ([AV]) Let M be a hyperkähler manifold, and $S \subset H^{1,1}(M)$ the set of all MBM classes of type $(1,1)$. Consider the corresponding set of hyperplanes $S^\perp := \{z^\perp \ \mid\ z \in S\}$ in $H^{1,1}(M)$. Then the Kähler cone of M is a connected component of $\operatorname{Pos}(M)\backslash \cup S^\perp$, where $\operatorname{Pos}(M)$ is the positive cone of M. Moreover, for any connected component K of $\operatorname{Pos}(M)\backslash \cup S^\perp$, there exists $\gamma \in O(H^2(M,\mathbb{Z}))$ in the monodromy group of M and a birational model M' such that $\gamma(K)$ is the Kähler cone of M'. ■

Remark 2.19: The main point of this theorem is that for a negative integral class $z \in H^{1,1}(M)$, the orthogonal hyperplane $z^\perp$ either passes through the interiour of some Kähler-Weyl chamber and then it contains no face of a Kähler-Weyl chamber (that is, z is not MBM), or its intersection with the positive cone is a union of faces of such chambers (when z is MBM). This is illustrated by a picture taken from [AV]:



Allowed partition **Prohibited partition**

2.2 Global Torelli theorem and deformations

In this subsection, we recall a number of results about deformations of hyperkähler manifolds used further on in this paper. For more details and references, see [V2].

Let M be a hyperkähler manifold (as usual, we assume M to be simple). Any deformation M' of M is also a simple hyperkähler manifold, because the Hodge numbers are constant in families and thus $H^{2,0}(M')$ is one-dimensional. Let us view M' as a couple (M, J), where J is a new complex structure on M, that is, a point of the Teichmüller space Teich.

Definition 2.20: Let

$$\mathsf{Per}: \ \mathsf{Teich} \longrightarrow \mathbb{P}H^2(M, \mathbb{C})$$

map J to the line $H^{2,0}(M, J) \in \mathbb{P}H^2(M, \mathbb{C})$. The map Per is called **the period map**.

Remark 2.21: The period map Per maps Teich into an open subset of a quadric, defined by

$$\mathbb{P}\mathsf{er} := \{l \in \mathbb{P}H^2(M, \mathbb{C}) \ \mid \ q(l, l) = 0, q(l, \bar{l}) > 0\}.$$

It is called **the period domain** of M. Indeed, any holomorphic symplectic form l satisfies the relations $q(l, l) = 0, q(l, \bar{l}) > 0$, as follows from (2.1).

Definition 2.22: Let M be a topological space. We say that $x, y \in M$ are **non-separable** (denoted by $x \sim y$) if for any open sets $V \ni x, U \ni y$, $U \cap V \neq \emptyset$.

By a result of Huybrechts [H1], non-separable points of Teich correspond to birational hyperkähler manifolds.

Definition 2.23: The space $\mathsf{Teich}_b := \mathsf{Teich}/\sim$ is called **the birational Teichmüller space** of M.

Remark 2.24: This terminology is slightly misleading since there are non-separable points of the Teichmüller space which correspond to biregular, not just birational, complex structures. Even for K3 surfaces, the Teichmüller space is non-Hausdorff.

Theorem 2.25: (Global Torelli theorem; [V2]) The period map $\mathsf{Teich}_b \xrightarrow{\mathsf{Per}} \mathbb{P}\mathsf{er}$ is an isomorphism on each connected component of Teich_b. ■

By a result of Huybrechts ([H3]), Teich has only finitely many connected components. We shall fix the component Teich^0 containing the parameter point for our initial complex structure, and denote by $\widetilde{\Gamma}$ the subgroup of finite index in the mapping class group fixing this component.

It is natural to view the quotient of Teich by the mapping class group as a moduli space for M and the quotient of Teich_b by the mapping class group as a "birational moduli space": indeed its points are in bijective correspondence with the complex structures of hyperkähler type on M up to a bimeromorphic equivalence.

Remark 2.26: The word "space" in this context is misleading. In fact, outside of a countable subset, the quotient $\mathsf{Teich}_b^0/\widetilde{\Gamma}$ has codiscrete topology. ([V3]).

The Global Torelli theorem can be stated as a result about the birational moduli space.

Theorem 2.27: ([V2, Theorem 7.2, Remark 7.4, Theorem 3.5]) Let (M, I) be a hyperkähler manifold, and W a connected component of its birational moduli space. Then W is isomorphic to $\mathbb{P}\mathsf{er}/\Gamma$, where Γ is an arithmetic subgroup in $O(H^2(M,\mathbb{R}), q)$, called **the monodromy group** of (M, I). In fact Γ is the image of $\widetilde{\Gamma}$ in $O(H^2(M,\mathbb{R}), q)$. ■

Remark 2.28: As we have already mentioned, the monodromy group of (M, I) can be also described as a subgroup of the group $O(H^2(M,\mathbb{Z}), q)$ generated by monodromy transform maps for Gauss-Manin local systems obtained from all deformations of (M, I) over a complex base ([V2, Definition 7.1]). This is how this group was originally defined by Markman ([M2], [M3]).

Definition 2.29: Let $z \in H^2(M,\mathbb{Z})$ be an integral cohomology class. The space Teich_z is the part of Teich where the class z is of type $(1, 1)$.

The following proposition is well-known.

Proposition 2.30: Teich_z is the inverse image under the period map of the subset $\mathbb{P}\mathsf{er}_z \subset \mathbb{P}\mathsf{er}$ which consists of l with $q(l, z) = 0$.

Proof: This is clear since $H^{1,1}(M)$ is the orthogonal, under q, to $H^{2,0}(M) \oplus H^{0,2}(M)$. ■

By a theorem of Huybrechts, a holomorphic symplectic manifold M is projective if and only if it has an integral $(1,1)$-class with strictly positive Beauville-Bogomolov square. In this case, the Picard lattice $H^{1,1}_{\mathbb{Z}}(M) = H^2(M, \mathbb{Z}) \cap H^{1,1}(M)$, equipped with the Beauville-Bogomolov form q, is a lattice of signature $(+, -, -, \dots, -)$. If M is not projective, the Picard lattice can be either negative definite, or degenerate negative semidefinite with one-dimensional kernel. In both cases, its rank cannot be maximal (i.e. equal to the dimension of $H^{1,1}(M)$), since the signature of q on $H^{1,1}(M)$ is $(+, -, -, \dots, -)$. Together with this observation, Proposition 2.30 easily implies the following

Proposition 2.31: Let M be an irreducible holomorphic symplectic manifold. There exists a deformation M' of M which is projective and such that all integral $(1,1)$-classes on M remain of type $(1,1)$ on M'. Moreover one can take M' of maximal Picard rank $h^{1,1}(M)$.

Proof: By Proposition 2.30, the locus C where all integral $(1,1)$-classes on M remain of type $(1,1)$ is the preimage of the intersection of N complex hyperplanes and $\mathbb{P}\mathsf{er}$, where N is strictly less than the (complex) dimension of $\mathbb{P}\mathsf{er}$. It is therefore strictly positive-dimensional. For M' representing a general point of this locus, the Picard lattice is the same as that of M, but at a special point the Picard number jumps. Namely it jumps along the intersection with each hyperplane of the form $z^\perp$, where z is an integral $(1,1)$-class. In particular, there are isolated points inside C where the Picard rank is maximal. By the observations above, the corresponding variety M' must be projective. ■

This proposition shall be useful in reducing the cone conjecture to the projective case with high Picard number (see Theorem 5.3).

3 Ergodic theory and its applications

3.1 Ergodic theory: basic definitions and facts

Definition 3.1: Let (M, μ) be a space with a measure, and G a group acting on M preserving μ. This action is **ergodic** if all G-invariant measurable subsets $M' \subset M$ satisfy $\mu(M') = 0$ or $\mu(M \backslash M') = 0$.

The following claim is well-known and its proof is straightforward (cf. e.g. [V3]).

Claim 3.2: Let M be a manifold (with a countable base), μ a Lebesgue measure, and G a group acting on (M, μ) ergodically. Then the set of points with non-dense orbits has measure 0.

Definition 3.3: Let G be a Lie group, and $\Gamma \subset G$ a discrete subgroup. Consider the pushforward of the Haar measure to G/Γ. Here, by abuse of terminology, "taking the pushforward" of a measure means measuring the intersection of the inverse image with a fixed fundamental domain. We say that Γ **has finite covolume** if the Haar measure of G/Γ is finite. In this case Γ is called **a lattice**, or sometimes **a lattice subgroup** (to distinguish it from free $\mathbb{Z}$-modules with a quadratic form, which are also often mentioned in this paper).

Remark 3.4: Borel and Harish-Chandra proved that an arithmetic subgroup of a reductive group G defined over $\mathbb{Q}$ is a lattice whenever G has no non-trivial characters over $\mathbb{Q}$ (see [BHCh], Theorem 7.8 for the semisimple case and Theorem 9.4 for the general case). In particular, all arithmetic subgroups of a semi-simple group are lattices. Therefore the monodromy and the Hodge monodromy groups from the previous section are lattices in the corresponding orthogonal groups, which is a very important point for us.

In this paper, we deal with the following example of an ergodic action.

Theorem 3.5: (Calvin C. Moore, [Moo, Theorem 4]) Let Γ be a lattice subgroup (such as an arithmetic subgroup) in a non-compact simple Lie group G with finite center, and $H \subset G$ a Lie subgroup. Then the left action of H on G/Γ is ergodic if and only if the closure of H is non-compact. ■

Let us also state the following classical result.

Theorem 3.6: (Birkhoff ergodic theorem, see for example [W], 1.6) Let μ be a probability measure on a manifold X, and let g_t be an ergodic flow preserving μ. Then for almost all $x \in X$ and any $f \in L^1(\mu)$, the limit of $m_T(f) = \frac{1}{T}\int_0^T f(g_t x)dt$ as $T \to +\infty$ exists and equals $\int_X f d\mu$. In particular, for any measurable subset K and almost all x, the part of time that the orbit of x spends in K is equal to $\mu(K)$. ■

3.2 Lie groups generated by unipotents

Here we state some of the main results of Ratner theory. We follow [KSS] and [Mor].

Definition 3.7: Let G be a Lie group, and $g \in G$ any element. We say that g is **unipotent** if $g = e^h$ for a nilpotent element h in its Lie algebra. A group G is **generated by unipotents** if G is multiplicatively generated by unipotent one-parameter subgroups.

Theorem 3.8: (Ratner orbit closure theorem, [R1])
Let $H \subset G$ be a Lie subroup generated by unipotents, and $\Gamma \subset G$ a lattice. Then the closure of any H-orbit Hx in G/Γ is an orbit of a closed, connected subgroup $S \subset G$, such that $S \cap x\Gamma x^{-1} \subset S$ is a lattice in S.

For an accessible account, see [Mor], especially theorem 1.1.15 (the formulation slightly differs from the above, but see Remark 3.12).

For arithmetic groups Ratner orbit closure theorem can be stated in a more precise way, as follows.

Theorem 3.9: Let G be a real algebraic group defined over $\mathbb{Q}$ and with no non-trivial characters, $W \subset G$ a subgroup generated by unipotents, and $\Gamma \subset G$ an arithmetic lattice. For a given $g \in G$, let H be the smallest real algebraic $\mathbb{Q}$-subgroup of G containing $g^{-1}Wg$. Then the closure of Wg in G/Γ is $(gHg^{-1})g$.

Proof: See [KSS, Proposition 3.3.7] or [Sh1, Proposition 3.2]. ■

Ratner orbit closure theorem is a consequence of her fundamental result on ergodic measures [R2], known as Ratner measure classification theorem, which we recall below.

Definition 3.10: Let G be a Lie group, Γ a lattice, and G/Γ the quotient space, considered as a space with Haar measure. Consider an orbit $Sx \subset G$ of a closed subgroup $S \subset G$, put the Haar measure (of S) on Sx, and assume that the Haar measure of its image in G/Γ is finite (this means that $S \cap x\Gamma x^{-1}$ is a lattice in S). A measure on G/Γ is called **algebraic** if it is proportional to the pushforward of the Haar measure on some orbit Sx/Γ to G/Γ with S and x satisfying the above assumption.

Let G be a non-compact simple Lie group with finite center and $H \subset G$ a Lie subgroup with non-compact closure, as in Moore's theorem (Theorem 3.5). If H is generated by unipotents, consider the algebraic measure on G/Γ which is proportional to the pushforward of the Haar measure of S, where S is taken from the Ratner's orbit closure theorem. It follows from Moore's theorem that the action of H on G/Γ is ergodic. Ratner's measure classification theorem states that all invariant ergodic measures under the action of subgroups **generated by unipotents** arise in this way.

Theorem 3.11: (Ratner's measure classification theorem, [R2])
Let G be a connected Lie group, Γ a lattice, and G/Γ the quotient space, considered as a space with Haar measure. Consider a finite measure μ on G/Γ. Assume that μ is invariant and ergodic with respect to an action of a subgroup $H \subset G$ generated by unipotents. Then μ is algebraic.

Proof: see [Mor, 1.3.7]. ■

Remark 3.12: In most texts, Ratner theorems are formulated for **unipotent flows**, that is, H is assumed to be a one-parameter unipotent subgroup $\{u(t)|t \in \mathbb{R}\}$. One gets rid of this assumption using the following lemma.

Lemma 3.13: ([MS, Lemma 2.3] or [KSS, Corollary 3.3.5]) Let H be a subgroup of G generated by unipotent one-parameter subgroups. Then any finite H-invariant H-ergodic measure on G/Γ is ergodic with respect to some one-parameter unipotent subgroup of H.

4 Algebraic measures on homogeneous spaces

The main result of this section (Theorem 4.11) follows from a theorem of Mozes and Shah [MS, Theorem 1.1].

4.1 Limits of ergodic measures

Definition 4.1: Recall that **a Polish topological space** is a metrizable topological space with countable base. Let V be the set of all finite Borel measures on a Polish topological space M, and $C^0(M)$ the space of bounded continuous functions. The **weak topology** on V is the weakest topology in which for all $f \in C^0(M)$ the map $V \longrightarrow \mathbb{R}$ given by $\mu \longrightarrow \int_M f\mu$ is continuous. If one identifies V with a subset in $C^0(M)^*$, the weak topology is identified with the weak-* topology on $C^0(M)^*$. This is why it is also called **the weak-* topology**.

Remark 4.2: It is not hard to prove that the space of probability measures on a compact Polish space is compact in weak topology. This explains the usefulness of this notion.

Theorem 4.3: (Mozes-Shah theorem)
Let G be a connected Lie group, Γ a lattice, u_i a sequence of unipotent one-parameter subgroups in G, and μ_i a sequence of u_i-invariant, u_i-ergodic probability measures on G/Γ, associated with orbits $S_i x_i \subset G/\Gamma$ as in Definition 3.10. Assume that $\lim \mu_i = \mu$ with respect to the weak topology, with μ a probability measure on X, and let $x \in \operatorname{Supp}(\mu)$. Then

(i) μ is an algebraic measure, associated with an orbit Sx as in Definition 3.10.

(ii) Let $g_i \in G$ be elements which satisfy $g_i x_i = x$, and assume that $g_i \to e$ in G (so that x_i converge to x). Then there exists $i_0 \in \mathbb{N}$ such that for all $i > i_0$, $S \cdot x \supset g_i S_i \cdot x_i$.

Proof: The statement (i) follows from [MS, Theorem 1.1 (3)] and Ratner measure classification theorem, and (ii) is [MS, Theorem 1.1 (2)]. ■

Remark 4.4: More precisely, in [MS, Theorem 1.1] there is an additional condition that the trajectories $\{u_i(t)\}x_i,\ t > 0$ should be uniformly distributed with respect to μ_i. But this is automatic by another theorem of Ratner (Ratner equidistribution theorem, see e.g. [Mor], Theorem 1.3.4), and in fact already by Birkhoff ergodic theorem (Theorem 3.6), which states the uniform distribution of orbits of one-parameter subgroups for almost all starting points.

The following theorem is an interpretation of Dani-Margulis theorem as stated in [DM, Theorem 6.1] obtained by applying Birkhoff ergodic therorem.

Theorem 4.5: (Dani-Margulis theorem).
Let G be a connected Lie group, Γ a lattice, $X := G/\Gamma$, $C \subset X$ a compact subset, and $\varepsilon > 0$. Then there exists a compact subset $K \subset X$ such that for any algebraic probability measure μ on X, satisfying $\mu(C) \neq 0$ and associated with a group generated by unipotents (as in Ratner theorems), one has $\mu(K) \geqslant 1-\varepsilon$.
■

Proof: By Lemma 3.13, μ is invariant and ergodic with respect to a one-parameter unipotent subgroup $u(t)$. Now apply [DM, Theorem 6.1] to a starting point x which is one of "almost all points" of $\operatorname{Supp}(\mu)\cap C$ in the sense of Birkhoff theorem. ■

Combining Dani-Margulis theorem and Mozes-Shah theorem, one gets the following useful corollary ([MS, Corollary 1.1, Corollary 1.3, Corollary 1.4]).

Corollary 4.6: Let G be a connected Lie group, Γ a lattice, $\mathcal{P}(X)$ be the space of all probability measures on $X = G/\Gamma$, and $\mathcal{Q}(X) \subset \mathcal{P}(X)$ the space of all algebraic probability measures associated with all subgroups $H \subset G$ generated by unipotents (as in Ratner theorems). Then $\mathcal{Q}(X)$ is closed in $\mathcal{P}(X)$ with respect to weak topology. Moreover, let $X \cap \{\infty\}$ denote the one-point compactification of X, so that $\mathcal{P}(X \cap \{\infty\})$ is compact. If for a sequence $\mu_i \in \mathcal{Q}(X)$, $\mu_i \to \mu \in \mathcal{P}(X \cap \{\infty\})$, then either $\mu \in \mathcal{Q}(X)$, or μ is supported at infinity.
■

4.2 Rational hyperplanes intersecting a compact set

Definition 4.7: Let $V_{\mathbb{Z}}$ be an $n+1$-dimensional lattice with a scalar product of signature $(+,-,-,...,-)$, $V_{\mathbb{Q}} = V_{\mathbb{Z}} \otimes \mathbb{Q}$ and $V := V_{\mathbb{Z}} \otimes \mathbb{R}$. We consider the projectivization of the positive cone $\mathbb{P}^+V$ as the hyperbolic space of dimension n. Given a $k+1$-dimensional subspace $W_{\mathbb{Q}} \subset V_{\mathbb{Q}}$ such that the restriction of the scalar product to $W_{\mathbb{Q}}$ still has signature $(1,k)$, we may associate the projectivized positive cone $\mathbb{P}^+W \subset \mathbb{P}^+V$ with $W = W_{\mathbb{Q}} \otimes_{\mathbb{Q}} \mathbb{R}$. When $k = n-1$, we shall call $\mathbb{P}^+W \subset \mathbb{P}^+V$ **a rational hyperplane** in $\mathbb{P}^+V$.

Let Γ be an arithmetic lattice subgroup in the group of isometries of $\mathbb{P}^+V$, and $\{S_i\}$ a set of rational hyperplanes. We are interested in the images of S_i

in $\mathbb{P}^+V/\Gamma$. The following theorem can be used to show that these images all intersect a compact subset of $\mathbb{P}^+V/\Gamma$.

Theorem 4.8: Let $\{S_i\}$ be a set of rational hyperplanes in $\mathbb{P}^+V$, $P_{\mathbb{Q}} \subset V_{\mathbb{Q}}$ a rational subspace of signature $(1,2)$, and $\mathbb{P}^+P \subset \mathbb{P}^+V$ the corresponding 2-dimensional hyperbolic subspace. Consider an arithmetic lattice $\Gamma \subset SO(V,\mathbb{Z})$, and let Γ_P be the stabilizer of $P_{\mathbb{Q}}$ in Γ. Then there exists a compact subset $K \subset \mathbb{P}^+P$ such that $\Gamma_P \cdot K$ intersects all the hyperplanes S_i.

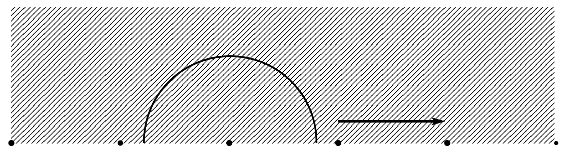
Proof: Since Γ has finite index in $O(V,\mathbb{Z})$, Γ_P has finite index in $O(P,\mathbb{Z})$. One may view Γ_P as a multi-dimensional analogue of Fuchsian or Kleinian group, acting properly discontinuously on the hyperbolic plane. Then Γ_P acts with finite stabilizers, and the quotient of the hyperbolic plane by Γ_P is a hyperbolic orbifold X. We must prove that there is a compact subset of X such that its intersection with the image of any line $L_i = S_i \cap \mathbb{P}^+P$ is non-empty. But any arithmetic lattice has a finite index subgroup which is torsion-free (for instance, the congruence subgroup formed by integer matrices which are identity modulo N for N big enough). Therefore, our orbifold X has a finite covering $\widetilde{X}$ which is a hyperbolic Riemann surface, and it suffices to prove that there is a compact $\widetilde{K} \subset \widetilde{X}$ such that $\pi(L_i)$ intersects $\widetilde{K}$ for any i, where $\pi : \mathbb{P}^+P \longrightarrow \widetilde{X}$ denotes the projection (quotient by a finite index subgroup $\widetilde{\Gamma_P} \subset \Gamma_P$).

Let Γ_{L_i} be the stabilizer of L_i in Γ_P. Since Γ_{L_i} has finite index in the isometry group of the sublattice underlying to L_i, the images $\pi(L_i)$ have finite length in $\widetilde{X}$. On the other hand, $\pi(L_i)$ are isometric images of L_i. Therefore, $\pi(L_i)$ are compact; in other words, these are closed geodesics on $\widetilde{X}$. We have reduced Theorem 4.8 to the following well-known lemma.

Lemma 4.9: Let S be a complete hyperbolic Riemann surface (of constant negative curvature and finite volume). Then there exists a compact subset $K \subset X$ intersecting each closed geodesic $l \subset S$.

Proof: To obtain K, it suffices to remove from S a neighbourhood of each cusp: indeed, there are no closed geodesics around cusps. This is an elementary exercise, apparently well known; see e.g. [MR, Theorem 1.2] which is in the same spirit. For the convenience of the reader, we sketch an argument here.

Let $\mathbb{H} = \{x \in \mathbb{C} \mid \operatorname{Im}(x) > 0\}$ be a hyperbolic half-plane, equipped with a Poincare metric, $t > 0$ a real number, and $\mathbb{H}_t = \{x \in \mathbb{C} \mid t > \operatorname{Im}(x) > 0\}$ a strip consisting of all $x \in \mathbb{C}$ with $0 < \operatorname{Im}(x) < t$. In a neighbourhood of a cusp point, S is isometric to a quotient $\mathbb{H}_t/\mathbb{Z}$, where the action of $\mathbb{Z}$ is generated by the parallel transport $\gamma_r(x) = x + r$, where $r \in \mathbb{R}$ is a fixed number. A geodesic is a half-circle perpendicular to the line $\operatorname{Im} x = 0$; closed geodesic in $\mathbb{H}_t/\mathbb{Z}$ is a half-circle which is mapped to itself by a power of γ_r. Such half-circles clearly do not exist (see the picture).



A neighbourhood of a cusp point in dimension 2

■

Remark 4.10: The result of this subsection shall be used in the next one to justify that a certain sequence of ergodic measures does not have a subsequence going to infinity. Since all the measures in question come from orbits of the same subgroup, this is also a consequence of [EMS], Corollary 1.10. We prefer nevertheless to keep our simple observations on hyperbolic geometry which might have some independent interest.

4.3 Measures and rational hyperplanes in the hyperbolic space

The hyperbolic space, that is, the projectivization of the positive cone in a real vector space $V = V_{\mathbb Z} \otimes \mathbb R$ with a quadratic form of signature $(1, n)$, is a homogeneous space in an obvious way. Indeed it is an orbit of any positive line by the connected component of the unity of $SO(1, n)$, and the stabilizer is isomorphic to $SO(n)$. If z is a negative vector, then $z^\perp$ is a hyperplane which intersects the positive cone; as in the previous paragraph, by a hyperplane in the hyperbolic space we shall mean the projectivization of this intersection.

Theorem 4.11: Let G be the connected component of the unity $SO^+(1, n)$ of $SO(1, n)$, where $n \geqslant 3$, $H := SO(n)$, and $\Gamma \subset G_{\mathbb Z}$ a discrete subgroup of finite index (and therefore of finite covolume, Remark 3.4). Consider the hyperbolic space $\mathbb H = H\backslash G = SO(n)\backslash SO^+(1, n)$. Let $Y := \bigcup S_i$ be a Γ-invariant union of rational hyperplanes. Then either Γ acts on $\{S_i\}$ with finitely many orbits, or Y is dense in $\mathbb H$.

Proof: Let $V = \mathbb R^{1,n}$ be a real vector space of signature $(1, n)$, $G = SO^+(V)$, and $\mathbb H = H\backslash G$, where $H \subset G$ is the stabilizer of an oriented positive hyperplane. We may identify $\mathbb H$ with the space of positive vectors $x \in V$, $(x, x) = 1$. In order to apply ergodic theory, we replace $\mathbb H$ by the incidence variety X of pairs $(\mathbb H_W \subset \mathbb H, x \in \mathbb H_W)$, where $\mathbb H_W$ is an oriented hyperplane in the hyperbolic space and $x \in \mathbb H_W$. Clearly, a point of X is uniquely determined by a pair of orthogonal vectors $x, y \in V$, where x is positive, $(x, x) = 1$, and $(y, y) = -1$. Therefore, $X = H_0\backslash G$, where $H_0 = SO(n - 1)$. The important point is that X is a quotient of G by a compact group (and so is $\mathbb H$). Moreover X is fibered over $\mathbb H$ in spheres of dimension $n - 1$.

We can lift our hyperplanes S_i to X in the tautological way. To make the picture transparent, we first treat the case $n = 2$, where $\mathbb H$ is the hyperbolic plane (since in the theorem we have $n \geqslant 3$, this is just to describe the lifting and see a certain well-known analogy). Here $X = SO^+(1,2)$ is the unit tangent bundle over $\mathbb{H}$, and a point $x \in S_i$ lifts as (x, z) where z is a unit tangent vector to S_i (there are two choices, we take one by fixing an orientation on S_i). If we lift all possible (not necessarily rational) hyperplanes to X in this way, we obtain a foliation known as **the geodesic flow**: our liftings never intersect and are tangent to an invariant vector field on $X = SO^+(1,2)$. Therefore all the lifting are orbits of the same Lie subgroup $H_1 \subset G$ (this one-parameter subgroup, isomorphic to $SO^+(1,1)$, can be identified with the group of diagonal two-by-two matrices with e^t and e^{-t} on the diagonal under an isomorphism between $SO^+(1,2)$ and $PSL(2,\mathbb{R})$, see the first chapter of Morris' book [Mor]).

For $n \geqslant 3$, we first tautologically lift the hyperplanes to X and then take preimages under the projection from G to X. Again, we obtain a translation-invariant foliation on G, which means that the liftings and their preimages are orbits of the same subgroup $H_1 \subset G$ (containing H_0). This subgroup is isomorphic to $SO^+(1, n-1)$, that is, generated by unipotents (in contrast with $n = 2$ case), so that ergodic theory applies.

Let us denote by R_i the preimage in G of the lifting of S_i to X. Each R_i is an orbit of H_1. By Theorem 4.8, there is a compact set C in $\mathbb{H}$ such that the Γ-orbit of any S_i intersects C. Since the projection from G to $\mathbb{H}$ is proper, the same is true for the set of R_i. Suppose that Γ acts on the set of S_i (and thus R_i) with infinitely many orbits. Consider the homogeneous space G/Γ. Each Γ-orbit on the set of R_i corresponds to an algebraic probability measure μ_i on G/Γ (note that since the hyperplanes S_i are rational, the quotient of each H_1-orbit R_i over its stabilizer in Γ has finite Haar volume by Borel and Harish-Chandra theorem). The support of μ_i is the image of R_i in G/Γ. Since the union of R_i is Γ-invariant, to prove Theorem 4.11, it suffices to show that the union of $\operatorname{Supp}(\mu_i)$ is dense in G/Γ: this will imply the density of R_i in G and therefore the density of S_i in $\mathbb{H}$.

By Corollary 4.6, the sequence μ_i has an accumulation point which is either a probability measure, or is supported at infinity. But the latter option is impossible. Indeed, by Theorem 4.8 all $\operatorname{Supp}(\mu_i)$ intersect the same compact subset of G/Γ. Thus there is a (slightly larger) compact C' in G/Γ such that $\mu_i(C') > 0$ for all i, and by Dani-Margulis theorem, for another compact K_ε and all i, $\mu_i(K_\varepsilon) > 1 - \varepsilon$.

Taking a suitable subsequence, we may therefore suppose that $\lim \mu_i = \mu$ where μ is an algebraic probability measure.

We have reduced Theorem 4.11 to the following lemma.

Lemma 4.12: Let G be the connected component $SO^+(1,n)$ of $SO(1,n)$, where $n \geqslant 3$, and $\Gamma \subset G_{\mathbb{Z}}$ a discrete subgroup of finite index (and therefore of finite covolume). Let μ_i be a sequence of algebraic probability measures on G/Γ associated with the orbits of a subgroup $H_1 \subset G$ isomorphic to $SO^+(1, n-1)$. Suppose μ_i converges to an algebraic probability measure μ. Then either μ_i are finitely many, or $\operatorname{Supp}(\mu)$ is G/Γ, so that $\operatorname{Supp}(\mu_i)$ are dense in G/Γ.

Proof: By Theorem 4.3 (ii), the support of μ contains a right translate by $g_i \to e$ of the support of infinitely many of μ_i. Moreover, μ is an algebraic measure associated with an orbit of a closed subgroup $F \subset G$. But there are no closed intermediate connected subgroups between $G = SO^+(1,n)$ and H_1, which stabilizes a hyperplane. Therefore, F is either equal to G, or is the stabilizer H of a hyperplane $\mathbb{H}_W$.

In the first case, the support of $\mu = \lim \mu_i$ is G/Γ and thus $\operatorname{Supp}(\mu_i)$ are dense in G/Γ.

In the second case, for $i \gg 0$, $\operatorname{Supp}(\mu) = g_i \operatorname{Supp}(\mu_i)$, that is, $Hx = g_i H_1 x_i$ where $g_i x_i = x$ (where x, x_i, g_i are as in Theorem 4.3). That is, $Hx = H_1^{g_i} x$ and therefore $H_1^{g_i} = H = H_1$. Since H_1 has finite index in its normalizer, this means that there are only finitely many μ_i. ■

5 The proof of Morrison-Kawamata cone conjecture

5.1 Morrison-Kawamata conjecture for the Kähler cone

The following theorem is an immediate consequence of Theorem 4.11.

Theorem 5.1: Let M be a projective simple hyperkähler manifold which has Picard number at least 4. Then the Hodge monodromy group acts with finitely many orbits on the set of MBM classes of type $(1,1)$.

Proof: This is the same as to say that the Hodge monodromy group acts with finitely many orbits on the set of their orthogonal hyperplanes, which by [AV] are exactly the hyperplanes supporting the faces of the Kähler chambers.

Since the Hodge monodromy group is of finite index in the orthogonal group of the Picard lattice, which is of signature $(+,-,\ldots,-)$, one can apply Theorem 4.11 to the Picard lattice, with Γ the Hodge monodromy group. One concludes that if the number of Γ-orbits is infinite, then the hyperplanes orthogonal to MBM classes should be dense in the positive cone. This is clearly absurd, as they should bound the ample cone, so the number of Γ-orbits is finite. ■

Corollary 5.2: On an M as above, the primitive MBM classes of type $(1,1)$ have bounded Beauville-Bogomolov square.

Proof: Indeed, the monodromy acts by isometries. ■

Theorem 5.3: Let M be a simple hyperkähler manifold such that $b_2(M) \geqslant 6$. Then the primitive MBM classes of type $(1,1)$ have bounded Beauville-Bogomolov square.

Proof: If M is not projective or the Picard number of M is less than four, apply Proposition 2.31 to get a projective deformation M' with Picard number at least four such that all MBM classes of type $(1,1)$ on M remain of type $(1,1)$ on M'. Then use the deformation invariance of MBM property proved in [AV] to conclude that these MBM classes remain MBM on M' and therefore the primitive ones must have bounded square by the preceding theorem.

The Morrison-Kawamata conjecture for the Kähler cone now follows in the same way as in [AV].

Theorem 5.4: Let M be a simple hyperkähler manifold with $b_2(M) \geqslant 6$. Then the automorphism group of M acts with finitely many orbits on the set of faces of its Kähler cone.

Proof: The argument is the same as in [AV] where the theorem has been obtained under the boundedness assumption on squares of primitive MBM classes, which we have just proved: see [AV, Theorem 6.6] there for an outline of the argument and [AV, Theorem 3.14, 3.29] for technicalities. ■

5.2 Morrison-Kawamata conjecture for the ample cone

Recall from e.g. [MY] that the classical Morrison-Kawamata cone conjecture is formulated in the projective case and treats the ample cone rather than the Kähler cone. It also states something *a priori* stronger than the finiteness of the number of orbits of the action of automorphism group on the set of faces of the cone, namely the existence of a finite polyhedral fundamental domain.

More precisely, following [Mo], [MY], let $\operatorname{Nef}(M)$ be the nef cone (that is, the closure of the ample cone of M) and define the cone $\operatorname{Nef}^+(M)$ as the convex hull of $\operatorname{Nef}(M) \cap H^{1,1}_{\mathbb{Q}}(M)$ in $H^{1,1}_{\mathbb{Q}}(M) \otimes \mathbb{R}$. One has $\operatorname{Amp}(M) \subset \operatorname{Nef}^+(M) \subset \operatorname{Nef}(M)$; the cone $\operatorname{Nef}^+(M)$ is just $\operatorname{Amp}(M)$ to which one has attached the rationally defined part of the boundary (such as the boundary given by the hyperplanes orthogonal to MBM classes).

Conjecture 5.5: (Morrison-Kawamata cone conjecture for the ample cone) The automorphism group $\operatorname{Aut}(M)$ has a finite polyhedral fundamental domain on $\operatorname{Nef}^+(M)$.

We shall see in this subsection that this in fact follows from our version of the cone conjecture, and therefore is true for all simple hyperkähler manifolds with $b_2 \neq 5$.

Suppose first that the Picard number of X is at least three. Denote by $\mathcal{C}(M)$ the intersection of $\operatorname{Pos}(M)$ with $\mathsf{NS}(X) \otimes \mathbb{R}$. The Hodge monodromy group Γ acts on $\mathbb{P}\mathcal{C}(M)$ with finite stabilizers (since the stabilizer of a point x in $\mathbb{P}\mathcal{C}(M)$ must also stabilize the orthogonal hyperplane to the line corresponding to x, and our form is negative definite on such a hyperplane). By its arithmeticity, replacing if necessary the group Γ by a finite index subgroup, we may assume there are no stabilizers at all. Indeed, an arithmetic lattice has a finite index torsion-free subgroup, which can be obtained by taking a congruence subgroup formed by integer matrices which are identity modulo N for N big enough. Consider the quotient $S := \mathbb{P}\mathcal{C}(M)/\Gamma$. Since Γ is arithmetic, Borel and Harish-Chandra theorem implies that S is a complete hyperbolic manifold of finite volume. The image of $\operatorname{Amp}(M)$ in S is a hyperbolic manifold T with finite (that is, consisting of finitely many geodesic pieces) boundary, by Theorem 5.4. It is known (see [Bow, Proposition 4.7 and 5.6] or [K, Theorem 2.6]) that such manifolds are *geometrically finite*, that is, they admit a finite cell decomposition with finite piecewise geodesic boundary (in fact one even has a decomposition with a single cell of maximal dimension, the *Dirichlet-Voronoi decomposition*). Thus T becomes a union of finitely many cells with finite piecewise geodesic boundary. Now compactify S to $\overline{S}$ by adding a point at each of the finitely many cusps. Since the cusps correspond to Γ-orbits of rational points on the boundary of $\mathbb{P}\mathcal{C}(M)$, the closed cells in $\overline{T}$ (the closure of T) naturally lift to $\mathbb{P}\operatorname{Nef}^+(M)$. Taking the union of suitable liftings, we obtain a finite polyhedron which is a fundamental domain for the subgroup of Γ preserving $\operatorname{Amp}(M)$, that is, of the automorphism group of M. We thus have proved the following

Theorem 5.6: Let M be a projective simple hyperkähler manifold with $b_2 \neq 5$ and Picard number at least three. The automorphism group has a finite polyhedral fundamental domain on $\operatorname{Nef}^+(M)$.

It remains to treat the manifolds with Picard number two, where Borel and Harish Chandra theorem does not apply. For this one observes (cf. [O]) that the boundary of the ample cone is either rational, and then the cone conjecture is a tautology; or it is irrational, meaning that there are no MBM classes, the ample cone is equal to the positive cone and $Aut(M) \cong \mathbb{Z}$ is the same as the Hodge monodromy (a finite index subgroup in the orthogonal group of the Neron-Severi lattice) and acts by translations on $\mathbb{P}\mathcal{C}(M)$. In this case, the existence of a fundamental domain which is an interval is also clear.

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