# The bend-and-break method; Mori's characterisation of projective space

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The bend-and-break method is invented by S. Mori in order to produce rational curves on a manifold $X$ with non-positive canonical class. All that follows is done in [M1] (with some precisions in [M2], [MM]). The starting observation is the following "rigidity lemma":

**Lemma:** *Let $U, V, X$ be algebraic varieties. Assume that $U$ is connected and $V$ is proper. Let $f : U \times V \to X$ be a morphism such that $f(u_0 \times V)$ is a point. Then $f(u \times V)$ is a point for any $u \in U$.*

In other words, if a regular $f$ contracts one fiber of $U \times V$, it contracts them all.

**Corollary:***Let $C$ be a smooth proper curve, $p \in C$, $g_0 : C \to X$ non-constant. Assume that $g$ deforms with one point fixed: that is, there is a family of maps $g_t : C \to X$ parametrized by a curve $D$, such that $g_t(p) = g_0(p)$ for all $t$. Then there is a rational curve on $X$, passing through $g_0(p)$.*

Indeed, our family gives rise to a rational map $f : C \times D \dashrightarrow X$. From the lemma, one deduces this map cannot be regular (in fact, under the assumption that $C$ is non-rational; but otherwise our statement is trivial). The image of its indeterminacy locus is a union of rational curves.

A slightly more involved argument gives the following

**Proposition:***Suppose that some rational curve on $C$ deforms non-trivially with two points fixed. Then one of those deformaions is reducible.*

By the *bend-and-break method*, one usually inderstands the Corollary and the Proposition together: once we have a curve which deforms a lot, use the deformations to get rational curves; if those curves, in turn, deform a lot, use the Proposition to break them into curves of smaller degree. Note that for a morphism $f : X \to C$, the dimension of its space of deformations is at least

$$h^0(C, f^*T_X) - h^1(C, f^*T_X) = deg(f^*(-K_X)) + (1 - g(C))dim(X), \quad (*)$$

so, rational curves of large anticanonical degree do deform a lot. The Proposition then says that we can lower their anticanonical degree until it reaches $dim(X) + 1$ (because if $deg(f^*(-K_X)) > dim(X) + 1$, the curve shall deform with two fixed points: fixing the image of one point imposes $dim(X)$ conditions on $f$).

In order to make the Corollary applicable, it remains to produce some curves with many deformations. Apriori, they do not have to exist: it can happen that the anticanonical degree of every $C \subset X$ is small compared to its genus, so we cannot get them from $(*)$. Here, Mori's beautiful idea is to consider the reduction in characteristics $p$. Indeed, in positive characteristics any curve $C$ has the Frobenius endomorphism $Fr : C \to C$ of degree $p^l$ for some $l > 0$. Replacing $f : C \to X$ by $f' = f \circ Fr^k$ for large $k$, we obtain $C'$ such that $deg(f'^*(-K_X))$ is very large, that is, a curve which deforms. The bend-and-break gives then rational curves of bounded degrees on the reductions in positive characteristics. Finally, one proves that these curves lift back on $X$ (we skip the details but see [CKM] for a very accessible exposition).

Similar ideas yield Mori's famous characterization of $\mathbb{P}^n$ ([M1], see also [Ko], V.3):

**Theorem:** *Let $X$ be a smooth $n$-dimensional Fano variety (that is, $-K_X$ is ample). Assume that for some $x \in X$ and any non-constant $f : (\mathbb{P}^1, 0) \to (X, x)$, the bundle $f^*T_X$ is ample (an ample vector bundle on $\mathbb{P}^1$ is simply the sum of line bundles of strictly positive degree). Then $X \cong \mathbb{P}^n$.*

**Corollary:** *If $T_X$ is ample, then $X \cong \mathbb{P}^n$.*

The idea of the proof is as follows: by bend-and-break, one obtains curves $f : \mathbb{P}^1 \to X$ (through $x$) of anticanonical degree at most $n+1$. Then one proves that every such $f$ is an embedding: the ampleness hypothesis is translated into $f$ being an immersion, of anticanonical degree exactly $n+1$, and one reduces to this by yet another version of the bend-and-break. Finally, one proves that the resulting "lines" (smooth curves of anticanonical degree $n+1$) through $x$ make the blow-up of $X$ at $x$ into a $\mathbb{P}^1$-bundle over an exceptional $\mathbb{P}^{n-1}$, and deduces $X \cong \mathbb{P}^n$.

# References

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[M1] S. Mori: Projective manifolds with ample tangent bundle, Ann. of Math. 110 (1979), 593-606.

[M2] S. Mori: Threefolds whose canonical bundles are not numerically effective, Ann. of Math. 116 (1982), 133-176

[MM] Y. Miyaoka, S. Mori: A numerical criterion for uniruledness, Ann. of Math. 124 (1986), 65-69.