# Fano varieties; Iskovskih's classification

Ekaterina Amerik

For details and extensive bibliography, we refer to [2], chapter V, and [1].

A *Fano variety* is a projective manifold $X$ such that the anticanonical line bundle $K_X^{-1}$ is ample. By Kodaira vanishing, the Hodge numbers $h^{p,0}(X) = h^{0,p}(X)$ are zero for $p \neq 0$. Furthermore, Fano manifolds are simply connected (this is implied for example by their property to be rationally connected; see the main article on rational curves and uniruled varieties).

Simplest examples are obtained by taking smooth complete intersections of type $(m_1, m_2, \dots, m_k)$ in $\mathbb{P}^n$. By adjunction formula, such a complete intersection is Fano if and only if $\sum_i m_i \leq n$. A larger class of examples is that of complete intersections in a *weighted projective space* $\mathbb{P}(a_0, a_1, \dots, a_n)$ (this is $(\mathbb{C}^{n+1} - 0)/\mathbb{C}^*$, where $\mathbb{C}^*$ acts with weights $a_0, a_1, \dots, a_n$; it is singular when not isomorphic to a usual projective space, but we consider complete intersections avoiding the singularities) : the Fano condition amounts then to $\sum_i m_i < \sum_i a_i$. Rational homogeneous varieties $G/H$ ($G$ semisimple, $H$ parabolic) are Fano, too.

A Fano curve is, obviously, $\mathbb{P}^1$. If $n = dim(X) = 2$ and $X$ is Fano, then $X$ is called a *Del Pezzo surface*. Such surfaces have been classically studied, and it is well-known that any such $X$ is isomorphic either to $\mathbb{P}^2$, or to $\mathbb{P}^1 \times \mathbb{P}^1$, or to $\mathbb{P}^2$ blown up in $d$ points ($1 \leq d \leq 8$) in general position, "general position" meaning here that no three points are on a line and no six on a conic. For $1 \leq d \leq 6$, the anticanonical map is an embedding. It realizes a blow-up of $\mathbb{P}^2$ in $d$ points ($1 \leq d \leq 6$) as a surface $X_l$ of degree $l = K_X^2 = 9 - d$ in $\mathbb{P}^l$. For $d = 7$, one obtains $X_2$ which is a double cover of $\mathbb{P}^2$ ramified along a quartic, and for $d = 8$, a hypersurface of degree 6 in the weigthed projective space $\mathbb{P}(1, 1, 2, 3)$ (which is the same as a double covering of a quadratic cone ramified in its section by a cubic; the double covering is given by the space of sections of $K_X^{\otimes -2}$.)

There are many more types of Fano threefolds. Even under the restriction $Pic(X) \cong \mathbb{Z}$, one obtains 18 families. Fano threefolds of Picard number one have been classified by Iskovskih, and Fano threefolds of higher Picard number, by Mori and Mukai (see [1]). Below we mention a few generalities on Fano manifolds and give an outline of Iskovskih's classification.

A basic invariant of a Fano manifold is its *index*: this is the maximal integer $r$ such that $K_X$ is divisible by $r$ in $Pic(X)$.

**Theorem 1** ([3]) *Let $X$ be Fano, $dim(X) = n$. Then the index $ind(X)$ is at most $n+1$; moreover, if $ind(X) = n+1$, then $X \cong \mathbb{P}^n$, and if $ind(X) = n$, then $X$*

*is a quadric.*

(Note that $ind(X) \leq n+1$ follows immediately from bend-and-break; but in fact the result of Kobayashi and Ochiai is much older, and the proof of their first statement is quite elementary.)

**Theorem 2** (Kollar-Miyaoka-Mori, [2]): *For any positive integer $n$, there is only finitely many deformation types of Fano manifolds of dimension $n$.*

The number of families probably grows very fast together with $n$.

A Fano threefold $X$ can have index 4 (if $X = \mathbb{P}^3$), 3 (if $X = Q^3$), 2 or 1. Suppose that the Picard number of $X$ is 1. Then $K_X = H_X^{-ind(X)}$, where $H_X$ is the ample generator of $Pic(X)$. Iskovskih's classification asserts the following:

If $ind(X) = 2$, then $1 \leq H_X^3 \leq 5$, and:

- if $H_X^3 = 1$, $X$ is a hypersurface of degree 6 in a weighted projective space $\mathbb{P}(1,1,1,2,3)$;
- if $H_X^3 = 2$, $X$ is a double covering of $\mathbb{P}^3$ ramified in a quartic surface (in other words, a hypersurface of degree 4 in a weighted projective space $\mathbb{P}^(1,1,1,1,2)$);
- if $H_X^3 = 3$, $X$ is a cubic in $\mathbb{P}^4$;
- if $H_X^3 = 4$, $X$ is a complete intersection of type $(2,2)$ in $\mathbb{P}^5$;
- if $H_X^3 = 5$, $X$ is a linear section of the Grassmannian $G(2,5)$ in the Plücker embedding.

If $ind(X) = 1$, then:

- $H_X^3$ takes all even values between 2 and 22, except 18;
- low values of $H_X^3$ correspond to double covers: of $\mathbb{P}^3$ ramified in a sextic if $H_X^3 = 2$, of a quadric ramified in a quartic section if $H_X^3 = 4$;
- for $H_X^3 = 4$ there is of course one more family - that of quartics in $\mathbb{P}^4$;
- and also for all other families, $H_X$ is very ample and embeds $V_d$ (that is, a Fano threefold with $H_X^3 = d$) in $\mathbb{P}^{d/2+2}$;
- $V_6$ and $V_8$ are obvious complete intersections, $V_{10}$ is a section of a cone over $G(2,5)$ by three hyperplanes and a quadric, $V_{14}$ is a linear section of $G(2,6)$;
- other $V_d$ ($d = 12, 16, 18, 22$) are more complicated, but there is a relatively simple description in terms of vector bundles on homogeneous varieties, due to Mukai. For instance, a $V_{22}$ is the zero set of a section of the sum of three copies of $\Lambda^2 U^*$ on $G(3,7)$ (where $U$ is the universal bundle).

There are more families of threefolds of higher Picard number; according to Mori and Mukai, its maximal value is 10.

# References


[1] V.A. Iskovskih, Yu. G. Prokhorov: Algebraic Geometry V: Fano varieties. Encyclopaedia of Math. Sciences 47, Springer-Verlag, Berlin 1999.

[2] J. Kollar: Rational curves on algebraic varieties, Springer-Verlag, Berlin 1996.

[3] S. Kobayashi, T. Ochiai: Characterizations of complex projective spaces and hyperquadrics. J. Math. Kyoto Univ. 13 (1973), 31-47.