# The theorem of Graber-Harris-Starr

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In 2001, Graber, Harris and Starr proved the following theorem, which is crucial in the study of uniruled and rationally connected varieties:

**Theorem 1**([GHS]) *Let $f : X \to B$ be a morphism from a smooth projective complex variety to a smooth projective curve. If the general fiber of $f$ is rationally connected, then $f$ has a section.*

Equivalently, a rationally connected variety over a function field of a curve always has a point. In this form, this is a generalization of Tsen's theorem which handles the case when $X$ a smooth hypersurface of degree $\leq n$ in $\mathbb{P}^n$.

This theorem has several important consequences (all listed in [GHS]):

**Corollary 1** *Let $f : X \to Y$ be a dominant morphism. If $Y$ and the general fiber of $f$ are rationally connected, then $X$ is rationally connected.*

Let $X$ be uniruled. Recall (from the article on uniruled varieties) that there is a rational map $\phi : X \dashrightarrow Z$, unique up to birational equivalence, such that the general fiber of $\phi$ is rationally connected and any rational curve meeting the general fiber is contained in this fiber. This map is called the *rational quotient* or the *maximally rationally connected fibration.*

**Corollary 2** *The rational quotient $Z$ is not uniruled.*

**Corollary 3** *Conjecture 1 from the article on uniruled varieties implies Conjecture 2: if we assume that any non-uniruled variety carries a non-zero pluricanonical form, then for any $X$ which is not rationally connected, $H^0(X, (\Omega^1_X)^{\otimes m}) \neq 0$ for some $m > 0$.*

Indeed, consider the rational quotient $Z$. By Corollary 2, $Z$ is not uniruled, so Conjecture 1 says that $H^0(Z, K_Z^{\otimes d}) \neq 0$ for some positive $d$. Thus $H^0(Z, (\Omega^1_Z)^{\otimes nd}) \neq 0)$ (where $n = dim(Z)$), and pulling this back to $X$, we get the non-vanishing.

Finally, the Theorem implies a somewhat surprising fact that the special fiber of a family of rationally connected varieties (i.e. of a family with rationally connected general fiber) always has a reduced irreducible component (in particular, no "multiple fibers" are possible).

The Theorem was generalized in positive characteristics by J. de Jong and J. Starr ([JS]). The only change in the statement is that one has to replace "rationally connected" by "separably rationally connected".

Graber, Harris, Mazur and Starr provided a kind of the "converse":

**Theorem 2**([GHMS]) *Let $f : X \to B$ be a family of projective manifolds. Suppose that its restriction on a generic curve in $B$ has a section. Then $X$ contains a subfamily of rationally connected varieties (by convention, here points are included in the class of rationally connected varieties).*

At the moment, some efforts, motivated by the theorem of Graber-Harris-Starr, are being made to define "higher rational connectedness": in particular, one would like a family of "$k-1$-rationally connected varieties" over a base of dimension $k$ to have rational sections, at least under certain natural conditions (a *rational section* is a section over an open part; one cannot of course require a section over the whole base as this requirement is not satisfied even by projective bundles). But this still seems to be very far from being finished.

# References


[GHS] T. Graber, J. Harris, J. Starr: Families of rationally connected varieties, Journal of AMS 16 (2003), no.5, 57-67.

[GHMS] T. Graber, J. Harris, B. Mazur, J. Starr: Ann. Sci. ENS (4) 38 (2005), no.5, 671-692.

[JS] J. de Jong, J. Starr: Every rationally connected variety over the function field of a curve has a rational point, Amer. J. Math. 125 (2003), 567-580.