Construction of automorphisms of hyperkähler manifolds

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Abstract

Let M be an irreducible holomorphic symplectic (hyperkähler) manifold. If $b_2(M) \geqslant 5$, we construct a deformation M' of M which admits a symplectic automorphism of infinite order. This automorphism is hyperbolic, that is, its action on the space of real $(1,1)$-classes is hyperbolic. If $b_2(M) \geqslant 14$, similarly, we construct a deformation which admits a parabolic automorphism.

Contents

1 Introduction

1.1 Sublattices and automorphisms

Since early 2000's, K3 surfaces were one of the prime subjects of holomorphic dynamics ([C], [McM]). By now, the dynamics of automorphism group act-

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ing on K3 surfaces is pretty much understood ([CD]). Some of these results are already generalized to irreducible holomorphic symplectic manifolds of Kähler type (simple hyperkähler manifolds) in any dimension ([O]).

The purpose of the present paper is to construct sufficiently many interesting automorphisms on a deformation of an arbitrary hyperkähler manifold (see Section 2 for basic definitions and properties of hyperkähler manifolds).

For known examples of hyperkähler manifolds, associated with a K3 surface or an abelian surface, it is not hard to find a deformation which admits a large automorphism group. Indeed, we can lift an automorphism of a K3 or a torus, or use some other explicit construction. However, the classification problem for hyperkähler manifolds still looks out of reach, and finding deformations with interesting automorphism groups without referring to the explicit geometry is much less obvious.

Even more complicated problem is to find n to 1 rational correspondences ("rational isogenies") from a manifold to itself or to some other hyperkähler manifold. Such constructions are of considerable importance, but the visible ways to approach this problem look rather difficult at the moment.

What makes possible the study of automorphisms, rather than isogenies, is that the group of automorphisms of a hyperkähler manifold can be understood in terms of its period lattice (that is, the Hodge structure on the second cohomology and the BBF form, see Subsection 2.1) and the Kähler cone. The later is described in terms of certain cohomology classes called **MBM classes** (Definition 2.8, Definition 2.9), which are, roughly speaking, cohomology classes of negative BBF-square whose duals are represented by minimal rational curves on a deformation of M.

This description is most easy to explain for a K3 surface. In this case, MBM classes are integral classes of self-intersection -2, commonly called (-2)**-classes**.

Let M be a projective K3 surface, and $\operatorname{Pos}(M) \subset H^{1,1}(M)$ the positive cone, that is, the one of two connected components of the set $\{v \in H^{1,1}(M,\mathbb{R}) \mid (v,v) > 0\}$ which contains the Kähler classes. Denote by R the set of all (-2)-classes on M, and let $R^\perp$ be the union of all orthogonal hyperplanes to all $v \in R$. Then the Kähler cone $\operatorname{Kah}(M)$ is one of the connected components of $\operatorname{Pos}(M)\backslash R^\perp$, and

$$\operatorname{Aut}(M) = \{g \in SO^+(H^2(M,\mathbb{Z})) \mid g(\operatorname{Kah}(M)) = \operatorname{Kah}(M)\}.$$

This gives an explicit description of the automorphism group, which becomes quite simple when $\operatorname{Kah}(M) = \operatorname{Pos}(M)$, and this happens when M has no (-2)-classes of Hodge type $(1,1)$. When $\operatorname{Kah}(M) = \operatorname{Pos}(M)$, the group

$\operatorname{Aut}(M)$ is identified with the subgroup $\Gamma_M \subset SO^+(H^2(M,\mathbb{Z}))$,

$$\Gamma_M = \{g \in SO^+(H^2(M,\mathbb{Z})) \ \mid \ g(H^{1,1}(M)) \subset g(H^{1,1}(M))$$

("the group of Hodge isometries of $H^2(M,\mathbb{Z})$"). It is not hard to see that Γ_M is mapped onto a finite index subgroup of the group of isometries of the Picard lattice $\operatorname{Pic}(M) = H^{1,1}(M,\mathbb{Z})$, hence it is infinite whenever this lattice has infinite automorphism group. Now $\operatorname{Pic}(M)$ has signature $(1,k)$ by Hodge index theorem. It is well-known (see e.g. [Di]) that Γ_M is infinite when $k > 1$ and also when $k = 1$ and the Picard lattice does not represent zero (that is, there is no nonzero $v \in \operatorname{Pic}(M)$ with $v^2 = 0$).

Therefore, to produce K3 surfaces with infinite automorphism group, it would suffice to find a primitive sublattice of rank $\geqslant 3$, signature $(1,k)$, $k \geqslant 2$ and without (-2)-vectors in $H^2(M,\mathbb{Z})$. This can be done using the work of V. Nikulin, [N], which implies that any lattice of signature $(1,k)$, $k < 10$, admits a primitive embedding to the K3 lattice (that is, an even unimodular lattice of signature $(3,19)$; such a lattice is unique up to isomorphism and isomorphic to $H^2(M,\mathbb{Z})$).

The argument above produces **symplectic automorphisms**, that is, automorphisms which preserve the holomorphic symplectic structure.

This approach is generalized in the present paper. In [AV3] it is shown that for each hyperkähler manifold M there exists $N > 0$, depending only on the deformation class of M, such that for all MBM classes v one has $-N < q(v,v) < 0$. In the present paper, we prove that the lattice $H^2(M,\mathbb{Z})$ of a hyperkähler manifold M satisfying $b_2(M) \geqslant 5$ (this is believed to hold always, but no proof exists today) contains a primitive sublattice $\Lambda \subset H^2(M,\mathbb{Z})$ which does not represent numbers smaller than N (that is, for any nonzero $v \in \Lambda$, one has $|q(v,v)| \geqslant N$). . Using the global Torelli theorem, we find a deformation M_1 of M with $\operatorname{Pic}(M_1) = \Lambda$. In this case, the Picard lattice of M_1 contains no MBM classes, the Kähler cone coincides with the positive cone, and the symplectic automorphism group is mapped onto a finite index subgroup of the isometry group $O(\Lambda)$ (Corollary 2.12).

This allows us to prove the following theorem.

Theorem 1.1: Let M be a hyperkähler manifold with $b_2(M) \geqslant 5$. Then M admits a projective deformation M' with infinite group of symplectic automorphisms and Picard rank 2.

Proof: Corollary 3.7 ■

The automorphisms obtained in Theorem 1.1 are **hyperbolic**: they act on $H^{1,1}(M)$ with one real eigenvalue $\alpha > 1$, another α^{-1}, and the rest of eigenvalues have absolute value 1. In fact, the symplectic automorphisms of hyperkähler manifolds can be classified in the same way as automorphism of the hyperbolic plane (see Theorem 3.4). There are hyperbolic, or, more precisely, loxodromic automorphisms (ones which act on $H^{1,1}(M)$ with two real eigenvalues of absolute value $\neq 1$), elliptic ones (automorphisms of finite order) and **parabolic** (quasiunipotent with a non-trivial rank 3 Jordan cell).

If we want to produce a parabolic automorphism of a deformation of a given hyperkähler manifold, more work is necessary. We need to find a primitive sublattice $\Lambda \subset H^2(M,\mathbb{Z})$ of signature $(1,k)$, $k \geqslant 2$, such that $q(v,v) \notin]-N,0[$ for $v \in \Lambda$, and Λ admits a parabolic isometry. In order to produce such a sublattice we rely on the classification of rational vector spaces with a quadratic form by the signature, discriminant and the collection of p-adic invariants, and on Nikulin's work on lattice embeddings. Our method works under a stronger restriction on b_2.

The main problem (and the main reason for the strong restriction on b_2) is that the second cohomology lattice $H^2(M,\mathbb{Z})$ of a hyperkähler manifold M is not necessarily unimodular. In this case, one cannot apply Nikulin's theorem directly. To construct the sublattice we need, we first embed our lattice $H^2(M,\mathbb{Z})$ into $H_{\mathbb{Q}} := H \otimes_{\mathbb{Z}} \mathbb{Q}$, where H is a unimodular lattice. Then we apply Nikulin's theorem to H, obtaining a primitive sublattice $\Lambda \subset H$, and take the intersection of Λ with the image of $H^2(M,\mathbb{Z}) \subset H_{\mathbb{Q}}$. This is no longer primitive in $H^2(M,\mathbb{Z})$, but we have a good control over the extent to which it is not, sufficient to assure that the "primitivization" does not have vectors of small nonzero square.

Theorem 1.2: Let M be a hyperkähler manifold with $b_2(M) \geqslant 14$. Then M has a deformation with $\operatorname{rk}\operatorname{Pic}(M) \geqslant 3$, such that its group of symplectic automorphisms contains a parabolic element.

Proof: Corollary 3.12. ■

Remark 1.3: Using the main result of [V2], one sees that under the conditions of each of the two theorems, the points corresponding to hyperkähler manifolds with a hyperbolic resp. parabolic automorphism are dense in the Teichmüller space.

2 Hyperkähler manifolds: basic results

In this section, we recall the definitions and basic properties of hyperkähler manifolds and MBM classes.

2.1 Hyperkähler manifolds

Definition 2.1: A **hyperkähler manifold** M, that is, a compact Kähler holomorphically symplectic manifold, is called **simple**, or **maximal holonomy** hyperkähler manifold (alternatively, **irreducible holomorphically symplectic (IHS)**), if $\pi_1(M) = 0$ and $H^{2,0}(M) = \mathbb{C}$.

This definition is motivated by the following theorem of Bogomolov.

Theorem 2.2: ([Bo1]) Any hyperkähler manifold admits a finite covering which is a product of a torus and several simple hyperkähler manifolds. ■

Remark 2.3: Further on, we shall tacitly assume that the hyperkähler manifolds we consider are of maximal holonomy (simple, IHS).

The second cohomology $H^2(M, \mathbb{Z})$ of a simple hyperkähler manifold M carries a primitive integral quadratic form q, called **the Bogomolov-Beauville-Fujiki form**. It generalizes the intersection product on a K3 surface: its signature is $(3, b_2 - 3)$ on $H^2(M, \mathbb{R})$ and $(1, b_2 - 3)$ on $H^{1,1}_{\mathbb{R}}(M)$. It was first defined in [Bo2] and [Bea], but it is easiest to describe it using the Fujiki theorem, proved in [F] and stressing the topological nature of the form.

Theorem 2.4: (Fujiki) Let M be a simple hyperkähler manifold, $\eta \in H^2(M)$, and $n = \frac{1}{2} \dim M$. Then $\int_M \eta^{2n} = cq(\eta, \eta)^n$, where q is a primitive integral nondegenerate quadratic form on $H^2(M, \mathbb{Z})$, and $c > 0$ is a rational number depending only on M. ■

Consider M as a differentiable manifold and denote by I our complex structure on M (we shall use notations like $\operatorname{Pic}(M, I)$, $H^{1,1}(M, I)$ etc., to stress that we are working with this particular complex structure). We call **the Teichmüller space** Teich the quotient $\mathsf{Comp}(M)/\operatorname{\mathsf{Diff}}_0(M)$, where $\mathsf{Comp}(M)$ denotes the space of all complex structures of Kähler type on M and $\mathsf{Diff}_0(M)$ is the group of isotopies. It follows from a result of Huybrechts (see [H]) that for an IHSM M, Teich has only finitely many connected components. Let Teich_I denote the one containing our given complex structure I. Consider the subgroup of the mapping class group $\mathsf{Diff}(M)/\mathsf{Diff}_0(M)$ fixing Teich_I.

Definition 2.5: The **monodromy group** $\mathsf{Mon}(M)$ is the image of this subgroup in $O(H^2(M,\mathbb{Z}),q)$. The **Hodge monodromy group** $\mathsf{Mon}_I(M)$ is the subgroup of $\mathsf{Mon}(M)$ preserving the Hodge decomposition.

Theorem 2.6: ([V1], Theorem 3.5) The monodromy group is a finite index subgroup in $O(H^2(M,\mathbb{Z}),q)$.

The image of the Hodge monodromy is therefore an arithmetic subgroup of the orthogonal group of the Picard lattice $\operatorname{Pic}(M,I)$. Notice that the action of $\mathsf{Mon}_I(M)$ on the Picard lattice can have a kernel; when (M,I) is projective, it is easy to see that the kernel is a finite group (just use the fact that it fixes a Kähler class and therefore consists of isometries), but it can be infinite in general ([McM]). By a slight abuse of notation, we sometimes also call the Hodge monodromy this arithmetic subgroup itself; one way to avoid such an abuse is to introduce the **symplectic Hodge monodromy group** $\mathsf{Mon}_{I,\Omega}(M)$ which is a subgroup of $\mathsf{Mon}_I(M)$ fixing the symplectic form Ω. Its representation on the Picard lattice is faithful and the image is the same as that of $\mathsf{Mon}_I(M)$, so that $\mathsf{Mon}_{I,\Omega}(M)$ is identified to an arithmetic subgroup of $O(\operatorname{Pic}(M,I),q)$.

Theorem 2.7: (Markman's Hodge-theoretic Torelli theorem, [Ma]) The image of $\operatorname{Aut}(M,I)$ acting on $H^2(M)$ is the subgroup of $\mathsf{Mon}_I(M)$ preserving the Kähler cone $\operatorname{Kah}(M,I)$.

In this paper, we construct hyperkähler manifolds with the Kähler cone equal to the positive cone, and use this construction to find manifolds admitting interesting automorphisms of infinite order.

2.2 MBM classes

We call a cohomology class $\eta \in H^2(M,\mathbb{R})$ **positive** if $q(\eta,\eta) > 0$, and **negative** if $q(\eta,\eta) < 0$. The **positive cone** $\operatorname{Pos}(M,I) \in H^{1,1}_{\mathbb{R}}(M,I)$ is the connected component of the set of positive classes on M which contains the Kähler classes. The Kähler cone is cut out inside the positive cone by a certain, possibly infinite, number of rational hyperplanes (by a result of Huybrechts, we may take for these the orthogonals to the classes of rational curves).

In [AV1], we have introduced the following notion.

Definition 2.8: An integral $(1,1)$-class z on (M,I) is called **monodromy birationally minimal (MBM)**, if for some $\gamma \in \mathsf{Mon}_I(M)$, the hyperplane $\gamma(z)^\perp$ supports a (maximal-dimensional) face of the Kähler cone of a birational model of (M,I).

We have shown in [AV1] the invariance of the MBM property under all deformations of complex structure which leave z of type $(1,1)$. Moreover we have observed that a negative class z generating the Picard group $\operatorname{Pic}(M,I)$ is MBM if and only if a rational multiple of z is represented by a curve (in fact automatically rational; when we speak about curves representing $(1,1)$-classes in cohomology, it means that we identify the integral classes of curves to certain rational $(1,1)$-classes by the obvious isomorphism provided by the BBF form). This leads to a simple extension of the notion to the classes in the whole $H^2(M,\mathbb{Z})$ rather than the Picard lattice. By writing M rather than (M,I), we let a complex structure I vary in its deformation class; this class is not uniquely determined by the topology, but there are finitely many of them by the already-mentioned finiteness result of Huybrechts ([H]).

Definition 2.9: A negative class $z \in H^2(M,\mathbb{Z})$ on a hyperkähler manifold is called **an MBM class** if there exist a deformation of M with $\operatorname{Pic}(M) = \langle z \rangle$ such that λz is represented by a curve, for some $\lambda \neq 0$.

Theorem 2.10: ([AV1], Section 6) Let (M,I) be a hyperkähler manifold, and S the set of all its MBM classes of type $(1,1)$. The Kähler cone of (M,I) is a connected component of $\operatorname{Pos}(M,I) \backslash \cup_{z \in S} z^\perp$.

Remark 2.11: As follows from an observation by Markman, the other connected components ("the Kähler chambers") are the monodromy transforms of the Kähler cones of birational models of (M,I). The Hodge monodromy group permutes the Kähler chambers.

From Theorem 2.10 and Hodge-theoretic Torelli we easily deduce the following

Corollary 2.12: Let (M,I) be a hyperkähler manifold which has no MBM classes of type (1,1). Then any element of $\mathsf{Mon}_I(M)$ lifts to an automorphism of (M,I).

Proof: Indeed, for such manifolds $\operatorname{Kah}(M,I) = \operatorname{Pos}(M,I)$ and therefore the whole group $\mathsf{Mon}_I(M)$ preserves the Kähler cone. ∎

2.3 Morrison-Kawamata cone conjecture, MBM bound and automorphisms

The following theorem has been proved in [AV2].

Theorem 2.13: ([AV2]) Suppose that (M,I) is projective and the Picard number $\rho(M,I) > 3$. Then the Hodge monodromy group has only finitely many orbits on the set of MBM classes of type $(1,1)$ on M. ■

This result is a version of Morrison-Kawamata cone conjecture for hyperkähler manifolds. Its proof is based on ideas of homogeneous dynamics (Ratner theory, Dani-Margulis, Mozes-Shah theorems).

Since the Hodge monodromy group acts by isometries, it follows that the primitive MBM classes in $H^{1,1}(M)$ have bounded square. Using deformations, one actually obtains the boundedness without the projectivity assumption and with the condition $\rho(M,I) > 3$ replaced by $b_2(M) > 5$. One of the main tools of this paper is a subsequent generalization of this statement.

Theorem 2.14: ([AV3]) Let M be a hyperkähler manifold with $b_2 \geqslant 5$. Then there exists a number $N > 0$, called **the MBM bound**, such that any MBM class z satisfies $|q(z,z)| < N$.

Let us explain how this theorem permits one to construct hyperkähler manifolds with large automorphism groups.

Definition 2.15: A **lattice**, or a **quadratic lattice**, is a free abelian group $\Lambda \cong \mathbb{Z}^n$ equipped with an integer-valued quadratic form q. When we speak of an embedding of lattices, we always assume that it is compatible with their quadratic forms.

Definition 2.16: A sublattice $\Lambda' \subset \Lambda$ is called **primitive** if Λ/Λ' is torsion-free. A number a is **represented** by a lattice (Λ, q) if $a = q(x,x)$ for some nonzero $x \in \Lambda$.

Now let M be a hyperkähler manifold. Consider the lattice $H^2(M,\mathbb{Z})$ equipped with the BBF form q. By Torelli theorem, for any primitive sublattice $\Lambda \subset H^2(M,\mathbb{Z})$ of signature $(1,k)$, there exists a complex structure I such that $\Lambda = \operatorname{Pic}(M,I)$ is the Picard lattice of (M,I). The key remark is that as soon as we succeed in finding such a primitive sublattice which does not represent small nonzero numbers, the corresponding hyperkähler manifold has fairly large automorphism group.

Theorem 2.17: Let M be a hyperkähler manifold, and $\Lambda \subset H^2(M,\mathbb{Z})$ a primitive sublattice of signature $(1,k)$ which does not represent any number $a, 0 \leqslant |a| \leqslant N$, where N is the MBM bound (we sometimes say in this case that Λ "satisfies the MBM bound"). Let (M,I) be a deformation of M such that $\Lambda = \operatorname{Pic}(M,I)$. Then the Kähler cone of (M,I) is equal to the positive cone and the group of holomorphic symplectic automorphisms $\operatorname{Aut}(M,\Omega)$ is projected with finite kernel to $\mathsf{Mon}_{I,\Omega}(M)$, which is a subgroup of finite index in $O(\Lambda)$.

Proof: For the finiteness of the kernel of the natural map from $\operatorname{Aut}(M,\Omega)$ to $\mathsf{Mon}_{I,\Omega}(M) \subset GL(H^2(M))$ see e. g. [V1]. Since $\Lambda = H^{1,1}_I(M,\mathbb{Z})$ satisfies the MBM bound, it contains no MBM classes, so the Kähler cone is equal to the positive cone. By Corollary 2.12, $\operatorname{Aut}(M,\Omega)$ maps onto $\mathsf{Mon}_{I,\Omega}(M)$. Now, $\mathsf{Mon}_{I,\Omega}(M)$ is a finite index subgroup in $O(\Lambda)$, as follows from Theorem 2.6. ■

3 Sublattices and automorphisms

3.1 Classification of automorphisms of a hyperbolic space

Remark 3.1: The group $O(m,n), m,n > 0$ has 4 connected components. We denote the connected component of 1 by $SO^+(m,n)$. We call a vector v **positive** if its square is positive.

Definition 3.2: Let V be a vector space with a quadratic form q of signature $(1,n)$, $\operatorname{Pos}(V) = \{x \in V \ \ | \ \ q(x,x) > 0\}$ its **positive cone**, and $\mathbb{P}^+V$ the projectivization of $\operatorname{Pos}(V)$. Denote by g any $SO(V)$-invariant Riemannian structure on $\mathbb{P}^+V$. Then $(\mathbb{P}^+V, g)$ is called **hyperbolic space**, and the group $SO^+(V)$ **the group of oriented hyperbolic isometries**.

Remark 3.3: Since the isotropy group (the stabilizer of a point $x \in \mathbb{P}^+V$ for $SO^+(V)$-action on $\mathbb{P}^+V$) is $SO(n)$ acting on $T_x\mathbb{P}^+V = x^\perp$, the hyperbolic metric on $\mathbb{P}^+V$ is unique up to a constant.

Theorem 3.4: (Classification of isometries of $\mathbb{P}^+V$)
Let $n > 0$, and $\alpha \in SO^+(1,n)$ is an isometry acting on V. Then one and

only one of these three cases occurs

(i) α has an eigenvector x with $q(x,x) > 0$ (α is "an elliptic isometry")

(ii) α has an eigenvector x with $q(x,x) = 0$ and real eigenvalue λ_x satisfying $|\lambda_x| > 1$ (α is "hyperbolic isometry").

(iii) α has a unique eigenvector x with $q(x,x) = 0$ and eigenvalue 1, and no fixed points on $\mathbb{P}^+V$ (α is "parabolic isometry").

Proof: This is a standard textbook result; see, for instance, [Ka]. ∎

Definition 3.5: Recall that the BBF form has signature $(1, b_2 - 3)$ on $H^{1,1}(M)$. An automorphism of a hyperkähler manifold (M, I) is called **elliptic (parabolic, hyperbolic)** if it is elliptic (parabolic, hyperbolic) on $H^{1,1}_I(M, \mathbb{R})$.

3.2 Rank-two sublattices and existence of hyperbolic automorphisms

In this Section, we prove the following theorem.

Theorem 3.6: Let L be a non-degenerate indefinite lattice of rank $\geqslant 5$, and N a natural number. Then L contains a primitive rank 2 sublattice Λ of signature $(1,1)$ which does not represent numbers of absolute value less than N.

This theorem immediately gives examples of hyperkähler manifolds with hyperbolic automorphisms.

Corollary 3.7: Let M be a hyperkähler manifold with $b_2(M) \geqslant 5$. Then M has a deformation admitting a hyperbolic automorphism.

Proof: Consider the lattice $L = H^2(M, \mathbb{Z})$ and let N be the MBM bound for deformations of M. Take a sublattice Λ as in Theorem 3.6 and a deformation of M such that $\Lambda = H^{1,1}_I(M, \mathbb{Z})$. Up to a finite index (meaning that the natural maps between these groups have finite kernel and image of finite index), $\operatorname{Aut}(M) = \mathsf{Mon}_I(M) = O(\Lambda)$. But Λ does not represent zero, and then it is well-known that $O(\Lambda)$ has a hyperbolic element (one way to view this is to interpret Λ, up to a finite index, as a ring of integers in a real quadratic extension of $\mathbb{Q}$, and notice that the units provide automorphisms; so there is an automorphism of infinite order, and it must automatically be hyperbolic as it cannot be parabolic or elliptic). ∎

To prove Theorem 3.6, we need the following proposition.

Proposition 3.8: Let Λ be a diagonal rank 2 lattice with diagonal entries α_1, α_2 divisible by an odd power of p, $\alpha_i = \beta_i p^{2n_i+1}$, and such that the numbers β_i are not divisible by p and the equation $\beta_1 x^2 + \beta_2 y^2 = 0$ has no solutions modulo p. Let $v \in \Lambda \otimes \mathbb{Q}$ be such that the value of the quadratic form on v is an integer. Then this integer is divisible by p.

Proof: A direct computation, which is especially straightforward when one works in $\mathbb{Q}_p$ instead of $\mathbb{Q}$. ∎

Proof of Theorem 3.6

By Meyer's Theorem [Me], L has an isotropic vector (that is, a vector v with $q(v) = 0$). The **isotropic quadric** $\{v \in L \mid q(v) = 0\}$ has infinitely many points if it has one, and not all of them are proportional. Take two of such non-proportional points v and v', and let $v_1 := av + bv'$. Then $q(v_1) = 2abq(v, v')$. We may chose $2ab$ to be of any sign and such that it has arbitrary large prime divisors in odd powers. Concretely, consider the lattice $M = \langle v, v' \rangle^\perp$ of signature $(r-1, s-1)$ (here (r, s) denotes the signature of L). It is always possible to find a vector $w \in \langle v, v' \rangle^\perp$ such that $q(w)$ is divisible by an odd power of a suitable sufficiently large prime number p, but not by an even one (for instance consider a rank-two sublattice where the form q is equivalent to $x^2 - dy^2$ over $\mathbb{Q}$ and pick a large p such that d is not a square modulo p; then one can choose a suitable w in such a sublattice). Now choose the multipliers a, b in such a way that the lattice $\Lambda := \langle v_1, w \rangle$ satisfies assumptions of Proposition 3.8 with this p and has signature $(1, 1)$. ∎

3.3 Sublattices of large rank and existence of parabolic automorphisms

The purpose of this section is to construct, in $H^2(M, \mathbb{Z})$, primitive sublattices of larger rank not representing small numbers (except possibly zero). We use Nikulin's theorem on primitive embeddings into unimodular lattices. As $H^2(M, \mathbb{Z})$ is not always unimodular, we have to provide a trick which allows a reduction to the unimodular case. The trick consists in remarking that we can embed $H^2(M, \mathbb{Z})$ into a lattice of rank $b_2(M) + 3$ which over $\mathbb{Q}$ is equivalent to a "standard" lattice $L_{st} = \sum \pm z_i^2$ (Theorem 3.9). Then we take a suitable Λ not representing numbers of absolute value less than N (except possibly zero) and embed it into L_{st} using Nikulin's results from [N]. The intersection $\Lambda \cap H^2(M, \mathbb{Z})$ is not necessarily primitive in $H^2(M, \mathbb{Z})$, but we can control the extent to which it is not primitive in terms depending only on the embedding of $H^2(M, \mathbb{Z})$ into L_{st}, and not on N; so, increasing N if necessary, we eventually get a primitive sublattice satisfying the MBM bound.

For a lattice Λ, we sometimes denote $\Lambda \otimes_{\mathbb{Z}} \mathbb{Q}$ by $\Lambda_{\mathbb{Q}}$. Recall that the **Hilbert symbol** $(a, b)_p$ of two p-adic numbers is equal to 1 if the equation $ax^2 + by^2 = z^2$ has nonzero solutions in $\mathbb{Q}_p$ and -1 otherwise. If a and b are nonzero rational numbers, one has $(a, b)_p = 1$ for all p except finitely many, and $\prod_p (a, b)_p = 1$ ([Se] chapter III, theorem 3).

Theorem 3.9: For any non-degenerate lattice (H, q) there is an embedding of rational vector spaces with a quadratic form $(H \otimes_{\mathbb{Z}} \mathbb{Q}, q) \subset (L_{\mathbb{Q}}, q_{st})$, where $q_{st} = \sum \pm z_i^2$, the rank of $L_{\mathbb{Q}}$ is equal to $rk(H) + 3$, and the signature of $L_{\mathbb{Q}}$ can be taken arbitrary among the possible ones $(r+3, s)$; $(r+2, s+1)$; $(r+1, s+2)$; $(r, s+3)$, where (r, s) is the signature of H.

Proof: The form q diagonalizes over the rationals; let $a_1, \ldots, a_n$, $n = r + s$, be its diagonal entries and $d = a_1 \ldots a_n$. It is well-known that a rational quadratic form is determined by its signature, its discriminant d as an element of $\mathbb{Q}^*/(\mathbb{Q}^*)^2$ and its collection of **p-adic signatures** $\varepsilon_p(q) = \Pi_{i<j}(a_i, a_j)_p$ for all primes p, where $(a_i, a_j)_p$ is the Hilbert symbol ([Se], Chapter IV.2, Theorem 9, Theorem 7). Let t be the number of desired negative diagonal entries for $(L_{\mathbb{Q}}, q_{st})$. We wish to add three dimensions to H to make the quadratic form equivalent to q_{st}, that is, we are looking for three extra diagonal entries b_0, b_1, $b_2 = (-1)^t d b_0 b_1$ such that for all primes p,

$$\varepsilon_p(q_{st}) = \varepsilon_p(q)(d, b_0 b_1 b_2)_p (b_0, b_1 b_2)_p (b_1, b_2)_p.$$

Using obvious identities like $(x, -x)_p = 1$ and $(x, y^2 z)_p = (x, z)_p$, we see that this amounts to asking that the quantities $(b_0, (-1)^{t-1} d)_p (b_1, (-1)^{t-1} d b_0)_p$ have prescribed signs for all p, and finally that $((-1)^{t-1} d b_0, (-1)^{t-1} d b_1)_p$ have prescribed signs for every p. The prescribed signs must satisfy the usual identities for Hilbert symbols: only finitely many are equal to -1 and the product is equal to one.

In other words, to prove the theorem, it is sufficient to produce two rational numbers x, y with arbitrarily defined signs such that for any p, $(x, y)_p$ are equal to prescribed δ_p (satisfying the usual identities). To do this, we use the following claim:

Claim 3.10: ([Se], Chapter III, Theorem 4) Let $x \in \mathbb{Q}$. To find an $y \in \mathbb{Q}$ with $(x, y)_p = \delta_p$ for all p, it suffices to find for each p, an $y_p \in \mathbb{Q}_p$ with $(x, y_p) = \delta_p$.

Now take x which is not a square modulo all p such that $\delta_p = -1$, then it is easy to find y_p using the explicit formulae for the Hilbert symbol ([Se], Chapter III, Theorem 1). ■

Let now H be $H^2(M, \mathbb{Z})$ of signature $(3, b_2 - 3)$, $L_{\mathbb{Q}}$ as in Theorem 3.9, say, of signature $(3, b_2)$, and let L be the standard integral lattice in $L_{\mathbb{Q}}$. This is an odd unimodular lattice. By the work of Nikulin, a lattice Λ of signature $(1, s)$ and the discriminant group $\Lambda^\vee/\Lambda$ without 2-torsion has a primitive embedding into L as soon as $s \leqslant b_2$ and $2(1+s) = 2\operatorname{rk}\Lambda < \operatorname{rk} L = b_2 + 3$ (in fact Nikulin only gives an embedding result for even lattices, but the crucial place in the argument is the construction of a lattice Λ' with appropriate signature and the discriminant form opposite to that of Λ; then the direct sum $\Lambda \oplus \Lambda'$ has a unimodular overlattice corresponding to a maximal isotropic subgroup in the direct sum of the discriminant groups. This construction is done by Nikulin also in the odd case ([N], section 1.16), and yields a unimodular overlattice of $\Lambda \oplus \Lambda'$ exactly in the same way).

In particular we can primitively embed into L a lattice Λ of signature $(1, [b_2/2])$, without 2-torsion in the discriminant and not representing numbers of small absolute value other than zero (this is achieved for instance by multiplying a unimodular lattice by a large prime). We are actually looking for a primitive sublattice in H, whereas the intersection $\Lambda \cap H$, of signature $(1, [b_2/2] - 3)$, is not necessarily primitive in H. But its non-primitivity can be controlled in terms of the embedding of H into $L_{\mathbb{Q}}$ and thus does not depend on Λ. In this way we obtain the following

Theorem 3.11: Let M be a hyperkähler manifold and N a natural number. Then $H := H^2(M, \mathbb{Z})$ contains a primitive sublattice of signature $(1, [b_2/2] - 3)$ which does not represent non-zero numbers of absolute value smaller than N. In particular, there is a deformation of M of Picard rank $[b_2/2] - 2$ which does not have any MBM classes.

Proof: Consider the embedding from H into $L_{\mathbb{Q}}$ of dimension b_2+3 and signature $(3, b_2)$ as in Theorem 3.9. Set $d = |H/H \cap L|$. Then for any primitive sublattice Λ of L, the group $(H \cap \Lambda_{\mathbb{Q}})/(H \cap L \cap \Lambda_{\mathbb{Q}}) = (H \cap \Lambda)_{\mathbb{Q}})/(H \cap \Lambda)$ embeds into $H/H \cap L$ and so has cardinality at most d. By Nikulin's results, if we take a lattice Λ of signature $(1, [b_2/2])$, without two-torsion in the discriminant and not representing any non-zero number of absolute value less than d^2N, it admits a primitive embedding to L. Then the "primitivization" of $\Lambda \cap H$ (that is, $H \cap \Lambda_{\mathbb{Q}} \subset H$), is a lattice of signature $(1, [b_2/2]-3)$ not representing nonzero numbers of absolute value less than N. Taking as N the MBM bound for our manifold M, we obtain deformations with relatively large Picard number and no MBM classes. ■

Corollary 3.12: Let M be a hyperkähler manifold with $b_2(M) \geqslant 14$. Then M has a deformation admitting a parabolic automorphism.

Proof. Step 1: Let $\Lambda = H^{1,1}_I(M, \mathbb{Z})$ be a primitive lattice of corank 2 and signature $(1, n)$ in $H^2(M, \mathbb{Z})$ satisfying the MBM bound. Then $\operatorname{Aut}(M, \Omega)$ has finite index in $O(\Lambda)$. It suffices to show that the Lie group $O(\Lambda \otimes_{\mathbb{Z}} \mathbb{R})$ contains a rational unipotent subgroup U. Then $U \cap \operatorname{Aut}(M, \Omega)$ is Zariski dense in U by another application of Borel and Harish-Chandra, and all its elements are parabolic.

Step 2: Suppose that there exists a rational vector v with $q(v, v) = 0$, and let $P \subset O(\Lambda \otimes_{\mathbb{Z}} \mathbb{R})$ be the stabilizer of v. This subgroup is clearly rational and parabolic; its unipotent radical is the group U which we require.

Step 3: Such a rational vector exists for any indefinite lattice of rank $\geqslant 5$ by Meyer's theorem [Me], therefore as soon as $[b_2(M)/2] - 2 \geqslant 5$, Λ has parabolic elements in its orthogonal group. ■

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