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# Teichmüller space for hyperkähler and symplectic structures

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ABSTRACT

Let $S$ be an infinite-dimensional manifold of all symplectic, or hyperkähler, structures on a compact manifold $M$, and $\mathsf{Diff}_0$ the connected component of its diffeomorphism group. The quotient $S/\mathsf{Diff}_0$ is called the Teichmüller space of symplectic (or hyperkähler) structures on $M$. MBM classes on a hyperkähler manifold $M$ are cohomology classes which can be represented by a minimal rational curve on a deformation of $M$. We determine the Teichmüller space of hyperkähler structures on a hyperkähler manifold, identifying any of its connected components with an open subset of the Grassmannian variety $SO(b_2-3,3)/SO(3)\times SO(b_2-3)$ consisting of all Beauville–Bogomolov positive 3-planes in $H^2(M,\mathbb{R})$ which are not orthogonal to any of the MBM classes. This is used to determine the Teichmüller space of symplectic structures of Kähler type on a hyperkähler manifold of maximal holonomy. We show that any connected component of this space is naturally identified with the space of cohomology classes $v\in H^2(M,\mathbb{R})$ with $q(v,v)>0$, where $q$ is the Bogomolov–Beauville–Fujiki form on $H^2(M,\mathbb{R})$.



## 1. Introduction

### 1.1. Statement of the main result

Denote by $\Gamma(\Lambda^2M)$ the space of all 2-forms on a manifold $M$, and let $\mathsf{Symp}\subset\Gamma(\Lambda^2M)$ be the space of all symplectic 2-forms. We equip $\Gamma(\Lambda^2M)$ with $C^\infty$-topology of uniform convergence on compacts with all derivatives. Then $\Gamma(\Lambda^2M)$ is a Fréchet vector space, and $\mathsf{Symp}$ a Fréchet manifold. We consider the group of diffeomorphisms, denoted $\mathsf{Diff}$ or $\mathsf{Diff}(M)$, as a Fréchet Lie group, and denote its connected component (known sometimes as the group of isotopies) by $\mathsf{Diff}_0$. The quotient group $\Gamma:=\mathsf{Diff}/\mathsf{Diff}_0$ is called **the mapping class group** of $M$.

Teichmüller space of symplectic structures on $M$ is defined as a quotient $\mathsf{Teich}_s:=\mathsf{Symp}/\mathsf{Diff}_0$. It was studied in [1] and [2] together with its quotient $\mathsf{Teich}_s/\Gamma=\mathsf{Symp}/\mathsf{Diff}$, known as **the moduli space** of symplectic structures.

Notice that by Moore's theorem the action of $\Gamma$ on the space $\mathsf{Teich}_s(V)$ of symplectic forms with fixed volume $V$ in $\mathsf{Teich}_s$ is ergodic e.g. for a compact torus of dimension $>2$ or a hyperkähler manifold.[1] Therefore, $\Gamma$ acts on $\mathsf{Teich}_s(V)$ with dense orbits, and the quotient "space" has a topology not much different from the codiscrete one; in particular, it is not "a manifold" even in the most general sense of this word. However, as shown in [4] (see also [2], Proposition 3.1), the space $\mathsf{Teich}_s$ is

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[1] See [3], Theorem 9.2, where the argument is given in all detail.

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a manifold, possibly non-Hausdorff, and the symplectic period map $\mathsf{Per}_s : \mathsf{Teich}_s \longrightarrow H^2(M, \mathbb{R})$, associating to $\omega \in \mathsf{Teich}_s$ its cohomology class, is locally a diffeomorphism.

In the present paper we study the Teichmüller space $\mathsf{Teich}_k$ of all symplectic structures of Kähler type on a hyperkähler manifold. Our main result is the following theorem.

**Theorem 1.1.** *Let $M$ be a hyperkähler manifold of maximal holonomy, and $\mathsf{Teich}_k$ the space of all symplectic forms admitting a compatible hyperkähler structure (this is equivalent to being Kähler in a certain complex structure, see Theorem 3.2). Then the symplectic period map $\mathsf{Per}_s : \mathsf{Teich}_k \longrightarrow H^2(M, \mathbb{R})$ is an open embedding on each connected component, and its image is determined by quadratic inequality*

$$\operatorname{im} \mathsf{Per}_s = \{v \in H^2(M, \mathbb{R}) \mid q(v, v) > 0\},$$

*where $q$ is the BBF form 2.1.*

**Proof.** See Theorem 5.1. ■

The idea is to relate $\mathsf{Teich}_k$ to the Teichmüller space of hyperkähler structures (Definition 4.5) and to remark that the latter has an explicit description in terms of **MBM classes** defined in [5].

### *1.2. The space of symplectic structures*

The study of the space of symplectic structures was initiated by Moser in [4]. Moser proved the following beautiful theorem, which lies in foundation of symplectic topology.

**Theorem 1.2.** *Let $\omega_t$ be a continuous family of symplectic structures on a compact manifold $M$. Assume that the cohomology class of $\omega_t$ is constant (that is, independent on $t$). Then all $\omega_t$ are related by diffeomorphisms.* ■

It is not hard to see that this theorem implies that the period map from the symplectic Teichmüller space to cohomology is a local diffeomorphism ([2], Proposition 3.1). However, further study of this space is very complicated, and in dimension $> 4$ almost nothing is known.

For a state of the art survey of the moduli of symplectic structures, please see [6]. For a particular interest to us is [6, Example 3.3], due to D. McDuff [7]. It shows that the Teichmüller space of symplectic structures on $S^2 \times S^2 \times T^2$ is non-Hausdorff.

The utility of our results for the general problem of studying the symplectic structures is somewhat restricted, because we consider only symplectic structures of Kähler type. It was conjectured, however, that all symplectic structures on K3 are of Kähler type (see e.g. [8]), hence this restriction could theoretically be lifted. However, this conjecture seems to be very hard. Theorem 1.1 implies the following weaker form of this conjecture.

**Corollary 1.3.** *Let $M$ be a hyperkähler manifold, and $\mathsf{Teich}_s$ a connected component of the Teichmüller space of symplectic structures containing a symplectic structure of hyperkähler type. Assume that $\omega \in \mathsf{Teich}_s$ is a symplectic structure which is not of hyperkähler type. Then $\omega$ is a non-Hausdorff point in $\mathsf{Teich}_s$, non-separable from a point of hyperkähler type.*

**Proof.** Let $\mathsf{Teich}_k \subset \mathsf{Teich}_s$ be the set of all points of Kähler type in $\mathsf{Teich}_s$. The map $\mathsf{Per}_s : \mathsf{Teich}_s \longrightarrow H^2(M, \mathbb{R})$ surjects to a connected component of all cohomology classes $\eta \in H^2(M)$ satisfying $\eta^{2n} \neq 0$, as follows from Theorem 2.4 and our main result. After gluing all non-separable points, the period map becomes an isomorphism, since it is étale by Moser theorem, and is an isomorphism on $\mathsf{Teich}_k$. This implies that all points of $\mathsf{Teich}_s \setminus \mathsf{Teich}_k$ are non-separable from points in $\mathsf{Teich}_k$. ■

However, this does not imply that $\mathsf{Teich}_s = \mathsf{Teich}_k$, because the symplectic Teichmüller space can be non-Hausdorff, as follows from [6, Example 3.3].

**Question 1.4.** *Let $\mathsf{Teich}_s$ be a connected component of the Teichmüller space of symplectic structures containing a symplectic structure of hyperkähler type. Is it true that $\mathsf{Teich}_s$ is Hausdorff?*

Notice that the Teichmüller space of complex structures on a hyperkähler manifold is non-Hausdorff in all known examples.

## 2. Hyperkähler manifolds: basic results

In this section, we recall the definitions and basic properties of hyperkähler manifolds and MBM classes.

### *2.1. Hyperkähler manifolds*

**Definition 2.1.** A **hyperkähler manifold** is a compact, Kähler, holomorphically symplectic manifold.

**Definition 2.2.** A hyperkähler manifold $M$ is called **simple**, or **maximal holonomy**, if $\pi_1(M) = 0$ and $H^{2,0}(M) = \mathbb{C}$.

This definition is motivated by the following theorem of Bogomolov.

**Theorem 2.3** ([9]). *Any hyperkähler manifold admits a finite covering which is a product of a torus and several simple hyperkähler manifolds.* ■

The second cohomology $H^2(M, \mathbb{Z})$ of a simple hyperkähler manifold $M$ carries an integral quadratic form $q$, called **the Bogomolov–Beauville–Fujiki form**. It was first defined in [10] and [11], but it is easiest to describe it using the Fujiki theorem, proved in [12].

**Theorem 2.4** (*Fujiki*). *Let $M$ be a simple hyperkähler manifold, $\eta \in H^2(M)$, and $n = \frac{1}{2}\dim M$. Then $\int_M \eta^{2n} = cq(\eta, \eta)^n$, where $c > 0$ is an integer.* ■

**Remark 2.5.** Fujiki formula (Theorem 2.4) determines the form $q$ uniquely up to a sign. For odd $n$, the sign is unambiguously determined as well. In general, there is an explicit formula due to Bogomolov and Beauville.

We shall not need it in the sequel; one should however keep in mind that $q$ is of signature $(3, b_2 - 3)$ on $H^2(M, \mathbb{R})$ and of signature $(1, b_2 - 3)$ on $H^{1,1}(M, \mathbb{R})$; a Kähler class on $M$ together with the real and imaginary parts of $\Omega$ generate a positive 3-space for $q$. This is going to be very much used.

### 2.2. MBM classes

**Definition 2.6.** A cohomology class $\eta \in H^2(M, \mathbb{R})$ is called **positive** if $q(\eta, \eta) > 0$, and **negative**, if $q(\eta, \eta) < 0$.

**Definition 2.7** ([13,14]). Let $M$ be a hyperkähler manifold. The **monodromy group** of $M$ is a subgroup of $O(H^2(M, \mathbb{Z}), q)$ generated by monodromy transforms for all Gauss–Manin local systems. This group can also be characterized in terms of the mapping class group action (Definition 3.14).

The Beauville–Bogomolov–Fujiki form allows one to identify $H^2(M, \mathbb{Q})$ and $H_2(M, \mathbb{Q})$. More precisely, it provides an embedding $H^2(M, \mathbb{Z}) \to H_2(M, \mathbb{Z})$ which is not an isomorphism (indeed $q$ is not necessarily unimodular) but becomes an isomorphism after tensoring with $\mathbb{Q}$. We thus can talk of classes of curves in $H^{1,1}(M, \mathbb{Q})$, meaning that the corresponding classes in $H_{1,1}(M, \mathbb{Q})$ are classes of curves and shall do this in what follows.

Recall that **a face** of a convex cone in a vector space $V$ is the intersection of its boundary and a hyperplane which has non-empty interior in the hyperplane.

If $f : (M, I) - - \to (M, I')$ is a birational isomorphism between hyperkähler manifolds, it is well-known to be an isomorphism in codimension one. Therefore the induced map $f^* : H^2(M, I) \to H^2(M, I')$ is an isomorphism. By the Kähler cone of a birational model of $(M, I)$ as a part of $H^2(M, I)$ we mean the inverse image by $f$ of the Kähler cone of such an $(M, I')$.

**Definition 2.8.** A non-zero negative integral cohomology class $z \in H^{1,1}(M, I)$ is called **monodromy birationally minimal** (MBM) if for some isometry $\gamma \in O(H^2(M, \mathbb{Z}))$ belonging to the monodromy group, $\gamma(z)^\perp \subset H^{1,1}(M, I)$ contains a face of the Kähler cone of one of birational models $(M, I')$ of $(M, I)$.

The following theorems summarize the main results about MBM classes from [5].

**Theorem 2.9** ([5], Corollary 5.13). *An MBM class $z \in H^{1,1}(M, I)$ is also MBM for any deformation of the complex structure $(M, I')$ where $z$ remains of type $(1, 1)$.*

**Theorem 2.10** ([5], Theorem 6.2). *The Kähler cone of $(M, I)$ is a connected component of $\operatorname{Pos}(M, I) \setminus \cup_{z \in S} z^\perp$, where $\operatorname{Pos}(M, I)$ is the positive cone of $(M, I)$ and $S$ is the set of MBM classes on $(M, I)$.*

Because of the deformation-invariance property of MBM classes, it is natural to fix a connected component $\mathsf{Comp}_0$ of the space of complex structures of hyperkähler type on $M$ and to call $z \in H^2(M, \mathbb{Z})$ an MBM class (relative to $\mathsf{Comp}_0$) when it is MBM in those complex structures where it is of type $(1, 1)$. One moreover has the following description of such classes.

**Corollary 2.11.** *A negative homology class $v \in H_2(M, \mathbb{Z})$ is MBM if and only $\lambda v$ can be represented by an irreducible rational curve on $(M, J)$ for some $J \in \mathsf{Comp}_0$ and $\lambda \in \mathbb{R}^{\neq 0}$.*

**Proof.** By deformation theory of hyperkähler manifolds (see next section for some details, in particular Proposition 3.15), there is a deformation $(M, J)$ of $(M, I)$ where only the multiples of $v$ survive as integral $(1, 1)$-classes. By Theorem 5.15 of [5], $v$ is MBM if and only if a multiple of $v$ is represented by a rational curve on $(M, J)$ (for reader's convenience we recall that the reason behind this is that by the results of Huybrechts [15] and Boucksom [16] the faces of the Kähler cone are given as orthogonals to classes of rational curves). ■

## 3. Global Torelli theorem, hyperkähler structures and monodromy group

In this section, we recall a number of results about hyperkähler manifolds, used further on in this paper. For more details and references, please see [17] and [18].

### 3.1. Hyperkähler structures

**Definition 3.1.** Let $(M, g)$ be a Riemannian manifold, and $I, J, K$ endomorphisms of the tangent bundle $TM$ satisfying the quaternionic relations

$$I^2 = J^2 = K^2 = IJK = -\operatorname{Id}_{TM}.$$

The triple $(I, J, K)$ together with the metric $g$ is called **a hyperkähler structure** if $I, J$ and $K$ are integrable and Kähler with respect to $g$.

Consider the Kähler forms $\omega_I, \omega_J, \omega_K$ on $M$:

$$\omega_I(\cdot,\cdot) := g(\cdot, I\cdot), \qquad \omega_J(\cdot,\cdot) := g(\cdot, J\cdot), \qquad \omega_K(\cdot,\cdot) := g(\cdot, K\cdot).$$

An elementary linear-algebraic calculation implies that the 2-form $\Omega := \omega_J + \sqrt{-1}\omega_K$ is of Hodge type $(2, 0)$ on $(M, I)$. This form is clearly closed and non-degenerate, hence it is a holomorphic symplectic form.

In algebraic geometry, the word "hyperkähler" is essentially synonymous with "holomorphically symplectic", due to the following theorem, which is implied by Yau's solution of Calabi conjecture [17,11].

**Theorem 3.2.** *Let $M$ be a compact, Kähler, holomorphically symplectic manifold, $\omega$ its Kähler form,* $\dim_{\mathbb{C}} M = 2n$. *Denote by $\Omega$ the holomorphic symplectic form on $M$. Suppose that $\int_M \omega^{2n} = \int_M (\operatorname{Re}\Omega)^{2n}$. Then there exist a unique hyperkähler metric $g$ with the same Kähler class as $\omega$, and a unique hyperkähler structure $(I, J, K, g)$, with $\omega_J = \operatorname{Re}\Omega$, $\omega_K = \operatorname{Im}\Omega$.* ■

Further on, we shall speak of "hyperkähler manifolds" meaning "holomorphic symplectic manifolds of Kähler type", and "hyperkähler structures" meaning the quaternionic triples together with a metric.

Every hyperkähler structure induces a whole 2-dimensional sphere of complex structures on $M$, as follows. Consider a triple $a, b, c \in R$, $a^2 + b^2 + c^2 = 1$, and let $L := aI + bJ + cK$ be the corresponding quaternion. Quaternionic relations imply immediately that $L^2 = -1$, hence $L$ is an almost complex structure. Since $I, J, K$ are Kähler, they are parallel with respect to the Levi-Civita connection. Therefore, $L$ is also parallel. Any parallel complex structure is integrable, and Kähler. We call such a complex structure $L = aI + bJ + cK$ **a complex structure induced by the hyperkähler structure**. There is a 2-dimensional holomorphic family of induced complex structures, and the total space of this family is called **the twistor space** of a hyperkähler manifold, its base being **the twistor line** in the Teichmüller space $\mathsf{Teich}$ which we are going to define next.

### 3.2. Global Torelli theorem and monodromy

**Definition 3.3.** Let $M$ be a compact complex manifold, and $\mathsf{Diff}_0(M)$ a connected component of its diffeomorphism group (**the group of isotopies**). Denote by $\mathsf{Comp}$ the space of complex structures on $M$, equipped with its structure of a Fréchet manifold, and let $\mathsf{Teich} := \mathsf{Comp}/\mathsf{Diff}_0(M)$. We call it **the Teichmüller space.**

**Remark 3.4.** In many important cases, such as for manifolds with trivial canonical class [19], $\mathsf{Teich}$ is a finite-dimensional complex space; usually it is non-Hausdorff.

**Definition 3.5.** The **mapping class group** is $\Gamma = \mathsf{Diff}(M)/\mathsf{Diff}_0(M)$. It naturally acts on $\mathsf{Teich}$ (the quotient of $\mathsf{Teich}$ by $\Gamma$ may be viewed as the "moduli space" for $M$, but in general it has very bad properties; see below).

**Remark 3.6.** Let $M$ be a hyperkähler manifold (as usually, we assume $M$ to be simple). For any $J \in \mathsf{Teich}$, $(M, J)$ is also a simple hyperkähler manifold, because the Hodge numbers are constant in families. Therefore, $H^{2,0}(M, J)$ is one-dimensional.

**Definition 3.7.** Let

$$\mathsf{Per}: \mathsf{Teich} \longrightarrow \mathbb{P}H^2(M, \mathbb{C})$$

map $J$ to the line $H^{2,0}(M, J) \in \mathbb{P}H^2(M, \mathbb{C})$. The map $\mathsf{Per}$ is called **the period map**.

**Remark 3.8.** The period map $\mathsf{Per}$ maps $\mathsf{Teich}$ into an open subset of a quadric, defined by

$$\mathbb{P}\mathsf{er} := \{l \in \mathbb{P}H^2(M, \mathbb{C}) \mid q(l, l) = 0, q(l, \bar{l}) > 0\}.$$

It is called **the period domain** of $M$.

**Proposition 3.9** (*[18], section 2.4*). *The period domain* $\mathbb{P}\mathrm{er}$ *is identified with the quotient* $SO(b_2 - 3, 3)/SO(2) \times SO(b_2 - 3, 1)$, *which is the Grassmannian of positive oriented* 2*-planes in* $H^2(M, \mathbb{R})$.

**Definition 3.10.** Let $M$ be a topological space. We say that $x, y \in M$ are **non-separable** (denoted by $x \sim y$) if for any open sets $V \ni x, U \ni y, U \cap V \neq \emptyset$.

**Definition 3.11.** The space $\mathrm{Teich}_b := \mathrm{Teich}/\sim$ is called **the birational Teichmüller space** of $M$.

**Remark 3.12.** This terminology is explained by a result of Huybrechts [20], which affirms that non-separable points $I, I' \in \mathrm{Teich}$ correspond to bimeromorphic complex manifolds. One should however keep in mind that these manifolds can even be biregular, so that the Teichmüller space is non-Hausdorff even for K3 surfaces.

**Theorem 3.13** (*Global Torelli theorem; [18]*). *The period map* $\mathrm{Teich}_b \xrightarrow{\mathrm{Per}} \mathbb{P}\mathrm{er}$ *is an isomorphism on each connected component of* $\mathrm{Teich}_b$. ■

As mentioned in the Introduction, the monodromy group (Definition 2.7) can be defined in terms of the Teichmüller space as follows. By a result of Huybrechts [21], $\mathrm{Teich}$ has only finitely many connected components. Let $\mathrm{Teich}_I$ be the component containing the parameter point for the complex structure $I$, and $\Gamma_I$ the subgroup of the mapping class group $\Gamma$ fixing this component.

**Definition 3.14.** The monodromy group of $(M, I)$ is the image of $\Gamma_I$ in the orthogonal group $O(H^2(M, \mathbb{Z}), q)$.

Finally, recall the following well-known fact which shall be used in the sequel.

**Proposition 3.15.** *Let* $z \in H^2(M, \mathbb{Z})$ *be a cohomology class. The part of* $\mathrm{Teich}$ *corresponding to the complex structures where* $z$ *is of type* $(1, 1)$ *is the inverse image of* $z^\perp \subset \mathbb{P}H^2(M, \mathbb{C})$ *(the orthogonal being taken with respect to* $q$*).* ■

## 4. Teichmüller space of hyperkähler structures

**Definition 4.1.** Let $(M, I, J, K, g)$ and $(M, I', J', K', g')$ be two hyperkähler structures. We say that these structures are **equivalent** if the corresponding quaternionic algebras in $\mathrm{End}(TM)$ coincide.

**Proposition 4.2.** *Let* $M$ *be a hyperkähler manifold, and* $(M, I, J, K, g)$ *and* $(M, I', J', K', g')$ *be two hyperkähler structures of maximal holonomy. Then the following conditions are equivalent.*

(i) $g$ *is proportional to* $g'$
(ii) $(M, I, J, K, g)$ *is equivalent to* $(M, I', J', K', g')$.

**Proof of (i) ⇒ (ii):** If $g$ is proportional to $g'$, the corresponding Levi-Civita connections coincide. The corresponding holonomy group is $Sp(n)$, and its stabilizer in $\mathrm{End}(TM)$ is a quaternionic algebra, generated by $I, J, K$ and also by $I', J', K'$.

**Proof of (ii) ⇒ (i):** Conversely, assume that the algebras generated by $I, J, K$ and by $I', J', K'$ coincide. Recall that a manifold is called **hypercomplex** if it is equipped with a triple of complex structures $I, J, K$ satisfying quaternionic relations. By Obata's theorem, there exists a unique torsion-free connection preserving $I, J, K$ on any hypercomplex manifold, [22]; this connection is called **the Obata connection**. Since the Levi-Civita connection on $(M, I, J, K, g)$ satisfies this condition, it coincides with the Obata connection for $(M, I, J, K)$ and for $(M, I', J', K')$ (the latter is true because the corresponding hypercomplex manifolds are equivalent). However, $g$ and $g'$ are invariant with respect to the holonomy of Levi-Civita connection, which is equal to $Sp(n)$. The space of $Sp(n)$-invariant symmetric 2-forms is 1-dimensional [23], hence $g$ and $g'$ are proportional. ■

**Remark 4.3.** It is easy to see that two hyperkähler structures of maximal holonomy $(M, I, J, K, g)$ and $(M, I', J', K', g')$ are equivalent if and only if there exists a unitary quaternion $h$ such that $hIh^{-1} = I', hJh^{-1} = J', hKh^{-1} = K'$, and a positive constant $\lambda$ with $\lambda g = g'$.

Consider the infinite-dimensional space $\mathrm{Hyp}$ of all quaternionic triples $I, J, K$ on $M$ which are induced by some hyperkähler structure, with the same $C^\infty$-topology of convergence with all derivatives. The quotient $\mathrm{Hyp}/SU(2)$ (which is probably better to write as $\mathrm{Hyp}/SO(3)$, since $-1$ acts trivially on the triples) is naturally identified with the set of equivalence classes of hyperkähler structures, up to changing the metric $g$ by a constant.

**Remark 4.4.** By Proposition 4.2, for manifolds with maximal holonomy the quotient $\mathrm{Hyp}_m := \mathrm{Hyp}/SU(2)$ is also identified with the space of all hyperkähler metrics of fixed volume, say, volume 1.

**Definition 4.5.** Define **the Teichmüller space of hyperkähler structures** as the quotient $\mathrm{Hyp}_m/\mathrm{Diff}_0$, where $\mathrm{Diff}_0$ is the connected component of the group of diffeomorphisms $\mathrm{Diff}$, and **the moduli of hyperkähler structures** as $\mathrm{Hyp}_m/\mathrm{Diff}$.

**Remark 4.6.** For most geometric structures, the Teichmüller spaces and especially the moduli spaces are non-Hausdorff. However, for manifolds of maximal holonomy the moduli space of hyperkähler structures is Hausdorff. This is because $\mathsf{Hyp}_m$ is the space of all hyperkähler metrics of fixed volume (Remark 4.4). However, there is a metric on the moduli space of all metrics, known as Gromov–Hausdorff metric [24], and a metric space is necessarily Hausdorff.

**Remark 4.7.** Let $(\omega_I, \omega_J, \omega_K)$ be a triple of classes obtained from a hyperkähler structure in a given component of $\mathsf{Teich}_h$. Then the space $\langle\omega_I, \omega_J, \omega_K\rangle$ is oriented. Indeed, $\langle\omega_J, \omega_K\rangle$ determines a complex structure $I$ by global Torelli theorem, and the sign of $\omega_I$ is determined by a component of $\operatorname{Pos}(M, I)$ containing its Kähler cone.

The main result of this section is the following theorem.

**Definition 4.8.** Let $M$ be a hyperkähler manifold of maximal holonomy, and $\mathsf{Teich}_h := \mathsf{Hyp}_m/\mathsf{Diff}_0$ the Teichmüller space of hyperkähler structures. Consider the space $\mathbb{P}\mathrm{er}_h = Gr_{+++}(H^2(M, \mathbb{R}))$ of all positive oriented 3-dimensional subspaces in $H^2(M, \mathbb{R})$, naturally diffeomorphic to $\mathbb{P}\mathrm{er}_h \cong SO(b_2 - 3, 3)/SO(3) \times SO(b_2 - 3)$. Let $\mathsf{Per}_h : \mathsf{Teich}_h \longrightarrow \mathbb{P}\mathrm{er}_h$ be the map associating to a hyperkähler structure $(M, I, J, K, g)$ the 3-dimensional space generated by the three Kähler forms $\omega_I, \omega_J, \omega_K$. This map called **the period map for the Teichmüller space of hyperkähler structures**, and $\mathbb{P}\mathrm{er}_h$ **the period space of hyperkähler structures**.

**Theorem 4.9.** *Let $M$ be a hyperkähler manifold of maximal holonomy, and $\mathsf{Per}_h : \mathsf{Teich}_h \longrightarrow \mathbb{P}\mathrm{er}_h$ the period map for the Teichmüller space of hyperkähler structures. Then the period map $\mathsf{Per}_h : \mathsf{Teich}_h \longrightarrow \mathbb{P}\mathrm{er}_h$ is an open embedding for each connected component. Moreover, its image is the set of all spaces $W \in \mathbb{P}\mathrm{er}_h$ such that the orthogonal complement $W^\perp$ contains no MBM classes.*

**Proof.** First we describe the image of the period map. Let $W \in \mathbb{P}\mathrm{er}_h = Gr_{+++}H^2(M, \mathbb{R})$ be the positive three-dimensional space corresponding to a point of $\mathbb{P}\mathrm{er}_h$. Consider the set of positive planes in $W$: each plane $V$ in this set is an element of $Gr_{++}H^2(M, \mathbb{R}) = \mathbb{P}\mathrm{er}$, thus a period point of an irreducible holomorphic symplectic manifold $(M, I)$. If $W$ comes from a hyperkähler structure $(I, J, K)$, that is, $W = \langle\omega_I, \omega_J, \omega_K\rangle$, then the orthogonal to $V$ in $W$ is generated by the class of the Kähler form $\omega_I$, $V$ itself being generated by $Re(\Omega)$ and $Im(\Omega)$, where $\Omega$ is the holomorphic symplectic form on $(M, I)$ (see Theorem 3.2). If an integral class $z$ is orthogonal to $W$, it means that it is of type $(1, 1)$ on $(M, I)$ for any $I$ corresponding to $V$ as above, by Proposition 3.15. In particular $z$ is orthogonal to $\omega_I$. Therefore $z$ cannot be MBM since the MBM classes are never orthogonal to Kähler classes.

Conversely, suppose that $W$ is not orthogonal to any MBM class. Then the same is true for a sufficiently general plane $V \subset W$. This plane corresponds to the period point of an irreducible holomorphic symplectic manifold $(M, I)$. Now take $v \in W$ orthogonal to $V$. Up to a sign, this is an element of $\operatorname{Pos}(M, I)$.

We remark that a cohomology class $\eta \in H^2(M)$ is of type $(1, 1)$ with respect to $(M, I)$ if and only if $\eta$ is orthogonal to $V$. Since $W$ is not orthogonal to any MBM class, and $V \subset W$ is generic, no MBM class is of type $(1, 1)$ on $(M, I)$. This means that the Kähler cone of $(M, I)$ is equal to the positive cone, that, is, up to a constant, $v$ is a Kähler class, and thus that there is a hyperkähler metric $g$ such that $v$ is proportional to $\omega_I$ and $W = \langle\omega_I, \omega_J, \omega_K\rangle$ (by Theorem 3.2). Therefore $W$ is in the image of $\mathsf{Per}_h$.

Finally, let us show that the period map is injective. As we have already mentioned, the planes $V \subset W$ are period points of irreducible holomorphic symplectic manifolds, parameterized by $S^2 = \mathbb{P}^1$. Hyperkähler structures correspond one-to-one to the twistor lines in the Teichmüller space $\mathsf{Teich}$. If two hyperkähler structures $g_1, g_2$ give the same vector space $W \in Gr_{+++}H^2(M, \mathbb{R})$, the corresponding twistor lines $L_{g_1}, L_{g_2}$ have the same image in $\mathbb{P}\mathrm{er} = Gr_{++}H^2(M, \mathbb{R})$. However, by the global Torelli (Theorem 3.13), this is only possible when each point of $L_{g_1}$ is unseparable from some point of $L_{g_2}$. It is easy to see that this never happens: indeed, as $W$ is not orthogonal to MBM classes, the Kähler cone is equal to the positive cone at a general point of $L_{g_1}$ (as well as of $L_{g_2}$), and such points are Hausdorff points of $\mathsf{Teich}$ (see for example [14]). ■

## 5. Teichmüller space of symplectic structures

Our goal in this section is to describe the Teichmüller space for symplectic structures of Kähler type $\mathsf{Teich}_k$.

**Theorem 5.1.** *The symplectic period map $\mathsf{Per}_s : \mathsf{Teich}_k \to H^2(M, \mathbb{R})$ is an embedding on each connected component, and its image is the set of positive vectors in $H^2(M, \mathbb{R})$.*

**Proof. Step 1:** The period map is locally a diffeomorphism by Moser's theorem ([4]; see also [2], Proposition 3.1), so we only have to show that it has connected fibers and describe the image. Describing the image is easy: let $v$ be a positive vector in $H^2(M, \mathbb{R})$, then we can always choose a positive 3-subspace $W \subset H^2(M, \mathbb{R})$ which contains $v$ and is not orthogonal to an MBM class. By the proof of Theorem 4.9, $W$ gives rise to a hyperkähler structure such that $v$ is the class of $\omega_I$, therefore $v$ must be in the image of $\mathsf{Per}_s$; the inverse inclusion is clear since Kähler classes are positive.

**Step 2:** To show that $\mathsf{Per}_s$ has connected fibers, we consider the following diagram:

$$\begin{array}{ccc} \widetilde{\mathsf{Teich}}_h & \xrightarrow{P} & \mathsf{Teich}_k \\ \big\downarrow \tilde{\mathsf{Per}}_h & & \mathsf{Per}_s \big\downarrow \\ \begin{array}{c}\{x, y, z \in H^2(M) | x^2 = y^2 = z^2 > 0, \\ x, y, z \text{ is an oriented, orthogonal triple} \\ \langle x, y, z\rangle^\perp \text{ contains no MBM classes}\}\end{array} & \xrightarrow{F} & \{x \in H^2(M) | x^2 > 0\} \end{array} \tag{5.1}$$

where $\widetilde{\mathsf{Teich}}_h$ is the Teichmüller space for hyperkähler triples $(I, J, K)$ together with the metric $g$, $P$ the forgetful map putting $(I, J, K, g)$ to the symplectic form $\omega_I$, and $F$ the forgetful map putting $(x, y, z)$ to $x$.

**Remark 5.2.** The space $\widetilde{\mathsf{Teich}}_h$ can be considered as an $SO(3) \times \mathbb{R}^+$-bundle over the Teichmüller space $\mathsf{Teich}_h$ introduced in Definition 4.8.

**Step 3:** The horizontal arrows of (5.1) are surjective (the upper one by Calabi–Yau theorem), and $\widetilde{\mathsf{Per}}_h$ is an isomorphism by Theorem 4.9. Therefore, fibers of $F$ are surjectively projected to the fibers of $\mathsf{Per}_s$. To prove that the latter are connected, it suffices to show that the fibers of the forgetful map $F$ are connected.

**Step 4:** The fibers of $F$ can be described as follows. Let $x \in H^2(M, \mathbb{R})$ be a positive vector, and $x^\perp \subset H^2(M, \mathbb{R})$ its orthogonal complement. Then $F^{-1}(x)$ is the space of oriented orthogonal pairs $y, z \in w^\perp$ such that $x^2 = y^2 = z^2$, and $\langle x, y, z\rangle^\perp$ contains no MBM classes. This space is an $S^1$-fibration over an open subset $X \subset \mathrm{Gr}_{++}(x^\perp)$ of the corresponding oriented Grassmannian $\mathrm{Gr}_{++}(x^\perp)$ consisting of all 2-planes not orthogonal to any of MBM classes in $x^\perp$. Therefore, $X$ is a complement to a codimension 2 subset of those planes $W \in \mathrm{Gr}_{++}(x^\perp)$ which are orthogonal to some MBM class in $x^\perp$. A complement to a codimension 2 subset in a connected manifold is also connected. This implies that $X$, and hence $F^{-1}(x)$, is connected, finishing the proof of Theorem 5.1. ■

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## References

[1] P.M.H. Wilson, Some remarks on moduli of symplectic structures, Manuscripta Math. 116 (1) (2005) 93–98.
[2] Jan Fricke, Lutz Habermann, On the geometry of moduli spaces of symplectic structures, Manuscripta Math. 109 (4) (2002) 405–417.
[3] M. Entov, M. Verbitsky, Full symplectic packing for tori and hyperkähler manifolds, in press.
[4] J. Moser, On the volume elements on a manifold, Trans. Amer. Math. Soc. 120 (1965) 286–294.
[5] E. Amerik, M. Verbitsky, Rational curves on hyperkähler manifolds, arXiv:1401.0479.
[6] Dietmar Salamon, Uniqueness of symplectic structures, Acta Math. Vietnam. 38 (1) (2013) 123–144.
[7] D. McDuff, Examples of symplectic structures, Invent. Math. 89 (1987) 13–36.
[8] S.K. Donaldson, in: S.S. Chern (Ed.), Two-forms on Four-manifolds and Elliptic Equations, World Sci. Publ, Hackensack, NJ, 2006, pp. 153–172.
[9] F.A. Bogomolov, On the decomposition of Kähler manifolds with trivial canonical class, Math. USSR-Sb. 22 (1974) 580–583.
[10] F.A. Bogomolov, Hamiltonian Kähler manifolds, Sov. Math. Dokl. 19 (1978) 1462–1465.
[11] A. Beauville, Varietes Kähleriennes dont la première classe de Chern est nulle., J. Differential Geom. 18 (1983) 755–782.
[12] A. Fujiki, On the de Rham cohomology group of a compact Kähler symplectic manifold, Adv. Stud. Pure Math. 10 (1987) 105–165.
[13] E. Markman, Integral constraints on the monodromy group of the hyperkahler resolution of a symmetric product of a K3 surface, Int. J. Math. 21 (2) (2010) 169–223. arXiv:math/0601304.
[14] E. Markman, A survey of Torelli and monodromy results for holomorphic-symplectic varieties, in: Proceedings of the Conference "Complex and Differential Geometry", in: Proceedings in Mathematics, vol. 8, Springer, 2011, pp. 257–322. arXiv:math/0601304.
[15] D. Huybrechts, The Kähler cone of a compact hyperkähler manifold, Math. Ann. 326 (3) (2003) 499–513. arXiv:math/9909109.
[16] S. Boucksom, Le cône kählérien d'une variété hyperkählérienne, C. R. Acad. Sci. Paris Sér. I Math. 333 (10) (2001) 935–938.
[17] A. Besse, Einstein Manifolds, Springer-Verlag, New York, 1987.
[18] M. Verbitsky, A global Torelli theorem for hyperkähler manifolds, Duke Math. J. 162 (15) (2013) 2929–2986.
[19] F. Catanese, A Superficial Working Guide to Deformations and Moduli, arXiv:1106.1368, 56 pages.
[20] D. Huybrechts, Compact hyperkähler manifolds: Basic results, Invent. Math. 135 (1999) 63–113. alg-geom/9705025.
[21] D. Huybrechts, Finiteness results for hyperkähler manifolds, J. Reine Angew. Math. 558 (2003) 15–22. arXiv:math/0109024.
[22] M. Obata, Affine connections on manifolds with almost complex, quaternionic or Hermitian structure, Japan J. Math. 26 (1955) 43–79.
[23] H. Weyl, The Classical Groups, Their Invariants and Representations, Princeton Univ. Press, New York, 1939.
[24] Gromov Misha, Metric structures for Riemannian and non-Riemannian spaces, Based on the 1981 French original. With appendices by M. Katz, P. Pansu and S. Semmes. Translated from the French by Sean Michael Bates, in: Progress in Mathematics, vol. 152, Birkhäuser, Boston, Inc., Boston, MA, 1999, p. xx+585.