The ubiquity of Configurations in Model Theory

Gödel Lecture 2020!!

LC'21, Poznan, 21 July 2021

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The subject

What to talk about ?????

For many years : Applications to algebraic Geometry

– Model theory of fields

– Zariski Geometries and Zilber's trichotomy

– Geometric stability or Geometric Model Theory

More recently

Back to older type of questions linked to Shelah's Classifictaion theory and his "Main Gap" result but the solution comes from geometric stability configurations and dichotomies

A little too technical for here but inspiration for the (necessarily) brief survey I will try to do today which hopefully will convince you that there is a continuity in our subject and that this continuity is centered around "configurations" . may seem a little artificial?

In all cases, we study the combinatorial properties of definable sets;

The inspiring work

The inspiration I will probably not get to talk about :
Joint work with B. Hart, E. Hrushovski and M.C. Laskowski
–on ArXiv (2020) "Non locally modular regular types in classifiable theories"
– nearly finished "Locally modular regular groups in classifiable theories"
– more to come, work in progress ".... Isolation vs Domination in classifiable theories..."
Continuity Questions stemming from
B. Hart, E. Hrushovski and M.C. Laskowski, The uncountable spectra of countable theories, Annals of Mathematics, 152, 2000,
Itself putting the finishing touch to the colossal work of S.Shelah on Classification Theory and the Main Gap Theorem (70's - 90)

Classification Theory, going back in time

A major achievement of "pure" model theory, the development by Saharon Shelah mid 70's to mid 80's of Classification Theory and the proof of the Main Gap Theorem.
Very very sketchy :
What is the idea : Consider all complete first order theories T (countable language to simplify) and find some (useful) dividing lines via which infinite combinatorial structures or configurations one can define in their models.
First some very different classes of examples
(I) –the theory of all infinite sets (language just equality)
– fix a division ring k , the theory of all infinite k-vector spaces
– the theory of all algebraically closed fields of fixed characteriscti.

More exemples

(II) Orderings :
– $(\mathbb{Q}, <)$, the rationals, dense linear ordering without end points
– $(\mathbb{R}, +, -, 0, 1, >)$ the reals , real closed fields
(III) The random graph (V, R) a set V with a binary symmetric relation Given any $n + m$ elements $a_1, \ldots, a_n, b_1, \ldots, b_m \in V$, there is a vertex c in V that is adjacent to each of $a_1, \ldots, an$ and is not adjacent to any of $b_1, \ldots, b_m$.

The dividing lines

Find conditions which constitute dividing lines between classes of theories
ex: there is a “definable” ordering or not
so that
on one side as a first approximation, you have chaos and complications, witnesses by “many models“ but more subtly , by the impossibility to find good invariants , to describe the isomorphism type of the models , in particular , “simpler “ and better than “the isomorphism type itself. On the other side, you have good properties which allow for better analysis of the models.
To formalize the question, study the function $I(T, \kappa)$ = the number of non isomorhic models of T of cardinality κ

Stability

First such dividing line :
Stable = "no" Ordering / Unstable = presence of an ordering
"No" means the following definable configuration does not exist :
in no model M of T can we find a formula $\phi(x, y)$ and an infinite $\{a_1, \ldots, a_n, \ldots\}$ such that $M \models \phi(a_i, a_j)$ iff $i < j$.
A formula "representing " an infinite ordered chain
Unstable T : Orderings, real closed fields , random graph, valued fields
$I(T, \kappa)$ = the maximum = 2^κ
No way to assign good invariants

Stability

Good properties; can define abstractly an independence relation on elements , genralizing the classical algebraic ones
easy case , over a model : let $M < N$ be models of T $a, b \in N$,a is independent from b over M iff for every formula $\phi(x, y)$, if $\phi(a, b)$, then already for some $m \in M$, $\phi(a, m)$ holds.
All exemples (I) are stable ; infinite sets, vector spaces, algebraically closed fields . Here for all κ uncountable $I(T, \kappa) = 1$ Note Morley's theorem (65) says; $I(T, \kappa) = 1$ for some uncountable κ iff $I(T, \kappa) = 1$ for all uncountable κ
The best possible invariants.
A very slightly more complicated exemple : the equivalence relation with infinitely many classes all infinite then only one countable model $\aleph_0$ classes , all of cardinality, $\aleph_0$, but in cardinality $\kappa \geq \aleph_1$ need to say how many classes of cardinality λ for every $\lambda \leq \kappa$.

Classifiable

one needs to have further dividing lines inside stable theories
– superstable : a certain type of trees cannot be defined or contion on the number of types.
stable non superstable $I(T, \kappa) = 2^K$ and no good invariants .
superstable $\rightarrow$ independence is nice and one can begin to define dimensions which one hopes to use as invariants .
But not enough: need more dividing lines NDOP (not dimensional order property), PMOP (prime models over pairs)
Then in the end Classifiable = Superstable NDOP PMOP and
Theorem Every model of a classifiable theory is prime and minimal over a tree of independent countable models . Consider this tree as a coloured tree with colour of node = isomorphism of the countable model, then the isomorphism type of this "simple coloured tree" is a good invariant.

Back to unstability

Unstable theories were the utmost non classifiable theories but already Shelah identified some dividing lines there also .
– Simplicity : a theory is simple if no formula has the tree property.
Stable $\subset$ simple.
Regain in interest in simple theories in mid 90's when Kim, Kim-Pillay showed good properties of independence in simple theories: in particular symmetry!
at around same time, Hrushovski and others started to develop tools for analysing particular structures (pseudo finite fields , pseudo algebraically closed fields, algebraically closed fields with an automorphism ...) which turned out to be simple.

Modern times and NIP

Again in the 70's, 80's :
Shelah defined the independence property : A formula has the IP if in some model of T there are $\{a_n; n \in \mathbb{N}\}$ and $\{b_X; X \subset \mathbb{N}\}$ such that $\phi(a_i; b_X)$ iff $i \in X$. NIP (or dependent) = not the independence property
SIMPLE $\cap$ NIP = STABLE .
Keisler (85) in NIP theories, one can define measures on the boolean algebra of definable sets which behave well.
Nowadays different presentation: link with Vapnik Chervonenkis dimension, probabilities .... Many very nice applications to combinatorics, using both NIP and measures and Incidence configurations, see Artem Chernikov's talk .

Link with applications and algebra

Two directions :

1. Looking at structures and theories, groups, fields, algebras etc and finding where they are in the spectrum of "instability"or stabilityOne can then use all the sophisticated tools developped in the "pure" theory to study these structures with a new point of view ...

–algebraically closed fields are very stable (in fact omega-stable and even strongly minimal)

– separably closed fields are stable non superstable

– For R a ring, R-modules are stable

– ordered groups, fields are unstable

– ordered abelian groups, real closed fields , more generally o-minimal structures , are NIP

– algebraically closed valued fields , closed p-adic fields are NIP

– random graph, algebraically closed field with an automorphism, pseudo finite fields are simple.

Link with applications, algebra

In the other direction

Theorem 1. (MacIntyre, 71) If K is an infinite field whose theory is $\aleph_1$-categorical, then K is algebraically closed .

2. (Cherlin-Shelah, 1980) if K is an infinite division ring whose theory is superstable, K is algebraically closed.

The stable field conjecture If K is an infinite field whose theory is stable , then K is separably closed.

also: if K is an ordered field whose theory is o-minimal, then K is real closed ...

Some results and open questions about NIP fields ....

Getting algebraic information from “abstract combinatorial” behaviour of definable sets, or presence or absence of certain configurations.

Geometric Stability Theory

Defining or discovering new algebraic structure from combinatorial configurations

Let us go back to a classical theorem of Geometry :

Incidence axioms , the projective case

P a set of points, lines , $D \subset P$

axioms : (1) Through any two distinct points of P, there is one unique line

(2) Two distinct lines intersect in exactly one point

"projective plane "

(3) another condition about lines and points called Desargues axioms

Then a classical theorem says :
there exists a division ring K and a vector space V over K of dimension 3 such that P is (isomorphic to)$P(V)$, the projective plane of V over K.
The classical proof, first constructs a group $(V, +)$ on P , then the division ring K is obtained as ring of endomorhisms of this group.

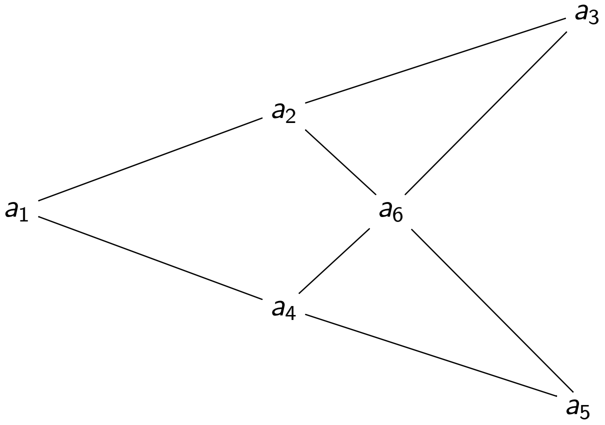
Vast generalization in Model theory of these ideas mid 80's
First B. Zilber , then E. Hrushovski
"geometric model theory" and its configurations.
Note that Zilber developped originally these ideas in order to answer a "classical" model theory question: can totally categorical theories be finitely axiomatized? No.

There are two fundamental related incidence or combinatorial configurations at the the basis of geometric stability .
1. the fact that on certain types (minimal types, regular types) the relation of dependence gives rise to a pregeometry in the sense of combinatorial geometry , that is a closure relation which staisfies the exchange principal.
2. the fact that groups and fields can be “definably” constructed from abstract configurations.

The group configuration

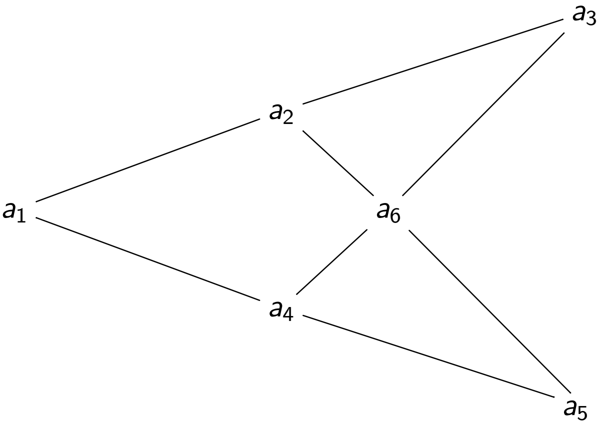
Consider $(V, +)$ the additive group of a vector space. Pick a_1, a_5, a_2 linearly independent

$a_4 = a_5 - a_1, a_3 = a_1 + a_2, a_6 = a_4 + a_3 = a_2 + a_5.$



In an adequate framework, the presence of such a configuration, the group configuration allows the construction of a "definable" group (might be infinitely definable, interpretable)
There are different contexts and hence different theorems : Here is one
Let T be stable, recall that a is algebraic over b if there is a formula $\theta(x, b)$ such that $\theta(a, b)$ holds and there are only finitely many x's such that $\theta(x, b)$ holds .

Suppose we have the configuration below where any two elements on a line are independent, any non aligned 3 elements are independent any element is algebraic over the other two elements on the same line



Suppose we have the configuration below where any two elements on a line are independent, any non aligned 3 elements are independent any element is algebraic over the other two elements on the same line, then there is a “definable” group G such that the “generic” elements of the group is interalgebraic with the elements in the configuration.
Under stricter contexts, there are also some “field” configurations , which witness the presence of a definable field.
Very relevant for applications to algebraic geometry.

If time permits

When algebraic closure (or dependence) forms a pregeometry
trivial : $cl(x) = \{x\}$ the infinite sets
modular $dim(cl(X \cup Y) = dim(cl(X)) + dim(cl(Y)) - dim(X \cap Y))$
vector spaces
not modular algebraically closed fields .
very approximative version of Theorem: if a type is modular non trivial, then it "is" the generic type of a definable abelian group G, the "quasi-endomorphisms" of this group form a division ring K, and the whole model theoretic structure is given by the vector space structure of G over K.

the work inspiring this talk

When a theory is classifiable, each model is prime over an independent tree of countable models . Interested in the isomorphism type of the models at the end of a branch, the leaves ....
They are of the form $M[a]$(“generated” by a over M, dominated) where a is a type where dependence is a pregeometry so can ask if it is trivial, modular. ?
question when does generated mean that $M[a]$ is the prime model over M and a? trivial know nothing
counterexamples when modular, but non modular always true
What really happens when modular ?

THANK YOU