Some applications of model theory

An illustration of the strength of the model theory of groups of finite rank

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In the background: the proof, in 1993, by Hrushovski of the function field Mordell-Lang conjecture, which remained, in Char. p the only known proof until 2013.

In the present: A series of 3 papers with Franck Benoist and Anand Pillay:

- BBP1: Semiabelian varieties over separably closed fields, maximal divisible subgroups and exact sequences , Journal of I.M.J. (2016).
- BBP2: On function field Mordell-lang and Manin-Mumford, J. of Math. Logic (2016)
- BBP3: On function field MordellLang: the semiabelian case and the socle theorem", Proceedings of the L. M.S. (2018).

Aims : to find alternate Model theoretic proof of Mordell-Lang, in characteristic p avoiding Zariski geometries and the trichotomy principle.

WHY?

Understand why Hrushovski's model theoretic proof circumvents the difficulties encountered in char.p by geometers and what exactly the trichotomy for Zariski geometries says in this particular context.

Around 2013 /2014 both a geometric proof (Rossler) and a new model theoretic proof [BBP], without Zariski geometries, but **for the case of abelian varieties** by reducing Mordell-Lang to Manin-Mumford; other geometric proofs since also for semiabelian.

Finally [BBP3] we deduce ML for semiabelian varieties, from ML for abelian varieties, using model theory of finite rank groups.

This will be the main subject of this talk

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Statement of Mordell-Lang

Theorem *Function field Mordell-Lang, weak version for simplicity*
$k \subset K$ two algebraically closed fields, G semiabelian variety over K, X irreducible subvariety of G and $\Gamma \subset G(K)$ finite rank subgroup of $G(K)$. Suppose that $X \cap \Gamma$ is Zariski dense in X and that the stabilizer of X in G is finite.
Then a translate of X is contained in a semiabelian subvariety of G which descends to k.

Don't Panic!!!

(from "The Hitchhiker's Guide to Galaxy")

First we explain the words/objects in the statement

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– Context for the moment: the theory of algebraically closed fields in any characteristic $k \subset K$.
We have some definable/geometric objetc : G, X
A **non definable object**: Γ is just a subgroup of $G(K)$, the (definable) group of K-rational points of the algebraic group G.
Start with X defined over K conclusion under the assumptions: situation descends to k.

Semiabelian varieties

G semiabelian variety over K?
First : Abelian Varieties are connected algebraic groups which are complete ie for every Y $\pi : A \times Y \mapsto Y$ is closed.
No non trivial group homomorphism to any affine group
Ex: Elliptic curves, Jacobians of curves.

G is a semiabelian variety : $G \in Ext(A, T)$ i.e.
$0 \to T \to G \to A \to 0$,
with $T = \mathbb{G}_m^r$ torus and A abelian variety .

Properties of semiabelian varieties

Simple examples : $T \times A$, or semi-split (G isogenous to $T \times A$)
But also non semi-split complicated examples: can have $0 \to T \to G \to A \to 0$ with A defined over k (T always is), BUT G is not.
(semi-split correspond to torsion elements in $Ext(A, T)$ and $Ext(A, T)$ is parametrized by $\hat{A}$, the dual of A, an L-rational point of $\hat{A}$ corresponding to G defined over L.)

Good properties: $G(K)$ is a commutative divisible group, for every n the n-torsion is finite, the torsion is infinite and dense in G.

Remarks: in char.p, they are exactly the divisible commutative algebraic groups.

The objects

X irreducible subvariety of G

G as an algebraic group has an induced topology, its Zariski topology. And X is a closed irreducible subset of G in this sense. In particular X is definable.

The statement again :

Theorem *Function field Mordell-Lang, weak version for simplicity*
$k \subset K$ two algebraically closed fields, G semiabelian variety over K, X irreducible subvariety of G and $\Gamma \subset G(K)$ finite rank subgroup of $G(K)$. Suppose that $X \cap \Gamma$ is Zariski dense in X and that the stabilizer of X in G is finite.
Then a translate of X is contained in a semiabelian subvariety of G which descends to k.

The objects

A semiabelian subvariety of G is a closed connected subgroup H of G.

H descends to k : H is isogenous with a group defined over k.

The full statement of ML says there is $h < G$, $\pi : H \mapsto H_k$ isogeny, $a + X \subset H$ and for some $Y \subset H_k$ defined over k, $(a + X) = \pi^{-1}(Y)$.

the group Γ is not definable or algebraic !
Γ is just a subgroup of $G(K)$.
finite rank : Γ is contained in the (p')-divisible hull of a finitely generated.
Exemples: finitely generated, the p'-torsion, the torsion in char. 0.

Hrushovski's proof from 1993

In char. p it was new (and remained the only proof until 2013).

The features in Hrushovski's proof that surprised algebraic geometers

1. The proof was "uniform " in char. 0 and char. p.
Model theory!

2. He just did not seem to run into the difficulties that geometers ran into: issues of separability related to the behviour of p-torsion.

3. His proof treated in exactly the same way the abelian and semiabelian case.

In his proof Udi proves and uses at various places some model theory of finite rank (Morley or U-rank)) groups: existence of a socle and "the (weak) socle theorem".

At first glance, the socle is used to pass from finite U-rank groups to minimal groups for which the Zilber dichotomy applies via Zariski structures .

But also permits to circumvent the "bad" (compared to abelian varieties) features of semiabelian varieties!!

Now by using this same theorem and similar ideas, one can show directly that Mordell-lang for semiabelian varieties reduces to the case of abelian varieties. And this reduction can be stated precisely in algebraic terms but have no equivalent algebraic proof

The algebraic statements

Let $k \subset K$, both algebraically closed, G semiabelian variety over K. Define the k-socle of G: $S_k(G) := G_k + A_k$, where

• G_k is the largest closed connected subgroup which descends to k

• A_k is the largest closed connected subgroup with no non trivial connected closed subgroup which descends to k.

Note that A_k must be an abelian variety and we say of such an abelian variety that it has k-trace zero.

G_k and A_k have finite intersection and any closed subgroup of $G_k + A_k$ is a product of a subgroup from G_k and a subgroup from A_k.

Note that : – ML is obviously true if $G = G_k$ ie ,G itself descends to k

– if G is a k-trace 0 abelian variety ($G = A_k$) ML says that $X = a$.

The reductions

Proposition

In order to prove Mordell-Lang for all semiabelian varieties, it suffices to prove it for semiabelian varieties of the form: $G_0 + A$, where G_0 is a semiabelian variety which descends to k and A is an abelian variety with k-trace zero.

We call such varieties, “socle-like” varieties.

We assume ML for all abelian varieties with k-trace zero

Proposition

Mordell-Lang holds for all socle-like varieties.

Corollary

Mordell-lang holds for all semiabelian varieties.

Algebraic proofs????

No model theoretic free proof. The second Proposition could seem easy .
Problem: would want to divide $X \cap \Gamma$ onto each of the factors. A_k and G_k have finite intersection, there is no non trivial group homomorphism from one to the other, so connected subgroups of G are products of subgroups but there may be some definable subset which is not a rectangle in other words A_k and G_k are **not orthogonal** .

Some Model Theory: orthogonality

Let $T = T^{eq}$ be a stable theory, $\mathbb{U}$ a saturated model of T
Definition of orthogonality : E, F two infinitely definable sets are orthogonal if every definable subset of $E \times F$ is a boolean combination of rectangles, i.e. of sets of the form $C \times D$, C definable subset of $E$and D definable subset of F(or if $a \subset E$ and $b \subset F$, a and b are independent)
Note that in an algebraically closed field, any two definable sets are non orthogonal.

Model-theoretic Socle

G a commutative infinitely definable group of finite U-rank (defined over some M_0).
Recall, if Q is a type, then Q is minimal if for all definable D, $D(\mathbb{U}) \cap Q(\mathbb{U})$ is finite or co-finite $=$ U-rank 1.
Immediate consequence of Zilber's indecomposability theorem (finite U-rank version):
Proposition Let $\{Q_i : i \in I\}$ be a family of minimal types in G. Then the subgroup generated by the Q_i's is infinitely definable and connected.

Definition Q a minimal type. G is almost Q-minimal if $G \subset acl(F \cup Q)$ for some finite set F.
G is almost pluriminimal if there are minimal types $Q_1, \ldots, Q_n$ and a finite set F such that $G \subset acl(F \cup Q_1 \cup \ldots \cup Q_n)$.

Model-theoretic Socle

Definition of the socle

Proposition

1. Q a minimal type of G. There is a largest connected infinitely definable almost Q-minimal subgroup B_Q of G.
2. There is a largest connected infinitely definable almost pluriminimal subgroup of G. We denote it by $S(G)$, the socle of G

The structure of the socle

Proposition

$S(G) = B_{Q_1} + \ldots + B_{Q_n}$ for some minimal types $Q_1, \ldots, Q_n$, which can be assumed to be pairwise orthogonal. Every minimal type in G is nonorthogonal to one of the Q_i's.
Almost direct sum : (the intersection $B_{Q_i} \cap B_{Q_j}$ is finite set for $i \neq j$).

How to use the socle?

Definition G, defined over C, is rigid if all connected infinitely definable subgroups (with extra parameters) are infinitely definable over $acl(C)$.

True for one-based groups, true for Semiabelian varieties!.

Proved in Hrushovski's Mordell-Lang paper (for finite Morley Rank)

Theorem (Socle theorem)

Let G be an infinitely definable group over $\emptyset$, finite U-rank, commutative and rigid. Let p be a complete stationary type in G, over $\emptyset$. Assume $Stab_G(p)$ is finite. Then there is a translate of $S(G)$ containing (the realizations of) p.

$=$ if $Stab_G(p) = \{g \in G : g + p = p\}$ is finite, then for some $a \in G$, $p \subset a + S(G)$.

Now back to the proof of the reduction to socle-like semiabelian varieties .

Note : semiabelian varieties are rigid (Every connected closed subgroup is already defined over K).

Need a model theoretic framework where to use the theorem and orthogonaloty :

The model-theoretic framework in char. p

Recall the statement: **Theorem** *Function field Mordell-Lang, weak version for simplicity* $k \subset K$ two algebraically closed fields, G semiabelian variety over K, X irreducible subvariety of G and $\Gamma \subset G(K)$ finite rank subgroup of $G(K)$, SUppose that $X \cap \Gamma$ is dense in X and that the stabilizer of X in G is finite.
Then a translate of X is contained in a sub-semiabelian variety of G which descends to k.

In char.p, reduce to : K is a (sufficiently saturated) separably closed field of degree of imperfection 1, ie $[K : K^p] = p$, K model of the theory $SCF_{p,1}$ (stable non superstable), and $k = K^{p^\infty} = \bigcap_n K^{p^n}$ (biggest algebraically closed subfield of K).
Different ambiant structure, more definable sets (SCF does not have quantifier elimination in language of rings), k has become infinitely definable, its generictype is a minimal type.

The model-theoertic framework in char. p

G is defined over K, $G(K)$ is a definable group in K, but no longer divisible.

Fact:(after maybe translating X) One can replace Γ by an infinitely definable subgroup of $G(K)$, $p^\infty G(K) = \bigcap_n p^n G(K)$ the biggest divisible subgroup of $G(K)$ denoted usually by $G^\sharp$, which has finite U-rank, and still satisfies that $X \cap G^\sharp$ is dense in X.

By the socle theorem, can in fact replace $G^\sharp$ by its socle $S(G^\sharp)$.

We know $S(G^\sharp) = B_{Q_1} + \ldots + B_{Q_n}$, Q_i minimal types pairwise orthogonal.

Nonorthogonality classes of minimal types in $G^\sharp$: the generic type of k and the rest.

Q minimal type in $G^\sharp$, $B_Q = H^\sharp$ for some H semiabelian subvariety of G:

Facts: • if Q is nonorthogonal to k, H is isogenous to some H_0 semiabelian variety over k
• if Q is orthogonal to k, H is an abelian variety with k-trace 0

Conclusion:

Recall our algebraic socle $S_k(G) = G_k + A_k$

$S(G^\sharp) = G_k^\sharp + A_k^\sharp = S_k(G)^\sharp$ with $G_k^\sharp$ and $A_k^\sharp$ orthogonal.

Conclusion

Recall the socle theorem:

Theorem (Socle theorem)

Let H be an infinitely definable group over $\emptyset$, finite U-rank, commutative and rigid. Let p be a complete stationary type in G, over $\emptyset$. Asume that $Stab_H(p)$ is finite. Then there is a translate of $S(H)$ containing (the realizations of) p.

Taking $H = G^\sharp$, p a type (for the theory SCF) generic in $X \cap G^\sharp$ and passing to Zariski closures, the socle theorem gives :

Proposition

Let $k < K_1$ be any pair of algebraically closed fields, G a semiabelian variety over K_1, X an irreducible subvariety of G such that the stabilizer of X in G is finite. Let Γ be a subgroup of $G(K_1)$ of finite rank, If $X \cap \Gamma$ is dense in X, then for some $a \in G(K_1)$, $a + X \subset S_k(G_k)$. It follows that Mordell-Lang for G reduces to Mordell-Lang for $S_K(G)$.

Algebraic socles

$0 \mapsto T \mapsto G \mapsto A \mapsto 0$

1. If G is almost split $G = TxA$ (up to isogeny), then $S_k(G) = G$.
2. If A has k-trace 0, then $S_k(G) = G$ if and only if G is almost split.
3. If A descends to k, $S_k(G) = G$ if and only if G descends to k. If moreover A is ismple, then $S_k(G) \neq G$ iff $S_k(G) = T$.

THANK YOU

Happy Birthday Dave!!!!