A stroll through some important tools of model theory illustrated by the example of the Mordell-Lang conjecture

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In the background: the groundbreaking proof, in 1993, by Hrushovski of the Mordell-Lang conjecture, which remained, in Char. p the only known proof until 2013.

In the present: Talk partly inspired by a series of papers joint work with Franck Benoist and Anand Pillay on the model theory of semiabelian varieties although this won't be completley visible here.
Aims : to find alternate Model theoretic proof of Mordell-Lang, avoiding Zariski geometries and the trichotomy principle.

WHY?

Success? Better undersatnding? Yes and no.
Around 2013 /2014 both a geometric proof (Rossler) and a model theoretic proof (BBP) for the case of abelian varieties ; other geometric proofs since.
Finally (BBP 2017) for semiabelian varieties.

The notions of model theory

- induced structure
- enriched structure
- orthogonality
- groups of finite Morley Rank

– Characterizing classical algebraic structures abstractly
or reconstructing algebraic structures (groups, fields) from abstract or combinatorial data .
In the spirit of the very old classical theorem in geometry which says that a Desarguesian projective geometry of dimension at least 3 is the projective geometry over a division ring.

– Zariski Geometries

Mordell-Lang?

Theorem *Function field Mordell-Lang*
$k \subset K$ two algebraically closed fields, G semiabelian variety over K (= *a divisible commutative algebraic group*), X irreducible subvariety of G (= *an irreducible zariski closed subset of* G) and $\Gamma \subset G(K)$ finitely generated subgroup.
Then,
– either $X \cap \Gamma = a_1 + \Gamma_0 \cup \ldots \cup a_n + \Gamma_n$), where, for each i, Γ_i is a subgroup of Γ
– or $H = \overline{\Gamma}$ (the zariski closure of Γ in G) is isomorphic to a group defined over k.

Don't Panic!!!

(from "The Hitchhiker's Guide to Galaxy")

This is NOT how we begin

Algebraically closed fields

Use the case of algebraically closed fields to guide us through a brief and very biased history of some basic notions of model theory and algebra

Recall :

Definition A field K is algebraically closed if every polynomial $P(X)$ in one variable in $K[X]$, of degree ≥ 1, has a solution in K

Ex: $\mathbb{C}$ the complex numbers, but not the reals $\mathbb{R}$.

K as a first-order model theory structure the language L_{ring}:

$$(K, +, ., -, 0, 1)$$

There is a theory T_{acf} (a set of sentences) such that a field is algebraically closed iff it is a model of T_{acf}:

– K is a field and

– for every $n > 1$

$$\forall y_0 \ldots \forall y_{n-1} \exists x \left(x^n + \Sigma_{i=0}^{n-1} y_i . x^i \right) = 0.$$

Model theory of algebraically closed fields

For $p = 0$ or p prime, the theory ACF_p of algebraic closed fields of characteristic p is complete.

Theorem(Tarski, Chevalley) Algebraically closed fields admit quantifier elimination.

Theorem (Macintyre, 1971) If an infinite field K has quantifier elimination, then K is algebraically closed.

Modern model theory : Get algebraic information from abstract data.

Definable sets in T_{ACF}

A set $D \subset K^n$ is definable if there is a formula $\phi(x)$ such that $D = \{a \in K^n; K \models \phi(a)\}$. Then we write $D = \phi(K)$.

– Zariski closed sets: solutions of polynomial equations $\{a \in K^n; f_1(a) = \ldots = f_s(a) = 0\}$ for $f_1, \ldots, f_s$ in $K[X_1, \ldots, X_n]$.

– quantifier free formulas $\longrightarrow$ finite boolean combinations of closed sets $=$ constructible sets

– Quantifier elimination $\longrightarrow$ definable $=$ constructible

Algebraic groups

Recall a group $(G, .)$ is definable in K if
– G is a definable subset of some K^n
– the multiplication and inverse maps are definable (ie their graphs are definable sets in $K^n \times K^n \times K^n$ and $K^n \times K^n$.)

Obvious definable groups in K : the additive group, the multiplicative group, the affine groups = closed subgroups of $GL_n(K)$ (definable in $K^n x K^n$).

Less obvious but true : any algebraic group G is definable or rather the K-rational points of G ($G(K)$) form a definable group in K.

Further properties

The theory T_{ACF_p} is $\aleph_1$-categorical (= categorical in every uncountable cardinality, Morley, 65).

More: The theory T_{ACF_p} is strongly minimal

A definable subset D in K^n is strongly minimal if for any definable $E \in K^n$, $D \cap E$ is finite or its complement in D is finite .

The algebraically closed field K itself is strongly minimal: any definable set in one variable = boolean combination of sets which are the solution set of a polynomial equation in one variable .

Strongly minimal structures

Examples of strongly minimal structures

1. Infinite set with no structure (only equality in the language)

2. An infinite vector spaces over a fixed division ring. It has the property that any definable subset is a (finite boolean combination of translates of subgroups

3. Algebraically closed fields

or “avatars“ of these.

Zilber trichotomy principle

Conjecture proposed by Boris Zilber in the 1980's : every strongly minimal theory "is" of one of these forms.

Disproved by Hrushovski in 90'S.

But proved (Hrushovski-Zilber, 93) to hold for a class of strongly minimal sets with extra properties, the Zariski structures.

Zariski Structures and Trichotomy

A Zariski structure is a strongly minimal set D where the atomic sets form the closed sets of a noetherian topology on each D^n,
the definable sets are the constructible sets and the dimension (given by the noetherian topology) satisfies some particular "good" properties (the dimension theorem)

Trichotomy

Then Dichotomy for groups :
G a strongly minimal group which is a zariski structure
1. D "is an abelian group'' and the structure on D is of linear (vector space) type: for every n every definable $X \subset D^n$ is a Boolean combiantion of translates of definable subgroups of D^n.
D is one-based, locally modular
or

2. in D there is an algebraically closed field K which is definable in some D^n , and D is homeomorphic to a projective curve over K , or "nearly" so.

The dichotomy

The function field Mordell Lang is also a dichotomy about certain algebraic groups
BUT in K algebraically closed, no infinite definable group is one-based..... so not directly such a dichotomy.

Mordell-Lang. A special case

Theorem *Function field Mordell-Lang*
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Then,
– either $X \cap \Gamma = a_1 + \Gamma_0 \cup \ldots \cup a_n + \Gamma_n$, where, for each i, Γ_i is a subgroup of Γ
– or $H = \overline{\Gamma}$ (the zariski closure of Γ in G) is isomorphic to a group defined over k.

Mordell-Lang continued, the definable objects

G is a semiabelian variety:
Commutative algebraic groups (*definable*)
– divisible, finite n-torsion for all n, but torsion is infinite.
Built from two extreme cases:
1. Abelian varieties: *Complete* connected algebraic group.
Ex: Elliptic curves, Jacobians of curves, never affine
No non trivial group homomorphism to any affine group.
2. (affine) tori $T = \mathbb{G}_m^n$
G is a semiabelian variety : $G \in Ext(A, T)$ i.e.
$0 \rightarrow T \rightarrow G \rightarrow A \rightarrow 0$,
with $T = \mathbb{G}_m^r$ torus and A abelian variety .
Examples : $T \times A$, or semi-split (G isogenous to $T \times A$)
But also non split complicated examples.

Remark: in char.p, exactly the divisible commutative algebraic groups.

Mordell-Lang. A special case

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X irreducible subvariety of G?

G as an algebraic group has an induced topology, its Zariski topology. And X is a closed irreducible subset of G in this sense. In particular X is definable.

$H = \overline{\Gamma}$ is a closed subgroup of G so also definable, so defined with parameters in K, but isomorphic to a group defined over k.
Say that H descends to k

Γ is not definable or algebraic !
Γ is just a finitely generated subgroup of $G(K)$. Even if Γ is generated by one element g_0 ,$x \in \Gamma$ iff
$x = g_0 \vee x = g_0 + g_0 \vee x = g_0 + g_0 + g_0 \ldots$

The Dichotomy

The dichotomy :
in the first case: note that the conclusion talks about the topology induced on Γ by the topology of G, it is induced by the closed subgroups only:
the closed subsets of Γ are the sets of the form $X \cap \Gamma$ for X closed in G, and it says they are just given by translates of subgroups. OR :

Γ descends to k.

Adding new definable sets

Adding Γ to the language?
Then ML becomes indeed:
Either Γ is a one-based group or $\overline{\Gamma}$ descends to k .
But
– not easier than the original statement
– k is not definable .....
MUST do it differently

Adding a derivation

Add more definable sets by adding a derivation on the field K,
a map δ from K to K such that
- $\delta(x+y) = \delta(x) + \delta(y)$
- $\delta(x.y) = x.\delta(y) + y.\delta(y)$

in characteristic 0 .

A little more complicated in Char. p, must add a family of strange derivations (Hasse derivations).

DCF_0

Can do this so that k becomes the field of constants of δ so definable as $\{a \in K; \delta(a) = 0\}$. we can suppose that K is differentally closed

(=existentally closed) and k is the field of constants in K.

The theory of differentially closed fields of char. 0, DCF_0 is richer than ACF_0 but still good from model theoretic point of view.
It is ω-stable : Every definable set has Morley Rank
We know a lot about definable groups, and *the strongly minimal subsets are Zariski structures !*

Working in DCF_0

Replace Γ by : $G^\sharp$ which is *the smallest definable subgroup of G which is zariski dense in G* or also the smallest δ-definable subgroup containing the torsion of G. (Infinitely definable in characteristic p)
$G^\sharp$ is nice, it is a group with finite Morley rank, but it is not strongly minimal.

The remaining model theory tools

Orthogonality :

Definition Let $D, E \subset K^n$ be δ-definable. They are orthogonal ($D \perp E$) if any infinite δ-definable subset $Y \subset D^r \times E^m$ is a rectangle : $Y = Y_D \times Y_E$, $Y_D \subset D^r, Y_E \subset E^m$ both δ-definable.

In K as a pure algebraically closed field, by $\aleph_1$-categoricity, any two definable subsets are non orthogonal.

In DCF_0 : Then for any H δ-definable group in K, strongly minimal, the Zariski structure Dichotomy says:
Either H is one based or H is non orthogonal to k, the field of constants
and then H descends to k.
But $G^\sharp$ is not always strongly minimal?

Saved!!

Any abelian variety A is a sum of simple abelian varieties pairwise not homomorphic, it follows that $A^\sharp = J_1 + \ldots + J_n$ a sum of (almost)strongly minimal groups which are pairwise orthogonal.

And the Torus $T = \mathbb{G}_m{}^n$ is already defined over k .
so for $G = T \times A$, $G^\sharp = T^\sharp \times A^\sharp = T^\sharp + J_1 + \ldots + J_n$, we manage.

Semiabelian varieties and model theory

But what about the general case ? $0 \to T \to G \to A \to 0$ when it is not split?

Have examples where the induced sequence $0 \to T^\sharp \to G^\sharp \to A^\sharp \to 0$ is not exact.
So one cannot deduce good properties for $G^\sharp$ from the same properties for $A^\sharp$ and $T^\sharp$.

Saved by the Socle theorem

G is no longer a sum of "simple" subgroups .

But model theory of finite rank groups shows that there is a maximal δ-definable subgroup of $G^\sharp$, its socle, $S(G^\sharp)$ which is a finite sum of pairwise orthogonal strongly minimal groups

The Socle theorem: for any X definable irreducible zariski closed in $\overline{G^\sharp}$, then a translate of X is contained in $\overline{S(G^\sharp)}$.

So can replace $G^\sharp$ by its socle, sum of pairwise orthogonal strongly minimal groups

and reduce the question to the good cases, when $G = TxA$.
The reduction uses the socle but not zariski geometries .
So Mordell-Lang for abelian varieties implies Mordell Lang for semiabelian varieties, via the theorem of the socle.
New "algebraic object ? the zariski closure of the socle of $G^\sharp$ ....

THANK YOU !