Erratum: Deconvolution with unknown noise distribution is possible for multivariate signals

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The erratum offers another way to obtain an inequality similar to that given in Proposition A.2 in [Gassiat et al., 2022], since an error has been found in its proof.

In this note we use the setting and the notations introduced in [Gassiat et al., 2022]. We first show how their proof works, and where the issue is.

For any ϕ and $\Phi_{R^\star}$ in $\Upsilon_{\rho,S}$, define $\Delta\phi = \phi - \Phi_{R^\star}$, and $N : \mathbb{R}^{d_1} \times \mathbb{R}^{d_2} \to \mathbb{C}$ such that for all $(t_1, t_2) \in \mathbb{R}^{d_1} \times \mathbb{R}^{d_2}$,

$$N(t_1, t_2; \Delta\phi, \Phi_{R^\star}) = \Delta\phi(t_1, , 0t_2)\Phi_{R^\star}(t_1)\Phi_{R^\star}(0, t_2) - \Phi_{R^\star}(t_1, t_2)\Delta\phi(t_1, 0)\Phi_{R^\star}(0, t_2) \\ - \Phi_{R^\star}(t_1, t_2)\Phi_{R^\star}(t_1, 0)\Delta\phi(0, t_2),$$

so that

$$M^{\text{lin}}(\Delta\phi, \Phi_{R^\star}; \nu) = \int_{[-\nu,\nu]^{d_1} \times [-\nu,\nu]^{d_2}} |N(t_1, t_2; \Delta\phi, \Phi_{R^\star})|^2 \, dt_1 dt_2.$$

If it is possible to prove that there exists a constant c depending only on ρ_0, ν, and S such that for all $\rho \leqslant \rho_0$, for all ϕ and $\Phi_{R^\star}$ in $\Upsilon_{\rho,S} \cap \mathcal{H}$, for all integer m,

$$\int_{[-\nu,\nu]^{d_1} \times [-\nu,\nu]^{d_2}} |N(t_1, t_2; T_m\Delta\phi, T_m\Phi_{R^\star})|^2 \, dt_1 dt_2 \geqslant c^m \|T_m\Delta\phi\|_{2,\nu}^2, \tag{1}$$

then, following the proof of [Gassiat et al., 2022], it holds that for all $\delta > 0$, there exists $\eta > 0$ and a constant $\widetilde{c}$ depending only on δ, ρ_0, ν, and S such that for all $\rho \leqslant \rho_0$, for all ϕ and $\Phi_{R^\star}$ in $\Upsilon_{\rho,S} \cap \mathcal{H}$, as soon as $\|\phi - \Phi_{R^\star}\|_{2,\nu} \leqslant \eta$, then

$$M_\star(\phi; \nu) \geqslant \widetilde{c}\|\phi - \Phi_{R^\star}\|_{2,\nu}^{2(1+\delta)}. \tag{2}$$

The error in [Gassiat et al., 2022] lies in the proof of Lemma B.1, which aims to show (1). It relates N with a matrix A with properties described in Lemma I.1 of Appendix I in the supplementary material of [Gassiat et al., 2022]. In particular, this Lemma entails that A restricted to the polynomials of degree at most m is injective (lower triangular with diagonal coefficients equal to -1), with explicit inverse, thus allowing the control of its lowest singular value. The problem lies in point iv) of this Lemma: A is actually not injective. Its diagonal entries with coordinates $((i_1,0),(i_1,0))$ and $((0,i_2),(0,i_2))$ for $i_1,i_2 \geqslant 1$ are zero, and its entry with coordinate $((0,0),(0,0))$ is +1.

For some structured submodels, (1) may be proved directly. To deal with more general cases, we introduce an alternative assumption which allows to prove (2) directly.

1 Structured submodels

Equation (1) may be proved by lower bounding

$$\int_{[-\nu,\nu]^{d_1}\times[-\nu,\nu]^{d_2}} \left|N(t_1,t_2;T_m\Delta\phi,T_m\Phi_{R^\star})\right|^2 dt_1 dt_2$$

with something satisfying (1). This is what is done in [Capitao-Miniconi et al., 2024] for the repeated measurements model, in which $X^{(1)} = X^{(2)} = X$, so that $\mathcal{H}$ is the set of functions ϕ such that $\phi(t_1,t_2) = \Phi(t_1+t_2)$. It is proved in [Capitao-Miniconi et al., 2024] that there exists a constant c_1 depending only on ρ_0, ν, and S such that for all $\rho \leqslant \rho_0$, for all ϕ and $\Phi_{R^\star}$ in $\Upsilon_{\rho,S}\cap\mathcal{H}$, for all integer m,

$$\int_{[-\nu,\nu]^{2d}} \left|N(t_1,t_2;T_m\Delta\phi,T_m\Phi_{R^\star})\right|^2 dt_1 dt_2 \geqslant c_1^m \int_{[-\nu,\nu]^{d}} \left|N(t,t;T_m\Delta\phi,T_m\Phi_{R^\star})\right|^2 dt,$$

see Lemma 8.1, and then that there exists a constant c_2 depending only on ρ_0, ν, and S such that for all $\rho \leqslant \rho_0$, for all ϕ and $\Phi_{R^\star}$ in $\Upsilon_{\rho,S}$, for all integer m,

$$\int_{[-\nu,\nu]^{d}} \left|N(t,t;T_m\Delta\phi,T_m\Phi_{R^\star})\right|^2 dt \geqslant c_2^m \|T_m\Delta\phi\|_{2,\nu}^2,$$

see Lemma 8.2, so that (1) holds.

2 General case

Consider now the following assumption. Let $\delta \in (0,1)$.

H(δ): There exists $\nu_0 > 0$ such that for all $\nu \in (0,\nu_0]$, there exists $c(\nu,\delta) > 0$ such that for all $R^\star$ such that $\Phi_{R^\star} \in \mathcal{H}\cap\Upsilon_{\kappa,S}$,

$$\forall \phi \in \mathcal{H}\cap\Upsilon_{\kappa,S} \quad \text{s.t.} \quad \phi \neq \Phi_{R^\star}, \quad \frac{\left\| \dfrac{\phi}{\Phi_{R^\star}} - \dfrac{\phi^{(1)}}{\Phi_{R^\star}^{(1)}} - \dfrac{\phi^{(2)}}{\Phi_{R^\star}^{(2)}} + 1 \right\|_{2,\nu}}{\max\left(\left\| \dfrac{\phi^{(1)}}{\Phi_{R^\star}^{(1)}} - 1 \right\|_{2,\nu}, \left\| \dfrac{\phi^{(2)}}{\Phi_{R^\star}^{(2)}} - 1 \right\|_{2,\nu} \right)^{1+\delta}} \geqslant c(\nu,\delta)\,. \tag{3}$$

Proposition 1. *Assume* **H**(δ) *holds. There exists* $\overline{\nu} > 0$ *depending only on* S *and* ρ *such that for all* $\nu \leqslant \overline{\nu}$*, there exists a constant* $\widetilde{c} > 0$ *depending only on* c_ν *and* $c(\nu, \delta)$ *such that for all* $R^\star$ *such that* $\Phi_{R^\star} \in \mathcal{H} \cap \Upsilon_{\kappa,S}$*, all* $Q^\star \in \mathbf{Q}(\nu, c_\nu, c_Q)$ *for some* $c_Q \in (0, \infty]$*, for all* $\phi \in \mathcal{H} \cap \Upsilon_{\kappa,S}$*, as soon as*

$$\left\| \frac{\phi}{\Phi_{R^\star}} - \frac{\phi^{(1)}}{\Phi^{(1)}_{R^\star}} - \frac{\phi^{(2)}}{\Phi^{(2)}_{R^\star}} + 1 \right\|_{2,\nu} \leqslant \min\left\{ \left(c(\nu,\delta)^2 2^{-(1+\delta)} (2\nu)^{d(1+\delta)} \right)^{\frac{1}{1-\delta}} ; 1 \right\}, \tag{4}$$

it holds

$$M_\star(\phi; \nu) \geqslant \widetilde{c} \|\phi - \Phi_{R^\star}\|_{2,\nu}^{2(1+\delta)} . \tag{5}$$

In [Gassiat et al., 2022], $\mathcal{H}$ is chosen as a closed subset of $\mathbf{L}^2(B^d_{\nu_{\text{est}}})$ such that all elements of $\mathcal{H}$ satisfy $\mathbf{H2}$. In the choice of $\mathcal{H}$ we may require that **H**(δ) holds, since this choice comes from the prior modeling that allows to fix $\mathbf{H2}$, see examples in Section 2 of [Gassiat et al., 2022]. If we add **H**(δ) in the choice of $\mathcal{H}$, then Theorem 3.2 and Theorem 3.3 follow from the arguments developed in [Gassiat et al., 2022] with no modification.

Notice that the uniform consistency of the estimator (Section A.1 in [Gassiat et al., 2022]) holds without any change, that is without assuming **H**(δ) which is only used to get rates.

Let us now prove Proposition 1. First, notice that for all $\phi \in \Upsilon_{\rho,S}$, $|\phi(t) - 1| \leqslant \sup_{u:\|u\| \leqslant \|t\|} \|\phi'(u)\| \|t\|$, where $\phi'(u)$ denotes the gradient of ϕ at u (recall that ϕ is multivariate analytic), so that it is possible to choose $\overline{\nu}$ depending only on S and d such that for all ρ, for all $\phi \in \Upsilon_{\rho,S}$, for all $\nu \leqslant \overline{\nu}$, all $t \in B^d_\nu$, $|\phi(t)| \geqslant 1/2$. For instance, one can choose $\overline{\nu} = 1/2Sd^2$. Let now $R^\star$ be such that $\Phi_{R^\star} \in \mathcal{H} \cap \Upsilon_{\kappa,S}$, and let $Q^\star \in \mathbf{Q}(\nu, c_\nu, c_Q)$ for some $c_Q \in (0, \infty]$. Let ϕ be any function in $\mathcal{H} \cap \Upsilon_{\kappa,S}$ such that (4) holds. Denote

$$g_1 = \frac{\phi^{(1)}}{\Phi^{(1)}_{R^\star}} - 1, \qquad g_2 = \frac{\phi^{(2)}}{\Phi^{(2)}_{R^\star}} - 1 \quad \text{and} \quad G = \frac{\phi}{\Phi_{R^\star}} - 1 - g_1 - g_2 .$$

Then, $M_\star$ rewrites

$$M_\star(\phi; \nu) = \|\Phi_{R^\star} \Phi^{(1)}_{R^\star} \Phi^{(2)}_{R^\star} (G - g_1 g_2) \Phi^{(1)}_{Q^\star} \Phi^{(2)}_{Q^\star}\|^2_{2,\nu} , \tag{6}$$

and we get $M_\star(\phi; \nu) \geqslant \frac{c_\nu^4}{2^6} \|G - g_1 g_2\|^2_{2,\nu}$. Using **H**($\delta$) and (4), we get

$$\|g_1\|_{2,\nu} \|g_2\|_{2,\nu} \leqslant \left(\frac{\|G\|_{2,\nu}}{c(\nu,\delta)} \right)^{2/(1+\delta)} \leqslant \frac{\|G\|_{2,\nu}}{2} (2\nu)^d$$

(recall $0 < \delta < 1$). We then get

$$\begin{aligned} M_\star(\phi; \nu) &\geqslant \frac{c_\nu^4}{2^6} (\|G\|_{2,\nu} - \|g_1 g_2\|_{2,\nu})^2 \\ &= \frac{c_\nu^4}{2^6} (\|G\|_{2,\nu} - (2\nu)^{-d} \|g_1\|_{2,\nu} \|g_2\|_{2,\nu})^2 \\ &\geqslant \frac{c_\nu^4}{2^8} \|G\|^2_{2,\nu}. \end{aligned}$$

On the other hand,

$$\begin{aligned}\|\phi - \Phi_{R^\star}\|_{2,\nu} &= \|G + g_1 + g_2\|_{2,\nu} \\ &\leqslant \|G\|_{2,\nu} + \|g_1\|_{2,\nu} + \|g_2\|_{2,\nu} \\ &\leqslant \|G\|_{2,\nu} + 2(c(\nu,\delta)^{-1}\|G\|_{2,\nu})^{1/(1+\delta)} \\ &\leqslant (1 + 2c(\nu,\delta)^{-1/(1+\delta)})\|G\|_{2,\nu}^{1/(1+\delta)}\end{aligned}$$

since $\|G\|_{2,\nu} \leqslant 1$. Hence, we may conclude

$$M_\star(\phi;\nu) \geqslant \frac{c_\nu^4}{2^8(1 + 2c(\nu,\delta)^{-1/(1+\delta)})^{2(1+\delta)}} \|\phi - \Phi_{R^\star}\|_{2,\nu}^{2(1+\delta)} .$$

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References

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