*

# Efficient semi-parametric estimation of the periods in a superposition of periodic functions with unknown shape

By Elisabeth Gassiat and Céline Lévy-Leduc[†]

*Université Paris-Sud Orsay France*

[‡§] We consider the estimation of the periods from a sum of periodic functions of unknown shape corrupted by Gaussian white noise. In the case of a single periodic signal, we propose a consistent and asymptotically efficient semiparametric estimator of the period. We then study the case of a sum of two periodic signals of unknown shape with different periods. For a large class of signals, we propose semiparametric estimators of the two periods that are consistent and asymptotically Gaussian.

**1. Introduction** Let us assume that we have $n$ observations of a sampled signal corrupted by white noise:

$$X(j) = s_0(jt_n) + \varepsilon(j)\ ;\ 1 \leq j \leq n, \tag{1.1}$$

where $s_0$ is an unknown real function (the signal), the $\varepsilon(j)$ are independent centered Gaussian random variables of unknown variance $\sigma_0^2$ and $t_n$ is the sampling period. We consider the situation where the signal $s_0$ can be decomposed into a sum of unknown periodic signals, and are interested in the estimation of the periods of the individual periodic components.

The issue of frequency estimation of periodic signals embedded in additive noise has already been addressed by several authors. The problem arises in a wide variety of areas such as communications, radio location of objects, seismic signal processing and computer assisted medical diagnosis. It has been addressed mainly according to a parametric point of view, that is using models where the number of unknown parameters is finite. The literature on this topic is so abundant that Stoica and Söderström (1992), Kootsookos (1999), Quinn and Hannan (2001) and more recently Kundu (2004) have written reviews on the subject. We just quote here some of them.
Following Kootsookos (1999), we can mainly distinguish three different problems of interest in the frequency estimation field: the single tone, the multi-harmonic and the multi-tone frequency estimation problems. They consist in estimating the parameters $\mu$, $\rho_j$, $\omega_j$, $\phi_j$, $\sigma^2$ and $p$ from the observations:

$$y_t = \mu + \sum_{j=1}^{p} \rho_j \cos(\omega_j(t-\nu) + \phi_j) + \varepsilon_t,\ t = 0, \ldots, T-1.$$

where $\varepsilon_t$ is some zero-mean random noise sequence with variance $\sigma^2$. Single tone frequency estimation is the case where $p = 1$, multi-harmonic the case where $\omega_j = j\omega_0$, and multi-tone the more general case. This issue arises in speech signal processing where, according to Kundu (2004), the accurate determination of the resonant frequencies of the vocal tract in various configurations is of interest both in synthesis and in the analysis of speech.
These different problems have been addressed both on theoretical and practical points of view by many authors. Assuming that $p$ is known, the maximum likelihood estimators may be proved consistent and asymptotically Gaussian in various settings, see Walker (1971), Hannan (1973), Whittle (1952), Quinn and Thomson (1991). Methods can also be proposed for estimating $p$, see Kavalieris and Hannan (1994). In the single tone model, Quinn and Fernandes (1991) have also proposed asymptotically optimal estimators but with a smaller computational cost than maximum likelihood estimators, see also Pisarenko (1973), Kay (1989), and the MUSIC estimator due to Schmidt (1986) for other algorithms.

*Received 2004.
[†]supported by Thalès Optronique, Guyancourt (France)
[‡]*AMS 2000 subject classifications.* Primary 00
[§]*Key words and phrases.* periodic signals, semiparametric estimation, efficiency

But when we are faced with a model which is described by too many parameters then a natural solution is to consider that the regression function is unknown and to estimate only some significant parameters: this is the semiparametric approach. This point of view has only been considered by Golubev (1988) and by Hall et al. (2000) where, in model (1.1), $s_0$ is periodic and one wants to know the period. Golubev (1988) considers the white noise model (that is the continuous version of the model (1.1)) and gives a consistent and asymptotically efficient estimator of the period. Hall et al. (2000) consider observations obtained with random design.

If now $s_0$ is the sum of two periodic signals, that is

$$s_0 = s_1^0 + s_2^0 \tag{1.2}$$

where $s_1^0$ and $s_2^0$ are periodic with distinct periods $\tau_1^0$ and $\tau_2^0$, then the problem appears to be harder. It is not difficult to see that if the ratio $\tau_1^0/\tau_2^0$ is a rational number, the signal is still periodic, but when this ratio is not a rational number, the signal is no longer periodic. At first glance one would think that to be able to infer two different periods, one would need the ratio of the periods to be non rational. But if one knows that one has to search for two distinct periods, then the problem of estimating the two periods is solvable for a large class of signals (see section 4). In general, for a sum of periodic signals, it is necessary to know the number of periodic signals to be able to infer the different periods.
The problem of estimating periods in a sum of periodic signals has various applications. For instance, it can be a step towards target identification in laser vibrometry. Indeed, after emission of a continuous wave by a coherent laser, reflection on the target of interest and demodulation, the received signal is of the form (1.1) with $s_0$ decomposed as a sum of complicated periodic components. Thus, estimating the periods can be used to achieve long range target identification because they characterize the vibrations of the target, see Prenat (2001) and Lavielle and Lévy-Leduc (2004).

To our knowledge, semiparametric estimation of two (or more) distinct periods has not yet been solved. We solve this problem in this paper. We propose estimators of the periods and prove that they are consistent and asymptotically Gaussian (efficient in many cases) in the semiparametric case when the shape of the signals is unknown. These results are not only of theoretical interest: the ideas of the present paper have been used in Lavielle and Lévy-Leduc (2004) where a frequency estimation algorithm is developed, and systematic simulations show that it is valid in great generality.

Let us now give an outline of the paper. In the next section, we give general ideas that lead to our estimators, some assumptions and some useful tools. In section 3, we first study the case where $s_0$ is itself periodic. This section is mainly inspired by ideas developed by Golubev (1988) and leads us to introduce some tools which will be useful for the study of the two periods problem. We prove that our estimator based on a maximization of an approximation of the profile likelihood is consistent and also asymptotically efficient thanks to the computation of the efficient Fisher information matrix. It is obtained by using LAN theory which is explained in van der Vaart (1998) and McNeney and Wellner (2000). In section 4, we solve the problem of efficient estimation of the two periods when $s_0$ is the sum of two periodic functions, this being the most interesting result of the paper. We prove that our estimators are often asymptotically efficient. In section 5, we give some indications about the practical implementation (all details are in Lavielle and Lévy-Leduc (2004)), and about further work. All technical proofs are given in section 6.

**2. General ideas and assumptions** If a real signal $s$ is periodic with period $\tau$, and if its restriction to $[0;\tau]$ is square integrable, then $s$ can be expanded in Fourier series:

$$s(t) = \sum_{k\in\mathbb{Z}} c_k(s)\exp(\frac{2ik\pi t}{\tau}),\ c_k(s) = \frac{1}{\tau}\int_0^\tau \exp(-\frac{2ik\pi t}{\tau})s(t)dt, \forall k \in \mathbb{Z},$$

where for each integer $k$, the conjugate of $c_k(s)$ is equal to $c_{-k}(s)$. Saying that the shape of the signal is unknown is the same as saying that the parameters $(c_k(s))_{k\in\mathbb{Z}}$ are unknown. A parameterization of periodic functions may thus be given as a period $\tau$ and a periodic function $f$ with period 1, so that $s(t) = f(t/\tau)$. For such $f$ one has $c_k(f) = c_k(s)$ for all $k$, and $f$ is called the shape of the signal. Notice that to define the shape parameters $(c_k)$ one has to know the period.

We shall consider periodic signals with period $\tau$ and not with period $\tau/m$ for some integer $m \geq 2$ which implies that $s$ satisfies

$$\mathbf{(S1)}: \exists h \in ]0,1[ \text{ such that } \forall m \geq 2, \sum_{k\geq 1} |c_{mk}(s)|^2 \leq (1-h) \sum_{k\geq 1} |c_k(s)|^2.$$

This is proved in section 6. We shall assume that the signal $s$ is smooth enough so that its Fourier coefficients $c_k(s)$ verify

$$\mathbf{(S2)}: \sum_{k\in\mathbb{Z}} k^2 \, |c_k(s)| < +\infty.$$

2.1. *A single periodic signal* Consider first the case where the signal $s_0$ itself is periodic with period $\tau_0$ without knowing any a priori bound for $\tau_0$. Then, to be able to find the period even without noise, one needs to have observations on an interval which is larger than the period, and with asymptotically vanishing sampling parameter. We thus assume throughout the paper that

$$\mathbf{(N1)}: \lim_{n\to+\infty} t_n = 0 \text{ and } \lim_{n\to+\infty} nt_n = +\infty. \tag{2.3}$$

To obtain an efficient estimator in semiparametric problems, various strategies may be tried, such as estimating the efficient score functions, or using the profile likelihood (see van der Vaart (1998) chapter 25 in the i.i.d. setting). We shall use here an approximation of the profile likelihood. To obtain the criterion, one maximizes the likelihood with respect to the nuisance parameters, which are here the $c_k(s)$'s. Let us approximate a signal $s$ by $\sum_{|k|\leq m_n} c_k \, e^{2ik\pi t/\tau}$ where $c_k$ and $\tau$ are unknown. Then maximization of the log-likelihood leads to minimize

$$\sum_{j=1}^{n} \left( X(j) - \sum_{|k|\leq m_n} c_k e^{\frac{2ik\pi j t_n}{\tau}} \right)^2 \tag{2.4}$$

with respect to the $c_k$'s to obtain an approximation of the profile log-likelihood. This provides approximately the following estimator for the $c_k$'s:

$$\hat{c}_k \approx \frac{1}{n} \sum_{j=1}^{n} X(j) \; e^{-\frac{2ik\pi j t_n}{\tau}}.$$

If we replace the $c_k$'s in (2.4) by the $\hat{c}_k$'s so obtained, we find that the estimator of the period should minimize the following function of $\tau$:

$$\sum_{j=1}^{n} \left( X(j) - \sum_{|k|\leq m_n} \hat{c}_k e^{\frac{2ik\pi j t_n}{\tau}} \right)^2$$

$$= \sum_{j=1}^{n} X(j)^2 - 2 \sum_{|k|\leq m_n} \hat{c}_k \sum_{j=1}^{n} X(j) e^{\frac{2ik\pi j t_n}{\tau}} + \sum_{j=1}^{n} \left( \sum_{|k|\leq m_n} \hat{c}_k e^{\frac{2ik\pi j t_n}{\tau}} \right)^2$$

$$= \sum_{j=1}^{n} X(j)^2 - 2n \sum_{|k|\leq m_n} |\hat{c}_k|^2 + \sum_{j=1}^{n} \left| \sum_{|k|\leq m_n} \hat{c}_k e^{\frac{2ik\pi j t_n}{\tau}} \right|^2$$

$$= \sum_{j=1}^{n} X(j)^2 - n \sum_{|k|\leq m_n} |\hat{c}_k|^2 + n \sum_{|k|,|k'|\leq m_n, k\neq k'} \hat{c}_k \hat{c}_{-k'} \varphi_n \left[ \frac{t_n}{\tau}(k-k') \right]$$

$$\approx \sum_{j=1}^{n} X(j)^2 - \frac{1}{n} \sum_{|k|\leq m_n} \left| \sum_{j=1}^{n} X(j) e^{\frac{2ik\pi j t_n}{\tau}} \right|^2 = \sum_{j=1}^{n} X(j)^2 - \frac{1}{n} \left| \sum_{j=1}^{n} X(j) \right|^2 - \frac{2}{n} \sum_{k=1}^{m_n} \left| \sum_{j=1}^{n} X(j) e^{\frac{2ik\pi j t_n}{\tau}} \right|^2,$$

where $\varphi_n(t) = n^{-1} \sum_{j=1}^{n} e^{2i\pi j t}$.
Thus we set:

$$\Lambda_n(\tau) = \sum_{k=1}^{m_n} \frac{1}{n^2} \left| \sum_{j=1}^{n} X(j) \exp\left(\frac{2ik\pi j t_n}{\tau}\right) \right|^2.$$

$\Lambda_n(\tau)$ is a cumulated periodogram where the cumulating parameter $m_n$ tends to infinity in an appropriate way (see assumptions below).
If we define $G_n(\tau)$ as the deterministic part of $\Lambda_n$ and $\eta_n(\tau)$ as the random part of $\Lambda_n$, that is

$$G_n(\tau) = \sum_{k=1}^{m_n} \frac{1}{n^2} \left| \sum_{j=1}^{n} s_0(jt_n) \exp(\frac{2ik\pi j t_n}{\tau}) \right|^2 , \; \eta_n(\tau) = \Lambda_n(\tau) - G_n(\tau),$$

we will see that $\eta_n$ is uniformly $o_P(1)$, and that asymptotically the maximum values for $G_n(\tau)$ are obtained at any multiple of the unknown period $\tau_0$, that is for $\tau = k\tau_0$, where $k$ is an integer. The maximum value of $G_n$ is of order $\sum_{p\geq 1} |c_p(s_0)|^2 = \frac{1}{2}\sum_{p\in\mathbb{Z}} |c_p(s_0)|^2 - |c_0(s_0)|^2$, which is up to the term $|c_0(s_0)|^2$ and by the Parseval's theorem half of the power of $s_0$.
Thus, to obtain a consistent estimator of $\tau_0$, one may choose the smallest value in the set of the $\tau$'s that roughly maximize $\Lambda_n(\tau)$. Moreover, since $\Lambda_n$ approximates the profile log-likelihood, a maximizer of $\Lambda_n$ in a neighborhood of $\tau_0$ should lead to an efficient estimator. So that to obtain an asymptotically efficient semiparametric estimator of the period, we follow the steps:

- define $\tilde{\tau}_n$ as the smallest value in some set of the $\tau$'s that roughly maximize $\Lambda_n$,
- then choose the estimator $\hat{\tau}_n$ as the maximizer of $\Lambda_n$ on a ball centered at $\tilde{\tau}_n$ and with small radius.

This definition will be made precise in section 3, and $\hat{\tau}_n$ thus defined will be proved to be asymptotically efficient. For technical reasons, we shall choose a parameter set which only asymptotically covers all positive numbers, that is we set

$$\Theta_n = [\delta_n, \Delta_n] ,$$

with

$$\textbf{(N2)} : \lim_{n\to+\infty} \delta_n = 0 \quad \lim_{n\to+\infty} \Delta_n = +\infty,$$

in an appropriate way. More precisely, we will use:

$$\textbf{(N3)} : \frac{m_n t_n}{\delta_n} \leq \frac{1}{8}, \; \lim_{n\to+\infty} m_n\delta_n = 0, \; \lim_{n\to+\infty} \frac{m_n}{\Delta_n} = 0, \; \lim_{n\to+\infty} \frac{m_n^{5/2}\Delta_n}{n t_n} = 0,$$
$$\lim_{n\to+\infty} \frac{m_n^2}{n t_n^2} = 0, \; \frac{n t_n}{m_n^5} = O(1).$$

This will be enough to obtain asymptotically efficient estimators without any assumption on the range of possible values for the periods.
**Remark**: Let us give sequences satisfying (N3). Let us denote $t_n$, $m_n$, $\delta_n$ and $\Delta_n$ by $t_n = n^{-\alpha}$, $m_n = n^{\beta}$, $\delta_n = n^{-\delta}$ and $\Delta_n = n^{l}$. If we take $0 < \alpha < 1/2$, $(1-\alpha)/5 < \beta < 1/2 - \alpha$, $\beta < \delta < \alpha - \beta$ and $\beta < l < 1 - \alpha - 5\beta/2$, then (N3) is fulfilled.

2.2. *Sum of two periodic signals* Let us now see how the ideas developed for a periodic signal may be extended to signals that consist of the sum of two periodic signals with distinct periods. Let us focus on the model (1.2), so that

$$X(j) = s_1^0(jt_n) + s_2^0(jt_n) + \varepsilon(j) = f_1^0\left(\frac{jt_n}{\tau_1^0}\right) + f_2^0\left(\frac{jt_n}{\tau_2^0}\right) + \varepsilon(j) \;\; ; \;\; 1 \leq j \leq n$$

where $s_1^0$ and $s_2^0$ are periodic functions respectively with period $\tau_1^0$, $\tau_2^0$, $f_1^0$ and $f_2^0$ are periodic functions with period 1 such that

$$f_1^0(t) = \sum_{k\in\mathbb{Z}} c_k^0 \exp(2ik\pi t), \; f_2^0(t) = \sum_{k\in\mathbb{Z}} d_k^0 \exp(2ik\pi t).$$

The first difficulty we are faced with is that there may be more than one way to split the signal into the sum of two periodic functions, so that the couple of periods in the decomposition is not well defined. This problem of identifiability occurs when $\tau_1^0/\tau_2^0$ is a rational number (which is the most frequent case in applications, since statistical data are stored in the computer memory in the floating point format with a finite precision). In this case, the signal $s_0$ is periodic, with period $\tau_0$, multiple of both $\tau_1^0$ and $\tau_2^0$. Let us give an example to illustrate the problem. If

$$s_1^0(t) + s_2^0(t) = \sum_{k=-13}^{13} c_k \; e^{2ik\pi\times 2t} + \sum_{l=-4}^{4} d_l \; e^{2il\pi\times 3t},$$

such that $c_k \neq 0$, $\forall k = -13, \ldots, 13$, $d_1 = d_{-1} = d_0 = 0$, the other $d_l$ being non equal to 0, one may also write

$$\begin{aligned} s_1^0(t) + s_2^0(t) = &\left( \sum_{k=-13}^{13} c_k e^{2ik\pi\times 2t} + d_2 \; e^{2i\times 2\pi\times 3t} + d_{-2} \; e^{-2i\times 2\pi\times 3t} + d_4 \; e^{2i\times 4\pi\times 3t} + d_{-4} \; e^{-2i\times 4\pi\times 3t} \right) \\ &+ \left( d_3 \; e^{2i\times 3\pi\times 3t} + d_{-3} \; e^{-2i\times 3\pi\times 3t} \right). \end{aligned}$$

With the first decomposition of the signal $s_1^0 + s_2^0$, the couple of periods is $(1/2, 1/3)$ but with the second decomposition, the couple of periods is $(1/2, 1/9)$.
To overcome this problem of identifiability, we shall define the couple of periods $(\tau_1^0, \tau_2^0)$ as the one, in all possible couples of periods, such that: in any sum of two periodic signals that is equal to the original signal $s_1^0 + s_2^0$, the couple of periods have: either the largest bigger than $\tau_2^0$, else the lowest bigger than $\tau_1^0$. Formally, let $\mathcal{T}(s_1^0, s_2^0)$ be the set of pairs of real numbers $(t_1, t_2)$ such that there exists a pair of periodic functions $(s_1, s_2)$ such that $s_1^0 + s_2^0 = s_1 + s_2$ and with respective periods $t_1$ and $t_2$. Define the couple of periods $(\tau_1^0, \tau_2^0)$ by

$$\begin{aligned} \tau_2^0 &= \inf \left\{ \max(t_1, t_2) \;:\; (t_1, t_2) \in \mathcal{T}(s_1, s_2) \right\}, \\ \tau_1^0 &= \min \left\{ t_1 \;:\; (t_1, \tau_2^0) \in \mathcal{T}(s_1, s_2) \right\}. \end{aligned} \tag{2.5}$$

Then by construction, the parameters of interest $\tau_1^0$ and $\tau_2^0$ are now identifiable. We shall also need some assumption saying that the signal is indeed the sum of exactly two periodic signals with distinct periods in some sense that will be made precise in section 4.

The second difficulty we are faced with is the choice of a criterion to define the estimator of the periods. We again use an approximation of the profile likelihood. Let us approximate $s_1^0$ and $s_2^0$ respectively by $\sum_{|k|\leq m_n} c_k \; e^{2ik\pi t/\tau_1}$ and $\sum_{|k|\leq m_n} d_k \; e^{2ik\pi t/\tau_2}$ where $c_k$, $d_k$, $\tau_1$ and $\tau_2$ are unknown. Maximization of the log-likelihood leads to minimize

$$\sum_{j=1}^{n} \left( X(j) - \sum_{|k|\leq m_n} c_k e^{\frac{2ik\pi jt_n}{\tau_1}} - \sum_{|k|\leq m_n} d_k e^{\frac{2ik\pi jt_n}{\tau_2}} \right)^2 \tag{2.6}$$

with respect to the $c_k$'s, $d_k$'s, $\tau_1$ and $\tau_2$. Two cases may occur:

- If $\tau_1/\tau_2 \notin \mathbb{Q}$ then

$$\hat{c}_{k_0} \approx \frac{1}{n} \sum_{j=1}^{n} X(j) e^{-\frac{2ik_0\pi jt_n}{\tau_1}},$$

  and in the same way

$$\hat{d}_{l_0} \approx \frac{1}{n} \sum_{j=1}^{n} X(j) e^{-\frac{2il_0\pi jt_n}{\tau_2}}.$$

- If $\tau_1/\tau_2 = p/q$ then

$$\hat{c}_{k_0} \approx \frac{1}{n} \sum_{j=1}^{n} X(j) e^{-\frac{2ik_0\pi jt_n}{\tau_1}}, \text{ if } p \text{ does not divide } k_0,$$

$$\hat{c}_{pu} \approx \frac{1}{n} \sum_{j=1}^{n} X(j) e^{-\frac{2ipu\pi jt_n}{\tau_1}} - \hat{d}_{qu}, \text{ if } k_0 = pu, \text{ where } u \text{ is an integer}$$

and

$$\hat{d}_{l_0} \approx \frac{1}{n}\sum_{j=1}^{n} X(j) e^{-\frac{2il_0\pi jt_n}{\tau_2}}, \text{ if } q \text{ does not divide } l_0.$$

In the first case, if we replace the $c_k$'s and $d_k$'s by $\hat{c}_k$ and $\hat{d}_k$ in (2.6), we obtain in the same way as in section 2.1 that to obtain maximum likelihood estimators of $\tau_1^0$ and $\tau_2^0$ one has to maximize with respect to $\tau_1$ and $\tau_2$ the function:

$$\Lambda_n(\tau_1) + \Lambda_n(\tau_2).$$

In the second case, if we replace the $c_k$'s and $d_k$'s by $\hat{c}_k$ and $\hat{d}_k$ in (2.6), we obtain

$$\sum_{j=1}^{n}\left(X(j) - \sum_{|k|\le m_n, p\nmid k} \hat{c}_k e^{\frac{2ik\pi jt_n}{\tau_1}} - \sum_{|l|\le m_n, q\nmid l} \hat{d}_l e^{\frac{2il\pi jt_n}{\tau_2}} - \sum_{u,\ k=pu,\ l=qu} (\hat{c}_{pu} + \hat{d}_{qu}) e^{\frac{2i\pi ujt_n}{\tau_*}}\right)^2,$$

where $a \nmid b$ means that $a$ does not divide $b$ and $\tau_* = \tau_2/p = \tau_1/q$. This quantity is approximately equal to

$$C_n - \frac{1}{n}\sum_{|k|\le m_n, p\nmid k}\left|\sum_{j=1}^{n} X(j) e^{\frac{2ik\pi jt_n}{\tau_1}}\right|^2 - \frac{1}{n}\sum_{|l|\le m_n, q\nmid l}\left|\sum_{j=1}^{n} X(j) e^{\frac{2il\pi jt_n}{\tau_2}}\right|^2 - \frac{1}{n}\sum_{u,\ |u|\le m_n/p,\ |u|\le m_n/q}\left|\sum_{j=1}^{n} X(j) e^{\frac{2i\pi ujt_n}{\tau_*}}\right|^2,$$

where $C_n$ does not depend neither on $\tau_1$ nor on $\tau_2$. This quantity may be rewritten as

$$\begin{aligned}
&C'_n - \Lambda_n(\tau_1) - \Lambda_n(\tau_2) + \frac{1}{2n}\sum_{u,\ |u|\le m_n/p,\ |u|\le m_n/q}\left|\sum_{j=1}^{n} X(j) e^{\frac{2i\pi ujt_n}{\tau_*}}\right|^2 \\
&= C'_n - \Lambda_n(\tau_1) - \Lambda_n(\tau_2) + \frac{1}{2n}\mathcal{R}e\left\{\sum_{u,\ |u|\le m_n/p,\ |u|\le m_n/q}\left(\sum_{j=1}^{n} X(j) e^{\frac{2i\pi pujt_n}{\tau_1}}\right)\left(\sum_{j=1}^{n} X(j) e^{-\frac{2i\pi qujt_n}{\tau_2}}\right)\right\} \\
&= C''_n - \Lambda_n(\tau_1) - \Lambda_n(\tau_2) + \frac{1}{n}\mathcal{R}e\left\{\sum_{1\le k,k'\le m_n, \frac{k}{k'}=\frac{\tau_1}{\tau_2}}\left(\sum_{j=1}^{n} X(j) e^{\frac{2i\pi kjt_n}{\tau_1}}\right)\left(\sum_{j=1}^{n} X(j) e^{-\frac{2i\pi k'jt_n}{\tau_2}}\right)\right\},
\end{aligned}$$

where $C'_n$ and $C''_n$ do not depend on $\tau_1$ nor $\tau_2$.

We are now in a position to define a criterion which will approximate the profile log-likelihood in all cases. Set

$$\mathcal{H}_n(\tau_1, \tau_2) = \Lambda_n(\tau_1) + \Lambda_n(\tau_2) - \Lambda_n^c(\tau_1, \tau_2) \tag{2.7}$$

where

$$\Lambda_n^c(\tau_1, \tau_2) = \frac{1}{n^2}\mathcal{R}e\left\{\sum_{(k,k')\in K_n(\tau_1,\tau_2)}\left(\sum_{j=1}^{n} e^{\frac{2ik\pi jt_n}{\tau_1}} X(j)\right)\left(\sum_{j=1}^{n} e^{-\frac{2ik'\pi jt_n}{\tau_2}} X(j)\right)\right\}$$

and

$$K_n(\tau_1, \tau_2) = \left\{(k_1, k_2) \in \{1, \ldots, m_n\}^2,\ |\frac{\tau_1}{\tau_2} - \frac{k_1}{k_2}| \le \beta_n\right\}$$

for some $\beta_n$ tending to 0.

In the same way as for a single periodic signal, we shall define the estimators by investigating the set of maximizers of the asymptotic value of the deterministic part of the criterion. Indeed, like for the single period case, one can prove that the random part of the criterion $\mathcal{H}_n$ is uniformly $o_P(1)$. We shall prove that the maximum values of the deterministic part of $\mathcal{H}_n$ are obtained for the:

- couples $(\tau_1, \tau_2)$ with $\tau_1$ around integer multiples of $\tau_1^0$ and $\tau_2$ around integer multiples of $\tau_2^0$,

- couples $(\tau_1, \tau_2)$ such that $\tau_1$ is an integer multiple of $\tau_0$, the period shared by $s_1^0$ and $s_2^0$ and $\tau_2$ is arbitrary, in the case where $\tau_1^0/\tau_2^0$ is a rational number.

- couples $(\tau_1, \tau_2)$ which are the symmetric of the previous ones with respect to the 45 degree line.

This maximal value is equal to half of the power of $s_1^0 + s_2^0$ up to $|c_0(s_1^0) + c_0(s_2^0)|^2$. We show on Figure 1 the couples $(\tau_1, \tau_2)$ for which the deterministic part of $\mathcal{H}_n$ is maximized in the case where $\tau_1^0/\tau_2^0 = 3/5$.

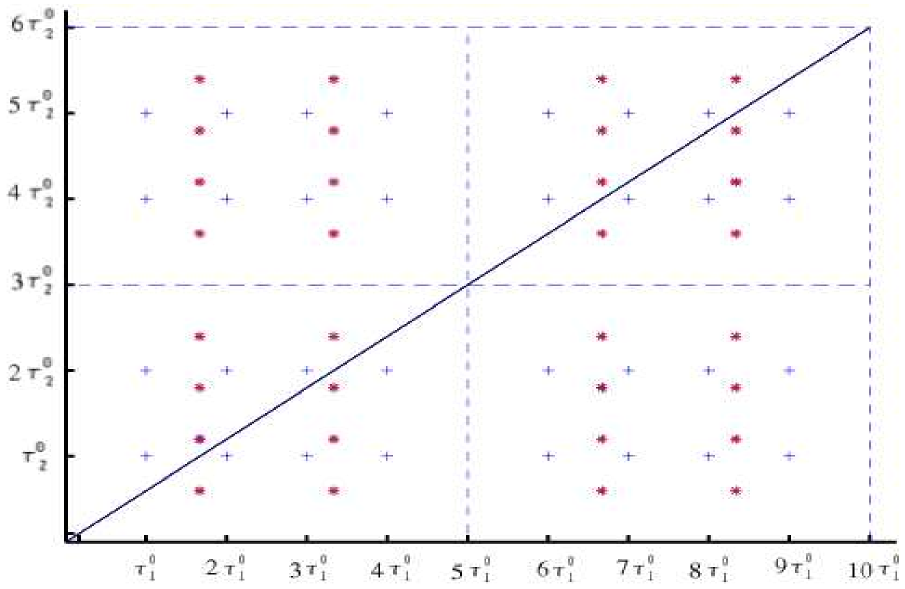


FIG. 1. *The points marked by '+', '∗' or by the dashed lines are the $(\tau_1, \tau_2)$ maximizing the deterministic part of $\mathcal{H}_n$ in the case where $\tau_1^0/\tau_2^0 = 3/5$.*

Let us define the set $\mathcal{U}_n$ of the $(\tau_1, \tau_2)$ which roughly maximize $\mathcal{H}_n$ over $\Theta = [\delta, \Delta]$ by

$$\mathcal{U}_n = \left\{ (\tau_1, \tau_2) \in \Theta^2,\ \mathcal{H}_n(\tau_1, \tau_2) \geq (1 - \varepsilon) \sup_{\Theta^2} \mathcal{H}_n(\theta_1, \theta_2) \right\},$$

where $\varepsilon$ is a small enough positive real number. It is now natural to define an estimator of the couple of periods as follows. Choose as an estimator of $\tau_2^0$: $\tilde{\tau}_2$ the smallest second coordinate of the points belonging to $\mathcal{U}_n$ that are above the 45 degree line, and choose as an estimator of $\tau_1^0$: $\tilde{\tau}_1$ the smallest first coordinate of the points belonging to $\mathcal{U}_n$ having $\tilde{\tau}_2$ as second coordinate. Thus defined, $\tilde{\tau}_1$ and $\tilde{\tau}_2$ may be proved to be consistent at the rate $O(1/(nt_n))$. The final estimators $\hat{\tau}_1$ and $\hat{\tau}_2$ are defined as the maximizers of $\mathcal{H}_n$ over balls of small radius centered in $\tilde{\tau}_1$ and $\tilde{\tau}_2$. These estimators are proved to be asymptotically Gaussian by following the ideas used for the single periodic case.

In the study of the criterions, the function $\varphi_n$ defined by $\varphi_n(t) = \frac{1}{n} \sum_{j=1}^{n} \exp(2i\pi jt)$ will be used, and we report some useful upper bounds. The roughest ones are:

$$|\varphi_n(t)| \leq 1,\ |\varphi'_n(t)| \leq \pi(n+1),\ |\varphi''_n(t)| \leq 4\pi^2(n+1)^2/3. \tag{2.8}$$

But, by using the concavity of the sinus function on $\left[0, \frac{\pi}{2}\right]$, we can deduce for some constant $C$ that for all $t$ in $\left[-\frac{1}{2}\ ;\ \frac{1}{2}\right]$ and $t \neq 0$:

$$|\varphi_n(t)| \leq \frac{C}{n|t|},\ |\varphi'_n(t)| \leq \frac{C}{|t|},\ |\varphi''_n(t)| \leq \frac{Cn}{|t|}. \tag{2.9}$$

These inequalities come from

$$\varphi_n(t) = \frac{\exp(i\pi(n+1)t)}{n}\frac{\sin(n\pi t)}{\sin(\pi t)}, \text{ if } t \neq 0 \text{ and } \varphi_n(0) = 1,$$

straightforward computations and use of concavity of the sinus function.

**3. The single period model** We consider in this section model (1.1) where $s_0$ is periodic with period $\tau_0$, and $f_0$ is its shape. We propose an estimator of the period $\tau_0$ which does not need any a priori knowledge of the function $s_0$. We shall in this section introduce the tools and the ideas that will be also used for the two periods model.

For any $R > 0$, define

$$B(\tau; R) = \left\{\theta \in \Theta_n : \left|\frac{\tau}{\theta} - 1\right| < R\right\}.$$

Let $\varepsilon_n$ be a sequence decreasing to 0 such that

$$\textbf{(N4)} \qquad \lim_{n\to+\infty} \frac{n t_n \varepsilon_n}{m_n \Delta_n} = +\infty, \ \lim_{n\to+\infty} \frac{\varepsilon_n}{n^{-1/5}} = +\infty.$$

Define

$$\mathcal{T}_n = \left\{\theta \in \Theta_n \ : \ \Lambda_n(\theta) \geq (1-\varepsilon_n) \sup_{\Theta_n} \Lambda_n(\tau)\right\},$$

and

$$\tilde{\tau}_n = \inf \mathcal{T}_n.$$

The estimator $\hat{\tau}_n$ is defined as the maximizer of $\Lambda_n$ over $B(\tilde{\tau}_n, 1/8)$.

The estimator $\hat{\tau}_n$ is asymptotically efficient:

**Theorem 1.** *Under the assumptions (S2), (N1), (N2), (N3),(N4) $\hat{\tau}_n$ is consistent and satisfies the asymptotic linearization*

$$n^{3/2} t_n(\hat{\tau}_n - \tau_0) = \frac{I_*^{\ -1}}{\sigma_0^2 n^{3/2} t_n} \sum_{j=1}^{n} \left(\frac{n}{2} - j\right) \frac{t_n}{\tau_0^2} f_0'\left(\frac{j t_n}{\tau_0}\right) \varepsilon(j) + o_{\mathbb{P}_n^0}(1), \tag{3.10}$$

*where*

$$I_* = \frac{1}{12\sigma_0^2 \tau_0^4} \int_0^1 f_0'^2(t)dt$$

*is the efficient Fisher information and $\mathbb{P}_n^0$ is the distribution of $(X(1), \ldots, X(n))$ when the parameter is $(\tau_0, f_0, \sigma_0)$. Moreover, $\hat{\tau}_n$ is asymptotically efficient, and $n^{3/2} t_n(\hat{\tau}_n - \tau_0)$ converges in distribution to the centered Gaussian distribution with variance $1/I_*$.*

**Remark**: One may compare with computations when the shape of the signal is known, and see that the efficient Fisher information is four times smaller when the function $s_0$ is unknown than it would be if it was known. But, this loss is due to the fact that the observations are $X(j)$ for $j = 1, \cdots, n$. Indeed, if we had $j = -n/2, \cdots, n/2$, this loss would not appear. In other words, it is not due to the unknown shape of the signal, but to the choice of the centering of the coordinates.

Proof. To begin with, we prove that the random part of the criterion is asymptotically negligible in probability, and precise analysis of the deterministic part $G_n$ will lead to convergence properties of $\tilde{\tau}_n$ and $\bar{\tau}_n$. All intermediate results are proved in section 6.

**Lemma 1.** *Under (S2), (N1), (N2), and (N3), for all $x_n$ such as $\lim_{n\to+\infty} x_n n^{1/2-\beta} = +\infty$ for some $\beta > 0$,*

$$\lim_{n\to+\infty} P\left(\sup_{\tau\in\Theta_n} |\eta_n(\tau)| \geq x_n\right) = 0.$$

A consequence of this Lemma is that $\sup_{\tau\in\Theta_n} |\eta_n(\tau)| = o_P(n^{-1/5})$.

Let us state some results on the deterministic part $G_n$.
For any $\tau$,

$$G_n(\tau) = \sum_{k=1}^{m_n} \frac{1}{n^2} \left| \sum_{j=1}^{n} s_0(jt_n) \exp(-\frac{2ik\pi jt_n}{\tau}) \right|^2 = \sum_{k=1}^{m_n} \left| \sum_{p\in\mathbb{Z}} c_p(s_0)\varphi_n\left(\frac{pt_n}{\tau_0} - \frac{kt_n}{\tau}\right) \right|^2,$$

since $s_0$ is a real valued function. One has, uniformly for $\tau \in \Theta_n$, using (2.8),

$$\sum_{k=1}^{m_n} \left| \sum_{|p|\geq m_n} c_p(s_0)\varphi_n\left(\frac{pt_n}{\tau_0} - \frac{kt_n}{\tau}\right) \right| \leq \frac{1}{m_n} \sum_{|p|\geq m_n} |p^2 c_p(s_0)| = O\left(\frac{1}{m_n}\right) = o(1) \qquad (3.11)$$

using (S2). Also, uniformly for $\tau \in \Theta_n$, for any $k \geq 1$, if $p \leq 0$, $|\frac{pt_n}{\tau_0} - \frac{kt_n}{\tau}| \geq t_n/\Delta_n$, so that using (2.9),

$$\left| \sum_{-m_n\leq p\leq 0} c_p(s_0)\varphi_n\left(\frac{pt_n}{\tau_0} - \frac{kt_n}{\tau}\right) \right| = O\left(\frac{\Delta_n}{nt_n}\right). \qquad (3.12)$$

Assume now that $\alpha_n$ tends to 0 in such a way that

$$\lim_{n\to+\infty} \alpha_n m_n^2 = 0, \ \lim_{n\to+\infty} \frac{m_n}{(nt_n\alpha_n)^2} = 0 \qquad (3.13)$$

(this is possible since $m_n = o((nt_n)^{2/5})$).

Then: if $0 < p \leq m_n$, for any $1 \leq k \leq m_n$, $\frac{k\tau_0}{p} \in \Theta_n$; thus if $\tau \notin \left(\cup_{l\leq m_n,j} B(\frac{j\tau_0}{l}, \alpha_n)\right) \cap \Theta_n$, then for any $1 \leq k \leq m_n$ and $0 < p \leq m_n$, $|p - \frac{k\tau_0}{\tau}| \geq |p|\alpha_n$. Use of (2.9) leads to

$$\sup_{\tau\notin\left(\underset{l\leq m_n,\,j}{\cup} B(\frac{j\tau_0}{l},\alpha_n)\right)\cap\Theta_n} \left| \sum_{0<p\leq m_n} c_p(s_0)\varphi_n\left(\frac{pt_n}{\tau_0} - \frac{kt_n}{\tau}\right) \right| = O\left(\frac{1}{nt_n\alpha_n}\right). \qquad (3.14)$$

(3.11), (3.12) and (3.14) imply that

$$\sup\left\{ G_n(\tau) \ : \ \tau \notin \left(\underset{l\leq m_n,\,j}{\cup} B(\frac{j\tau_0}{l}, \alpha_n)\right) \cap \Theta_n \right\} = o(1),$$

since by (3.11), $\sum_{k=1}^{m_n} \left| \sum_{|p|\geq m_n} c_p(s_0)\varphi_n\left(\frac{pt_n}{\tau_0} - \frac{kt_n}{\tau}\right) \right|^2 = o(1)$.
If now $l \leq m_n$ and $j$ is such that $j$ and $l$ are relatively prime, and $\frac{j\tau_0}{l} \in \Theta_n$, for any $\tau \in B(\frac{j\tau_0}{l}, \alpha_n)$, uniformly in $l, j, \tau$,

$$\sum_p c_p(s_0)\varphi_n\left(\frac{pt_n}{\tau_0} - \frac{kt_n}{\tau}\right) = \begin{cases} O\left(\frac{m_n\Delta_n}{nt_n}\right) & \text{if } j \nmid k \\ c_{k'l}(s_0)\varphi_n\left(\frac{k'lt_n}{\tau_0} - \frac{k'jt_n}{\tau}\right) + O\left(\frac{1}{nt_n}\right) & \text{if } k = jk' \end{cases} \qquad (3.15)$$

(for a proof, see Section 6), where the notation $u \mid v$ for $v \in \mathbb{Z}$ and $u \in \mathbb{Z}^*$ means that there exists $k \in \mathbb{Z}$ such that $v = ku$ and $u \nmid v$ is the negation of $u \mid v$.
Thus, uniformly on $B(\frac{j\tau_0}{l}, \alpha_n)$ by (S1),

$$G_n(\tau) \leq \sum_{k\geq 1} |c_{kl}(s_0)|^2 + o(1) \leq (1-h) \sum_{k\geq 1} |c_k(s_0)|^2 + o(1) \qquad (3.16)$$

when $l \neq 1$. This implies that with a probability tending to 1, the supremum of $\Lambda_n$ is attained on $\underset{j}{\cup} B(j\tau_0, \alpha_n)$ since

$$G_n(\frac{j\tau_0}{l}) = \sum_{k\geq 1} |c_{kl}(s_0)|^2 + o(1) \text{ when } j \text{ and } l \text{ are relatively prime}$$

and

$$G_n(j\tau_0) = \sum_{k\geq 1} |c_k(s_0)|^2 + o(1).$$

But (3.15) and Lemma 1 lead to:

$$\sup_{\tau\in\cup_j B(j\tau_0,\alpha_n)} \Lambda_n(\tau) \leq \sum_{1\leq k\leq m_n} |c_k(s_0)|^2 + O_P\left(\frac{m_n\Delta_n}{nt_n} + n^{-1/5}\right)$$

and

$$\Lambda_n(\tau_0) = \sum_{1\leq k\leq m_n} |c_k(s_0)|^2 + O_P\left(\frac{m_n\Delta_n}{nt_n} + n^{-1/5}\right),$$

so that with a probability tending to 1, $\tau_0 \in \mathcal{T}_n$. Straightforward consequences of these results together with Lemma 1 are that, with a probability tending to 1,

$$\mathcal{T}_n \subset \cup_{j\leq\frac{\Delta_n}{\tau_0}} B(j\tau_0,\alpha_n),\ \tilde{\tau}_n \in B(\tau_0,\alpha_n), \tag{3.17}$$

since $\tilde{\tau}_n$ is the infimum of $\mathcal{T}_n$.
Let now $\bar{\tau}_n$ be the maximizer of $\Lambda_n$ over $B(\tau_0, 1/4)$. $\bar{\tau}_n$ will only be a theoretical tool since of course one needs to know $\tau_0$ to compute it. We have again:

$$\bar{\tau}_n \in B(\tau_0,\alpha_n).$$

But for any $\rho > 0$, for any $\tau \in B(\tau_0,\alpha_n)$ and $\tau \notin B(\tau_0, \rho/nt_n)$ one has

$$\begin{aligned}
G_n(\tau) - G_n(\tau_0) &= \sum_{k=1}^{m_n} |c_k(s_0)|^2 \left\{\left|\varphi_n\left(\frac{kt_n}{\tau_0}(1-\frac{\tau_0}{\tau})\right)\right|^2 - 1\right\} + o(1)\\
&= \sum_{k=1}^{m_n} |c_k(s_0)|^2 \left\{\left(\frac{\sin\left[\frac{nk\pi t_n}{\tau_0}(1-\frac{\tau_0}{\tau})\right]}{\frac{nk\pi t_n}{\tau_0}(1-\frac{\tau_0}{\tau})}\right)^2 - 1\right\} + o(1)\\
&\leq -\sum_{k=1}^{m_n} |c_k(s_0)|^2 \int_0^1 \left(\sin\left[t\frac{nk\pi t_n}{\tau_0}(1-\frac{\tau_0}{\tau})\right]\right)^2 dt + o(1)\\
&\leq -\min\left\{C\frac{\rho^2}{\tau_0}; \frac{1}{2}\left(1-\frac{1}{\pi}\right)\right\}\sum_{k\geq 1} |c_k(s_0)|^2 + o(1)
\end{aligned}$$

for some constant $C > 0$. One thus easily obtains that for any $\rho > 0$,

$$\lim_{n\to+\infty} P\left(\tilde{\tau}_n \notin B(\tau_0, \frac{\rho}{nt_n})\right) = 0, \tag{3.18}$$

and

$$\lim_{n\to+\infty} P\left(\bar{\tau}_n \notin B(\tau_0, \frac{\rho}{nt_n})\right) = 0,$$

which implies that there exists $\rho_n$ tending to 0 such that,

$$\lim_{n\to+\infty} P\left(\bar{\tau}_n \notin B(\tau_0, \frac{\rho_n}{nt_n})\right) = 0. \tag{3.19}$$

Notice that the refinement of the speed of convergence of $\bar{\tau}_n$ to $\tau_0$ as $o(1/nt_n)$ will be needed in the proof of the linearization below.
Recall that $\hat{\tau}_n$ is the maximizer of $\Lambda_n$ over $B(\tilde{\tau}_n, 1/8)$, so that these results lead to

$$\lim_{n\to+\infty} P\left(\hat{\tau}_n = \bar{\tau}_n\right) = 1. \tag{3.20}$$

We now may write:

$$\Lambda_n'(\bar{\tau}_n) = 0 = \Lambda_n'(\tau_0) + (\bar{\tau}_n - \tau_0)\,\Lambda_n''(\tau')$$

for some random $\tau' \in B(\tau_0, \frac{\rho_n}{nt_n})$. We prove in section 6 that:

$$\Lambda_n'(\tau_0) = \left(\frac{1}{\tau_0^2}\sum_{j=1}^n t_n\left(\frac{1}{2}-\frac{j}{n}\right) f_0'\left(\frac{jt_n}{\tau_0}\right)\varepsilon(j)\right)(1 + o_{\mathbb{P}_n^0}(1)), \tag{3.21}$$

and uniformly for $\tilde{\tau} \in B(\tau_0, \frac{\rho_n}{nt_n})$

$$\Lambda_n''(\tilde{\tau}) = -n^2 t_n^2 \sigma_0^2 I_* (1 + o_{\mathbb{P}_n^0}(1)). \tag{3.22}$$

But this leads to

$$n^{3/2} t_n (\bar{\tau}_n - \tau_0) = \frac{{I_*}^{-1}}{\sigma_0^2 n^{3/2} t_n} \sum_{j=1}^{n} \left(\frac{n}{2} - j\right) \frac{t_n}{\tau_0^2} f_0'\left(\frac{jt_n}{\tau_0}\right) \varepsilon(j) + o_{\mathbb{P}_n^0}(1)$$

which by equation (3.20) implies that

$$n^{3/2} t_n (\hat{\tau}_n - \tau_0) = \frac{{I_*}^{-1}}{\sigma_0^2 n^{3/2} t_n} \sum_{j=1}^{n} \left(\frac{n}{2} - j\right) \frac{t_n}{\tau_0^2} f_0'\left(\frac{jt_n}{\tau_0}\right) \varepsilon(j) + o_{\mathbb{P}_n^0}(1).$$

By Proposition 1 below, $\hat{\tau}_n$ is asymptotically efficient and $n^{3/2} t_n(\hat{\tau}_n - \tau_0)$ converges in distribution to the centered Gaussian distribution with variance $1/I_*$.

We introduce some notations in order to find a condition which ensures the asymptotic efficiency of an estimator of $\tau_0$. In order to define efficiency, we shall use LAN theory like in Le Cam (1986), van der Vaart (1998), McNeney and Wellner (2000). Denote $\mathbb{P}_{n,(\tau,f,\sigma)}$ the distribution of $(X(1), \ldots, X(n))$ when the parameter is $(\tau, f, \sigma)$, and $\mathbb{P}_n^0$ when the parameter is $(\tau_0, f_0, \sigma_0)$.
Let $L_n(\tau, f, \sigma)$ be the log-likelihood ratio, that is

$$L_n(\tau, f, \sigma) = \log \frac{d\mathbb{P}_{n,(\tau,f,\sigma)}(X(1), \cdots, X(n))}{d\mathbb{P}_{n,(\tau_0,f_0,\sigma_0)}(X(1), \cdots, X(n))}.$$

Define $\mathcal{T}$ the set of periodic functions $f$ with period 1, with square integrable restriction on $[0, 1]$ and such that $\sum_{k\in\mathbb{Z}} |c_k(f)| < +\infty$. We shall follow the semiparametric analysis developed in McNeney and Wellner (2000).

**Proposition 1.** *If $f_0' \in \mathcal{T}$, under (N1), the sequence of statistical models $\left(\mathbb{P}_{n,(\tau,f,\sigma)}\right)$ is locally asymptotically normal at $(\tau_0, f_0, \sigma_0)$ with tangent space $\mathcal{H} = \mathbb{R} \times \mathcal{T} \times \mathbb{R}_+$ endowed with the inner product*

$$\langle (a, f, b), (c, g, d) \rangle_{\mathcal{H}} = \frac{1}{\sigma_0^2} \int_0^1 \left(f(t) - \frac{a}{2\tau_0^2} f_0'(t)\right) \left(g(t) - \frac{c}{2\tau_0^2} f_0'(t)\right) dt + \frac{ac}{12\sigma_0^2 \tau_0^4} \int_0^1 {f_0'}^2(t) dt + 2\frac{bd}{\sigma_0^2},$$

*that is*

$$L_n(\tau_0 + \frac{a}{n^{3/2} t_n}, f_0 + \frac{f}{\sqrt{n}}, \sigma_0 + \frac{b}{\sqrt{n}}) = W_n(a, f, b) - \frac{1}{2} \parallel (a, f, b) \parallel_{\mathcal{H}}^2 + o_{\mathbb{P}_n^0}(1)$$

*where*

$$W_n(a, f, b) = \sum_{j=1}^{n} \left(\frac{1}{\sqrt{n}} f\left(\frac{jt_n}{\tau_0}\right) - \frac{a}{n^{3/2} t_n} \left(\frac{jt_n}{\tau_0^2}\right) f_0'\left(\frac{jt_n}{\tau_0}\right)\right) \frac{\varepsilon(j)}{\sigma_0^2} + \frac{1}{\sqrt{n}} \sum_{j=1}^{n} \frac{b}{\sigma_0} \left(\left(\frac{\varepsilon(j)}{\sigma_0}\right)^2 - 1\right),$$

*and for any finite subset $h_1, \ldots h_d \in \mathcal{H}$, $(W_n(h_1), \ldots, W_n(h_d))$ converges in distribution under $\mathbb{P}_n^0$ to the centered Gaussian vector with covariance $\langle h_i, h_j \rangle_{\mathcal{H}}$.*
*The parameter $\tau = \nu_n(\mathbb{P}_{n,(\tau,f,\sigma)})$ is differentiable with derivative $\dot{\nu}(a, f, b) = a$, and*

$$\dot{\nu}^T(a) = \left(\frac{a}{I_*}, \frac{a}{2\tau_0^2 I_*} f_0', 0\right),$$

*where $I_*$, the efficient Fisher information for $\tau$, is given by*

$$I_* = \frac{1}{12\sigma_0^2 \tau_0^4} \int_0^1 f_0'^2(t) dt.$$

*In particular if $\hat{\tau}$ satisfies:*

$$n^{3/2} t_n (\hat{\tau} - \tau_0) = \frac{{I_*}^{-1}}{\sigma_0^2 n^{3/2} t_n} \sum_{j=1}^{n} \left(\frac{n}{2} - j\right) \frac{t_n}{\tau_0^2} f_0'\left(\frac{jt_n}{\tau_0}\right) \varepsilon(j) + o_{\mathbb{P}_n^0}(1) \tag{3.23}$$

*then $\hat{\tau}$ is regular and efficient.*

The proof is given in section 6.

**4. The two periods model** We will now focus on model (1.2), so that

$$X(j) = s_1^0(jt_n) + s_2^0(jt_n) + \varepsilon(j) = f_1^0\left(\frac{jt_n}{\tau_1^0}\right) + f_2^0\left(\frac{jt_n}{\tau_2^0}\right) + \varepsilon(j) \quad ; \quad 1 \le j \le n$$

where $f_1^0$ and $f_2^0$ are periodic functions of period 1 and

$$s_1^0(t) = \sum_{k\in\mathbb{Z}} c_k^0 \exp(\frac{2ik\pi t}{\tau_1^0}),\ s_2^0(t) = \sum_{k\in\mathbb{Z}} d_k^0 \exp(\frac{2ik\pi t}{\tau_2^0}).$$

In this section, we assume for technical reasons that an upper bound and a lower bound are known for both periods, that is we set $\Theta_n = \Theta = [\delta, \Delta]$ for some fixed $\delta > 0$ and $\Delta > 0$.
We shall now make precise the fact that a signal is the sum of exactly two periodic signals. Let $s$ be a signal such that there exist two periodic functions $s_1$, $s_2$ with respective periods $\tau_1$ and $\tau_2$ with $s = s_1 + s_2$. Define the functions $f_1(s_1, s_2)$, $f_2(s_1, s_2)$, $g(s_1, s_2)$ as follows. If $\tau_1/\tau_2 = l/m$ where $l$ and $m$ are relatively prime integers,

$$\begin{cases} g(s_1, s_2)(x) &= \sum_{k\in\mathbb{Z}} (c_{lk}(s_1) + c_{mk}(s_2))e^{2ik\pi x}, \\ f_1(s_1, s_2)(x) = & \sum_{k\in\mathbb{Z}, l\nmid k} c_k(s_1)e^{2ik\pi x}, \\ f_2(s_1, s_2)(x) = & \sum_{k\in\mathbb{Z}, m\nmid k} c_k(s_2)e^{2ik\pi x}, \end{cases} \tag{4.24}$$

where the notation $u \mid v$ for $v \in \mathbb{Z}$ and $u \in \mathbb{Z}^*$ means that there exists $k \in \mathbb{Z}$ such that $v = ku$ and $u \nmid v$ is the negation of $u \mid v$. If $\tau_1/\tau_2 \notin \mathbb{Q}$,

$$\begin{cases} g(s_1, s_2)(x) &= 0, \\ f_1(s_1, s_2)(x) = & \sum_{k\in\mathbb{Z}} c_k(s_1)e^{2ik\pi x}, \\ f_2(s_1, s_2)(x) = & \sum_{k\in\mathbb{Z}} c_k(s_2)e^{2ik\pi x}. \end{cases} \tag{4.25}$$

We shall say that a signal $s$ can be split into two periodic functions if and only if there exists two periodic functions $s_1$, $s_2$ such that $f_1(s_1, s_2) \neq 0$ and $f_2(s_1, s_2) \neq 0$, and that a signal $s$ can not be split into two periodic functions if and only if for all periodic functions $s_1$, $s_2$ such that $s = s_1 + s_2$, $f_1(s_1, s_2) = 0$ or $f_2(s_1, s_2) = 0$.
In this section, we assume that $s_0$ can be split into two periodic functions, so that in (1.2), the couple of periods $(\tau_1^0, \tau_2^0)$ is given by (2.5). Let $\tilde{f}_1 = f_1(s_1^0, s_2^0)$, $\tilde{f}_2 = f_2(s_1^0, s_2^0)$, and $g(s_1^0, s_2^0) = g_0$. We shall also assume that $\tilde{f}_1$, $\tilde{f}_2$, and $g_0$ can not be split into two periodic functions. This is what will be meant here by "$s_0$ is the sum of exactly two periodic signals".
Note that when the ratio $\tau_1^0/\tau_2^0$ tends to a non rational number then the numbers $l$ and $m$ tend to infinity. Thus, $g_0$ tends to zero, $\tilde{f}_1$ tends to $f_1^0$ and $\tilde{f}_2$ tends to $f_2^0$. Note also that this assumption excludes periods with integer ratio.

The fact that $s_0$ is the sum of exactly two periodic signals implies that there exists $h \in ]0, 1[$ such that for any integers $r, r' \ge 2$,

$$(\mathbf{I}) \ : \ \sum_{k\ge1} |c_{kr'}(f)|^2 + \sum_{k\ge1} |c_{kr}(f)|^2 - \sum_{k\ge1} |c_{ekr_1r_1'}(f)|^2 \le (1-h) \sum_{k\ge1} |c_k(f)|^2$$

for $f = \tilde{f}_1$, $f = \tilde{f}_2$ and $f = g_0$, where $e$ is the greatest common divisor of $r$ and $r'$ and $r_1$, $r_1'$ are such that $r = er_1$, $r' = er_1'$. This is proved in section 6.
We shall now define the estimator of the couple of periods.
In the criterion $\mathcal{H}_n$, we shall use $\beta_n$ verifying

$$(\mathbf{N5}) : \lim_{n\to+\infty} m_n\beta_n = 0,\ \beta_n \ge \left(\frac{\Delta}{\delta}\right)\left(\frac{1+\alpha_n}{1-\alpha_n} - 1\right)$$

for the sequence $\alpha_n$ defined in section 3.
Let $\varepsilon \in ]0; h[$, and define

$$\mathcal{U}_n = \left\{(\theta_1, \theta_2) \in \Theta^2 \ : \ \mathcal{H}_n(\theta_1, \theta_2) \ge (1-\varepsilon)\sup_{\Theta^2} \mathcal{H}_n(\tau_1, \tau_2)\right\}.$$

Notice that of course, $\mathcal{U}_n$ is symmetrical with respect to the 45 degree line (that is $(\tau_1,\tau_2)\in\mathcal{U}_n$ if and only if $(\tau_2,\tau_1)\in\mathcal{U}_n$).
Let

$$\tilde{\tau}_2=\inf\left\{\tau_2\ :(\tau_1,\tau_2)\in\mathcal{U}_n,\ \tau_1<\tau_2\right\}$$

and

$$\tilde{\tau}_1=\inf\left\{\tau_1\ :(\tau_1,\tilde{\tau}_2)\in\mathcal{U}_n\right\}.$$

The estimator $(\hat{\tau}_1,\hat{\tau}_2)$ is defined as the maximizer of $\mathcal{H}_n$ over $(B(\tilde{\tau}_1,1/8)\times B(\tilde{\tau}_2,1/8))\cap\{\tau_1<\tau_2\}$.

**Theorem 2.** *Assume (N1), (N3), (N5), and (I). Assume also that $s_1^0$ and $s_2^0$ satisfy (S2). Let $\tau_*=\tau_1^0/l=\tau_2^0/m$ if $\tau_1^0/\tau_2^0=l/m$. Then*

$$n^{3/2}t_n\begin{pmatrix}\hat{\tau}_1-\tau_1^0\\ \hat{\tau}_2-\tau_2^0\end{pmatrix}=\frac{{K_*}^{-1}}{n^{3/2}t_n}\begin{pmatrix}\sum\limits_{j=1}^{n}\left(\frac{n}{2}-j\right)t_n\left(\frac{1}{(\tau_1^0)^2}\tilde{f}_1'\left(\frac{jt_n}{\tau_1^0}\right)+\frac{g_0'\left(\frac{jt_n}{\tau_*}\right)}{2\tau_1^0\tau_*}\right)\varepsilon(j)\\ \sum\limits_{j=1}^{n}\left(\frac{n}{2}-j\right)t_n\left(\frac{1}{(\tau_2^0)^2}\tilde{f}_2'\left(\frac{jt_n}{\tau_2^0}\right)+\frac{g_0'\left(\frac{jt_n}{\tau_*}\right)}{2\tau_2^0\tau_*}\right)\varepsilon(j)\end{pmatrix}+o_{\mathbb{P}_n^0}(1),$$

*where*

$$K_*=\begin{pmatrix}\frac{1}{12}\left(\frac{1}{(\tau_1^0)^4}\int_0^1{\tilde{f}_1'}^2(t)dt-\frac{1}{(\tau_1^0)^2(\tau_*)^2}\int_0^1{g_0'}^2(t)dt\right) & \frac{1}{8(\tau_*)^2\tau_1^0\tau_2^0}\int_0^1{g_0'}^2(t)dt\\ \frac{1}{8(\tau_*)^2\tau_1^0\tau_2^0}\int_0^1{g_0'}^2(t)dt & \frac{1}{12}\left(\frac{1}{(\tau_2^0)^4}\int_0^1{\tilde{f}_2'}^2(t)dt-\frac{1}{(\tau_2^0)^2(\tau_*)^2}\int_0^1{g_0'}^2(t)dt\right)\end{pmatrix}.$$

*Let*

$$\begin{aligned}J_*&=\frac{1}{12\sigma_0^2}\lim_{T\to+\infty}\frac{1}{T}\begin{pmatrix}\frac{1}{(\tau_1^0)^4}\int_{-T/2}^{T/2}({f_1^0}')^2(\frac{t}{\tau_1^0})dt & \frac{1}{(\tau_1^0)^2(\tau_2^0)^2}\int_{-T/2}^{T/2}{f_1^0}'(\frac{t}{\tau_1^0}){f_2^0}'(\frac{t}{\tau_2^0})dt\\ \frac{1}{(\tau_1^0)^2(\tau_2^0)^2}\int_{-T/2}^{T/2}{f_1^0}'(\frac{t}{\tau_1^0}){f_2^0}'(\frac{t}{\tau_2^0})dt & \frac{1}{(\tau_2^0)^4}\int_{-T/2}^{T/2}({f_2^0}')^2(\frac{t}{\tau_2^0})dt\end{pmatrix}\\ &=\frac{1}{12\sigma_0^2}\begin{pmatrix}\frac{1}{(\tau_1^0)^4}\int_0^1{\tilde{f}_1'}^2(t)dt+\frac{1}{4(\tau_1^0)^2(\tau_*)^2}\int_0^1{g_0'}^2(t)dt & \frac{1}{4(\tau_*)^2\tau_1^0\tau_2^0}\int_0^1{g_0'}^2(t)dt\\ \frac{1}{4(\tau_*)^2\tau_1^0\tau_2^0}\int_0^1{g_0'}^2(t)dt & \frac{1}{(\tau_2^0)^4}\int_0^1{\tilde{f}_2'}^2(t)dt+\frac{1}{4(\tau_2^0)^2(\tau_*)^2}\int_0^1{g_0'}^2(t)dt\end{pmatrix}.\end{aligned}$$

*Then*

$$n^{3/2}t_n\begin{pmatrix}\hat{\tau}_1-\tau_1^0\\ \hat{\tau}_2-\tau_2^0\end{pmatrix}\to\mathcal{N}\left(0,K_*^{-1}J_*(K_*^{-1})^T\right).$$

**Remark**: By proposition 2 below, we see that $(\hat{\tau}_1,\hat{\tau}_2)$ is asymptotically efficient when $g_0=0$. This occurs in particular when $\tau_1^0/\tau_2^0$ is a non rational number. In particular, one sees that:

- If $\tau_1^0/\tau_2^0\notin\mathbb{Q}$, $J_*$ is diagonal and linearized efficient estimators of the periods are asymptotically independent,
- and if $\tau_1^0/\tau_2^0\in\mathbb{Q}$, $J_*$ is in general not diagonal and linearized efficient estimators of the periods are asymptotically correlated.

Note also that if $\tau_1^0/\tau_2^0$ is a rational number then $\varepsilon$ in the definition of $\mathcal{U}_n$ can be replaced by a sequence $\varepsilon_n$ having the same properties as the sequence $\varepsilon_n$ used in the single period case.

We introduce some notations in order to find a condition which ensures the asymptotic efficiency of estimators of $\tau_1^0$ and $\tau_2^0$. Let $\mathbb{P}_{n,(\tau_1,\tau_2,f_1,f_2,\sigma)}$ be the distribution of $(X(1),\cdots,X(n))$ when the parameter is $(\tau_1,\tau_2,f_1,f_2,\sigma)$, and $\mathbb{P}_n^0$ when the parameter is $(\tau_1^0,\tau_2^0,f_1^0,f_2^0,\sigma_0)$. Let $L_n(\cdot)$ be the log-likelihood ratio.
Let $\mathcal{H}$ be the quotient space of $\mathbb{R}^2\times\mathcal{T}^2\times\mathbb{R}_+$ by the equivalence relation

$$(a_1,b_1,f_1,g_1,c_1)\sim(a_2,b_2,f_2,g_2,c_2)$$

if and only if for all $t$

$$f_1(\frac{t}{\tau_1^0})+g_1(\frac{t}{\tau_2^0})-\frac{a_1}{2(\tau_1^0)^2}f_1^{0\prime}(\frac{t}{\tau_1^0})-\frac{b_1}{2(\tau_2^0)^2}f_2^{0\prime}(\frac{t}{\tau_2^0}) = f_2(\frac{t}{\tau_1^0})+g_2(\frac{t}{\tau_2^0})-\frac{a_2}{2(\tau_1^0)^2}f_1^{0\prime}(\frac{t}{\tau_1^0})-\frac{b_2}{2(\tau_2^0)^2}f_2^{0\prime}(\frac{t}{\tau_2^0}).$$

**Proposition 2.** *If $f_1^{0\prime}$, $f_2^{0\prime}$ are in $\mathcal{T}$, under (N1), the sequence of statistical models $\left(\mathbb{P}_{n,(\tau_1,\tau_2,f_1,f_2,\sigma)}\right)$ is locally asymptotically normal at $(\tau_1^0,\tau_2^0,f_1^0,f_2^0,\sigma_0)$ with tangent space $\mathcal{H}$ endowed with the inner product*

$$\langle (a_1,b_1,f_1,g_1,c_1),(a_2,b_2,f_2,g_2,c_2)\rangle_{\mathcal{H}} =$$

$$\frac{1}{\sigma_0^2}\lim_{T\to+\infty}\frac{1}{T}\int_{-T/2}^{T/2}\left(f_1(\frac{t}{\tau_1^0})+g_1(\frac{t}{\tau_2^0})-\frac{a_1}{2(\tau_1^0)^2}f_1^{0\prime}(\frac{t}{\tau_1^0})-\frac{b_1}{2(\tau_2^0)^2}f_2^{0\prime}(\frac{t}{\tau_2^0})\right)$$
$$\left(f_2(\frac{t}{\tau_1^0})+g_2(\frac{t}{\tau_2^0})-\frac{a_2}{2(\tau_1^0)^2}f_1^{0\prime}(\frac{t}{\tau_1^0})-\frac{b_2}{2(\tau_2^0)^2}f_2^{0\prime}(\frac{t}{\tau_2^0})\right)dt$$
$$+\frac{1}{12\sigma_0^2}\lim_{T\to+\infty}\frac{1}{T}\int_{-T/2}^{T/2}\left(\frac{a_1}{(\tau_1^0)^2}f_1^{0\prime}(\frac{t}{\tau_1^0})+\frac{b_1}{(\tau_2^0)^2}f_2^{0\prime}(\frac{t}{\tau_2^0})\right)\left(\frac{a_2}{(\tau_1^0)^2}f_1^{0\prime}(\frac{t}{\tau_1^0})+\frac{b_2}{(\tau_2^0)^2}f_2^{0\prime}(\frac{t}{\tau_2^0})\right)dt+2\frac{c_1c_2}{\sigma_0^2},$$

*that is*

$$L_n(\tau_1^0+\frac{a}{n^{3/2}t_n},\tau_2^0+\frac{b}{n^{3/2}t_n},f_1^0+\frac{f}{\sqrt{n}},f_2^0+\frac{g}{\sqrt{n}}\sigma_0+\frac{c}{\sqrt{n}}) = W_n(a,b,f,g,c)-\frac{1}{2}\parallel (a,b,f,g,c)\parallel_{\mathcal{H}}^2+o_{\mathbb{P}_n^0}(1)$$

*where*

$$W_n(a,b,f,g,c)=\sum_{j=1}^n\frac{\varepsilon(j)}{\sigma_0^2}A_{j,n}+\frac{1}{\sqrt{n}}\sum_{j=1}^n\frac{c}{\sigma_0}\left((\frac{\varepsilon(j)}{\sigma_0})^2-1\right),$$

$$A_{j,n}=\left(\frac{1}{\sqrt{n}}f\left(\frac{jt_n}{\tau_1^0}\right)-\frac{a}{n^{3/2}t_n}\left(\frac{jt_n}{(\tau_1^0)^2}\right)f_1^{0\prime}\left(\frac{jt_n}{\tau_1^0}\right)+\frac{1}{\sqrt{n}}g\left(\frac{jt_n}{\tau_2^0}\right)-\frac{b}{n^{3/2}t_n}\left(\frac{jt_n}{(\tau_2^0)^2}\right)f_2^{0\prime}\left(\frac{jt_n}{\tau_2^0}\right)\right),$$

*and for any finite subset $h_1,\dots h_d\in\mathcal{H}$, $(W_n(h_1),\dots,W_n(h_d))$ converges in distribution under $\mathbb{P}_n^0$ to the centered Gaussian vector with covariance $\langle h_i,h_j\rangle_{\mathcal{H}}$.*
*The parameter $(\tau_1,\tau_2)=\nu_n(\mathbb{P}_{n,(\tau_1,\tau_2,f_1,f_2,\sigma)})$ is differentiable with derivative $\dot{\nu}(a,b,f,g,c)=(a,b)$, and*

$$\dot{\nu}^T(a,b)=\left(\tilde{a},\tilde{b},\frac{\tilde{a}}{2(\tau_1^0)^2}f_1^{0\prime}(\frac{t}{\tau_1^0}),\frac{\tilde{b}}{2(\tau_2^0)^2}f_2^{0\prime}(\frac{t}{\tau_2^0}),0\right),\ \begin{pmatrix}\tilde{a}\\ \tilde{b}\end{pmatrix}=J_*^{-1}\begin{pmatrix}a\\ b\end{pmatrix}.$$

*In particular if $(\hat{\tau}_1,\hat{\tau}_2)$ satisfies:*

$$n^{3/2}t_n\begin{pmatrix}\hat{\tau}_1-\tau_1^0\\ \hat{\tau}_2-\tau_2^0\end{pmatrix}=\frac{{J_*}^{-1}}{\sigma_0^2n^{3/2}t_n}\begin{pmatrix}\sum_{j=1}^n\left(\frac{n}{2}-j\right)\frac{t_n}{(\tau_1^0)^2}f_1^{0\prime}\left(\frac{jt_n}{\tau_1^0}\right)\varepsilon(j)\\ \sum_{j=1}^n\left(\frac{n}{2}-j\right)\frac{t_n}{(\tau_2^0)^2}f_2^{0\prime}\left(\frac{jt_n}{\tau_2^0}\right)\varepsilon(j)\end{pmatrix}+o_{\mathbb{P}_n^0}(1),$$

*then $(\hat{\tau}_1,\hat{\tau}_2)$ is regular and efficient.*

The proof is given in section 6.
The limits involved in the Proposition exist. Indeed, define

$$\varphi(\tau)=\int_{-1/2}^{1/2}e^{2i\pi u\tau}du=\frac{\sin(\pi\tau)}{\pi\tau},\quad |\varphi(\tau)|\le C\min\left(1,1/|\tau|\right).$$

One has for any $f_1$ and $f_2$ in $\mathcal{T}$ and positive real numbers $\tau_1$ and $\tau_2$

$$\frac{1}{T}\int_{-T/2}^{T/2}f_1\left(\frac{t}{\tau_1}\right)f_2\left(\frac{t}{\tau_2}\right)dt=\sum_{k,l}c_k(f_1)c_l(f_2)\varphi\left[T\left(\frac{k}{\tau_1}+\frac{l}{\tau_2}\right)\right]. \tag{4.26}$$

If $\tau_1/\tau_2$ is an irreducible fraction $p/q$,

$$\lim_{T\to+\infty}\frac{1}{T}\int_{-T/2}^{T/2}f_1\left(\frac{t}{\tau_1}\right)f_2\left(\frac{t}{\tau_2}\right)dt=\sum_m c_{mp}(f_1)c_{-mq}(f_2),$$

and if $\tau_1/\tau_2 \notin \mathbb{Q}$, it can be proved (see section 6) that

$$\lim_{T\to+\infty}\frac{1}{T}\int_{-T/2}^{T/2} f_1\left(\frac{t}{\tau_1}\right) f_2\left(\frac{t}{\tau_2}\right) dt = c_0(f_1)c_0(f_2) = \int_0^1 f_1(t)dt \int_0^1 f_2(t)dt.$$

**Proof of Theorem 2.** We shall use as intermediate tool $(\bar\tau_1, \bar\tau_2)$, the maximizer of $\mathcal{H}_n$ over $(B(\tau_1^0, 1/8)\times B(\tau_2^0, 1/8))\cap\{\tau_1<\tau_2\}$.

Denote by $H_n(\tau_1,\tau_2)$ the deterministic part of $\mathcal{H}_n(\tau_1,\tau_2)$ defined by

$$H_n(\tau_1,\tau_2) = G_n(\tau_1) + G_n(\tau_2) - G_n^c(\tau_1,\tau_2)$$

where

$$G_n^c(\tau_1,\tau_2) = \frac{1}{n^2}\mathcal{R}e\left\{\sum_{(k,k')\in K_n(\tau_1,\tau_2)}\left(\sum_{j=1}^n e^{\frac{2ik\pi jt_n}{\tau_1}}(s_1^0+s_2^0)(jt_n)\right)\left(\sum_{j=1}^n e^{-\frac{2ik'\pi jt_n}{\tau_2}}(s_1^0+s_2^0)(jt_n)\right)\right\}$$

and by

$$R_n(\tau_1,\tau_2) = \mathcal{H}_n(\tau_1,\tau_2) - H_n(\tau_1,\tau_2) = R_{n,1}(\tau_1,\tau_2) + R_{n,2}(\tau_1,\tau_2)$$

the random part of $\mathcal{H}_n$, with

$$R_{n,2}(\tau_1,\tau_2) = \zeta_n(\tau_1) + \zeta_n(\tau_2) - \zeta_n^c(\tau_1,\tau_2)$$

where

$$\zeta_n^c(\tau_1,\tau_2) = \frac{1}{n^2}\mathcal{R}e\left\{\sum_{(k,k')\in K_n(\tau_1,\tau_2)}\left(\sum_{j=1}^n e^{\frac{2ik\pi jt_n}{\tau_1}}\varepsilon(j)\right)\left(\sum_{j=1}^n e^{-\frac{2ik'\pi jt_n}{\tau_2}}\varepsilon(j)\right)\right\}.$$

As for the single period case, the first point is to obtain consistency of $(\tilde\tau_1,\tilde\tau_2)$ and $(\bar\tau_1,\bar\tau_2)$ at some speed, and this point relies on precise analysis of the deterministic part. We thus begin to prove that the random part is negligible in probability.

**Lemma 2.** *Under the assumptions of Theorem 2, for all $x_n$ such that $\lim_{n\to+\infty} x_n n^{1/4-\beta} = +\infty$ for some $\beta > 0$,*

$$\lim_{n\to+\infty} P\left(\sup_{(\tau_1,\tau_2)\in\Theta^2}|R_n(\tau_1,\tau_2)|\geq x_n\right) = 0$$

A consequence of this lemma is that $\sup_{(\tau_1,\tau_2)\in\Theta^2}|R_n(\tau_1,\tau_2)| = o_P(n^{-1/5})$.
A proof of this Lemma is given in section 6.

**Lemma 3.** *There exists $\rho_n$ tending to 0 such that with a probability tending to 1,*

$$(\tilde\tau_1,\tilde\tau_2)\in B(\tau_1^0,\frac{\rho_n}{nt_n})\times B(\tau_2^0,\frac{\rho_n}{nt_n}),\ (\bar\tau_1,\bar\tau_2)\in B(\tau_1^0,\frac{\rho_n}{nt_n})\times B(\tau_2^0,\frac{\rho_n}{nt_n}).$$

Proof. We shall consider separately the case of rational or non rational ratio of the two periods.
Consider first the case where $\frac{\tau_1^0}{\tau_2^0}\in\mathbb{Q}$, say $\frac{\tau_1^0}{\tau_2^0}=\frac{l}{m}$.
Notice that for any integers $j,r$, $B(\frac{j\tau_1^0}{r},\alpha_n) = B(\frac{jl\tau_2^0}{rm},\alpha_n)$.
Using $\lim_{n\to+\infty} m_n\beta_n = 0$, for large enough $n$,

$$K_n(\tau_1^0,\tau_2^0) = \left\{(k_1,k_2)\in\{1,\ldots,m_n\}^2, \frac{k_1}{k_2}=\frac{l}{m}\right\}. \tag{4.27}$$

Also, one obtains that for any $r_i\leq m_n$ and $j_i$ such that $(j_i\tau_i^0/r_i)\in\Theta$, $i=1,2$,

$$\forall\tau_1\in B(\frac{j_1\tau_1^0}{r_1},\alpha_n), \forall\tau_2\in B(\frac{j_2\tau_2^0}{r_2},\alpha_n), \left|\frac{\tau_1}{\tau_2}-\frac{j_1r_2l}{j_2r_1m}\right|\leq\left(\frac{\Delta}{\delta}\right)\left(\frac{1+\alpha_n}{1-\alpha_n}-1\right)$$

so that using (N5), for large enough $n$, for any $r_i \le m_n$ and $j_i$ such that $(j_i\tau_i^0/r_i) \in \Theta$, $i=1,2$, $\forall \tau_1 \in B(\frac{j_1\tau_1^0}{r_1}, \alpha_n), \forall \tau_2 \in B(\frac{j_2\tau_2^0}{r_2}, \alpha_n)$,

$$\left\{(k_1,k_2) \in \{1,\ldots,m_n\}^2 : \frac{k_1}{k_2} = \frac{j_1 r_2 l}{j_2 r_1 m}\right\} \subset K_n(\tau_1,\tau_2). \tag{4.28}$$

Now, for any integer $k$ and any $\tau$,

$$\frac{1}{n}\sum_{j=1}^{n} e^{-\frac{2ik\pi j t_n}{\tau}}(s_1^0+s_2^0)(jt_n) = \sum_p c_p^0 \varphi_n\left(\frac{pt_n}{\tau_1^0} - \frac{kt_n}{\tau}\right) + \sum_p d_p^0 \varphi_n\left(\frac{pt_n}{\tau_2^0} - \frac{kt_n}{\tau}\right). \tag{4.29}$$

Using (4.27) and equation (3.15) repeatedly, one easily obtains

$$\mathcal{H}_n(\tau_1^0,\tau_2^0) = \sum_{k\ge 1}\left|c_{kl}^0 + d_{km}^0\right|^2 + \sum_{k\ge 1, l\nmid k}\left|c_k^0\right|^2 + \sum_{k\ge 1, m\nmid k}\left|d_k^0\right|^2 + o_{\mathbb{P}_n^0}(1).$$

Also, using (4.28) and equation (3.15) repeatedly, one obtains that uniformly in $r_i \le m_n$ and $j_i$ such that $(j_i\tau_i^0/r_i)\in\Theta$, $i=1,2$, uniformly for $\tau_1 \in B(\frac{j_1\tau_1^0}{r_1},\alpha_n)$ and $\tau_2 \in B(\frac{j_2\tau_2^0}{r_2},\alpha_n)$,

$$\begin{aligned} H_n(\tau_1,\tau_2) \le &\sum_{k\ge 1}\left|c_{kmr_3}(s_2^0) + c_{klr_3}(s_1^0)\right|^2 + \sum_{k\ge 1, b|k, l\nmid k}\left|c_{kr_3}(s_1^0)\right|^2 + \sum_{k\ge 1, m\nmid k, m_4|k}\left|c_{kr_3}(s_2^0)\right|^2 \\ &+ \sum_{k\ge 1, r_5\nmid k}\left|c_{kmr_4}(s_2^0) + c_{klr_4}(s_1^0)\right|^2 + \sum_{k\ge 1, l_2|k, l\nmid k, l_2 r_7\nmid k}\left|c_{kr_4}(s_1^0)\right|^2 + \sum_{k\ge 1, m\nmid k, d|k, dr_{10}\nmid k}\left|c_{kr_3}(s_2^0)\right|^2 + o(1)\end{aligned}$$

where $(j_1;m)=a$ with $m=am_4$, $(r_1;l)=b$ with $l=bl_1$ and $r_1=br_3$, $(m;r_2)=d$ with $r_2=dr_4$, $(j_2;l)=c$ with $l=cl_2$, $(r_3;r_4)=e$ with $r_3=er_5$, $(r_1;l_2r_4)=f$ with $r_1=fr_7$ and $(m_4r_3;r_2)=g$ with $m_4r_3=gr_{10}$. Here, the notation $(a;b)$ for positive integers $a$ and $b$ refers to the greatest common divisor of $a$ and $b$. This implies that,

$$\sup_{r_1\nmid j_1, r_2\nmid j_2}\ \sup_{\tau_1\in B(\frac{j_1\tau_1^0}{r_1},\alpha_n), \tau_2\in B(\frac{j_2\tau_2^0}{r_2},\alpha_n)} H_n(\tau_1,\tau_2) \le (1-h)H_n(\tau_1^0,\tau_2^0) + o(1)$$

using (I). Same results hold exchanging $\tau_1$ and $\tau_2$.
Also, recalling that $\tau_0$ is the global period of the signal,

$$\sup_{j_1, r_2\nmid j_2}\ \sup_{\tau_1\in B(j_1\tau_0,\alpha_n), \tau_2\in B(\frac{j_2\tau_2^0}{r_2},\alpha_n)} \max\left\{H_n(\tau_1,\tau_2), H_n(\tau_2,\tau_1)\right\} \le H_n(\tau_1^0,\tau_2^0) + o(1)$$

and

$$\sup_{j_1,j_2}\ \sup_{\tau_1\in B(j_1\tau_1^0,\alpha_n), \tau_2\in B(j_2\tau_2^0,\alpha_n)} \max\left\{H_n(\tau_1,\tau_2), H_n(\tau_2,\tau_1)\right\} \le H_n(\tau_1^0,\tau_2^0) + o(1).$$

Now, using (3.11), (3.12) and (3.14),

$$\sup_{\tau_1,\tau_2\notin\left(\cup_{r\le m_n, j} B(j\tau_1^0/r,\alpha_n)\right)\cup\left(\cup_{r\le m_n, j} B(j\tau_2^0/r,\alpha_n)\right)} H_n(\tau_1,\tau_2) = o(1),$$

and if $r_1 \nmid j_1$,

$$\sup_{\tau_1\in B(j_1\tau_1^0/r_1,\alpha_n), \tau_2\notin\left(\cup_{r\le m_n, j} B(j\tau_1^0/r,\alpha_n)\right)\cup\left(\cup_{r\le m_n, j} B(j\tau_2^0/r,\alpha_n)\right)} H_n(\tau_1,\tau_2) \le (1-h)H_n(\tau_1^0,\tau_2^0) + o(1),$$

with similar inequality exchanging $\tau_1$ and $\tau_2$.
All these results imply that with a probability tending to 1,

$$\begin{aligned}\mathcal{U}_n \subset &\left(\cup_{j_1,j_2} B(j_1\tau_1^0,\alpha_n)\times B(j_2\tau_2^0,\alpha_n)\right) \cup \left(\cup_{j_1,j_2} B(j_2\tau_2^0,\alpha_n)\times B(j_1\tau_1^0,\alpha_n)\right) \\ &\cup \left(\cup_j B(j\tau_0,\alpha_n)\times\Theta\right) \cup \left(\Theta\times\cup_j B(j\tau_0,\alpha_n)\right).\end{aligned}$$

Moreover, since with a probability tending to 1, $(\tau_1^0,\tau_2^0)\in\mathcal{U}_n$, then with a probability tending to 1,

$$(\tilde\tau_1,\tilde\tau_2) \in B(\tau_1^0,\alpha_n)\times B(\tau_2^0,\alpha_n),\ (\bar\tau_1,\bar\tau_2) \in B(\tau_1^0,\alpha_n)\times B(\tau_2^0,\alpha_n).$$

Then, using computations similar than for (3.18) and (3.19), Lemma 3 follows for the case of rational ratio.

Consider now the case where $\frac{\tau_1^0}{\tau_2^0} \notin \mathbb{Q}$.
Let us notice the following simple facts concerning $K_n(\tau_1, \tau_2)$.
Using $\lim_{n\to+\infty} \beta_n = 0$,

$$\text{if } \frac{\tau_1}{\tau_2} \notin \mathbb{Q},\ \inf\{k\ : \exists k', (k,k') \in K_n(\tau_1,\tau_2) \text{ or } (k',k) \in K_n(\tau_1,\tau_2)\} \to +\infty. \tag{4.30}$$

Let $num(n)$ and $den(n)$ be the smallest integers such that

$$\frac{num(n)}{den(n)} - 2\left(\frac{\Delta}{\delta}\right)^3 \beta_n \leq \frac{\tau_1^0}{\tau_2^0} \leq \frac{num(n)}{den(n)} + 2\left(\frac{\Delta}{\delta}\right)^3 \beta_n. \tag{4.31}$$

Then

$$\lim_{n\to+\infty} num(n) = +\infty,\ \lim_{n\to+\infty} den(n) = +\infty. \tag{4.32}$$

Now, for any fixed $k$, using (3.11), (3.12), (3.14) and (3.15), one obtains, uniformly in all parameters:

$$\begin{gathered}\text{if } \tau \notin \left(\cup_{r\leq m_n,j} B(j\tau_1^0/r,\alpha_n)\right) \cup \left(\cup_{r\leq m_n,j} B(j\tau_2^0/r,\alpha_n)\right),\\ \sum_p c_p^0 \varphi_n\left(\frac{pt_n}{\tau_1^0} - \frac{kt_n}{\tau}\right) + \sum_p d_p^0 \varphi_n\left(\frac{pt_n}{\tau_2^0} - \frac{kt_n}{\tau}\right) = o\left(\frac{1}{m_n}\right),\end{gathered}$$

$$\begin{gathered}\text{if } \tau \in B(j_1\tau_1^0/r_1,\alpha_n) \text{ and } \tau \notin \left(\cup_{r\leq m_n,j} B(j\tau_2^0/r,\alpha_n)\right),\\ \sum_p c_p^0 \varphi_n\left(\frac{pt_n}{\tau_1^0} - \frac{kt_n}{\tau}\right) + \sum_p d_p^0 \varphi_n\left(\frac{pt_n}{\tau_2^0} - \frac{kt_n}{\tau}\right) = \begin{cases} o\left(\frac{1}{m_n}\right) & \text{if } j_1 \nmid k \\ c^0_{k'r_1}\varphi_n\left(\frac{k'r_1t_n}{\tau_1^0} - \frac{k'j_1t_n}{\tau}\right) + o\left(\frac{1}{m_n}\right) & \text{if } k = k'j_1. \end{cases}\end{gathered}$$

Notice now that,

$$\begin{gathered}\text{if } B\left(\frac{j_1\tau_1^0}{r_1},\alpha_n\right) \cap B\left(\frac{j_2\tau_2^0}{r_2},\alpha_n\right) \neq \emptyset,\\ \left(j_1 \geq \sqrt{den(n)} \text{ and } r_1 \geq \sqrt{num(n)}\right) \text{ or } \left(j_2 \geq \sqrt{num(n)} \text{ and } r_2 \geq \sqrt{den(n)}\right),\end{gathered} \tag{4.33}$$

so that, by using (4.32),

$$\begin{aligned}&\text{if } \tau \in B(j_1\tau_1^0/r_1,\alpha_n) \cap B(j_2\tau_2^0/r_2,\alpha_n),\\ &\sum_p c_p^0 \varphi_n\left(\frac{pt_n}{\tau_1^0} - \frac{kt_n}{\tau}\right) + \sum_p d_p^0 \varphi_n\left(\frac{pt_n}{\tau_2^0} - \frac{kt_n}{\tau}\right)\\ &= \begin{cases} o\left(\frac{1}{m_n}\right) & \text{if } j_1 \nmid k \text{ and } j_2 \nmid k \\ c^0_{k'r_1}\varphi_n\left(\frac{k'r_1t_n}{\tau_1^0} - \frac{k'j_1t_n}{\tau}\right) + o\left(\frac{1}{m_n}\right) & \text{if } k = k'j_1 \text{ and } j_2 \nmid k \\ d^0_{k'r_2}\varphi_n\left(\frac{k'r_2t_n}{\tau_2^0} - \frac{k'j_2t_n}{\tau}\right) + o\left(\frac{1}{m_n}\right) & \text{if } k = k'j_2 \text{ and } j_1 \nmid k \\ c^0_{r_1k/j_1}\varphi_n\left(\frac{r_1kt_n/j_1}{\tau_1^0} - \frac{kt_n}{\tau}\right) + d^0_{r_2k/j_2}\varphi_n\left(\frac{r_2kt_n/j_2}{\tau_2^0} - \frac{kt_n}{\tau}\right) + o\left(\frac{1}{m_n}\right) & \text{if } j_1 \mid k \text{ and } j_2 \mid k. \end{cases}\end{aligned}$$

First of all, we obtain

$$H_n(\tau_1^0,\tau_2^0) = \sum_{k\geq1} |c_k^0|^2 + \sum_{k\geq1} |d_k^0|^2 + o(1).$$

Then,

$$\sup_{\tau_1,\tau_2 \notin \left(\cup_{r\leq m_n,j} B(j\tau_1^0/r,\alpha_n)\right)\cup\left(\cup_{r\leq m_n,j} B(j\tau_2^0/r,\alpha_n)\right)} H_n(\tau_1,\tau_2) = o(1).$$

One has now to consider the cases where $\tau_1$, $\tau_2$ are or not in a ball around a fraction of $\tau_1^0$ and are or not in a ball around a fraction of $\tau_2^0$.
Notice that if $\tau_1 \in B(j_i\tau_i^0/r_i,\alpha_n)$ and $\tau_2 \in B(p_i\tau_i^0/q_i,\alpha_n)$, then

$$\left\{(k_1,k_2) \in \{1,\ldots,m_n\}^2\ :\ \frac{k_1}{k_2} = \frac{j_iq_i}{r_ip_i}\right\} \subset K_n(\tau_1,\tau_2),$$

and if $\tau_1 \in B(j_1\tau_1^0/r_1, \alpha_n)$ and $\tau_2 \in B(p_2\tau_2^0/q_2, \alpha_n)$, then for any $(k_1, k_2) \in K_n(\tau_1, \tau_2)$,

$$\left|\frac{k_1 r_1 p_2}{k_2 j_1 q_2} - \frac{\tau_1^0}{\tau_2^0}\right| \leq 2\left(\frac{\Delta}{\delta}\right)^3 \beta_n,$$

so that

$$k_2 j_1 q_2 \geq den(n) \text{ and } k_1 r_1 p_2 \geq num(n)$$

and one may use (4.32).
All these tools together with (I) lead to the fact that with a probability tending to 1,

$$\mathcal{U}_n \subset \left(\cup_{j,p} B(j\tau_1^0, \alpha_n) \times B(p\tau_2^0, \alpha_n)\right) \cup \left(\cup_{j,p} B(p\tau_2^0, \alpha_n) \times B(j\tau_1^0, \alpha_n)\right),$$

and that with a probability tending to 1,

$$(\tilde{\tau}_1, \tilde{\tau}_2) \in B(\tau_1^0, \alpha_n) \times B(\tau_2^0, \alpha_n), \; (\bar{\tau}_1, \bar{\tau}_2) \in B(\tau_1^0, \alpha_n) \times B(\tau_2^0, \alpha_n).$$

Now, for any $(\tau_1, \tau_2) \in B(\tau_1^0, \alpha_n) \times B(\tau_2^0, \alpha_n)$,

$$H_n(\tau_1, \tau_2) = \sum_k \left|c_k^0 \varphi_n\left(\frac{pt_n}{\tau_1^0} - \frac{kt_n}{\tau_1}\right)\right|^2 + \sum_k \left|d_k^0 \varphi_n\left(\frac{pt_n}{\tau_2^0} - \frac{kt_n}{\tau}\right)\right|^2 + o(1)$$

so that, using computations similar than for (3.18) and (3.19), Lemma 3 follows for the case of non rational ratio.

Since $(\hat{\tau}_1, \hat{\tau}_2)$ maximizes $\mathcal{H}_n$ over $(B(\tilde{\tau}_1, 1/8) \times B(\tilde{\tau}_2, 1/8)) \cap \{\tau_1 < \tau_2\}$ and $(\bar{\tau}_1, \bar{\tau}_2)$ maximizes $\mathcal{H}_n$ over $(B(\tau_1^0, 1/8) \times B(\tau_2^0, 1/8)) \cap \{\tau_1 < \tau_2\}$, Lemma 3 implies:

$$\lim_{n \to +\infty} P\left((\hat{\tau}_1, \hat{\tau}_2) = (\bar{\tau}_1, \bar{\tau}_2)\right) = 1. \tag{4.34}$$

As soon as one proves that

$$\frac{\partial \mathcal{H}_n}{\partial \tau_1}(\tau_1^0, \tau_2^0) = \left(\sum_{j=1}^n \left(\frac{1}{2} - \frac{j}{n}\right) t_n \left(\frac{1}{(\tau_1^0)^2} \tilde{f}_1'\left(\frac{jt_n}{\tau_1^0}\right) + \frac{g_0'\left(\frac{jt_n}{\tau_*}\right)}{2\tau_1^0 \tau_*}\right) \varepsilon(j)\right)(1 + o_{\mathbb{P}_n^0}(1)),$$

$$\frac{\partial \mathcal{H}_n}{\partial \tau_2}(\tau_1^0, \tau_2^0) = \left(\sum_{j=1}^n \left(\frac{1}{2} - \frac{j}{n}\right) t_n \left(\frac{1}{(\tau_2^0)^2} \tilde{f}_2'\left(\frac{jt_n}{\tau_2^0}\right) + \frac{g_0'\left(\frac{jt_n}{\tau_*}\right)}{2\tau_2^0 \tau_*}\right) \varepsilon(j)\right)(1 + o_{\mathbb{P}_n^0}(1)),$$

and

$$\begin{pmatrix} \frac{\partial^2 \mathcal{H}_n}{\partial \tau_1^2}(\tau_1, \tau_2) & \frac{\partial^2 \mathcal{H}_n}{\partial \tau_1 \partial \tau_2}(\tau_1, \tau_2) \\ \frac{\partial^2 \mathcal{H}_n}{\partial \tau_1 \partial \tau_2}(\tau_1, \tau_2) & \frac{\partial^2 \mathcal{H}_n}{\partial \tau_2^2}(\tau_1, \tau_2) \end{pmatrix} = -n^2 t_n^2 \sigma_0^2 K_* (1 + o_{\mathbb{P}_n^0}(1)),$$

uniformly for $\tau_1$ in $B(\tau_1^0, \rho_n/nt_n)$, $\tau_2$ in $B(\tau_2^0, \rho_n/nt_n)$, one obtains Theorem 2 by using (4.34) and a Taylor expansion for $(\bar{\tau}_1, \bar{\tau}_2)$. These results may be obtained by computations similar to those done for the single period model and by using arguments similar to those used in the proof of Lemma 3 to derive the asymptotic behavior of all terms in the computation. The proof is left to the reader.

## 5. Further comments

5.1. *Practical implementation* To implement the methods developed in this paper, one has to choose the value of $m_n$, the number of cumulated periodograms, and $\varepsilon_n$ (or $\varepsilon$), indicating the closeness of our criterion to the actual value of the maximum of $\Lambda_n$. As usual, such a choice is very important in practice:

- it has an influence on the observed performance of the algorithm,
- and it has to be completely data driven.

Lavielle and Lévy-Leduc (2004) have proposed and implemented such an algorithm. It is based on estimating the period of a general unknown periodic function by penalizing the cumulated periodogram $\Lambda_n$. They have also proposed a way to generalize it to the estimation of the periods of two unknown periodic functions based on a penalization of the criterion $\mathcal{H}_n$. These algorithms have been tested on signals arising in laser vibrometry. These signals may be modeled as $\sum_{m=1}^{M} a_m \exp(ic_m \cos(2\pi t/\tau_0))$, where all the parameters are unknown and $M$ is very large: $M \approx 200$. The algorithm has also been tested on sums of two such signals having distinct periods. These simulation studies show that these algorithms provide accurate estimators of the periods at a lower signal to noise ratio than algorithms classically used in laser vibrometry.

5.2. *Prospects* The theoretical results obtained in this paper, together with the practical implementation developed in Lavielle and Lévy-Leduc (2004), provide a significant contribution to the problem of semiparametric estimation of the periods from a sum of periodic functions corrupted by Gaussian white noise. There are now several ways in which such ideas can be extended.

- The performance of our algorithms could probably be improved if we used tapered versions of our criterions. This has to be tested in order to see whether this leads to practical results where the periods are well estimated at even lower signal to noise ratios.
- It would be interesting to extend our method to the estimation of more than two periods. Some preliminary simulations seem to be promising.
- If one is able to estimate any fixed number of unknown periods, one should be able, using penalization techniques, to estimate the "exact" number of periodic signals involved in a sum of periodic signals.

All these ideas will be investigated (practically and theoretically) in further work.

## 6. Proofs

6.1. *Proof of (S1)* Assume that

$$\forall h \in ]0,1[, \exists m_h \geq 2, \sum_{k\geq 1} |c_{mk}(s)|^2 > (1-h) \sum_{k\geq 1} |c_k(s)|^2.$$

Thus,

$$\forall p \in \mathbb{N}, \exists m_p \geq 2, \sum_{k\geq 1} |c_{km_p}(s)|^2 > \left(1 - \frac{1}{2^p}\right) \sum_{k\geq 1} |c_k(s)|^2.$$

As $(m_p)$ is a sequence of integers two cases may occur: either $(m_p)$ is upper bounded and admits a limit point $m$, or $(m_p)$ is not upper bounded and tends to infinity. The second case is impossible since $s$ is not the null function, and we deduce that $\exists m \geq 2$, $\sum_{k\geq 1} |c_{mk}(s)|^2 = \sum_{k\geq 1} |c_k(s)|^2$. As $s$ is real, we deduce that $\exists m \geq 2$, $\sum_{k\in\mathbb{Z}} |c_{mk}(s)|^2 = \sum_{k\in\mathbb{Z}} |c_k(s)|^2$. So, the only $c_p(s)$'s which are not equal to zero are such that $m$ divides $p$. Thus $s(t) = \sum_{p\in\mathbb{Z}} c_p(s) e^{\frac{2ip\pi t}{\tau}} = \sum_{k\in\mathbb{Z}} c_{km}(s) e^{\frac{2ik\pi t}{\tau/m}}$ and $s$ is periodic with period $\tau/m$ for some integer $m \geq 2$.

6.2. *Proof of Lemma 1* Note that for all $\tau \in \Theta_n$, $\eta_n(\tau) = \zeta_n(\tau) + \xi_n(\tau)$ with:

$$\zeta_n(\tau) = \sum_{k=1}^{m_n} \frac{1}{n^2} \left| \sum_{j=1}^{n} \varepsilon(j) \exp\left(\frac{2ik\pi j t_n}{\tau}\right) \right|^2$$

and

$$\xi_n(\tau) = \sum_{j=1}^{n} \left( \frac{1}{n^2} \sum_{k=1}^{n} \left[ \sum_{-m_n \leq l \leq m_n \ ;\ l\neq 0} e^{\frac{2i\pi l(j-k)t_n}{\tau}} \right] s_0(kt_n) \right) \varepsilon(j).$$

We use the following lemma due to Golubev (1988).

**Lemma 4.** ***Generalized Markov inequality***
*For $\mathcal{L}$ stochastic process and $A = [\alpha;\beta]$, we have $\forall\lambda > 0$, $\forall R > 0$,*

$$P\left(\sup_{\tau\in A}\mathcal{L}(\tau) > R\right) \le \exp(-\lambda R)\sup_{\tau\in A}\left(\sqrt{E\left[e^{2\lambda\mathcal{L}(\tau)}\right]}\right)\left[1+\lambda\int_{\tau\in A}\sqrt{E\left[|\mathcal{L}'(\tau)|^2\right]}d\tau\right].$$

Let us first control the Laplace transform of $\eta_n(\tau)$.

**Lemma 5.** *Under the assumptions of Lemma 1, for all $\lambda > 0$ such that $\lambda\sigma_0^2 m_n/n \le 1/4$, for some constant $C > 0$,*

$$E\left[e^{\lambda\eta_n(\tau)}\right] \le \exp\left[\frac{C\lambda^2}{n}\right].$$

By the Cauchy-Schwarz inequality, we just have to upper bound $E\left[e^{\lambda\xi_n(\tau)}\right]$ and $E\left[e^{\lambda\zeta_n(\tau)}\right]$. Now, $E\left[e^{\lambda\xi_n(\tau)}\right] = e^{\frac{\lambda^2V^2}{2}}$
where:

$$\begin{aligned}
V^2 &= \frac{\sigma_0^2}{n^2}\sum_{j=1}^{n}\left\{\sum_{-m_n\le l\le m_n;l\ne 0}\left(\sum_{p\in\mathbb{Z}}c_p(s_0)\varphi_n\left(\frac{pt_n}{\tau_0}-\frac{lt_n}{\tau}\right)\right)e^{\frac{2i\pi ljt_n}{\tau}}\right\}^2\\
&\le \frac{\sigma_0^2}{n}\left\{\sum_{-m_n\le l\le m_n;l\ne 0}\left|\sum_{p\in\mathbb{Z}}c_p(s_0)\varphi_n\left(\frac{pt_n}{\tau_0}-\frac{lt_n}{\tau}\right)\right|\right\}^2\\
&\le \frac{2\sigma_0^2}{n}\left(\sum_{k\in\mathbb{Z}}|c_k(s_0)|\right)^2
\end{aligned}$$

so that

$$E\left[e^{\lambda\xi_n(\tau)}\right] \le \exp\left[\frac{C\lambda^2}{n}\right],$$

for some $C > 0$.
Now, $\zeta_n(\tau) =< A(\tau)\varepsilon\ ;\ \varepsilon >$ where $A(\tau)$ is the $n\times n$ matrix such that for $1\le j,k\le n$:

$$A(\tau)_{j,k} = \frac{K_n\left[\frac{(j-k)t_n}{\tau}\right]}{2n^2}$$

where: $K_n(v) = \sum_{-m_n\le l\le m_n;l\ne 0} e^{2i\pi lv}$. Thus $A$ is real, symmetric and positive.
Furthermore,

$$K_n(v) = \begin{cases}\frac{2\sin[\pi v m_n]}{\sin(\pi v)}\cos[\pi v(m_n+1)],\ if\ v\ne 0,\\ 2m_n,\ if\ v = 0.\end{cases}$$

Now,

$$E\left[e^{\lambda\zeta_n(\tau)}\right] = E\left[e^{\lambda\sigma_0^2\sum_{i=1}^n\mu_i\varepsilon_i'^2}\right]$$

where: $\mu_i$ are the eigenvalues of $A(\tau)$ and $\varepsilon_i'$ are independent standard normal random variables, so that $E\left[e^{\lambda\zeta_n(\tau)}\right] = \exp\left[-\frac{1}{2}\sum_{i=1}^n\log\left(1-2\lambda\sigma_0^2\mu_i\right)\right]$ as soon as for all $i$, $2\lambda\sigma_0^2\mu_i < 1/2$, which holds when $\lambda\sigma_0^2 m_n/n < 1/4$ since $tr(A(\tau)) = m_n/n$. We thus have

$$E\left[e^{\lambda\zeta_n(\tau)}\right] \le \exp\left[\sigma_0^2\lambda\sum_{i=1}^n\mu_i + 2\sigma_0^4\lambda^2\sum_{i=1}^n\mu_i^2\right] \le \exp(\frac{1}{4}+\frac{1}{8})$$

since $\log(1+u) \ge u - u^2$ if $u \ge -1/2$.

**Lemma 6.** *Under the assumptions of Lemma 1, for some $C > 0$,*

$$E\left[|\zeta_n'(\tau)|^2\right] \leq \frac{Cm_n^4 t_n^2}{\tau^4}, \tag{6.35}$$

$$E\left[|\xi_n'(\tau)|^2\right] \leq C\left(\frac{nm_n^2 t_n^2}{\tau^4} + \frac{nt_n^2}{\tau^2\tau_0^2}\right). \tag{6.36}$$

Thus, applying Lemma 4 one has for some constant $C_2 > 0$ and if $\lambda\sigma_0^2 m_n/n < 1/4$ :

$$P\left(\sup_{\tau\in\Theta_n} \eta_n(\tau) > x_n\right) \leq e^{-\lambda x_n} e^{C\lambda^2/n}\left(1 + C_2\lambda\left[\frac{\sqrt{n}m_n t_n}{\delta_n} + \sqrt{n}t_n \log(\frac{\Delta_n}{\delta_n})\right]\right).$$

If $x_n m_n$ tends to 0, then by (N3), taking $\lambda = nx_n/2C$, for some $C" > 0$,

$$P\left(\sup_{\tau\in\Theta_n} \eta_n(\tau) > x_n\right) \leq C" x_n n\sqrt{n}(\log n)e^{-Cnx_n^2},$$

tends to 0. If now $x_n m_n$ does not tend to 0, for large enough $n$, $x_n \geq y_n$ for a sequence $y_n$ such that $y_n m_n$ tends to 0.
Lemma 1 follows since $\zeta_n$ is positive and $-\xi_n$ is normally distributed with the same mean and variance as $\xi_n$.
**Proof of Lemma 6** We first prove the inequality (6.35).
Straightforward computations lead to,

$$E\left[\zeta_n'(\tau)^2\right] = \left(\frac{4\pi t_n}{n^2\tau^2}\right)^2 E\left[\left(\sum_{l=1}^{m_n} l\left[Z_1(l)Z_2(l) - Z_3(l)Z_4(l)\right]\right)^2\right]$$

where:

$$\begin{cases} Z_1(l) = \sum_{j=1}^{n} j\sin\left(\frac{2\pi ljt_n}{\tau}\right)\frac{\varepsilon(j)}{\sigma_0}, \\ Z_2(l) = \sum_{k=1}^{n} \cos\left(\frac{2\pi lkt_n}{\tau}\right)\frac{\varepsilon(k)}{\sigma_0}, \\ Z_3(l) = \sum_{j=1}^{n} j\cos\left(\frac{2\pi ljt_n}{\tau}\right)\frac{\varepsilon(j)}{\sigma_0}, \\ Z_4(l) = \sum_{k=1}^{n} \sin\left(\frac{2\pi lkt_n}{\tau}\right)\frac{\varepsilon(k)}{\sigma_0}. \end{cases}$$

So:

$$E\left[\zeta_n'(\tau)^2\right] = \left(\frac{4\pi t_n}{n^2\tau^2}\right)^2 E\left[\sum_{1\leq l,k\leq m_n} klU(l)U(k)\right]$$

where: $U(l) = Z_1(l)Z_2(l) - Z_3(l)Z_4(l)$.
Thus: $U(l)U(k) = Z_1(l)Z_2(l)Z_1(k)Z_2(k) - Z_1(l)Z_2(l)Z_3(k)Z_4(k) - Z_3(l)Z_4(l)Z_1(k)Z_2(k) + Z_3(l)Z_4(l)Z_3(k)Z_4(k)$.
These different terms will be noted respectively: $\boxed{1}$, $\boxed{2}$, $\boxed{3}$, $\boxed{4}$.
We will focus on: $E\left[\sum_{1\leq k,l\leq m_n} lk\boxed{1}\right]$ because the others can be treated in the same way.

$$E\left[\sum_{1\leq k,l\leq m_n} lk\boxed{1}\right]$$

$$= \sigma_0^4 \sum_{1\leq l,k\leq m_n} lk \left\{\begin{array}{l} 3\sum_{j=1}^{n} j^2 \sin\left(\frac{2\pi ljt_n}{\tau}\right)\sin\left(\frac{2\pi kjt_n}{\tau}\right)\cos\left(\frac{2\pi ljt_n}{\tau}\right)\cos\left(\frac{2\pi kjt_n}{\tau}\right) \\ +\left(\sum_{j=1}^{n} j\sin\left(\frac{2\pi ljt_n}{\tau}\right)\cos\left(\frac{2\pi ljt_n}{\tau}\right)\right)\left(\sum_{m=1}^{n} m\sin\left(\frac{2\pi kmt_n}{\tau}\right)\cos\left(\frac{2\pi kmt_n}{\tau}\right)\right) \\ +\left(\sum_{j=1}^{n} j^2\sin\left(\frac{2\pi ljt_n}{\tau}\right)\sin\left(\frac{2\pi kjt_n}{\tau}\right)\right)\left(\sum_{m=1}^{n} \cos\left(\frac{2\pi lmt_n}{\tau}\right)\cos\left(\frac{2\pi kmt_n}{\tau}\right)\right) \\ +\left(\sum_{j=1}^{n} j\sin\left(\frac{2\pi ljt_n}{\tau}\right)\cos\left(\frac{2\pi kjt_n}{\tau}\right)\right)\left(\sum_{m=1}^{n} m\cos\left(\frac{2\pi lmt_n}{\tau}\right)\sin\left(\frac{2\pi kmt_n}{\tau}\right)\right) \end{array}\right\}.$$

Now, the only thing that remains to do is to prove that each term into the braces is less than the expected quantity. This can be done by saying that the absolute values of the sinus and the cosine are less than one.

For the inequality (6.36), one has after some straightforward computations, since $\xi_n(\tau)$ is real,

$$\tau^4 E\left[\xi_n'(\tau)^2\right]$$
$$=\frac{\sigma_0^2}{n^2}\sum_{j=1}^n\left\{\begin{array}{l} j\sum_{|l|\le m_n;l\neq 0}(2i\pi l t_n)e^{2i\pi l j t_n/\tau}\sum_{p\in\mathbb{Z}}c_p(s_0)\varphi_n\left[t_n\left(\frac{p}{\tau_0}-\frac{l}{\tau}\right)\right] \\ -\sum_{|l|\le m_n;l\neq 0}l t_n e^{2i\pi l j t_n/\tau}\sum_{p\in\mathbb{Z}}c_p(s_0){\varphi_n}'\left[t_n\left(\frac{p}{\tau_0}-\frac{l}{\tau}\right)\right]\end{array}\right\}^2.$$

Focus on the first sum into the braces and cut $\sum_p$ into $\sum_{|p|>m_n}$ and $\sum_{|p|\le m_n}$.

$$\sum_{|l|\le m_n;l\neq 0}|l|t_n\sum_{|p|>m_n}|c_p(s_0)|\left|\varphi_n\left[t_n\left(\frac{p}{\tau_0}-\frac{l}{\tau}\right)\right]\right|\le m_n^2 t_n\frac{1}{m_n}\sum_{|p|>m_n}|p||c_p(s_0)|.$$

Thanks to **(N3)**,

$$\sum_{|l|\le m_n;l\neq 0}|l|t_n\sum_{|p|\le m_n}|c_p(s_0)|\left|\varphi_n\left[t_n\left(\frac{p}{\tau_0}-\frac{l}{\tau}\right)\right]\right|$$
$$=\sum_{|l|\le m_n;l\neq 0}|l|t_n\, c_{[l\tau_0/\tau]}(s_0)+m_n^2 t_n\sum_{|p|\le m_n;p\neq[l\tau_0/\tau]}\frac{2\tau_0|c_p(s_0)|}{n t_n}$$
$$\le\frac{t_n\tau}{\tau_0}\sum_{m\in\mathbb{Z}^*}|m+1||c_m(s_0)|+\frac{C\tau_0 m_n^2}{n}\le\frac{C t_n\tau}{\tau_0}+\frac{C\tau_0 m_n^2}{n}$$

for some constant $C$. Thus the modulus of the first term into the braces is less than:

$$jC\left(m_n t_n+\frac{t_n\tau}{\tau_0}+\frac{\tau_0 m_n^2}{n}\right).$$

With almost the same arguments, the modulus of the second term into the braces is less than :

$$C\left(n m_n t_n+\frac{n t_n\tau}{\tau_0}+\tau_0 m_n^2\right).$$

6.3. *Proof of equation (3.15)* Write

$$\frac{p t_n}{\tau_0}-\frac{k t_n}{\tau}=\frac{p t_n}{\tau_0}-\frac{k l t_n}{j\tau_0}+\frac{k l t_n}{j\tau_0}-\frac{k t_n}{\tau}.$$

For $\tau\in B(\frac{j\tau_0}{l},\alpha_n)$, $|\frac{klt_n}{j\tau_0}-\frac{kt_n}{\tau}|\le\frac{\alpha_n t_n m_n^2}{j\tau_0}$ tends to 0.
For $k=jk'$ and $p\neq k'l$, $|p-k'l|\ge 1$, and for $k\neq jk'$, $|p-\frac{kl}{j}|\ge\frac{1}{j}$. For $p$, $|p|\le {m_n}^2$, we use that $\frac{1}{j}\ge\frac{\tau_0}{m_n\Delta_n}$ and (2.9). For $p$, $|p|>{m_n}^2$, we use that $\frac{n t_n}{m_n^5}=O(1)$ which implies that $\frac{1}{m_n^4}=O\left(\frac{m_n\Delta_n}{n t_n}\right)$.

6.4. *Proof of equation (3.21)*

$$\Lambda_n'(\tau_0)=\xi_n'(\tau_0)+\zeta_n'(\tau_0)+G_n'(\tau_0).$$

First, after some straightforward computations:

$$\xi_n'(\tau_0)=\sum_{k=1}^n\varepsilon(k)\left(\frac{1}{n^2}\sum_{l=1}^n-\frac{(k-l)t_n}{(\tau_0)^2}K_n'\left(\frac{(k-l)t_n}{\tau_0}\right)s_0(lt_n)\right).$$

But,

$$\begin{aligned}
&\frac{1}{n}\sum_{l=1}^{n} -(k-l)t_nK_n'\left(\frac{(k-l)t_n}{\tau_0}\right)s_0(lt_n)\\
&= -\sum_{|q|\le m_n}\sum_{p\in\mathbb{Z}} c_p(s_0)e^{\frac{2i\pi qkt_n}{\tau_0}}\left\{(2i\pi q)kt_n\varphi_n\left[\frac{t_n}{\tau_0}(p-q)\right]-t_nq\varphi_n{}'\left[\frac{t_n}{\tau_0}(p-q)\right]\right\}\\
&= t_n\left(k-\frac{n+1}{2}\right)\left(\sum_{|j|\le m_n} 2i\pi jc_j(s_0)e^{\frac{2i\pi jkt_n}{\tau_0}}\right)\\
&-\sum_{|j|\le m_n}\sum_{p\neq j,|p|\le m_n{}^2} c_p(s_0)e^{\frac{2i\pi jkt_n}{\tau_0}}\left\{(2i\pi j)kt_n\varphi_n\left[\frac{t_n}{\tau_0}(p-j)\right]-t_nj\varphi_n{}'\left[\frac{t_n}{\tau_0}(p-j)\right]\right\}\\
&-\sum_{|j|\le m_n}\sum_{p\neq j,|p|> m_n{}^2} c_p(s_0)e^{\frac{2i\pi jkt_n}{\tau_0}}\left\{(2i\pi j)kt_n\varphi_n\left[\frac{t_n}{\tau_0}(p-j)\right]-t_nj\varphi_n{}'\left[\frac{t_n}{\tau_0}(p-j)\right]\right\}.
\end{aligned}$$

Thus:

$$\xi_n'(\tau_0)=\frac{1}{\tau_0^2}\sum_{k=1}^{n}t_n\left(\frac{1}{2}-\frac{k}{n}\right)f_0'\left(\frac{kt_n}{\tau_0}\right)\varepsilon(k)+o_{\mathbb{P}_n^0}(\sqrt{n}t_n).$$

Then, $\zeta_n'(\tau_0) =< B(\tau_0)\varepsilon\ ;\ \varepsilon >$ where

$$B_{j,k}(\tau)=-\frac{(j-k)t_n}{2n^2\tau^2}\sum_{|l|\le m(n)\ ;\ l\neq 0} 2i\pi l e^{2i\pi lt_n\left(\frac{j-k}{\tau}\right)}.$$

With this definition, $B(\tau)$ is real, symmetrical, $Tr(B(\tau))=0$ and one has, as usual,

$$\operatorname{Var}\left(\zeta_n'(\tau_0)\right)=2\sigma_0^4Tr\left(B(\tau_0)^2\right)=O\left(t_n^2m_n^4\right),$$

so that by using the fact that $\lim_{n\to+\infty}\frac{m_n^4}{n}=0$ (by (N3)),

$$\zeta_n'(\tau_0)=o_{\mathbb{P}_n^0}(\sqrt{n}t_n).$$

At last,

$$\begin{aligned}
G_n'(\tau_0)&=\sum_{k=1}^{m_n}\sum_{p_1,p_2}c_{p_1}(s_0)\overline{c_{p_2}(s_0)}\left(\frac{t_nk}{(\tau_0)^2}\right)\left\{\begin{array}{c}\varphi_n'\left[t_n\left(\frac{p_1}{\tau_0}-\frac{k}{\tau_0}\right)\right]\overline{\varphi_n}\left[t_n\left(\frac{p_2}{\tau_0}-\frac{k}{\tau_0}\right)\right]\\ +\varphi_n\left[t_n\left(\frac{p_1}{\tau_0}-\frac{k}{\tau_0}\right)\right]\overline{\varphi_n'}\left[t_n\left(\frac{p_2}{\tau_0}-\frac{k}{\tau_0}\right)\right]\end{array}\right\}\\
&=\sum_{k=1}^{m_n}\left[\begin{array}{c}|c_k(s_0)|^2\left(\frac{t_nk}{(\tau_0)^2}\right)\left(\varphi_n'(0)\overline{\varphi_n}(0)+\varphi_n(0)\overline{\varphi_n'}(0)\right)\\ +\sum_{(p_1,p_2)\neq(k,k)}c_{p_1}(s_0)\overline{c_{p_2}(s_0)}\left(\frac{t_nk}{(\tau_0)^2}\right)\left\{\begin{array}{c}\varphi_n'\left[t_n\left(\frac{p_1}{\tau_0}-\frac{k}{\tau_0}\right)\right]\overline{\varphi_n}\left[t_n\left(\frac{p_2}{\tau_0}-\frac{k}{\tau_0}\right)\right]\\ +\varphi_n\left[t_n\left(\frac{p_1}{\tau_0}-\frac{k}{\tau_0}\right)\right]\overline{\varphi_n'}\left[t_n\left(\frac{p_2}{\tau_0}-\frac{k}{\tau_0}\right)\right]\end{array}\right\}\end{array}\right]\\
&=o(\sqrt{n}t_n).
\end{aligned}$$

by separating the cases $|p_1|\le m_n^2$, $|p_1|>m_n^2$, $|p_2|\le m_n^2$ and $|p_2|>m_n^2$. The result follows by using $\lim_{n\to+\infty} nt_n^{8/3}=+\infty$, which is a consequence of (N3), and by the use of (S2).

6.5. *Proof of equation (3.22)* First, $G_n''(\tilde\tau)=G_n''(\tau_0)+(\tilde\tau-\tau_0)G_n'''(\tau)$, for some $\tau\in B\left(\tau_0;\frac{\rho_n}{nt_n}\right)$.

$$\begin{aligned}
G_n''(\tau_0)&=\sum_{k=1}^{m_n}\sum_{p_1,p_2}c_{p_1}\overline{c_{p_2}}\left(-\frac{2t_nk}{(\tau_0)^3}\right)\left\{\begin{array}{c}\varphi_n'\left[t_n\left(\frac{p_1}{\tau_0}-\frac{k}{\tau_0}\right)\right]\overline{\varphi_n}\left[t_n\left(\frac{p_2}{\tau_0}-\frac{k}{\tau_0}\right)\right]\\ +\varphi_n\left[t_n\left(\frac{p_1}{\tau_0}-\frac{k}{\tau_0}\right)\right]\overline{\varphi_n'}\left[t_n\left(\frac{p_2}{\tau_0}-\frac{k}{\tau_0}\right)\right]\end{array}\right\}\\
&+\sum_{k=1}^{m_n}\sum_{p_1,p_2}c_{p_1}\overline{c_{p_2}}\left(\frac{t_nk}{(\tau_0)^2}\right)^2\left\{\begin{array}{c}\varphi_n''\left[t_n\left(\frac{p_1}{\tau_0}-\frac{k}{\tau_0}\right)\right]\overline{\varphi_n}\left[t_n\left(\frac{p_2}{\tau_0}-\frac{k}{\tau_0}\right)\right]\\ +2\varphi_n'\left[t_n\left(\frac{p_1}{\tau_0}-\frac{k}{\tau_0}\right)\right]\overline{\varphi_n'}\left[t_n\left(\frac{p_2}{\tau_0}-\frac{k}{\tau_0}\right)\right]\\ +\varphi_n\left[t_n\left(\frac{p_1}{\tau_0}-\frac{k}{\tau_0}\right)\right]\overline{\varphi_n''}\left[t_n\left(\frac{p_2}{\tau_0}-\frac{k}{\tau_0}\right)\right]\end{array}\right\}.
\end{aligned}$$

By the same arguments as those used to compute $G_n'(\tau_0)$,

$$
\begin{aligned}
G_n''(\tau_0) &= \sum_{k=1}^{m_n} k^2|c_k|^2\left(\frac{t_n}{\tau_0^2}\right)^2\left(-\frac{4\pi^2n^2}{3}+2\pi^2n^2-\frac{4\pi^2n^2}{3}\right)(1+o(1))\\
&= -\frac{2n^2\pi^2}{3\tau_0^4}t_n^2\sum_{k=1}^{m_n}k^2|c_k|^2(1+o(1)) = -n^2t_n^2\sigma_0^2I^*(1+o(1)).
\end{aligned}
$$

In the same way, it can be proved that $G_n'''(\tau)=o(n^3t_n^3)$.
Finally, $G_n''(\tilde\tau)=-n^2t_n^2\sigma_0^2I^*(1+o(1))$.
As for $\zeta_n''(\tilde\tau)$, we prove that $\zeta_n''(\tilde\tau)=o_p(n^2t_n^2)$.
For $\tau\in B\left(\tau_0;\frac{\rho_n}{nt_n}\right)$, $\zeta_n''(\tau)=<C(\tau)\varepsilon;\varepsilon>$ where

$$
C_{j,k}(\tau)=\frac{1}{n^2\tau^3}\sum_{|l|\le m_n;l\neq 0}2i\pi l(j-k)t_ne^{\frac{2i\pi l(j-k)t_n}{\tau}}+\frac{1}{2n^2\tau^4}\sum_{|l|\le m_n;l\neq 0}\left[2i\pi l(j-k)t_n\right]^2e^{\frac{2i\pi l(j-k)t_n}{\tau}}.
$$

By the generalized Markov inequality (Lemma 4)

$$
P\left(\sup_{\tau\in B\left(\tau_0;\frac{\rho_n}{nt_n}\right)}\zeta_n''(\tau)>n^2t_n^2\right)\le\exp(-\lambda n^2t_n^2)E\left(e^{2\lambda\zeta_n''(\tau)}\right)\left(1+\lambda\int_\tau\sqrt{E\left[|\zeta_n'''(\tau)|^2\right]}d\tau\right).
$$

As usual,

$$
E\left[e^{\lambda\zeta_n''(\tau)}\right]\le\exp\left(2\lambda^2\sigma^4\sum_{k=1}^n\mu_k^2\right)
$$

if $\lambda=O\left(\frac{1}{nm_n^3t_n^2}\right)$ (because the eigenvalues $\mu_k$ of the matrix $C$ satisfy $\max_k|\mu_k|=O\left(\frac{m_n^2t_n}{\tau^3}+\frac{nm_n^3t_n^2}{\tau^4}\right)$).
With $Tr(C(\tau)^2)=O\left(\frac{m_n^4t_n^2}{\tau^6}+\frac{n^2m_n^6t_n^4}{\tau^8}\right)$,

$$
E\left[e^{\lambda\zeta_n''(\tau)}\right]=\exp\left[\frac{2\lambda^2{\sigma_0}^4m_n^4t_n^2}{\tau^6}\left(1+\frac{m_n^2t_n^2n^2}{\tau^2}\right)\right].
$$

With usual arguments,

$$
E\left[|\zeta_n'''(\tau)|^2\right]=O\left(\frac{m_n^7n^5t_n^6}{\tau^{12}}\right)
$$

so that,

$$
P\left(\sup_{\tau\in B\left(\tau_0;\frac{\rho_n}{nt_n}\right)}\zeta_n''(\tau)>n^2t_n^2\right)\le C\exp(-\lambda n^2t_n^2)\exp(\lambda^2m_n^6t_n^4n^2)\left[1+\lambda\int_\tau O\left(\frac{m_n^7n^5t_n^6}{\tau^{12}}\right)d\tau\right].
$$

And,

$$
\lim_{n\to+\infty}P\left(\sup_{\tau\in B\left(\tau_0;\frac{\rho_n}{nt_n}\right)}\zeta_n''(\tau)>n^2t_n^2\right)=0.
$$

Let us prove that $\xi_n''(\tilde\tau) = o_p(n^2 t_n^2)$.
Straightforward computations lead to

$$\begin{aligned}
\xi_n''(\tau) &= \frac{1}{n\tau^4}\sum_{|l|\le m_n}\sum_{j=1}^{n}(2i\pi l j t_n)^2 e^{\frac{2i\pi l j t_n}{\tau}}\sum_{p\in\mathbb{Z}} c_p(s_0)\varphi_n\left[t_n\left(\frac{p}{\tau_0}-\frac{l}{\tau}\right)\right]\varepsilon(j)\\
&+\frac{1}{n\tau^4}\sum_{|l|\le m_n}\sum_{j=1}^{n}(l t_n)^2 e^{\frac{2i\pi l j t_n}{\tau}}\sum_{p\in\mathbb{Z}} c_p(s_0)\varphi_n''\left[t_n\left(\frac{p}{\tau_0}-\frac{l}{\tau}\right)\right]\varepsilon(j)\\
&-\frac{2}{n\tau^4}\sum_{|l|\le m_n}\sum_{j=1}^{n} j l t_n e^{\frac{2i\pi l j t_n}{\tau}}\sum_{p\in\mathbb{Z}} c_p(s_0)\varphi_n'\left[t_n\left(\frac{p}{\tau_0}-\frac{l}{\tau}\right)\right]\varepsilon(j)\\
&-\frac{2}{n\tau^3}\sum_{|l|\le m_n}\sum_{j=1}^{n} 2i\pi l j t_n e^{\frac{2i\pi l j t_n}{\tau}}\sum_{p\in\mathbb{Z}} c_p(s_0)\varphi_n\left[t_n\left(\frac{p}{\tau_0}-\frac{l}{\tau}\right)\right]\varepsilon(j)\\
&+\frac{2}{n\tau^3}\sum_{|l|\le m_n}\sum_{j=1}^{n} l t_n e^{\frac{2i\pi l j t_n}{\tau}}\sum_{p\in\mathbb{Z}} c_p(s_0)\varphi_n'\left[t_n\left(\frac{p}{\tau_0}-\frac{l}{\tau}\right)\right]\varepsilon(j).
\end{aligned}$$

So $E\left(e^{\lambda\xi_n''(\tau)}\right) = e^{\lambda^2\sigma^2/2}$ with $\sigma^2 = O\left(\frac{n^3 t_n^4}{\tau^8}\right)$.
It can also be proved that

$$E\left[\xi_n'''(\tau)^2\right] = O\left[\frac{n m_n^4 t_n^4}{\tau^{10}} + \frac{n t_n^4}{\tau^6} + \frac{m_n^6 t_n^2}{n\tau^{10}} + \frac{n m_n^6 t_n^6}{n\tau^{16}} + \frac{n^3 t_n^6}{n\tau^{10}} + \frac{n m_n^6 t_n^4}{\tau^{16}}\right]$$

so that using again the generalized Markov inequality (Lemma 4),

$$P\left(\sup_{\tau\in B\left(\tau_0;\frac{\rho_n}{n t_n}\right)} \xi_n''(\tau) > n^2 t_n^2\right) \le C\exp(-\lambda n^2 t_n^2)\exp(\lambda^2 n^3 t_n^4)\left[1+\lambda\int_\tau \sqrt{E\left[\xi_n'''(\tau)^2\right]}d\tau\right].$$

The expected result follows with $\lambda = 1/(2nt_n^2)$.

6.6. *Proof of Proposition 1* Easy computations lead to, with $u(j) = \varepsilon(j)/\sigma_0$,

$$\begin{aligned}
L_n(\tau_0 + \frac{a}{n^{3/2}t_n}, f_0 + \frac{f}{\sqrt{n}}, \sigma_0 + \frac{b}{\sqrt{n}}) &= \frac{1}{\sqrt{n}}\sum_{j=1}^{n}\frac{b}{\sigma_0}(u(j)^2-1) + \sum_{j=1}^{n} u(j)\frac{A_{j,n}}{\sigma_0}\\
&-\frac{3}{2n}\frac{b^2}{\sigma_0^2}\sum_{j=1}^{n}\left(u(j)^2 - \frac{1}{3}\right) - \frac{1}{2}\sum_{j=1}^{n}\left(\frac{A_{j,n}}{\sigma_0}\right)^2 + o_{\mathbb{P}_n^0}(1)
\end{aligned}$$

where:

$$A_{j,n} = \frac{1}{\sqrt{n}} f\left(\frac{j t_n}{\tau_0}\right) - \frac{a}{n^{3/2}t_n}\left(\frac{j t_n}{\tau_0^2}\right) f_0'\left(\frac{j t_n}{\tau_0}\right).$$

The result follows from the Lindeberg's theorem and from the fact that if $f$ and $g$ are in $\mathcal{H}$:

$$\lim_{n\to\infty}\frac{1}{n}\sum_{j=1}^{n} f\left(\frac{j t_n}{\tau_0}\right) g\left(\frac{j t_n}{\tau_0}\right) = \int_0^1 f(t)g(t)dt \tag{6.37}$$

$$\lim_{n\to\infty}\frac{1}{n^3}\sum_{j=1}^{n} j^2 f_0'^2\left(\frac{j t_n}{\tau_0}\right) = \frac{1}{3}\int_0^1 f_0'^2(t)dt \tag{6.38}$$

$$\lim_{n\to\infty}\frac{1}{n^2}\sum_{j=1}^{n} j f_0'\left(\frac{j t_n}{\tau_0}\right) f\left(\frac{j t_n}{\tau_0}\right) = \frac{1}{2}\int_0^1 f_0'(t)f(t)dt \tag{6.39}$$

To prove (6.37), (6.38) and (6.39), write

$$\frac{1}{n}\sum_{j=1}^{n} f\left(\frac{jt_n}{\tau_0}\right) g\left(\frac{jt_n}{\tau_0}\right) = c_0(fg) + \sum_{k\in\mathbb{Z}^*} c_k(fg)\varphi_n(\frac{kt_n}{\tau_0})$$

$$\frac{1}{n^3}\sum_{j=1}^{n} j^2 f_0'^2\left(\frac{jt_n}{\tau_0}\right) = \frac{c_0(f_0'^2)}{3} + o(1) - \sum_{k\in\mathbb{Z}^*} c_k(f_0'^2)\frac{1}{4\pi^2 n^2}\varphi_n''(\frac{kt_n}{\tau_0})$$

$$\frac{1}{n^2}\sum_{j=1}^{n} j f_0'\left(\frac{jt_n}{\tau_0}\right) f\left(\frac{jt_n}{\tau_0}\right) = \frac{c_0(f_0'f)}{2} + o(1) + \sum_{k\in\mathbb{Z}^*} c_k(f_0'f)\frac{1}{2\pi i n}\varphi_n'(\frac{kt_n}{\tau_0}),$$

use the fact that for $|k| \leq \tau_0/2t_n$, $|kt_n/\tau_0| \leq 1/2$, use (2.9) and

$$\sum_{|k|>\tau_0/2t_n} c_k(h) = o(1)$$

for $h = fg$ or $h = f_0'^2$ or $h = ff_0'$ since $t_n \to 0$.
$\dot{\nu}^T$ follows from direct computation, and efficiency of linearized estimators follows from McNeney and Wellner (2000).

6.7. *Proof of (I)* Remark that the assumption (I) can be rewritten as follows: there exists $h \in ]0,1[$ such that for any $r, r' \geq 2$,

$$\sum_{k\geq 1, e|k} |c_{kr_1'}(f)|^2 + \sum_{k\geq 1, e|k} |c_{kr_1}(f)|^2 - \sum_{k\geq 1, e|k} |c_{kr_1 r_1'}(f)|^2 \leq (1-h)\sum_{k\geq 1} |c_k(f)|^2$$

where $e$ is the greatest common divisor of $r$ and $r'$ and $r_1$, $r_1'$ are such that $r = er_1$, $r' = er_1'$. Thus, $r_1$ and $r_1'$ are relatively prime integers. Now, the negation of the previous statement implies that $\forall h \in ]0,1[$, there exists two relatively prime integers $p, q \geq 2$, such that

$$\sum_{k\geq 1} |c_{kp}(f)|^2 + \sum_{k\geq 1} |c_{qk}(f)|^2 - \sum_{k\geq 1} |c_{pqk}(f)|^2 > (1-h)\sum_{k\geq 1} |c_k(f)|^2 .$$

Thus, $\forall n \in \mathbb{N}, \exists p_n, q_n \geq 2$ such that

$$\sum_{k\geq 1} |c_{kp_n}(f)|^2 + \sum_{k\geq 1} |c_{q_n k}(f)|^2 - \sum_{k\geq 1} |c_{p_n q_n k}(f)|^2 > \left(1 - \frac{1}{2^n}\right)\sum_{k\geq 1} |c_k(f)|^2 .$$

Like in the proof of (S1), if neither $p_n$ nor $q_n$ is upper bounded, $f = 0$, which is impossible when $f = \tilde{f}_1$ or $f = \tilde{f}_2$. In case $g_0 = 0$, (I) still holds for $f = g_0$. If now only one of the two previous sequences is upper bounded ($p_n$ for instance) then the first coordinate of a limit point of $(p_n, q_n)$ is $p$. This implies that $\sum_{k\geq 1} |c_{kp}(f)|^2 = \sum_{k\geq 1} |c_k(f)|^2$ which is impossible because $f$ satisfies (S1). If the two sequences are upper bounded then the couple $(p_n, q_n)$ has limit points, let $(p, q)$ be one of them, and we obtain

$$\sum_{k\geq 1} |c_{kp}(f)|^2 + \sum_{k\geq 1} |c_{qk}(f)|^2 - \sum_{k\geq 1} |c_{pqk}(f)|^2 = \sum_{k\geq 1} |c_k(f)|^2 ,$$

which means that $f$ can be split into a sum of periodic functions with periods $1/p$ and $1/q$. Note that $p$ and $q$ are relatively prime integers since we take the same extraction of $(p_n)$ and $(q_n)$. This concludes the proof.

6.8. *Proof of Lemma 2* First of all, using the fact that $\lim_{n\to+\infty} m_n\beta_n = 0$, one obtains that for any $(\tau_1, \tau_2)$, for large enough $n$, for any $k \in \{1, \ldots, m_n\}$, the sets $\{k_1 : (k_1, k) \in K_n(\tau_1, \tau_2)\}$ and $\{k_2 : (k, k_2) \in K_n(\tau_1, \tau_2)\}$ have at most one element. This allows to use Cauchy-Schwarz inequality to obtain

$$|R_{n,2}(\tau_1, \tau_2)| \leq \zeta_n(\tau_1) + \zeta_n(\tau_2) + \sqrt{\zeta_n(\tau_1)}\sqrt{\zeta_n(\tau_2)},$$

$$|R_{n,1}(\tau_1, \tau_2)| \leq 2\sqrt{\zeta_n(\tau_1)}\sqrt{G_n(\tau_1)} + 2\sqrt{\zeta_n(\tau_2)}\sqrt{G_n(\tau_2)} + 2\sqrt{\zeta_n(\tau_1)}\sqrt{G_n(\tau_2)} + 2\sqrt{\zeta_n(\tau_2)}\sqrt{G_n(\tau_1)},$$

so that

$$\sup_{\tau_1,\tau_2} |R_{n,2}(\tau_1,\tau_2)| \le 3 \sup_{\tau} \zeta_n(\tau)$$

and

$$\sup_{\tau_1,\tau_2} |R_{n,1}(\tau_1,\tau_2)| \le 8 \sup_{\tau} \sqrt{\zeta_n(\tau)} \sup_{\tau} \sqrt{G_n(\tau)}.$$

The Lemma follows from Lemma 1 and $\sup_{\tau} G_n(\tau) = O(1)$.

6.9. *Proof of Proposition 2* The proof is similar to the one of Proposition 1 and one just needs to prove limits of sums of the form: $\frac{1}{n}\sum_{j=1}^{n} f_1\left(\frac{jt_n}{\tau_1^0}\right) f_2\left(\frac{jt_n}{\tau_2^0}\right)$, $\frac{1}{n^2}\sum_{j=1}^{n} j f_1\left(\frac{jt_n}{\tau_1^0}\right) f_2^{0\prime}\left(\frac{jt_n}{\tau_2^0}\right)$, $\frac{1}{n^2}\sum_{j=1}^{n} j f_2\left(\frac{jt_n}{\tau_2^0}\right) f_1^{0\prime}\left(\frac{jt_n}{\tau_1^0}\right)$, and $\frac{1}{n^3}\sum_{j=1}^{n} j^2 f_1^{0\prime}\left(\frac{jt_n}{\tau_1^0}\right) f_2^{0\prime}\left(\frac{jt_n}{\tau_2^0}\right)$. Now

$$\frac{1}{n}\sum_{j=1}^{n} f_1\left(\frac{jt_n}{\tau_1^0}\right) f_2\left(\frac{jt_n}{\tau_2^0}\right) = \sum_{k,l} c_k(f_1)c_l(f_2)\varphi_n\left[t_n\left(\frac{k}{\tau_1^0}+\frac{l}{\tau_2^0}\right)\right], \tag{6.40}$$

so that, if they exist, the limit of (6.40) as $n \to +\infty$ and (4.26) as $T \to +\infty$ are the same. If $\tau_1^0/\tau_2^0$ is an irreducible fraction $p/q$, this limit exists (and is equal to $\sum_m c_{mp}(f_1)c_{-mq}(f_2)$). If $\tau_1^0/\tau_2^0 \notin \mathbb{Q}$, write

$$\sum_{k,l} c_k(f_1)c_l(f_2)\varphi\left[T\left(\frac{k}{\tau_1^0}+\frac{l}{\tau_2^0}\right)\right] = c_0(f_1)c_0(f_2) + \sum_{k\neq 0, l\neq 0} c_k(f_1)c_l(f_2)\varphi\left[\frac{Tl}{\tau_1^0}\left(\frac{k}{l}+\frac{\tau_1^0}{\tau_2^0}\right)\right] + o(1).$$

For any $l \neq 0$, there exists $m(l)$ such that

$$\frac{m(l)}{l} < \frac{\tau_1^0}{\tau_2^0} < \frac{m(l)+1}{l},\ \alpha(l) = \min\left(\frac{\tau_1^0}{\tau_2^0} - \frac{m(l)}{l}; \frac{m(l)+1}{l} - \frac{\tau_1^0}{\tau_2^0}\right) > 0.$$

If $k < -m(l) - 1$ or $k > -m(l)$, $|\frac{k}{l} + \frac{\tau_1^0}{\tau_2^0}| > \frac{1}{|l|}$, so that

$$\sum_{k,l} c_k(f_1)c_l(f_2)\varphi\left[T\left(\frac{k}{\tau_1^0}+\frac{l}{\tau_2^0}\right)\right] = c_0(f_1)c_0(f_2) + o(1) + \sum_{l\neq 0} c_{-m(l)}(f_1)c_l(f_2)\varphi\left[\frac{Tl}{\tau_1^0}\left(\frac{-m(l)}{l}+\frac{\tau_1^0}{\tau_2^0}\right)\right]$$
$$+\sum_{l\neq 0} c_{-m(l)-1}(f_1)c_l(f_2)\varphi\left[\frac{Tl}{\tau_1^0}\left(\frac{-m(l)-1}{l}+\frac{\tau_1^0}{\tau_2^0}\right)\right].$$

Now, since $\tau_1^0/\tau_2^0 \notin \mathbb{Q}$, $m(l) \to +\infty$ as $l \to +\infty$, and for any $L$ such that, if $|l| \geq L$, $|c_{-m(l)-1}(f_1)| \le \epsilon/2$ and $|c_{-m(l)}(f_1)| \le \epsilon/2$:

$$\sum_{k,l} c_k(f_1)c_l(f_2)\varphi\left[T\left(\frac{k}{\tau_1^0}+\frac{l}{\tau_2^0}\right)\right] = c_0(f_1)c_0(f_2) + o(1) + O\left(\epsilon\sum_{|l|\geq L}|c_l(f_2)|\right) + o\left(\sum_{|l|<L, l\neq 0}\frac{1}{l\alpha(l)}\right)$$

so that the limits of (6.40) as $n \to +\infty$ and (4.26) as $T \to +\infty$ exist (and is equal to $c_0(f_1)c_0(f_2)$). In the same way:

$$\lim_{n\to+\infty}\frac{1}{n^2}\sum_{j=1}^{n} j f_1\left(\frac{jt_n}{\tau_1^0}\right) f_2'\left(\frac{jt_n}{\tau_2^0}\right) = \frac{1}{2}\lim_{T\to+\infty}\frac{1}{T}\int_{-T/2}^{T/2} f_1\left(\frac{t}{\tau_1^0}\right) f_2'\left(\frac{t}{\tau_2^0}\right) dt,$$

$$\lim_{n\to+\infty}\frac{1}{n^2}\sum_{j=1}^{n} j f_2\left(\frac{jt_n}{\tau_2^0}\right) f'_1\left(\frac{jt_n}{\tau_1^0}\right) = \frac{1}{2}\lim_{T\to+\infty}\frac{1}{T}\int_{-T/2}^{T/2} f'_1\left(\frac{t}{\tau_1^0}\right) f_2\left(\frac{t}{\tau_2^0}\right) dt,$$

$$\lim_{n\to+\infty}\frac{1}{n^3}\sum_{j=1}^{n} j^2 f'_1\left(\frac{jt_n}{\tau_1^0}\right) f'_2\left(\frac{jt_n}{\tau_2^0}\right) = \frac{1}{3}\lim_{T\to+\infty}\frac{1}{T}\int_{-T/2}^{T/2} f'_1\left(\frac{t}{\tau_1^0}\right) f'_2\left(\frac{t}{\tau_2^0}\right) dt.$$

## References


Golubev, G. K. (1988). Estimation of the period of a signal with an unknown form against a white noise background. *Problemy Peredachi Informatsii 24*(4), 38–52.

Hall, P., Reimann, J., and Rice, J. (2000). Nonparametric estimation of a periodic function. *Biometrika 87*(3), 545–557.

Hannan, E. J. (1973). The estimation of frequency. *J. Appl. Probability 10*, 510–519.

Kavalieris, L. and Hannan, E. (1994). Determining the number of terms in a trigonometric regression. *Journal of time series analysis 15*(6), 613–625.

Kay, S. M. (1989). A fast and accurate single frequency estimator. *IEEE Trans. on ASSP 37*, 1987–1990.

Kootsookos, P. J. (1999, February). A review of the frequency estimation and tracking problems. Frequency estimation and tracking project, CRC for Robust and Adaptive Systems, DSTO, Salisbury Site. www.itee.uq.edu.au/ kootsoop/comparaison-t.pdf.

Kundu, D. (2004). Computational aspects in statistical signal processing. *Statistical computing: existing methods and recent developments*.

Lavielle, M. and Lévy-Leduc, C. (2004). Semiparametric estimation of the frequency of unknown periodic functions and its application to laser vibrometry signals. Accepted for publication in IEEE Transactions on Signal Processing.

Le Cam, L. (1986). *Asymptotic methods in statistical decision theory*. Springer-Verlag.

McNeney, B. and Wellner, J. A. (2000). Application of convolution theorems in semiparametric models with non-i.i.d. data. *J. Statist. Plann. Inference 91*(2), 441–480.

Pisarenko, V. F. (1973). The retrieval of harmonics from a covariance function. *Geophys. J. R. Astr. Soc. 10*, 347–366.

Prenat, M. (2001). Vibration modes and laser vibrometry performance in noise. In *Proceedings of the Physics in Signal and Image Processing (PSIP2001) conference, Marseille, France*.

Quinn, B. G. and Fernandes, J. M. (1991). A fast efficient technique for the estimation of frequency. *Biometrika 78*(3), 489–497.

Quinn, B. G. and Hannan, E. J. (2001). *The estimation and tracking of frequency*. Cambridge Series in Statistical and Probabilistic Mathematics. Cambridge: Cambridge University Press.

Quinn, B. G. and Thomson, P. J. (1991). Estimating the frequency of a periodic function. *Biometrika 78*(1), 65–74.

Schmidt, R. O. (1986). Multiple emitter location and signal parameter estimation. *IEEE Trans. Antennas Propag. 34*, 276–280.

Stoica, P. and Söderström, T. (1992). List of references on spectral line analysis. *Signal Processing 35*, 1533–1538.

van der Vaart, A. W. (1998). *Asymptotic statistics*. Cambridge Series in Statistical and Probabilistic Mathematics. Cambridge: Cambridge University Press.

Walker, A. (1971). On the estimation of a harmonic component in a time series with stationary independent residuals. *Biometrika 58*, 21–36.

Whittle, P. (1952). The simultaneous estimation of a time series harmonic components and covariance structure. *Trab. Estadist 3*, 43–57.

LABORATOIRE DE MATHEMATIQUE
U.M.R C 8628
UNIVERSITÉ PARIS-SUD
BAT. 425
91405 ORSAY CEDEX
FRANCE
E-MAIL: ELISABETH.GASSIAT@MATH.U-PSUD.FR

LABORATOIRE DE MATHEMATIQUE
U.M.R C 8628
UNIVERSITÉ PARIS-SUD
BAT. 425
91405 ORSAY CEDEX
FRANCE
E-MAIL: CELINE.LEVY-LEDUC@MATH.U-PSUD.FR