Hyper-Kähler manifolds and Lagrangian fibrations

X compact complex manifold

Assume: $X \underset{\text{closed embedding}}{\hookrightarrow} \mathbb{P}^m_{\mathbb{C}}$ ($\underset{\text{Chow's Thm}}{\rightsquigarrow}$ X algebraic)

hyper-Kähler

Def Say X irreducible holomorphic symplectic if

- X simply-connected
- $\exists!$ holo symplectic 2-form ($\Rightarrow \dim_{\mathbb{C}} X = \underbrace{2n}_{\text{even}}$)

Q1 : Classify HK manifolds

Upshot: [w/ Debarre, Huybrechts, Voisin] Can we say something in dim 4.

K3 surfaces ($n=1$)

$S \subseteq \mathbb{P}^3_{\mathbb{C}}$ quartic surface $\leftarrow$ 0-locus of homog. poly of degree 4 in 4 variables.

- S simply conn. ($\leftarrow$ Lefschetz Hyperpl Thm. $\pi_k(\mathbb{P}^3, S) = 0$, $k=0,1$)

• symplectic form $\quad \eta = \mathrm{Res}\left(\dfrac{\sum (-1)^i x_i \, dx_0 \wedge \dots \wedge \widehat{dx_i} \wedge \dots dx_3}{f} \right)$

Thm [Kodaira]

All K3 surfaces are deformation equivalent

$$\begin{array}{ccc} \mathcal{X} & \hookleftarrow & \mathcal{X}_t \\ \downarrow & & \downarrow \\ \mathbb{D} & \ni & t \end{array}$$

$\mathcal{X}_0 =$ any K3

$\mathcal{X}_1 =$ quartic

Pf: • deformation th.

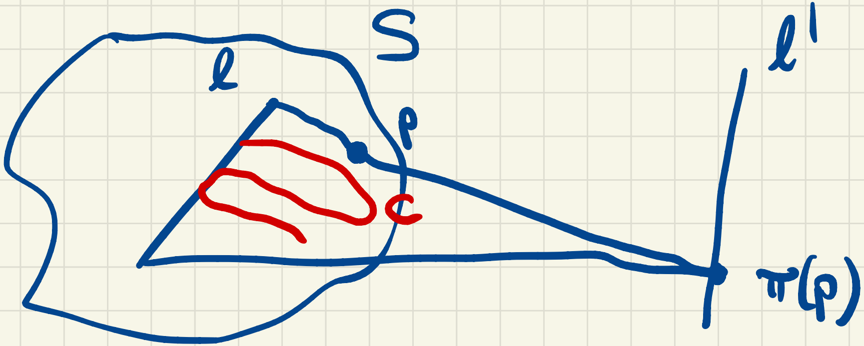
• being quartic surface is "cohomological" property.

$\blacksquare$

Ex ("interesting quartic surfaces")

Can vary the coeff. of quartic poly $\rightsquigarrow$ $\exists$ line $\cong \mathbb{P}^1_{\mathbb{C}} \hookrightarrow S$.

$\rightsquigarrow$ e.g. $f = x_0^4 - x_1^4 + x_2^4 - x_3^4$ $\quad \begin{cases} x_0 = x_1 \\ x_2 = x_3 \end{cases}$



$\rightsquigarrow \quad \pi : S \longrightarrow \mathbb{P}^1 \cong l'$ w/ fibers curves of deg 3

$\rightsquigarrow$ complex torus of dim 1.

generic fiber

$\eta|_{\text{gen. fiber}} \equiv 0$

$\rightsquigarrow$ Lagrangian fibration

Idea [O'Grady] Generalize this in dim ≥ 4.

$\cdot\ \cdot$

Conj Any HK manifold can be deformed into a HK mfd admitting Lagrangian fibration.

yes, in dim 4 under certain top. assptn.

Q 2: Can we use this conj towards classif. of HKs ?

Lagrangian fibrations

X HK $\rightsquigarrow$ $H^2(X,\mathbb{Z})$ free ab. gp. of finite rk.

$\alpha \in H^2(X,\mathbb{Z}) \rightsquigarrow \int_X \alpha^{2n} \in \mathbb{Z}$

$\pi : X \to B$ Lagrangian fibr.

$\dim B = n$

B proj.

$$\begin{array}{ccccc} B & \hookrightarrow & \mathbb{P}^N & & \alpha_B \\ \cup & & \cup & & \| \\ D = H \cap B & \hookrightarrow & H \cong \mathbb{P}^{N-1} & \rightsquigarrow & [D] \in H^2(B,\mathbb{Z}) \end{array}$$

$\rightsquigarrow$ $\alpha := \pi^* \alpha_B \in H^2(X, \mathbb{Z})$ $\in \langle [D] : D \subseteq X \text{ hyp. section} \rangle$

We have : (1) α "algebraic" , $\alpha \neq 0$

(2) $\int_X \alpha^{2m} = 0$

Say α <u>nef</u>

(3) $\forall \underset{\text{curve}}{C} \hookrightarrow X$ <u>closed</u> , $\int_C \alpha \geq 0$

<u>Conj</u> (SYZ Conj for HK / abundance Conj for HK / TBHTHS Conj)

X HK , $\dim X = 2m$

<u>Assume</u>: $\alpha \in H^2(X, \mathbb{Z})$ sat. (1), (2), (3).

Thm $\exists\ \pi : X \to B$ Lagr. fibr. st. $\alpha = \pi^* \alpha_B$

$\square$

<u>Q3</u>: What is B?

<u>Conj</u> $B \cong \mathbb{P}^n$ $\sim$ <u>True</u>: - if B smooth [Hwang]

- $\dim X = 4$ [Huybrechts-Xu]

<u>Fibers</u>: $b \in B$ gen. pt. $\rightsquigarrow f^{-1}(b)$ <u>torus</u> of dim n. (abelian variety)

$\rightsquigarrow \exists$ anoth. class $\beta \in H^2(X, \mathbb{Z})$ st. $\beta|_{f^{-1}(b)}$ lagr. sect'n.

$\rightsquigarrow \quad a = \frac{1}{n!}\int_X \alpha^n \beta^n \in \mathbb{Z}_{\geq 1}$

<u>Main Thm 1</u> [DHMV] X HK, $\dim X = 4$

<u>Assume</u>: $\alpha, \beta \in H^2(X, \mathbb{Z})$ s.t. α sat. (1), (2), (3)

and $a = 1$

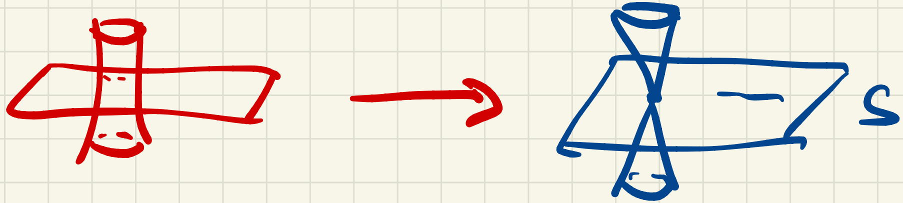
Then SYZ Conj holds.

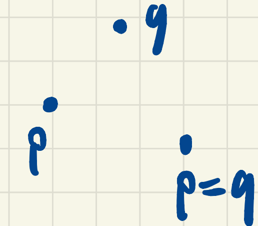
$\boxtimes$

<u>Examples</u> of HK4.

S K3 surface $\rightsquigarrow$ $S^{(2)} = S\times S/\mathfrak{S}_2$

$\wr$ loc.



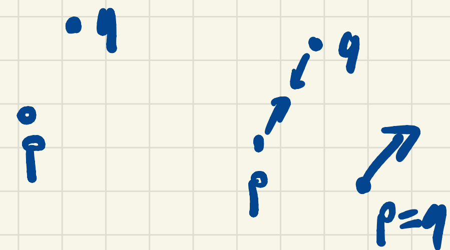


Hilbert square: $S^{[2]} \to S^{(2)}$

$\wr$

$T^*\mathbb{P}^1 \times S \to Q \times S$

sympl.



$S^{[2]}$ HK4

Main Thm 2 [DHMV] (O'Grady Conj.)

X HK 4 fold

Assume: $\exists\, \alpha, \beta \in H^2(X, \mathbb{Z})$ s.t.

$$\cdot \int_X \alpha^4 = 0$$

$$\cdot \frac{1}{2} \int_X \alpha^2 \beta^2 = 1$$

Thm X defo eq. to $S^{[2]}$.

$\blacksquare$

$b_2(X)$

$$\int_X \alpha^{2m} = q_X \cdot q_X(\alpha)^m$$

Fujiki const