ANTISYMPLECTIC INVOLUTIONS ON PROJECTIVE HKs

work in progress w/

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Motivation 1

Y Fano manifold of K3 type

$$\begin{tikzcd}
Y \arrow[r, bend left, "\text{“geometrically”}"] \arrow[r, bend right, "\text{“categorically”}"'] & (F, h) \arrow[l]
\end{tikzcd}$$

proj. HK → F

primitive polarization → h

Ex Y cubic 4 fold $\rightsquigarrow (F, h)$

geom: F = variety of lines $\subseteq Y$ [Beauville-Donagi]
h = Plücker pol.

Categ: $D^bY = \langle \mathcal{D}_Y, \mathcal{O}_Y, \mathcal{O}_Y(1), \mathcal{O}_Y(2) \rangle$

[Kuznetsov] $\mathcal{D}_Y$ non-comm. K3 surface

[BLMS] $\sigma \in \mathrm{Stab}(\mathcal{D}_Y)$

[Li-Pertusi-Zhao] $(F, h) \cong (M_\sigma(\lambda), \ell_\sigma)$

<u>Idea</u>: Realize Y as (connected cpt. of) fixed locus of antisymplectic involution

Construction :

F HK , $\dim F = 4$, defo equiv. to $\mathrm{Hilb}^2(K3)$
(K3-type)

$n \equiv 3 \bmod 4.$

h prim. pol. , $h^2 = 2n$, $\mathrm{div}(h) = 2$

wrt/ BBF form on $H^2(F,\mathbb{Z})$ $\rightsquigarrow$ $h.H^2(F,\mathbb{Z}) = 2\mathbb{Z} \subseteq \mathbb{Z}$

$$\rightsquigarrow \quad A_{2,n} = \begin{pmatrix} 2 & -1 \\ -1 & \frac{n+1}{2} \end{pmatrix}$$

rk 2, even, pos. def. lattice

λ_1 , λ_2 basis ($\cong \mathbb{Z}^2$)

Strange duality (Apostolov, Hulek, Le Potier)

$(F, h) \longleftrightarrow (X, \lambda)$ X HK, $\dim X = 2n+2$

K3 type

$\lambda^2 = 2$, $\operatorname{div}(\lambda) = 2$

Intuition :

$(\lambda_1 + 2\lambda_2)^2 = 2n$

$\lambda_1^2 = 2$

$$(M_\gamma(\lambda_1), \theta(\lambda_1 + 2\lambda_2)) \longleftrightarrow (M_\sigma(\lambda_1 + 2\lambda_2), \theta(\lambda_1))$$

$\theta : \lambda_1^\perp \xrightarrow{\sim} H^2(M, \mathbb{Z})$

$\boxed{(\lambda_1 . \lambda_1 + 2\lambda_2) = 0}$

Actual def. :

$$\begin{tikzcd}
(F,h) \arrow[r, rightsquigarrow] \arrow[dd, leftrightarrow] & H^2(F,\mathbb{Z})_{\mathrm{prim}} = \langle \lambda_1, h \rangle^{\perp} \subseteq \Lambda_F \\
 & U^4 \oplus E_8(-1)^2 + \text{Hodge str.} \arrow[u, "\wr"'] \\
(X,\lambda) & H^2(X,\mathbb{Z})_{\mathrm{prim}} = \langle h, \lambda_1 \rangle^{\perp} \subseteq \Lambda_X = \Lambda_F \arrow[l, rightsquigarrow]
\end{tikzcd}$$

$$(F, h) \rightsquigarrow (X, \lambda)$$

$\dim X = 2n+2$

$\lambda^2 = 2$

$\mathrm{div}(\lambda) = 2$

Ex Y cubic 4fold $\rightsquigarrow$ (F, h) var. of lines

$\rightsquigarrow$ (X, λ) LLSvS 8fold param. (eq. classes of) twisted cubic curves.

Fact: $\exists\, \tau : X \xrightarrow{\sim} X$, $\tau^2 = \mathrm{id}$, antisympl. s.t. Y conn. cpt. of $\mathrm{Fix}(\tau)$.

Fixed loci of antisympl. invol's

Setting : (X,λ) X HK, $\dim X = 2n+2$

K3 type $n \geq 0$

λ prim. pol., $\lambda^2 = 2$

↑ { div = 1, or 2,

Verbitsky Torelli Thm

+

Markmann Monodromy Thm

$\rightsquigarrow \rho_\lambda(x) := -x + (x,\lambda)\cdot\lambda$ on $H^2(X,\mathbb{Z})$

λ ample

$\exists\, \tau : X \xrightarrow{\sim} X$ $\tau^2 = \mathrm{id}$, antisympl

$\tau_* = \rho_\lambda$.

Main Thm (1) The number of connected cmpts of $\mathrm{Fix}(\tau)$ is equal to $\mathrm{div}(\lambda)$.

(2) If $\mathrm{div}(\lambda) = 2$, then one component is **Fano** manifold

↑ $\dim = n+1$

index $= 3$

Expect:
$$\left.\begin{array}{l} \rho = 1 \\ h^{3,1} = 1 \end{array}\right] \leftarrow \text{true for } n = 3, 7$$

Ex Y cubic 4 fold $(n = 3)$

$$\mathrm{Fix}(\tau) = Y \sqcup \tilde{Y}$$

cubic 4fold $\rightsquigarrow Y$

smoothening of $\bar{Y} \rightsquigarrow \tilde{Y}$

$\Gamma \subseteq \mathbb{P}^2$ deg 6

$$\begin{tikzcd} D \arrow[d, "2-1"'] & \subseteq & \mathrm{Bl}_{\mathrm{Sym}^2\Gamma}(\mathrm{Sym}^4\Gamma) \arrow[d, "\text{normaliz.}"] \\ \bar{D} & \subseteq & \bar{Y} \end{tikzcd}$$

Ex & Motiv. 2

4 fold

$\mathrm{div}(\lambda) = 1$ $\qquad (X, \lambda)$ double EPW sextics

$\lambda^2 = 2$

$n = 1$

(O'Grady)

$\tau \circlearrowright X \xrightarrow{2:1} Z \sim$ sextic $\subseteq \mathbb{P}^5$

$\mathrm{Sing}(Z) = W$ ← surface of gen. type

(Ferretti)

$\rightsquigarrow \boxed{\mathrm{Fix}(\tau) = W}$

Idea (O'Grady): $2W$ sweeps $X \rightsquigarrow$ covering family of Lagrangian cycles.

Sketch pf of Main Thm

Idea: Specialize to (X, λ) where τ can be understood ...

Only way is to look for (X, λ) singular

<u>Ex</u>

$$\begin{tikzcd} & \text{nodal cubic (A)} \\ Y \text{ cubic} \arrow[ur, squiggly] \arrow[dr, squiggly] & \\ & \text{chordal cubic (B)} \end{tikzcd}$$

(A) (C. Lehn) (S,h) S K3 $h^2=6$

$$\underline{v} = (0,h,3)$$

$$\begin{tikzcd}
M = M_h(\underline{v}) \arrow[loop left, "\tau"] \arrow[r, dashed, leftrightarrow, "\text{Mukai flop at } \check{\mathbb{P}}^4"'] \arrow[d, "\pi"] & M_0 \arrow[loop above, "\tau"] \arrow[r, dashed, leftrightarrow, "\text{Mukai flop at } \mathbb{P}^3\text{-bdle} / S\times S"'] & M_1 \arrow[loop above, "\tau"] \arrow[d, "\varphi"', "\text{div. contr}"] \\
\check{\mathbb{P}}^4 \arrow[u, bend left, "\Delta"] & & \overline{M} \arrow[loop right, "\tau"]
\end{tikzcd}$$

Then

- flops do not create/destroy any cpt. of $\mathrm{Fix}(\tau)$
- $(\overline{M}, \overline{\tau})$ specializ. we want !

e.g.

$$\begin{tikzcd}
\check{\mathbb{P}}^4 \arrow[r, dashed, leftrightarrow] & \mathbb{P}^4 \arrow[d, phantom, sloped, "\supset"] & Bl_S \mathbb{P}^4 \arrow[l] \arrow[r] \arrow[d, phantom, sloped, "\supset"] & \overline{Y} \text{ nodal cubic} \arrow[d, phantom, sloped, "\ni"] \\
 & S & LGr(2,4) \arrow[r] & p
\end{tikzcd}$$

In general, can generalize this:

(S,h) S K3 $h^2 = 2n$ $(g = n+1)$

$\underline{v} = (0, h, n)$ $4 \mid g$

$$\begin{tikzcd}
M = M_h(\underline{v}) \arrow[loop left, "\tau"] \arrow[d, "\pi"] \arrow[r, dashed, leftrightarrow, "\sim"] & M_0 \arrow[loop above, "\tau"] \arrow[r, dashed, leftrightarrow, "\sim"] & M_1 \arrow[loop above, "\tau"] \arrow[r, dashed, leftrightarrow, "\sim"] & \cdots \arrow[r, dashed, leftrightarrow, "\sim"] & M_{g/4} \arrow[loop above, "\tau"] \arrow[d, "\varphi\ \text{div. contr.}"] \\
\check{\mathbb{P}}^g \arrow[u, bend left, "s"] & \text{Fix has 2 \underline{cmpts}} \arrow[ul] & \text{speci\underline{aliz}.} \arrow[rr] & & \boxed{\overline{M} \circlearrowleft \tau}
\end{tikzcd}$$

e.g., Fano cmpt :

$$\begin{tikzcd}
 & & Y_0 \arrow[d, phantom, sloped, "="] & & & \\
\check{\mathbb{P}}^g \arrow[r, dashed, leftrightarrow] & \mathbb{P}^g \arrow[d, phantom, sloped, "\supset"] & Bl_S\mathbb{P}^g \arrow[l] \arrow[r, dashed, leftrightarrow, "\sim", "\text{anti flip at } \mathrm{Hilb}^2 S"'] & Y_1 \arrow[r, dashed, leftrightarrow, "\sim"] & \cdots \arrow[r, dashed, leftrightarrow, "\sim"] & Y_{g/4} \arrow[d, "\varphi"] \\
 & S & & & & \bar{Y} \arrow[d, phantom, sloped, "="] \\
 & & & & \text{specialization of } \underline{\text{Fano}} \arrow[ur, squiggly] & {}
\end{tikzcd}$$

Conj Y Fano cpt. $m = \frac{g}{4} = \frac{n+1}{4}$

K3 categ.

$D^bY = \langle \mathcal{D}^{[m]},$

"analogues of deriv. categ. of Hilbert schemes"

$\mathcal{O}_Y, \mathcal{D}, \mathcal{D}^{[2]}, \dots, \mathcal{D}^{[m-1]}$

$\mathcal{O}_Y(1), \dots\dots$

$\mathcal{O}_Y(2), \dots \rangle$

1 — 0

1 — 2

1 21 1 — 4

1 23 1 — 6

1 22 253 22 1 — 8

$$\langle D^b \mathrm{Hilb}^2 S,\ D^b S,\ D^b S,\ D^b S,\ \mathcal{O}_Y,\ \mathcal{O}_Y(1),\ \mathcal{O}_Y(2) \rangle$$

$$\begin{tikzcd}
 & \mathrm{LGr}(2,4) \arrow[d] \arrow[r, phantom, "\subseteq"] & Y \arrow[d] \\
A_1\ \underline{\underline{\text{sing.}}} \arrow[ru] & Z \arrow[r, phantom, "\subseteq"] & \overline{Y} \\
\text{smooth cubic 4fold of deg 14} \arrow[ru] & &
\end{tikzcd}$$

X <u>cohom. of K3-type</u>

$\lambda \quad \lambda^2 = 2$

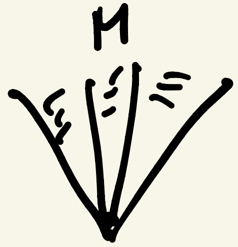
$$\rho_\lambda(n) = -n + (\lambda, n) \cdot \lambda$$

$$\in O^+(H^2(X, \mathbb{Z}))$$

$$\cup\!|$$

$$\in \mathrm{Mon}(X)$$

S ⑭ $\mathrm{Hilb}^4 S \overset{\sim}{\dashrightarrow} \mathrm{Hilb}^4 S$



$$M \xrightarrow[\sim]{\tau} M \qquad \tau^2 = id$$

$\underline{\mathrm{Fix}(\tau) = ?}$

$$\begin{tikzcd}
\mathbb{P}^M \arrow[dr, "{(a,b)}"'] & & \check{\mathbb{P}}^m \arrow[dl, "{(b,a)}"] \\
& \mathrm{Hilb}^m S \times \mathrm{Hilb}^m S \arrow[loop below, <->] &
\end{tikzcd}$$

$\mathrm{Fix} = \Delta$

$\boxed{\check{\mathbb{P}}^{m+1} \dashleftarrow\dashrightarrow \underset{\overset{\cup}{S}}{\mathbb{P}^m}}$ $\qquad V = (1, b) \qquad \mathbb{P}^{b-1}/S$