A walk through hyper-Kähler geometry

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2025 SRI in Algebraic Geometry — Fort Collins, CO, July 25, 2025

European Research Council
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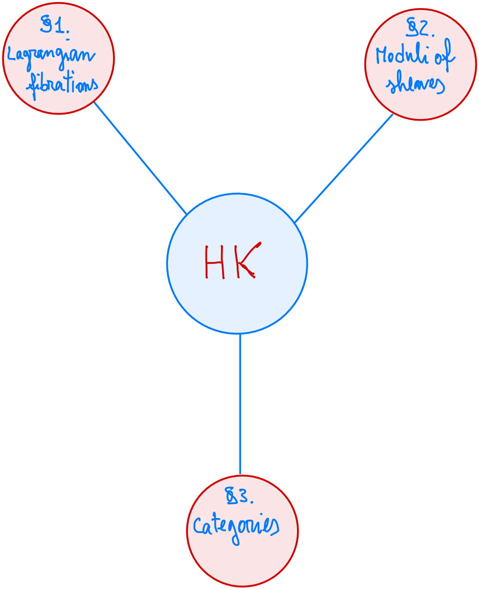
Plan


§1.
Lagrangian fibrations
§2.
Moduli of sheaves
HK
§3.
Categories

Hyper-Kähler manifolds

DEFINITION. M smooth projective variety over $\mathbb{C}$.
M is hyper-Kähler (HK) manifold if

- $H^0(M, \Omega_M^2) = \mathbb{C} \cdot \sigma$, where σ is a non-degenerate 2-form ($\rightsquigarrow \dim(M) = 2n$),
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MOTIVATION. The Beauville–Bogomolov Decomposition Theorem:
X smooth projective variety with $\mathrm{c}_1(X) = 0$ in $H^2(X, \mathbb{R})$.
Then there exists $X' \to X$ finite étale with

$$X' \cong A \times \prod_i Y_i \times \prod_j M_j,$$

where:

- A abelian variety,
- Y_i Calabi–Yau (CY) manifold,
- M_j HK manifold.

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EXTENSIONS:

- singular case $\rightsquigarrow$ (irreducible) symplectic variety,
- analytic case.

Examples

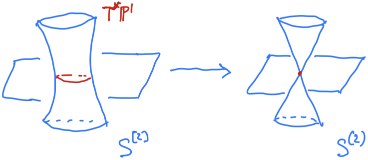
$n = 1$: HK surfaces ≡ K3 surfaces; they are all deformation equivalent [Kodaira].

Known deformation families of HK manifolds:

$$\left.\begin{array}{l} \mathrm{K3}^{[n]} \\ \mathrm{Kum}_n \end{array}\right\} n \geq 2, \text{ [Beauville]} \qquad\qquad \left.\begin{array}{l} \mathrm{OG10} \\ \mathrm{OG6} \end{array}\right\} \text{[O'Grady].}$$

Ex. [Fujiki]

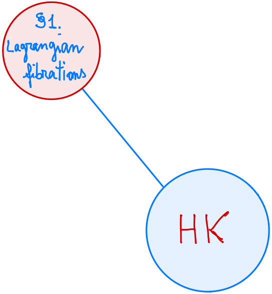
$$\begin{tikzcd} \mathrm{Bl}_\Delta(S\times S) \arrow[r] \arrow[d, "2-1"] \arrow[loop left, "\tau"] & S\times S \arrow[d, "2-1"] \arrow[loop right, "\tau"] \\ S^{[2]} \arrow[r] & S^{(2)} := \dfrac{S\times S}{\mathfrak{S}_2} \end{tikzcd}$$



S K3 surface

$S^{[2]}$ Hilbert square of S

§1. Lagrangian fibrations



MOTIVATION. Use the Lagrangian fibration structure to:

- construct examples of HK, by compactifying abelian fibrations [Markman],
- bound deformation classes of HK [Engel–Filipazzi–Greer–Mauri–Svaldi].

The SYZ Conjecture

CONJECTURE [Generalized Abundance Conjecture for HK].
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[Matsushita] $\rightsquigarrow$ the induced morphism is a Lagrangian fibration

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REMARKS.

- The SYZ Conjecture is true for all known deformation families of HK manifolds.
- If $b_2(M) \geq 5$, then the SYZ Conjecture implies that a HK manifold M can be deformed to a HK manifold admitting a Lagrangian fibration.

$\rightsquigarrow$ HK manifolds as "special" relative abelian varieties.

Construction – Examples

EXAMPLE [The Beauville–Mukai system, I].
(S, H) very general polarized K3 surface of genus $g \geq 2$. Consider

$$U_{\mathrm{sm}} := \{C \in |H| \ : \ C \text{ smooth}\} \subset |H| \cong \mathbb{P}^g$$

and the relative Jacobian fibration

$$f_{\mathrm{sm}} \colon \mathcal{J}_{\mathrm{sm}} \longrightarrow U_{\mathrm{sm}}, \qquad f_{\mathrm{sm}}^{-1}([C]) = \mathrm{Jac}(C).$$

Then there exists a HK manifold M of $\mathrm{K3}^{[g]}$-type, together with a Lagrangian fibration $f \colon M \to \mathbb{P}^g$ extending f_{sm}.

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Example [The Laza–Saccà–Voisin system, I].
$X \subset \mathbb{P}^5$ smooth cubic fourfold, $H := \mathcal{O}_{\mathbb{P}^5}(1)|_X$. Consider

$$U_{\mathrm{sm}} := \{Y \in |H| \ : \ Y \text{ smooth cubic threefold}\} \subset |H| \cong \mathbb{P}^5$$

and the relative intermediate Jacobian fibration

$$f_{\mathrm{sm}} \colon \mathcal{J}_{\mathrm{sm}} \longrightarrow U_{\mathrm{sm}}, \qquad f_{\mathrm{sm}}^{-1}([Y]) = \mathrm{J}(Y).$$

Then there exists a HK manifold M of OG10-type, together with a Lagrangian fibration $f\colon M \to \mathbb{P}^5$ extending f_{sm}.

Construction – Main Results

It is very hard to construct examples of HK manifolds!
In the singular case, the following results give a powerful construction method.

THEOREM [Saccà].
B normal projective, $U \subset B$ open.
M_U smooth quasi-projective symplectic variety ($\rightsquigarrow \sigma_U$ symplectic form).
$f_U \colon M_U \to U$ surjective with connected fibers such that there exists $\emptyset \neq U' \subset U$ open with

$$f_{U'} : M_{U'} \to U' \qquad \text{proper Lagrangian fibration.}$$

Assume:

- $\operatorname{codim}(B \setminus U) \geq 2$
- σ_U extends on a smooth compactification of M_U.

Then there exists M terminal $\mathbb{Q}$-factorial projective symplectic variety, together with a Lagrangian fibration $f \colon M \to B$ extending f_U.

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THEOREM [Y. Liu–Z. Liu–C. Xu].
Stronger assumptions on M_U, or on the fibers of $f_{U'} \Longrightarrow M$ irreducible symplectic.

EXAMPLE. We can deduce the two previous examples once we extend the fibration to, e.g., the locus of hypersurfaces with nodal singularities (we do not need the fiber to be compact there!).

Construction – Further examples

THEOREM [Y. Liu–Z. Liu–C. Xu].
$X \subset \mathbb{P}^5$ general smooth cubic fourfold.
Then there is an associated M projective irreducible symplectic variety with terminal $\mathbb{Q}$-factorial singularities, $\dim(M) = 42$, $b_2(M) \geq 24$, together with a Lagrangian fibration $f\colon M \to \mathbb{P}(1^{15}, 2^6, 3)$.

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THEOREM [M–O'Grady–Saccà].
$X \to \mathrm{Gr}(2,5)$ general Gushel–Mukai 6fold.
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IDEA OF THE PROOF.
Both results are based on Saccà's theorem and on a construction by Iliev–Manivel:

- [LLX] look at cubic fivefolds containing X and their intermediate Jacobians,
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QUESTION. Is the regular locus in these examples simply connected?

Results on the SYZ Conjecture

THEOREM [Debarre–Huybrechts–M–Voisin].
M smooth HK 4fold.
L nef line bundle on *M*.
Assume that there exists another line bundle L' on *M* such that

$$\int_M c_1(L)^4 = \int_M c_1(L')^4 = 0$$

$$\frac{1}{2}\int_M c_1(L)^2 c_1(L')^2 = 1.$$

Then the SYZ Conjecture holds for L.

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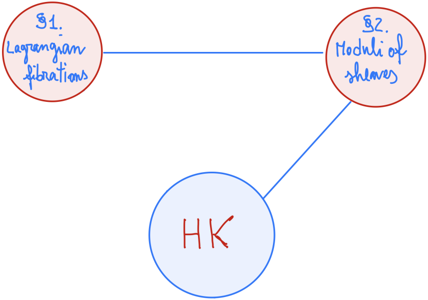
COROLLARY [O'Grady's Conjecture].
M smooth HK 4fold.
Assume there are classes $\mathrm{l}, \mathrm{l}' \in H^2(M, \mathbb{Z})$ such that

$$\int_M \mathrm{l}^4 = 0 \qquad \frac{1}{2}\int_M \mathrm{l}^2\mathrm{l}'^2 = 1.$$

Then M is of $\mathrm{K3}^{[2]}$-type.

§2. Moduli of sheaves



Motivation. Use moduli spaces to study:

- cycles and Hodge classes [Mukai, Markman]
- singularities [O'Grady]
- ample cones [Bayer-M]
- Lagrangian fibrations [Markman, Yoshioka, Bayer-M]

Stable sheaves and their moduli

IDEA. Reinterpret the previous results by using stable sheaves.

X projective variety, H ample on X, $\mathbf{v}$ numerical class

$$\rightsquigarrow \quad M_{X,H}(\mathbf{v}) := \left\{ \begin{array}{c} \text{moduli space of (equivalence classes of)} \\ H\text{-Gieseker semistable sheaves on } X \\ \text{with numerical class } \mathbf{v} \end{array} \right\}.$$

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KEY POINT. $M_{X,H}(\mathbf{v})$ is a projective variety and we can study its singularities by using:

- tangent space $T_{M_{X,H}(\mathbf{v}),[\mathcal{E}]} \equiv \mathrm{Ext}^1(\mathcal{E},\mathcal{E})$, and
- the explicit local structure, depending on $\mathrm{Ext}^2(\mathcal{E},\mathcal{E})$,

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EXAMPLE [The Beauville–Mukai system, II].
(S, H) very general polarized K3 surface of genus $g \geq 2$.
The Beauville–Mukai system can be reinterpreted as a moduli space of rank 1 torsion sheaves of fixed degree supported on curves in the linear system $|H|$.

Mukai's theory

S K3 surface $\rightsquigarrow$ Mukai Hodge structure:

- $\widetilde{H}(S,\mathbb{Z}) := H^0(S,\mathbb{Z}) \oplus H^2(S,\mathbb{Z}) \oplus H^4(S,\mathbb{Z}) \ni \mathbf{v} = (v_0, v_1, v_2)$.
- Mukai pairing
$$(\mathbf{v}, \mathbf{v}') := \int_S v_1 \cup v_1' - v_0\, v_2' - v_2\, v_0'.$$
- $H^{2,0}(S) \hookrightarrow H^2(S,\mathbb{C}) \hookrightarrow \widetilde{H}(S,\mathbb{C})$ weight 2 Hodge structure.
- Mukai vector
$$v(\mathcal{E}) := \mathrm{ch}(\mathcal{E})\sqrt{\mathrm{td}_S} \in \widetilde{H}_{\mathrm{alg}}(S,\mathbb{Z}), \quad \mathcal{E} \in \mathrm{Coh}(S).$$

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Theorem [Mukai, Beauville, Kuleshov, Huybrechts, O'Grady, Yoshioka, Perego–Rapagnetta].
Fix $\mathbf{v}$ Mukai vector. Then, for H general polarization on S with respect to $\mathbf{v}$, we have:

- numerical criterion for when $M_{S,H}(\mathbf{v})$ is non-empty in terms of Mukai structure;
- $M_{S,H}(\mathbf{v})$ is a irreducible symplectic variety with terminal $\mathbb{Q}$-factorial singularities, smooth of $\mathrm{K3}^{[n]}$-type if $\mathbf{v}$ primitive, and with one exception in the non-primitive case, where the moduli space has a smooth symplectic resolution, giving a HK manifold of OG10-type;
- explicit description of $H^2(M_{S,H}(\mathbf{v}),\mathbb{Z})$ in terms of $\mathbf{v}$ and $\widetilde{H}(S,\mathbb{Z})$.

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Important point. Local structure: moduli space is locally cut out by quadrics.

Problem. When trying to generalize this to higher dimensions, nothing works anymore. We expect though that all HK can be obtained by some "extension" of the above theorem.

Atomic sheaves, I

THEOREM [Voisin].
M HK manifold, $Z \subset M$ *smooth* Lagrangian submanifold.
Consider the restriction morphism in cohomology

$$H^2(M,\mathbb{Z}) \xrightarrow{\mathrm{res}} H^2(Z,\mathbb{Z}).$$

Then, while deforming *M* as a complex manifold:
Z deforms "as much as possible" $\Leftrightarrow$ its class stays of Hodge type "as much as possible"
$\Leftrightarrow$ the morphism res has one-dimensional image.

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IDEA. The definition of atomic sheaves generalizes this theorem to include non-commutative deformations of M as well. This extra condition corresponds (cohomologically) to asking that the canonical line bundle ω_Z lies in the image of res.

Atomic sheaves, I

DEFINITION [Beckmann, Markman].
M HK manifold, $\mathcal{E}$ sheaf on $M \rightsquigarrow v(\mathcal{E}) := \mathrm{ch}(\mathcal{E})\sqrt{\mathrm{td}_M}$ Mukai vector.
Say $\mathcal{E}$ is:

- atomic if
$$\mathrm{obs}_{\mathcal{E}} \colon \mathrm{HT}^2(M) \longrightarrow \mathrm{H}^*(M, \Omega^*_M) \qquad \mu \mapsto \mu \lrcorner v(\mathcal{E})$$
has one-dimensional image, where
$$\mathrm{HT}^2(M) := \mathrm{H}^2(M, \mathcal{O}_M) \oplus \mathrm{H}^1(M, T_M) \oplus \mathrm{H}^0(M, \Lambda^2 T_M).$$
- 1-obstructed if
$$\chi_{\mathcal{E}} \colon \mathrm{HH}^2(M) \longrightarrow \mathrm{Ext}^2(\mathcal{E}, \mathcal{E})$$
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EXPLANATION.

- $\mathrm{HT}^2(M) \cong \mathrm{HH}^2(M)$ measures first-order deformations of M, including non-commutative ones.
- $\mathrm{obs}_{\mathcal{E}}$ measures whether $v(\mathcal{E})$ stays of Hodge-type along the given first order deformation.
- $\chi_{\mathcal{E}}$ measures obstruction to deform $\mathcal{E}$ along the given first order deformation.

Atomic sheaves, II

KEY PROPERTIES:

- 1-obstructed $\Rightarrow$ atomic, and conjecturally the converse holds for simple sheaves [Beckmann, Markman],
- atomic $\longleftrightarrow$ Mukai Hodge structure on $\widetilde{H}(M,\mathbb{Q}) := H^2(M,\mathbb{Q}) \oplus U_{\mathbb{Q}}$ associated to M [Beckmann, Markman],
- wall-crossing behavior as for surfaces, when changing polarization [O'Grady],
- stable atomic vector bundles are projectively hyperholomorphic and their moduli space is locally cut out by quadrics [Beckmann, Markman, Verbitsky, Meazzini–Onorati],
- symplectic form, "unique" on the 1-obstructed vector bundles locus [Verbitsky, Kuznetsov–Markushevich, Bottini],
- derived invariant [Taelman, Beckmann, Markman].

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EXAMPLES.

- Sheaves supported on Lagrangian submanifolds [Beckmann, Bottini–M].
- Rigid vector bundles [O'Grady] ($\leftsquigarrow$ these are 1-obstructed!).

The Laza–Saccà–Voisin system, II

$X \subset \mathbb{P}^5$ smooth cubic fourfold, $H := \mathcal{O}_{\mathbb{P}^5}(1)|_X \rightsquigarrow$

$$F(X) := \left\{\text{lines contained in } X\right\}$$

is a HK 4fold of $\mathrm{K3}^{[2]}$-type [Beauville–Donagi].

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The Laza–Saccà–Voisin system, II

$X \subset \mathbb{P}^5$ smooth cubic fourfold, $H := \mathcal{O}_{\mathbb{P}^5}(1)|_X \rightsquigarrow$

$$F(X) := \left\{\text{lines contained in } X\right\}$$

is a HK 4fold of $\mathrm{K3}^{[2]}$-type [Beauville–Donagi].

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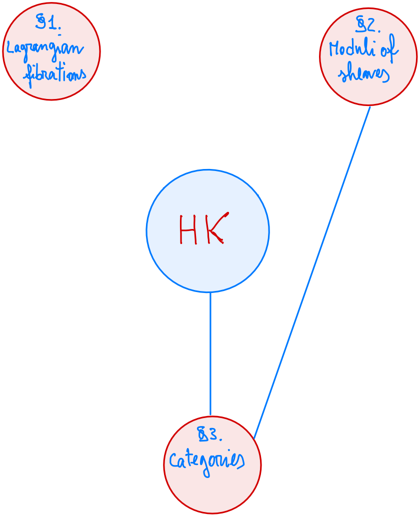
EXPECTATION. The singular examples of [LLX] and [MOS] should be described as moduli spaces of atomic sheaves associated to immersed Lagrangian subvarieties in $F(X)$ [H. Guo–Z. Liu].
$\rightsquigarrow$ need new examples/construction of Lagrangian submanifolds

§3. Categories



MOTIVATION. Use categories to understand:

- Fourier–Mukai techniques
- non-commutative deformations
- Lagrangian submanifolds

The Kuznetsov component for cubics

$X \subset \mathbb{P}^5$ smooth cubic fourfold, $H := \mathcal{O}_{\mathbb{P}^5}(1)|_X \rightsquigarrow$

$$\begin{aligned}\mathcal{D}_X &:= \langle \mathcal{O}_X, \mathcal{O}_X(H), \mathcal{O}_X(2H)\rangle^\perp \subset \mathrm{D}^\mathrm{b}(X)\\ &= \left\{E \in \mathrm{D}^\mathrm{b}(X) \,:\, \mathrm{Ext}^i(\mathcal{O}_X, E) = \mathrm{Ext}^i(\mathcal{O}_X(H), E) = \mathrm{Ext}^i(\mathcal{O}_X(2H), E) = 0,\ \text{for all } i\right\}.\end{aligned}$$

This is a CY2 category which should be thought as a “polarized” non-commutative deformation of a K3 surface [Kuznetsov, Bayer–Lahoz–M–Nuer–Perry–Stellari].

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the push-forward functor induces a spherical functor

$$\Phi_Y \colon \mathcal{C}_Y \longrightarrow \mathcal{D}_X,$$

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KEY POINT. $F(X)$ is a moduli space of stable objects in $\mathcal{D}_X$ and the Lagrangian surfaces $F(Y) \subset F(X)$ arise from moduli spaces of stable objects in $\mathcal{C}_Y$ via Φ_Y.

Hyper-Kähler manifolds and polarized CY categories

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- (M, λ) polarized HK manifold $\rightsquigarrow \mathcal{D}_M$ smooth and proper CY category (of lower dimension)?
- Polarization $\longleftrightarrow$ Bridgeland stability condition on $\mathcal{D}_M$?
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RESULTS.

- (M, λ) very general polarized HK manifold of $\mathrm{K3}^{[n]}$-type. Then:
  - there is $\mathcal{D}_M$ smooth and proper CY2 category associated to M [Markman–Mehrotra];
  - there is a Bridgeland stability condition on $\mathcal{D}_M$ and (M, λ) is a moduli space [C. Li–M–Perry–Stellari–X. Zhao].
- Stronger results hold for polarized HK manifolds of Kum_n-type [Bayer–Perry–Pertusi–X. Zhao].