Lagrangian fibrations on hyper-Kähler fourfolds

[w/ O. Debarre, D. Huybrechts, C. Voisin]

Def X compact Kähler manifold, $\dim X = 2n$

Say X hyper-Kähler if

- X simply connected
- $H^0(X, \Omega^2_X) \cong \mathbb{C} \cdot \eta$ ← nowhere deg. holom. 2-form

Ex $(n=1)$ X K3 surface

Q : $n > 1$?

Goal : First steps towards classif. of HK fourfolds $(n=2)$

Conj 1 Any HK manifold can be deformed into a HK manifold w/ Lagrangian fibration

$\curvearrowright$

$f : X \to B$ proper

w/ conn. fibers, B normal proj

w/ $f^{-1}(b) \subseteq X$ abelian variety

$\forall b \in X$ general pt.

[Matsushita]

Rmk $L := f^* \mathcal{O}_B(1)$ nef

w/ $\displaystyle\int_X c_1(L)^{2n} = 0$

Conj 2 (SYZ Conj for HK, abundance Conj. for HK, TBHTHS)

X HK, L nef line bdle st. $\displaystyle\int_X c_1(L)^{2n} = 0$

Then L semiample

$\blacksquare$

[Mats.] Conj 2 $\Rightarrow$ $|mL|$ $m \gg 0$ induces a Lagrangian fibration.

$\rightsquigarrow$ $\boxed{\text{Conj 2} \Rightarrow \text{Conj 1}}$

$b_2(X) \geq 5$

[Hwang] X proj., B smooth $\Rightarrow B \cong \mathbb{P}^n$

[Huybrechts-Xu] X fourfold $\Rightarrow B \cong \mathbb{P}^2$

<u>Conj</u>' • Always $B \cong \mathbb{P}^n$

• $f = \varphi_{|L|} : X \to \mathbb{P}^n$ L <u>primitive</u>

$\boxtimes$ $n+1 = h^0(X, L)$.

<u>Ex</u> All these conjs are proved in known deform. families of HK:

$K3^{[n]}$, Kum_n, $OG6$, $OG10$

X defo equiv. to $S^{[n]} = \mathrm{Hilb}^n S$
S K3 surface

e.g. (S, H) very gen. K3 surface of genus g

dim $2g$ $\rightsquigarrow$ $X := M_S(\underline{v}) :=$ mod. space of rk 1 t.free sheaves supp. on $C \in |H|$ of degree $g-1$

$\underline{v} = (0, H, 0)$

$$\theta : \underline{v}^{\perp} \subseteq H^*(S, \mathbb{Z}) \xrightarrow{\sim} H^2(X, \mathbb{Z})$$

$$L := \theta((0,0,-1)) \quad \text{nef}$$

$$M := \theta((1,0,0))$$

s.t. $\displaystyle\int_X c_1(L)^{2g} = 0$

$$\int_X c_1(L)^g c_1(M)^g = g!$$

$\rightsquigarrow$ $f = \varphi_{|L|} : X \longrightarrow \mathbb{P}^g$ Lagr. fibr.

w/ $f^{-1}(p) = J^{g-1} C_p$, $C_p \in |H|$ smooth

and $M|_{f^{-1}(p)}$ princ. polariz.

Rmk $H = \theta((1,0,1)) = M \otimes L^{-1}$ effective

($\rightsquigarrow$ M not nef)

$\triangle$

The Main Thm

$n=2$

X HK fourfold

Assume: $\exists\ \ell, m \in H^2(X,\mathbb{Z})$ st.

- $\int_X \ell^4 = 0$
- $\int_X \ell^2 \cdot m^2 = 2$

Conj. by O'Grady

Thm Either X is of $K3^{[2]}$-type

(or $\exists$ defo of X w/ "special geometry")

work in progress by Voisin
this does NOT happen

$\boxtimes$

Idea of Pf

St. 1 We can deform X to projective one st.

$$\ell = c_1(L)$$
$$m = c_1(M)$$
L, M line bdles.

and can assume $\int_X m^4 = 0$

and $NS(X) = \mathbb{Z}\cdot L \oplus \mathbb{Z}\cdot M$

Thm A The Huybrechts-Riemann-Roch poly of X is:

$$P_X(2k) := \chi(X, L^a \otimes M^b) \qquad k = ab$$

$$= \chi(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(k+1)) \qquad \forall a, b \in \mathbb{Z}.$$

$$= \frac{(k+2)(k+3)}{2} \qquad \boxtimes$$

St. 2 : SYZ Conj

Assume further that L nef

Thm B either $\exists$ Lagr. fibration

$$f : X \to \mathbb{P}^2, \quad f^*\mathcal{O}_{\mathbb{P}^2}(1) \cong L$$

$$(\to f = \varphi_{|L|})$$

or X "special geometry"

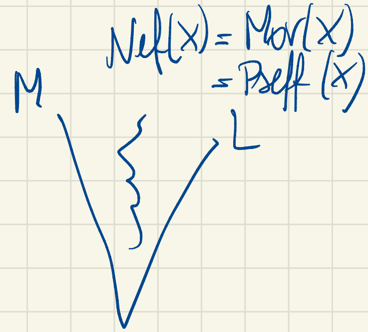
Explanations

Thm B vs Main Thm :

L, M , L nef

Can assume $M \in \mathrm{Pseff}(X)$

There are 2 possibilities :



(I) both L and M $\underline{\underline{\text{nef}}}$

$\rightsquigarrow$ $L \otimes M$ $\underline{\underline{\text{ample}}}$

Then:

(I.1) after possibly permuting L and M,

$h^0(L), h^0(M) \neq 0$ and the lin. syst. $|L|$

induces a Lagr. fibr

$$f = \varphi_{|L|} : X \to \mathbb{P}^2 \quad \underline{\underline{\text{flat}}}$$

$$f^*\mathcal{O}_{\mathbb{P}^2}(1) \simeq L$$

$\rightsquigarrow$ f_*M line bdle on $\mathbb{P}^2$

$\underset{\text{Thm A}}{\rightsquigarrow}$ $f_* M = \mathcal{O}_{\mathbb{P}^2}(1)$

$\rightsquigarrow$ $H^0(X, M \otimes L^\vee) \simeq H^0(\mathbb{P}^2, \mathcal{O}) \simeq \mathbb{C}$

$\rightsquigarrow$ M cannot be nef $\quad \underline{\underline{\lightning}}$

(I.2) on the image of rat'l map

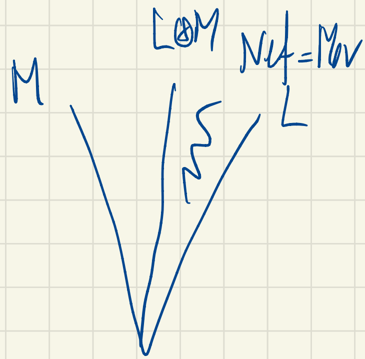
$$\varphi_{|L\otimes M|} : X \dashrightarrow \mathbb{P}^5$$

is <u>rationally connected</u>

"special geometry"

(and $h^0(L)$ or $h^0(M) = 0$
and $L\otimes M$ ample)

$\rightsquigarrow$ Case (I) does NOT arise.
(Voisin)



(II) [Markman] $\exists$ div. contr.

$$\varphi : X \to \bar{X} \quad \text{s.t.}$$

- $E = \mathrm{Exc}(\varphi)$ integral divisor
  w/ $[E] = m - l$

- $\varphi = \varphi_{|k\cdot(L\otimes M)|}$ $\quad k \gg 0$.

[Wierzba + Prop] $\varphi_E : E \to \varphi(E)$

$\mathbb{P}^1$-bdle over $S := \varphi(E)$ K3 surface

v. gen. of genus 2.

$\rightsquigarrow$ prove Thm B

$$\begin{tikzcd} & X \arrow[d, "f"] & \text{Lagr.} \\ E \arrow[ur, hook] \arrow[r, "\psi_E"'] \arrow[d, "\mathbb{P}^1"', "\varphi_E"] & \mathbb{P}^2 & \\ S & & \end{tikzcd}$$

$\rightsquigarrow$ Mukai th. of v. bdles on K3

$$\begin{tikzcd} E \cong H \arrow[r] \arrow[d] & \mathbb{P}^2 \\ S & \end{tikzcd}$$

$\rightsquigarrow$ $X \cong M_S(0, H, 0)$.