Weight of tautological classes of Character Varieties.

(d'après V. Shende)

Σ_g : (punctured) surface of genus g.

$$\begin{tikzcd} \{\pi_1 \longrightarrow GL_n\} = GL_n^{2g} \arrow[d] & \mathcal{U}_n \arrow[l] \arrow[d] \\ SL_n & \{\zeta_n\} \arrow[l] \end{tikzcd}$$

Monodromy around puncture

fixed root of unity.

$\mathcal{M}_n = \mathcal{U}_n / PGL_n$ " Twisted char. variety "

$\tilde{\mathcal{M}}_n = \mathcal{M}_n / \mathbb{G}_m^{2g}$

Thm (Hausel - Rodrigez-Villegas)

$$H^\bullet(\mathcal{M}_n ; \mathbb{Q}) \simeq H^\bullet(\tilde{\mathcal{M}}_n ; \mathbb{Q}) \otimes H^\bullet(\mathbb{G}_m^{2g} ; \mathbb{Q}).$$

Thm (Markman)

$\hookrightarrow$ generated by tautological classes $\alpha_k \in H^{2k-2}$; $\beta_k \in H^{2k}$.

$\phi_{kj} \in H^{2k-1}$; $k = 2, \cdots, n$

$j = 1, \cdots, 2g$.

Goal: $\alpha_k, \beta_k, \psi_{kj}$ is weight filtration.

Thm: (H-RV; Shende).

Denote ${}^m\mathrm{Hdg}^k = F^k H^m \cap \bar{F}^k H^m \cap W_{2k} H^m$. then:

$$\underbrace{\alpha_k \in {}^{2k-2}\mathrm{Hdg}^k}_{\text{Shende.}} ; \quad \underbrace{\beta_k \in {}^{2k}\mathrm{Hdg}^k ; \quad \phi_{kj} \in {}^{2k-1}\mathrm{Hdg}^k}_{\text{H-RV.}}$$

Problem: The universal bundle is NOT algebraic.

Think of $\widetilde{\mathcal{M}}_n$

$$GL_n^{2g} \longleftarrow \mathcal{U}_n \circlearrowleft PGL_n \times \mathbb{G}_m^{2g}$$

$$GL_n^{2g} / \mathbb{G}_m^{2n} \times PGL_n = PGL_n^{2g} / PGL_n = \{\pi_1 \longrightarrow PGL_n\} / PGL_n$$

DM-stack $\hookrightarrow$ $\widetilde{\mathcal{M}}_n \longrightarrow Loc_\Sigma (PGL_n)$ $\hookleftarrow$ Artin stack.

$$Loc_\Sigma(PGL_n) \times \Sigma \xrightarrow[\text{universal bundle.}]{} BPGL_n$$

Simplicial sheaves

Eg.: • G: group scheme $/\mathbb{C}$

$$BG = * \rightleftarrows G \;\substack{\leftarrow\\\rightarrow\\\leftarrow\\\rightarrow}\; G \times G \cdots \quad \in sSch$$

(simplicial Scheme)

• $\Sigma_B \rightsquigarrow$ simplicial set

• using a triangulation
• Using a (finite) Čech cover $(\mathcal{U}_\alpha)$

$$\Delta_n^\Sigma = \{(\alpha_0, \cdots, \alpha_n) \mid \mathcal{U}_{\alpha_0} \cap \cdots \cap \mathcal{U}_{\alpha_n} \neq \emptyset\}.$$
$$\alpha_0 \leq \cdots \leq \alpha_n$$

$$\Sigma = |\Delta_\bullet^\Sigma| ; \quad \Delta_\bullet^\Sigma = \Delta_0^\Sigma \rightleftarrows \Delta_1^\Sigma \cdots$$
$$\Delta_0^\Sigma = I, \quad \Delta_1^\Sigma \subset I^2$$

$$E \in Set \Rightarrow E^{Sch} = \coprod_{e \in E} \operatorname{Spec} \mathbb{C}. \rightsquigarrow \Delta_\bullet^\Sigma \in sSch$$

• $Loc_\Sigma(PGL_n)_\bullet :=$ internal Hom from $\Delta_\bullet^\Sigma$ to $BPGL_n$ in $sSch$.

$$Loc_\Sigma(G)_\cdot : [n] \longmapsto Hom_{sSch}(\Delta^n \times \Delta_\bullet^\Sigma, BG)$$

$$\Delta^n : [p] \longmapsto Hom_\Delta([p], [n])$$

$$
\begin{tikzcd}
Loc_\Sigma(G)_0 =: Loc_0 : & \{U_\alpha \cap U_\beta \cap U_\gamma \neq \emptyset\} \arrow[r] \arrow[d] \arrow[d, shift left=2] \arrow[d, shift right=2] & G \times G \arrow[d] \arrow[d, shift left=2] \arrow[d, shift right=2] & \text{cocycle condition.} \\
& \{U_\alpha \cap U_\beta \neq \emptyset\} \arrow[r] \arrow[d, shift left=1] \arrow[d, shift right=1] & G \arrow[d, shift left=1] \arrow[d, shift right=1] & \text{change of charts} \\
& \{U_\alpha\} \arrow[r] & * & \\
& & \text{trivial } G\text{-bdle on } U_\alpha. &
\end{tikzcd}
$$

$\rightsquigarrow$ $Loc_0 = \{$principal $G(\mathbb{C})$-bundle w/ a trivializat° along Čech cover$\}$

$$
\begin{tikzcd}
Loc_0 \arrow[r, shift left=2, leftarrow] \arrow[r] \arrow[r, shift right=2, leftarrow] & Loc_1 \arrow[r, shift left=2, leftarrow] \arrow[r, leftarrow] \arrow[r, shift right=2, leftarrow] & \cdots
\end{tikzcd}
$$

$\rightsquigarrow$

$$\{\{U_\alpha\} \to G = BG_1\} \overset{\text{Yoneda.}}{=\!=\!=} \{\{U_\alpha\} \times \Delta^1 \to BG\}.$$

$$Loc_\Sigma(G)_\bullet \times \Delta^\Sigma_\bullet \xrightarrow{ev} BG \quad \text{map of simplicial sheaves.}$$

Thm (Deligne [Hodge III])

- There is a Hodge theory of simplicial schemes.
- $H^\bullet(BG; \mathbb{Q})$ is pure. $\quad G$: linear reductive group.

(Dold-Kan) $\quad$ Also holds for PGL_n.

Lem: $H^\bullet(\Delta^\Sigma_\bullet, \mathbb{Q})$ is entirely of weight 0

$$H^\bullet(\Delta^\Sigma_\bullet, \mathbb{Q}) = H^\bullet(\Sigma, \mathbb{Q}).$$

$$H^\bullet(BG) \xrightarrow{ev^*} H^\bullet(Loc_\Sigma(G)_\bullet) \otimes H^\bullet(\Delta^\Sigma_\bullet)$$

c_k

Let $H^d(\Sigma, \mathbb{Q}) \underset{\tau}{\to} \mathbb{Q}$. $\quad H^1(\Sigma, \mathbb{Q}) = Vect(\tau_1, \cdots, \tau_{2g})$

$$\alpha_k = \int_\Sigma ev^* c_k \qquad \beta_k = \int_* ev^* c_k. \qquad \psi_{kj} = \int_{\tau_j} ev^* c_k.$$

Because $H^\bullet(\Sigma)$ is entirely of weight 0.

$\alpha_k, \beta_k,$ and ϕ_{kj} have weight $\geq k$ (= weight of C_k)

<u>Cor</u>: $\bigoplus_{m,k} {}^m\mathrm{Hdg}^k = H^\bullet(\check{\mathcal{M}}_n) \twoheadleftarrow \mathbb{Q}[\alpha_k, \beta_k, \phi_{k,j}]$