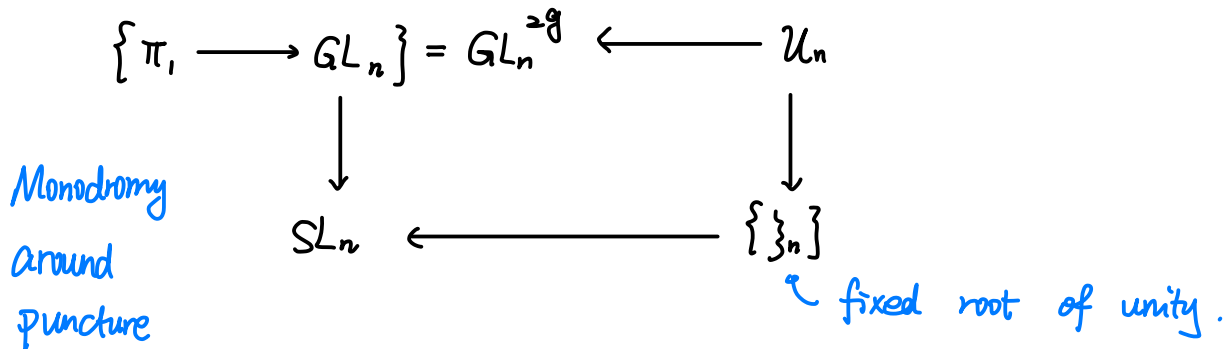


Weight of tautological classes of

Character Varieties.

(d'après V. Shende)

Σ_g : (punctured) surface of genus g .



$$\mathcal{M}_n = \mathcal{U}_n / PGL_n \quad \text{"Twisted char. variety"}$$

$$\tilde{\mathcal{M}}_n = \mathcal{M}_n / \mathbb{G}_m^{2g}$$

Thm (Hausel - Rodriguez-Villegas)

$$H^*(\mathcal{M}_n; \mathbb{Q}) \cong H^*(\tilde{\mathcal{M}}_n; \mathbb{Q}) \otimes H^*(\mathbb{G}_m^{2g}; \mathbb{Q}).$$

Thm (Martman)

generated by tautological classes $\alpha_k \in H^{2k-2}$; $\beta_k \in H^{2k}$.
 $\phi_{kj} \in H^{2k-1}$; $k = 2, \dots, n$
 $j = 1, \dots, 2g$.

Goal: $\alpha_k, \beta_k, \psi_{kj}$ is weight filtration.

Thm: (H-RV; Shende).

Denote ${}^m \text{Hdg}^k = F^k H^m \cap \bar{F}^k H^m \cap W_{2k} H^m$. then:

$$\alpha_k \in {}^{2k-2} \text{Hdg}^k; \quad \beta_k \in {}^{2k} \text{Hdg}^k; \quad \phi_{kj} \in {}^{2k-1} \text{Hdg}^k$$

Shende.

H-RV.

Problem: The universal bundle is NOT algebraic.

Think of $\tilde{\mathcal{M}}_n$

$$GL_n^{2g} \longleftarrow \mathcal{U}_n \curvearrowright PGL_n \times G_m^{2g}$$

$$GL_n^{2g} / G_m^{2n} \times PGL_n = PGL_n^{2g} / PGL_n = \{ \pi_1 \rightarrow PGL_n \} / PGL_n$$

$$\text{DM-stack } \tilde{\mathcal{M}}_n \longrightarrow \text{Loc}_\Sigma(PGL_n) \quad \text{Artin stack.}$$

$$\text{Loc}_\Sigma(PGL_n) \times \Sigma \xrightarrow{\text{universal bundle.}} BPGL_n$$

Simplicial sheaves

Ex: G : group scheme / \mathbb{C}

$$BG = * \begin{matrix} \rightleftarrows \\ \rightleftarrows \\ \rightleftarrows \end{matrix} G \begin{matrix} \rightleftarrows \\ \rightleftarrows \\ \rightleftarrows \end{matrix} G \times G \dots$$

(simplicial scheme) $\in \text{sSch}$

- $\Sigma_B \rightsquigarrow$ simplicial set
- using a triangulation (finite)
- using a Čech cover (U_α)

$$\Delta_n^\Sigma = \left\{ (\alpha_0, \dots, \alpha_n) \mid U_{\alpha_0} \cap \dots \cap U_{\alpha_n} \neq \emptyset, \alpha_0 \leq \dots \leq \alpha_n \right\}$$

$$\Sigma = |\Delta_\bullet^\Sigma|, \quad \Delta_\bullet^\Sigma = \Delta_0^\Sigma \begin{matrix} \longleftarrow \\ \rightleftarrows \\ \longrightarrow \end{matrix} \Delta_1^\Sigma \begin{matrix} \longleftarrow \\ \rightleftarrows \\ \longrightarrow \end{matrix} \dots$$

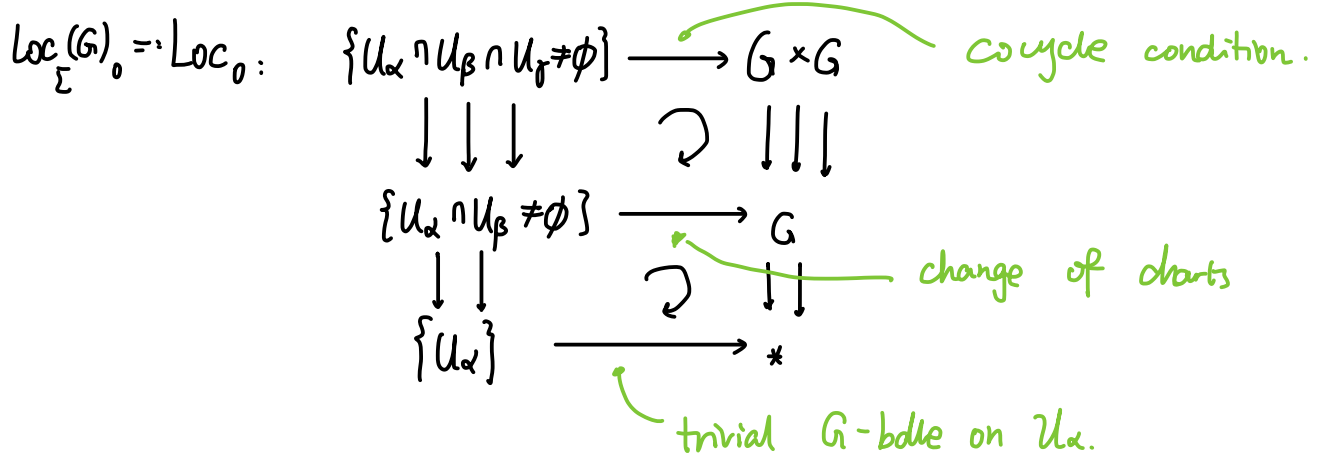
$\begin{matrix} \text{I} & & \text{I}^2 \end{matrix}$

$$E \in \text{Set} \Rightarrow E^{\text{Sch}} = \coprod_{e \in E} \text{Spec } \mathbb{C}. \rightsquigarrow \Delta_\bullet^\Sigma \in \text{sSch}$$

$\text{Loc}_\Sigma(PGL_n) :=$ internal Hom from Δ_\bullet^Σ to $BPGL_n$ in sSch .

$$\text{Loc}_\Sigma(G): [n] \longmapsto \text{Hom}_{\text{sSch}}(\Delta^n \times \Delta_\bullet^\Sigma, BG)$$

$$\Delta^n: [p] \longmapsto \text{Hom}_\Delta([p], [n])$$



$\hookrightarrow \text{Loc}_0 = \{ \text{principal } G(\mathbb{C})\text{-bundle w/ a trivializat}^\circ \text{ along Čech cover} \}$

$\text{Loc}_0 \begin{matrix} \longleftarrow \\ \rightleftarrows \\ \longleftarrow \end{matrix} \text{Loc}_1 \begin{matrix} \longleftarrow \\ \rightleftarrows \\ \longleftarrow \end{matrix} \dots$

\downarrow
 $\{ \{U_\alpha\} \rightarrow G = BG_1 \} \stackrel{\text{Yoneda}}{=} \{ \{U_\alpha\} \times \Delta^1 \rightarrow BG \}$

$\text{Loc}_\Sigma(G)_0 \times \Delta_\Sigma \xrightarrow{\text{ev}} BG$ *map of simplicial sheaves.*

Thm (Deligne [Hodge III])

- There is a Hodge theory of simplicial schemes.
- $H^*(BG, \mathbb{Q})$ is pure. G : linear reductive group.
(Dold-Kan) Also holds for PGL_n .

Lem: $H^*(\Delta_\Sigma, \mathbb{Q})$ is entirely of weight 0

$H^*(\Delta_\Sigma, \mathbb{Q}) = H^*(\Sigma, \mathbb{Q})$

$H^*(BG) \xrightarrow{\text{ev}^*} H^*(\text{Loc}_\Sigma(G)_0) \otimes H^*(\Delta_\Sigma)$

C_k

Let $H^d(\Sigma, \mathbb{Q}) \xrightarrow{\tau} \mathbb{Q}$. $H^1(\Sigma, \mathbb{Q}) = \text{Vect}(\tau_1, \dots, \tau_{2g})$

$\alpha_k = \int_\Sigma \text{ev}^* C_k \quad \beta_k = \int_* \text{ev}^* C_k \quad \psi_{kj} = \int_{\tau_j} \text{ev}^* C_k$

Because $H^*(\Sigma)$ is entirely of weight 0.

$\alpha_k, \beta_k,$ and $\phi_{k,j}$ have weight $\geq k$ (= weight of C_k)

Cor.: $\bigoplus_{m,k}^m \text{Hdg}^k = H^*(\tilde{M}_n) \longleftarrow \mathbb{Q}[\alpha_k, \beta_k, \phi_{k,j}]$