Antisymplectic involutions on projective hyper-Kähler manifolds

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(joint work with Laure Flapan, Kieran G. O'Grady, Giulia Saccà)

In this talk, I reported on the study of fixed loci of antisymplectic involutions on projective hyper-Kähler manifolds, induced by an ample class of square 2 in the Beauville-Bogomolov-Fujiki lattice. I presented results on how to determine the number of connected components of the fixed loci and how to study their geometry in lower dimensions.

Let X be a projective hyper-Kähler (HK) manifold, namely X is simply connected and $H^0(X, \Omega_X^2) = \mathbb{C} \cdot \eta$, where η is a non-degenerate symplectic form.

Definition 1. Let $\tau\colon X \xrightarrow{\cong} X$ be an involution, $\tau^2 = \mathrm{id}$. We say that τ is *antisymplectic* if $\tau^*\eta = -\eta$.

An immediate observation is that if τ is an antisymplectic involution, then its fixed locus $\mathrm{Fix}(\tau) \subset X$ is a closed lagrangian submanifold.

The goal is to understand the geometry of $\mathrm{Fix}(\tau)$; see [1, 14]. The motivation comes from several viewpoint in the theory of HK manifolds, including understanding the correspondence with Fano manifolds (currently only observed in special examples [2, 4, 5, 6]) and in the existence of covering families of lagrangian submanifolds and applications to the study of Chow groups [16]. The rich geometry of these fixed loci can be already observed in the lower dimensional case; for example, EPW sextics [15] and cubic fourfolds [12].

Notice also that for *symplectic* involutions, namely if $\tau^*\eta = \eta$, the fixed loci are well understood for two of the main families of examples of HK manifolds [11]: their connected components are symplectic submanifolds in that case.

Let (X, λ) be a polarized hyper-Kähler manifold of dimension $2n$. We assume that X is of $\mathrm{K3}^{[n]}$*-type*, namely it is deformation equivalent to the Hilbert scheme of n points on a K3 surface.

Let q_X denote the Beauville-Bogomolov-Fujiki quadratic form on $H^2(X;\mathbb{Z})$. We assume that the polarization λ satisfies $q_X(\lambda) = 2$. If we denote by $\mathrm{div}(\lambda)$ the positive generator of the ideal $\{q(\lambda, w) \,:\, w \in H^2(X;\mathbb{Z})\} \subset \mathbb{Z}$, the *divisibility* of λ, then we must have $\mathrm{div}(\lambda) \in \{1, 2\}$; moreover, if $\mathrm{div}(\lambda) = 2$, then $4 \mid n$.

By the Global Torelli Theorem [17, 13, 10], to such polarization λ we can associate an antisymplectic involution

$$\tau_\lambda\colon X \xrightarrow{\cong} X$$

which acts on $H^2(X;\mathbb{Z})$ as reflection at λ:

$$\tau_{\lambda,*}(x) = -x + q_X(\lambda, x), \qquad x \in H^2(X;\mathbb{Z})).$$

Equivalently, we are looking at involutions τ for which the invariant part of the action on $H^2(X;\mathbb{Z})$ is of rank 1, generated by an ample class of square 2.

The main result in [8] determines the number of connected components of $\mathrm{Fix}(\tau_\lambda)$:

Theorem 2. *The fixed locus* $\mathrm{Fix}(\tau_\lambda)$ *has exactly* $\mathrm{div}(\lambda)$ *connected components.*

We can then start looking at the geometry of such fixed loci in lower dimension. We start with the divisibility 1 case; by Theorem 2 the fixed locus $F := \mathrm{Fix}(\tau_\lambda)$ is connected in this case. The case $n = 2$ is now well-known: the general (X, λ) in the moduli space is a double EPW sextic, with the double cover involution coinciding with the involution τ_λ. Then F is a surface of general type, whose invariants are all known; see [7]. In the cases $n = 3$ and $n = 4$ we do expect a similar behavior: the fixed locus F should be of general-type with an explicit formula for its canonical bundle in terms of $\lambda|_F$.

In the divisibility 2 case, again by Theorem 2 the fixed locus $\mathrm{Fix}(\tau_\lambda)$ has exactly two connected components. The first case $n = 4$ is already not completely clear: all (X, λ) in the moduli spaces are isomorphic to the Lehn-Lehn-Sorger-van Straten HK 8-fold associated to a cubic fourfold (not containing a plane), with the involution coinciding with the involution coming from realizing X as moduli space of equivalence classes of twisted cubic curves in the cubic Y; see [12] and [3, Appendix B]. One component to the fixed locus is then isomorphic to the cubic fourfold Y itself. The second component is the closure of the locus parameterizing twisted cubics contained in a cubic surface with four A_1-singularities, but the global geometry of this component is still unknown (although we suspect it being of general type).

The main result in [9] deals with the next case $n = 8$.

Theorem 3. *Let* $n = 8$ *and let* (X, λ) *be a polarized HK manifold of* $\mathrm{K3}^{[8]}$*-type such that* $q_X(\lambda) = 2$ *and* $\mathrm{div}(\lambda) = 2$*. Then one connected component* Y *of* $\mathrm{Fix}(\tau_\lambda)$ *is a prime Fano manifold of dimension* 8 *and index 3.*

The odd cohomology of Y *vanishes and its Hodge diamond is*

$H^8(Y;\mathbb{C})$:	1	22	253	22	1
$H^6(Y;\mathbb{C})$:		1	22	1	
$H^4(Y;\mathbb{C})$:		1	22	1	
$H^2(Y;\mathbb{C})$:			1		
$H^0(Y;\mathbb{C})$:			1		

Some of the arguments in our proofs work for any n. In divisibility 2, we can always isolate a special component Y, by using the choice of a linearization of the action of the involution on the line bundle $\mathcal{O}_X(\lambda)$. Theorem 3 would then hold in any dimension, if we would be able to establish normality of a certain degeneration of the fixed component Y.

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